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EXAMPLES AND PROBLEMS ON CONICS
AND SOME OF THE HIGHER PLANE CURVES.

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A COLLECTION OF

EXAMPLES AND PROBLEMS ON CONICS

AND SOME OF THE

HIGHER PLANE CURVES.

BY

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PREFACE.

THE greater part of the Examples on Conics and Cubics was worked out by me during my study of Dr. Salmon's Treatises on *Conics* and the *Higher Plane Curves*. Several of those on Bicircular Quartics were suggested by Dr. Casey's Memoir* on those Curves, and Darboux's *Sur une classe remarquable de Courbes et de Surfaces Algebriques*.† I believe that either the Examples themselves or the methods adopted for their solution are original.

This Volume is addressed to those who take an interest in properties of Plane Curves which are not of a purely elementary character, and, as the results obtained are new, I am in hopes that it may prove acceptable. I have throughout

* *Transactions of the Royal Irish Academy*, vol. XXIV. p. 457, 1869.

† Paris, 1873.

assumed the reader to be familiar with Dr. Salmon's *Conics* and *Curves*; and, in fact, these Examples may be considered as an addition to those in his works. For this reason I have used little explanatory matter, and rendered the proofs of the Examples as brief as possible. Most of the problems can be solved by methods which have been employed in the proofs of Examples, and, in the case of the more difficult, solutions will be found at the end of the book.

Several of the Examples have appeared from time to time in the *Educational Times*, and some of the Theorems on Nodal Cubics formed a paper in the *Proceedings of the London Mathematical Society* (vol. XII. p. 99).

In conclusion, I have to thank the Board of Trinity College for their liberality in contributing to the expense of publication.

TRINITY COLLEGE,
April, 1882.

CONTENTS.

ERRATA.

Page 11 line 2, *for* centre *read* points of contact.

" 19 line 7, *for* $\left(\frac{x-x'}{a^2}\right)^2$ and $\left(\frac{y-y'}{b^2}\right)^2$ *read* $\frac{(x-x')^2}{a^2}$ and $\frac{(y-y')^2}{b^2}$.

" 52 bottom line *for* a^2-c^2, *read* a^2+c^2.

" 79 bottom line *for* e^2 *read* e^4.

" 81 Ex. 194 *for* similar *read* similarly situated.

EXAMPLES AND PROBLEMS ON CONICS, AND SOME OF THE HIGHER PLANE CURVES.

I. *Examples and Problems on Conics.*

1. *To find the equation of the circle circumscribing a triangle inscribed in a conic.*

Let the equation of the conic be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\equiv U=0,$$

and that of the circle

$$(x-x')^2+(y-y')^2-r^2\equiv S=0.$$

Then the discriminant of $S-h^2U$ is found to be

$$\frac{x'^2}{a^2-h^2}+\frac{y'^2}{b^2-h^2}+\frac{r^2}{h^2}-1=0,$$

or $$h^6-h^4(a^2+b^2+r^2-x'^2-y'^2)$$
$$+h^2\{a^2b^2+r^2(a^2+b^2)-b^2x'^2+a^2y'^2\}-a^2b^2r^2=0.$$

But comparing the coefficients of the identity

$$S-h^2U=k\left\{\frac{x}{a}\cos\tfrac{1}{2}(\alpha+\beta)+\frac{y}{b}\sin\tfrac{1}{2}(\alpha+\beta)-\cos\tfrac{1}{2}(\alpha-\beta)\right\}$$
$$\times\left\{\frac{x}{a}\cos\tfrac{1}{2}(\gamma+\delta)+\frac{y}{b}\sin\tfrac{1}{2}(\gamma+\delta)-\cos\tfrac{1}{2}(\gamma-\delta)\right\},$$

(Salmon's *Conics*, p. 208), we get

$$h^2 = a^2 \sin^2 \tfrac{1}{2} (\alpha + \beta) + b^2 \cos^2 \tfrac{1}{2} (\alpha + \beta)$$

= the square of the semi-diameter parallel to one of the sides of the triangle.

Hence, from the absolute term of the equation whose roots are $a^2 - h_1^2$, $a^2 - h_2^2$, $a^2 - h_3^2$, we obtain

$$x' = \frac{\sqrt{\{(a^2 - h_1^2)(a^2 - h_2^2)(a^2 - h_3^2)\}}}{a\sqrt{(a^2 - b^2)}}.$$

Similarly we have

$$y' = -\frac{\sqrt{\{(h_1^2 - b^2)(h_2^2 - b^2)(h_3^2 - b^2)\}}}{b\sqrt{(a^2 - b^2)}}.$$

Again, from the equation in h^2 we obtain $r = \frac{h_1 h_2 h_3}{ab}$, and

$$x'^2 + y'^2 - r^2 = a^2 + b^2 - h_1^2 - h_2^2 - h_3^2.$$

We may also find the coordinates of the centre thus: eliminating y between the equations of the conic and circle, we get

$$(a^2 - b^2)\, x^4 - 4a^2 x' x^3 + \&c. = 0,$$

hence $$x' = \frac{a^2 - b^2}{4a^2}(x_1 + x_2 + x_3 + x_4)$$

$$= \frac{a^2 - b^2}{4a}\{\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha + \beta + \gamma)\}$$

(Salmon's *Conics*, p. 218)

$$= \frac{a^2 - b^2}{a}\cos\tfrac{1}{2}(\alpha + \beta)\cos\tfrac{1}{2}(\beta + \gamma)\cos\tfrac{1}{2}(\gamma + \alpha).$$

In the same way we find

$$y = \frac{b^2 - a^2}{b}\sin\tfrac{1}{2}(\alpha + \beta)\sin\tfrac{1}{2}(\beta + \gamma)\sin\tfrac{1}{2}(\gamma + \alpha).$$

2. *To find the equation of the circle through the middle points of the sides of the same triangle.*

Let the equation of the circle be

$$(x-x'')^2+(y-y'')^2-r'^2=0,$$

and let x', y', be the coordinates of the centre of the circumscribing circle, and α, β those of the centroid; then, by a known geometrical relation,

$$2x''=3\alpha-x', \quad 2y''=3\beta-y',$$

and

$$3\alpha=a(\cos\alpha+\cos\beta+\cos\gamma), \quad 3\beta=b(\sin\alpha+\sin\beta+\sin\gamma);$$

hence,

$$2ax''=a^2(\cos\alpha+\cos\beta+\cos\gamma) \\ -(a^2-b^2)\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\gamma+\alpha),$$

$$2by''=b^2(\sin\alpha+\sin\beta+\sin\gamma) \\ +(a^2-b^2)\sin\tfrac{1}{2}(a+\beta)\sin\tfrac{1}{2}(\beta+\gamma)\sin\tfrac{1}{2}(\gamma+\alpha).$$

Again,

$$\begin{aligned} x''^2+y''^2-r'^2 &= \tfrac{1}{4}\{(3\alpha-x')^2+(3\beta-y')^2-4r'^2\} \\ &= \tfrac{1}{4}\{9(\alpha^2+\beta^2)-6(\alpha x'+\beta y')+x'^2+y'^2-r^2)\} \\ &= (a^2+b^2)\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha). \end{aligned}$$

3. *To find the locus of the centroid of an equilateral triangle inscribed in a conic.*

Equating the coordinates of the centroid and the centre of the circumscribing circle, we get, if

$$\alpha+\beta+\gamma=-\delta, \quad x=\frac{a(a^2-b^2)}{a^2+3b^2}\cos\delta, \quad y=-\frac{b(a^2-b^2)}{b^2+3a^2}\sin\delta.$$

Hence the locus is the conic

$$\frac{x^2}{a^2}(a^2+3b^2)^2+\frac{y^2}{b^2}(b^2+3a^2)^2=(a^2-b^2)^2.$$

4. If the centroïd of a triangle inscribed in a hyperbola is on the curve, prove that (1) the circle through the middle points of the sides cuts the director circle orthogonally, (2) the area of the triangle formed by the tangents to the curve at the vertices of the given triangle is equal to half the area of the given triangle, (3) the ellipse touching the sides of the triangle at their middle points passes through the centre of the curve.

(Salmon's *Conics*, p. 257, Ex. 3).

5. *To find the equation of the circle circumscribing a triangle formed by three tangents to a conic.*

The equation of the circle is

$$
\begin{aligned}
&(a^2 \sin^2\alpha + b^2 \cos^2\alpha) \sin(\beta-\gamma) \\
&\qquad \times \left(\frac{x}{a}\cos\beta + \frac{y}{b}\sin\beta - 1\right)\left(\frac{x}{a}\cos\gamma + \frac{y}{b}\sin\gamma - 1\right) \\
&+ (a^2 \sin^2\beta + b^2 \cos^2\beta) \sin(\gamma-\alpha) \\
&\qquad \times \left(\frac{x}{a}\cos\gamma + \frac{y}{b}\sin\gamma - 1\right)\left(\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha - 1\right) \\
&+ (a^2 \sin^2\gamma + b^2 \cos^2\gamma) \sin(\alpha-\beta) \\
&\qquad \times \left(\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha - 1\right)\left(\frac{x}{a}\cos\beta + \frac{y}{b}\sin\beta - 1\right) = 0,
\end{aligned}
$$

or, multiplying out, and reducing,

$$
\begin{aligned}
x^2 + y^2 &- \frac{x}{a}\{M(\cos\alpha + \cos\beta + \cos\gamma) + (a^2 - b^2)\cos(\alpha+\beta+\gamma)\} \\
&- \frac{y}{b}\{M(\sin\alpha + \sin\beta + \sin\gamma) - (a^2 - b^2)\sin(\alpha+\beta+\gamma)\} + M = 0,
\end{aligned}
$$

where

$$
M = \frac{a^2 + b^2 + (a^2 - b^2)\{\cos(\beta+\gamma) + \cos(\gamma+\alpha) + \cos(\alpha+\beta)\}}{4\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha)\cos\tfrac{1}{2}(\alpha-\beta)}.
$$

If A, B, C be the angles of the triangle, p_1, p_2, p_3 the perpendiculars from the centre on the sides, Δ the area of the triangle, and R the radius of the circumscribing circle,

we have $$\frac{\Delta}{R} = p_1 \sin A + p_2 \sin B + p_3 \sin C,$$

but $$p_1 = \frac{ab}{\sqrt{(a^2 \sin^2\alpha + b^2 \cos^2\alpha)}},$$

and $$\sin A = \frac{ab \sin(\beta - \gamma)}{\sqrt{(a^2 \sin^2\beta + b^2 \cos^2\beta)(a^2 \sin^2\gamma + b^2 \cos^2\gamma)}};$$

also $$\Delta = ab \tan \tfrac{1}{2}(\beta - \gamma) \tan \tfrac{1}{2}(\gamma - \alpha) \tan \tfrac{1}{2}(\alpha - \beta),$$

(Salmon's *Conics*, p. 209, Ex. 9).

Therefore

$$R = \frac{\sqrt{\{(a^2 \sin^2\alpha + b^2 \cos^2\alpha)(a^2 \sin^2\beta + b^2 \cos^2\beta)(a^2 \sin^2\gamma + b^2 \cos^2\gamma)\}}}{4ab \cos \frac{1}{2}(\beta - \gamma) \cos \frac{1}{2}(\gamma - \alpha) \cos \frac{1}{2}(\alpha - \beta)}.$$

If this circle pass through the centre of the conic, the locus of its centre is the conic

$$4(a^2x^2 + b^2y^2) = (a^2 - b^2)^2.$$

6. *To find the equation of the polar circle of the same triangle.*

We find

$$x^2 + y^2$$

$$-\frac{x}{a}\left\{\frac{a^2\{2\cos\frac{1}{2}(\beta+\gamma)\cos\frac{1}{2}(\gamma+\alpha)\cos\frac{1}{2}(\alpha+\beta) - \cos(\alpha+\beta+\gamma)\} + (a^2+b^2)\cos\alpha\cos\beta\cos\gamma}{\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\gamma-\alpha)\cos\frac{1}{2}(\alpha-\beta)}\right\}$$

$$-\frac{y}{b}\left\{\frac{b^2\{2\sin\frac{1}{2}(\beta+\gamma)\sin\frac{1}{2}(\gamma+\alpha)\sin\frac{1}{2}(\alpha+\beta) + \sin(\alpha+\beta+\gamma)\} + (a^2+b^2)\sin\alpha\sin\beta\sin\gamma}{\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\gamma-\alpha)\cos\frac{1}{2}(\alpha-\beta)}\right\}$$

$$+ a^2 + b^2 = 0.$$

If r be the radius of this circle, and t_1, t_2, t_3 the lengths of the tangents drawn from the vertices of the triangle to the director circle,

$$r^2 = -\frac{t_1^{\,2} t_2^{\,2} t_3^{\,2}}{4a^2b^2};$$

for
$$r^2 = -2\Delta \cot A \cot B \cot C,$$

and
$$\cot A = \frac{t_1^{\,2}}{2ab \tan \frac{1}{2}(\beta - \gamma)},$$

(Salmon's *Conics*, p. 161).

For the parabola $y^2 - 4mx = 0$,

$$r^2 = -\frac{p_1 p_2 p_3}{m},$$

where p_1, p_2, p_3 are the perpendiculars from the vertices of the triangle on the directrix.

7. *To find the equations of the circles touching the sides of a triangle formed by three tangents to a conic.*

Let the tangential equation of the conic be

$$a^2\lambda^2 + b^2\mu^2 - 1 \equiv V = 0,$$

and that of the circle

$$(x'\lambda + y'\mu - 1)^2 - r^2(\lambda^2 + \mu^2) \equiv \Sigma = 0.$$

Then the discriminant of $V + \frac{h^2}{r^2}\Sigma$ is found to be

$$\frac{x'^2}{a^2 - h^2} + \frac{y'^2}{b^2 - h^2} + \frac{r^2}{h^2} - 1 = 0.$$

But if $V + \frac{h^2}{r^2}\Sigma$ represent two points, they evidently lie on the conic $(a^2 - h^2)\lambda^2 + (b^2 - h^2)\mu^2 - 1 = 0$, or

$$\frac{x^2}{a^2 - h^2} + \frac{y^2}{b^2 - h^2} - 1 = 0,$$

hence, $h^2 = a^2 - a'^2$, where a' is equal to half the major axis of a confocal conic passing through a vertex of the triangle.

Hence if μ_1, μ_2, μ_3 be the semi-major axes of the confocal ellipses, and ν_1, ν_2, ν_3 of the confocal hyperbolæ through the vertices of the triangle, we have for the equation of the inscribed circle

$$x^2 + y^2 - \frac{2\nu_1\nu_2\nu_3}{ac}x + \frac{2}{bc}\sqrt{\{(c^2-\nu_1^2)(c^2-\nu_2^2)(c^2-\nu_3^2)\}}\,y$$
$$+\nu_1^2+\nu_2^2+\nu_3^2-a^2-c^2=0,$$
$$r = \frac{\sqrt{\{(a^2-\nu_1^2)(a^2-\nu_2^2)(a^2-\nu_3^2)\}}}{ab},$$

and for the equation of an exscribed circle

$$x^2 + y^2 - \frac{2\mu_1\mu_2\nu_3}{ac}x + \frac{2}{bc}\sqrt{\{(\mu_1^2-c^2)(\mu_2^2-c^2)(c^2-\nu_3^2)\}}\,y$$
$$+\mu_1^2+\mu_2^2+\nu_3^2-a^2-c^2=0,$$
$$r' = \frac{\sqrt{\{(\mu_1^2-a^2)(\mu_2^2-a^2)(a^2-\nu_3^2)\}}}{ab}.$$

From these expressions for the radii we deduce, if s be the semi-perimeter,

$$s = \frac{\sqrt{\{(\mu_1^2-a^2)(\mu_2^2-a^2)(\mu_3^2-a^2)\}}}{ab}.$$

8. *Find the equation of the circle touching an ellipse and the tangents to it from the point* (μ, ν).

It is

$$x^2+y^2-\frac{2\nu}{c}\,\frac{\{a\sqrt{(\mu^2-c^2)}-b\mu\}}{\sqrt{(\mu^2-c^2)}-b}\,x$$
$$+2\frac{\sqrt{(c^2-\nu^2)}}{c}\,\frac{\{a\mu-b\sqrt{(\mu^2-c^2)}-c^2\}}{\sqrt{(\mu^2-c^2)}-b}\,y$$
$$+\nu^2-a^2-c^2+2a\frac{\{a\sqrt{(\mu^2-c^2)}-b\mu\}}{\sqrt{(\mu^2-c^2)}-b}=0.$$

Hence, if the point move along a confocal ellipse, the locus of the centre of the circle is an ellipse.

9. Two vertices of a triangle circumscribed to an ellipse move along confocal hyperbolœ; prove that the locus of the centre of the inscribed circle is a concentric ellipse.

10. *To find the locus of the centroid of a triangle inscribed in one conic*

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \equiv U = 0\right)$$

and circumscribed to another whose tangential equation is

$$(A,\ B,\ C,\ F,\ G,\ H)(\lambda,\ \mu,\ \nu)^2 \equiv \Sigma = 0.$$

Writing down the condition that the chord

$$\frac{x}{a}\cos\tfrac{1}{2}(\alpha+\beta) + \frac{y}{b}\sin\tfrac{1}{2}(\alpha+\beta) - \cos\tfrac{1}{2}(\alpha-\beta) = 0$$

should touch Σ and two similar equations for the other sides, multiplying them by $\sin(\alpha-\beta)$, $\sin(\beta-\gamma)$, $\sin(\gamma-\alpha)$ and adding them together, we get, after dividing by

$$\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha),$$

$$C\{1+4\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha)\}$$
$$-\frac{2G}{a}(\cos\alpha+\cos\beta+\cos\gamma) - \frac{2F}{b}(\sin\alpha+\sin\beta+\sin\gamma)$$
$$+\frac{A}{a^2}+\frac{B}{b^2}=0.$$

But $\cos\alpha+\cos\beta+\cos\gamma = \dfrac{3x}{a}$, $\sin\alpha+\sin\beta+\sin\gamma = \dfrac{3y}{b}$,

$$1+8\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha) = 9\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right),$$

hence we have for the equation of the locus the conic

$$9C\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-12\left(\frac{Gx}{a^2}+\frac{Fy}{b^2}\right)+2\left(\frac{A}{a^2}+\frac{B}{b^2}\right)+C=0.$$

11. Prove that the locus of the centroid of a triangle inscribed in a conic and circumscribed to a parabola is a right line.

12. *To find the locus of the centre of the circumscribing circle of a triangle inscribed in one conic* $S'=0$, *and circum scribed to another conic* $S=0$.

It may be shown by the invariants that, if a triangle be inscribed in a conic $S'=0$ and circumscribed to a conic $S=0$, it is also self-conjugate with regard to another fixed conic

$$4F+\Theta S'=0 \text{ (Salmon's } \textit{Conics}\text{, Art. 376).}$$

Hence, if $C=0$ is the director circle of this conic, and r the radius of the circumscribing circle, we have $r^2=C$ (Salmon's *Conics*, Art. 375, Ex. 2).

Let $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$, then by the invariants we have (Salmon's *Conics*, Art. 376, Ex. 2),

$$r^4-2r^2(x^2+y^2+a^2+b^2)+(x^2+y^2)^2-2c^2(x^2-y^2)+c^4=0.$$

Hence we obtain for the equation of the locus, putting $x^2+y^2+a^2+b^2-C=L$ a line, the conic

$$L^2=4(a^2x^2+b^2y^2+a^2b^2).$$

Also, since the circumscribing circle cuts orthogonally a fixed circle and has its centre on a fixed conic, it has double contact with a bicircular quartic.

13. Prove that the foci of S are single foci of this quartic.

If the conic S become a circle, the locus of the centre of the circles has double contact with the circle C.

In this case the bicircular quartic breaks up into two circles.

These two circles have double contact with the conic S', and their chords of contact pass through the centre of the circle S.

14. A triangle is inscribed in a conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, and circumscribed to a circle S whose centre is on the directrix of the conic $\left(x=\frac{a^2}{c}\right)$; prove that the circle S has double contact with the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2x}{c}+\frac{a^2c^2-b^4}{c^4}=0,$$

and that the circumscribing circle of the triangle passes through the corresponding focus $(c, 0)$.

15. *To find the locus of the centre of the inscribed circle of a triangle circumscribed to a conic and inscribed in a circle S.*

Let r be the radius of the circle inscribed in the triangle and a the radius of S, then $S=2ar$ by Euler's equation, and $r^2=V$ a fixed conic (Salmon's *Conics*, Art. 371. Ex. 4), since the triangle is self-conjugate with regard to V. Hence the locus is the bicircular quartic $S^2-4a^2V=0$, having quartic contact with the conic U where it is met by the circle S.

16. If t be the length of the tangent drawn from any point δ on a conic to the circle passing through the points α, β, γ, we can prove that

$$t^2=4c^2\sin\tfrac{1}{2}(\delta-\alpha)\sin\tfrac{1}{2}(\delta-\beta)\sin\tfrac{1}{2}(\delta-\gamma)\sin\tfrac{1}{2}(\delta+\alpha+\beta+\gamma).$$

Let $\alpha=\beta=\gamma$, and the equation of the conic referred to three osculating circles S_1, S_2, S_3 which intersect in the same point on the curve, may be written $\sqrt[3]{S_1}+\sqrt[3]{S_2}+\sqrt[3]{S_3}=0$.

If these circles intersect at angles ϕ, χ, ψ, and A, B, C be the angles of the triangle formed by their centres, show that

$$\phi = 3A - \pi, \quad \chi = 3B - \pi, \quad \psi = 3C - \pi.$$

17. A circle touches the tangents drawn from points on a conic S to a confocal conic, and cuts orthogonally a circle C; prove that it has double contact with a bicircular quartic whose single foci are the points of intersection of S and C.

18. *A series of conics are circumscribed to a quadrilateral; prove that the director circles have double contact with a bicircular quartic, of which the intersections of diagonals and opposite sides are foci.*"

Since the locus of the centre of the variable director circle is a conic passing through the intersections of the diagonals and opposite sides of the quadrilateral, and since it also cuts orthogonally the circle passing through the same points, its envelope is a bicircular quartic whose foci are the intersections of the fixed circle and conic. Or thus, substituting for a, b, &c., $a + \lambda a'$, $b + \lambda b'$, &c., in the equation of the director circle

$$C(x^2 + y^2) - 2Gx - 2Fy + A + B \equiv S = 0,$$

we get

$$S\lambda^2 + \Sigma\lambda + S' = 0,$$

where

$$\Sigma = (a'b + b'a - 2bh')(x^2 + y^2) - 2x(fh' + f'h - bg' - b'g)$$
$$- 2y(gh' + g'h - af' - a'f) + ac' + c'a - 2gg' + bc' + b'c - 2ff' = 0.$$

Hence the envelope is $\Sigma^2 = 4SS'$.

Σ is evidently the director circle of the covariant conic Φ.

19. *A conic passes through the intersections of diagonals and opposite sides of the same quadrilateral and has a focus*

on the bicircular quartic; prove that its corresponding directrix will pass through a vertex of the quadrilateral.

Taking the intersections of diagonals and opposite sides for triangle of reference, and ρ_1, ρ_2, ρ_3 denoting the distances of a point from the vertices of this triangle, the director circle of the conic $a\alpha^2 + b\beta^2 + c\gamma^2 = 0$ is

$$bc \sin^2 A\rho_1^{\,2} + ca \sin^2 B\rho_2^{\,2} + ab \sin^2 C\rho_3^{\,2} = 0,$$

(Salmon's *Conics*, p. 339), and the envelope of this subject to the condition $a\alpha'^2 + b\beta'^2 + c\gamma'^2 = 0$ is

$$\alpha' \sin A\rho_1 + \beta' \sin B\rho_2 + \gamma'\rho_3 \sin C = 0.$$

But ρ_1, ρ_2, ρ_3 are evidently proportional to the perpendiculars from the vertices of the triangle of reference on the directrix of a conic which passes through the vertices of the triangle of reference, and has a focus coinciding with the point ρ_1, ρ_2, ρ_3 on the quartic. Hence the directrix of this conic passes through one or other of the points $(\alpha' \pm \beta' \pm \gamma')$.

Again, if a conic pass through three fixed points and have an asymptote parallel to a given line, the envelope of its director circle is a circular cubic.

20. *To find the equation of the circle circumscribing the common self-conjugate triangle of two conics.*

Form the equation of the circle cutting at right angles their director circles and the director circle of the covariant conic Φ.

21. Prove that the envelope of the director circles of a series of conics touching three fixed lines and passing through a fixed point is a bicircular quartic. (Salmon's *Conics*, p. 256.)

22. *A circle circumscribes a triangle circumscribed to a conic S and has its centre on S; prove that it touches the director circle of S.*

Let $$S = \sqrt{(l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0,$$
then the radical axis of the director circle and the circumscribing circle is

$$l\alpha \cot A + m\beta \cot B + n\gamma \cot C \text{ (Salmon's } \textit{Conics}\text{, p. 339)}$$

$= 0$, which touches the circumscribing circle if

$$\sqrt{(l \cos A)} + \sqrt{(m \cos B)} + \sqrt{(n \cos C)} = 0,$$

which is also the condition that the centre of the circumscribing circle should lie on S.

23. *Find the locus of the centre of a circle circumscribing a triangle circumscribed to a conic, if it cut orthogonally a fixed circle whose centre is a focus of the conic.*

Let r be the radius of the variable circle, ρ_1, ρ_2 the distances of its centre from the foci of the conic, and k the radius of the fixed circle; then

$$r^4 - (\rho_1^2 + \rho_2^2 + 4b^2)\, r^2 + \rho_1^2\rho_2^2 = 0, \quad r^2 = \rho_1^2 - k,$$

hence $$\frac{\rho_1^2}{k^2} - \frac{\rho_2^2}{k^2 + 4b^2} - 1 = 0 \text{ a circle.}$$

The envelope of the variable circle in this case is a Cartesian oval confocal with the conic.

24. *To find the locus of the centroid of an equilateral triangle self-conjugate with regard to a conic.*

Let R be the radius of the circumscribed circle, and r of the inscribed circle, then, by the invariants, we have

$$x^2 + y^2 = a^2 + b^2 + R^2,$$

$$b^2x^2 + a^2y^2 - a^2b^2 = (a^2 + b^2)\, r^2 = \tfrac{1}{4}(a^2 + b^2)\, R^2.$$

Hence the equation of the locus is

$$(a^2 - 3b^2)\, x^2 + (b^2 - 3a^2)\, y^2 = (a^2 - b^2)^2.$$

And the envelope of the circumscribing circle of the same triangle is the bicircular quartic

$$(x^2+y^2+a^2+b^2)^2=4(a^2-b^2)^2\left(\frac{x^2}{a^2-3b^2}+\frac{y^2}{b^2-3a^2}\right).$$

25. *To find the locus of the centroid of an equilateral triangle circumscribed to a conic.*

The invariants give us

$$R^4-2R^2(x^2+y^2+a^2+b^2)+(x^2+y^2)^2-2c^2(x^2-y^2)+c^4=0,$$

$$x^2+y^2=a^2+b^2-\tfrac{1}{2}R^2.$$

Hence the equation of the locus is the bicircular quartic

$$\{3(x^2+y^2)-a^2-b^2\}^2=4\{a^2x^2+b^2y^2+a^2b^2\}.$$

26. An equilateral triangle is inscribed in a parabola $y^2-px=0$; prove that the locus of its centroid is the parabola

$$9y^2+2p^2-px=0.$$

Prove that the circles circumscribing equilateral triangles circumscribed to a parabola have a common radical axis.

27. *To find the envelope of the director circles of a system of conics having double contact with two fixed conics.*

Let the tangential equations of the two fixed conics be

$$a\alpha^2+b\beta^2+c\gamma^2=0,\quad a'\alpha^2+b'\beta^2+c'\gamma^2=0,$$

where α, β, γ are the perpendiculars from the vertices of the triangle of reference on a line.

Then (Salmon's *Conics*, p. 251) the tangential equation of a conic having double contact with the two fixed conics is

$$\mu^2(l\alpha+m\beta)^2-2\mu\{(ac'+a'c)\alpha^2+(bc'+b'c)\beta^2+2cc'\gamma^2\}$$
$$+(l\alpha-m\beta)^2=0,$$

where $\qquad l^2=ac'-a'c,$ and $m^2=-(bc'-b'c),$

or, putting $$\mu = \tan\tfrac{1}{2}\theta,$$

$$\alpha^2(ac'-a'c)-\beta^2(bc'-b'c)-2\sqrt{\{-\{(ac'-a'c)(bc'-b'c)\}}\cos\theta\alpha\beta$$
$$-\sin\theta\{(ac'+a'c)\alpha^2+(bc'+b'c)\beta^2+2cc'\gamma^2\}=0.$$

But if ρ_1, ρ_2, ρ_3 be the distances of a point from the vertices of the triangle of reference, and Σ be the circle described on the side opposite γ as diameter, the equation of the director circle of

$$A\alpha^2+B\beta^2+C\gamma^2+2H\alpha\beta=0$$

is (Salmon's *Conics*, p. 258)

$$A\rho_1^2+B\rho_2^2+C\rho_3^2+2H\Sigma=0.$$

Hence the required envelope is the bicircular quartic

$$(ac'-a'c)(bc'-b'c)\Sigma^2-4cc'(a\rho_1^2+b\rho_2^2+c\rho_3^2)$$
$$\times(a'\rho_1^2+b'\rho_2^2+c'\rho_3^2)=0.$$

28. *To find the equation of the circle circumscribing the triangle formed by the tangents to the conic* $S\equiv\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$, *drawn from the point* (x', y') *and their chord of contact.*

The equation will be of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+(lx+my+n)\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}-1\right)=0.$$

Substituting in this equation the coordinates of the point (x', y'), we get $n=-(lx'+my'+1)$, and the conditions that the equation should represent a circle, give

$$\frac{ly'}{b^2}+\frac{mx'}{a^2}=0,\quad \frac{1+lx'}{a^2}=\frac{1+my'}{b^2}.$$

Hence the equation becomes

$$a^2b^2\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$
$$+c^2\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}-1\right)\left(\frac{xx'}{a^2}-\frac{yy'}{b^2}-\frac{(x'^2+y'^2)}{c^2}\right)=0,$$

or $$\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)(x^2+y^2)-\frac{xx'}{a^2}(x'^2+y'^2+c^2)$$
$$-\frac{yy'}{b^2}(x'^2+y'^2-c^2)+c^2\left(\frac{x'^2}{a^2}-\frac{y'^2}{b^2}\right)=0.$$

Also if R be the radius of the circle, ρ, ρ' the distances of (x', y') from the foci

$$2R=\frac{\rho\rho'}{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}}\sqrt{\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}\right)}.$$

To find the angle under which this circle cuts the director circle.

We have

$$c^2\frac{\left(\frac{x'^2}{a^2}-\frac{y'^2}{b^2}\right)}{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}}=a^2+b^2-2R\cos\theta\sqrt{(a^2+b^2)},$$

or $$(a^2+b^2)\,\rho^2\rho'^2\cos^2\theta=4\,(b^4x'^2+a^4y'^2);$$

or, again,

$$\rho^2\rho'^2\sin^2\theta=(x'^2+y'^2-a^2-b^2)\left(x'^2+y'^2-\frac{(a^2-b^2)^2}{a^2+b^2}\right).$$

Hence, if (x', y') lie on the director circle or the inverse of the director circle with respect to the circle described on the line joining the foci as diameter, the variable circle touches the director circle.

Again, if (x', y') lie on the inverse of S with respect to the circle described on the line joining the foci as diameter, the variable circle touches S.

29. *Given the rectangle under the segments of the perpendiculars of a triangle formed by two tangents to a conic and their chord of contact, find the locus of the intersection of the perpendiculars.*

If two conics can be reduced to the forms

$$2hxy + cz^2 = 0, \quad x^2 + y^2 + z^2 = 0,$$

then the invariant relation is satisfied,

$$\Theta\Theta' = \Delta\Delta'.$$

Hence, if the conic is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$, the locus is the curve of the fourth order

$$\left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 - r^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\right\}(x^2 + y^2 - a^2 - b^2 - r^2) = r^2.$$

If the polar circle cut orthogonally the fixed circle $x^2 + y^2 - k^2 = 0$, the locus of its centre is the conic

$$\frac{a^2x^2}{k^2 - b^2} + \frac{b^2y^2}{k^2 - a^2} = a^2 + b^2,$$

and its envelope is the bicircular quartic

$$(x^2 + y^2 + k^2)^2 - 4(a^2 + b^2)\left\{(k^2 - b^2)\frac{x^2}{a^2} + (k^2 - a^2)\frac{y^2}{b^2}\right\} = 0.$$

The two conics coincide if $k^2 = \dfrac{a^4 + b^4 + a^2b^2}{a^2 + b^2}$.

30. Tangents are drawn to the conic $S \equiv \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$ from any point on $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 4 = 0$; prove that they form with their chord of contact a triangle whose centroid is on $S = 0$.

31. To find the locus of a point such that if from it tangents be drawn to $S = 0$, they will form with their chord of contact a triangle whose intersection of perpendiculars lies on $S = 0$. Let (x', y') be the point, (α, β) the intersection of perpendiculars, $S \equiv \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1$,

then
$$\alpha = \frac{x'}{a^2}(a^2 - \lambda^2), \quad \beta = \frac{y'}{b^2}(b^2 - \lambda^2),$$

where
$$\lambda^2 = \frac{x'^2 + y'^2 - a^2 - b^2}{\frac{x'^2}{a^2} + \frac{y'^2}{b^2}}.$$

Hence, substituting (α, β) in $S = 0$, and dividing by $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1$, we obtain
$$a^2x'^2 + b^2y'^2 - (a^2 + b^2)^2 = 0,$$

which is the reciprocal of S with respect to its director circle.

The points (x', y'), (α, β) are evidently conjugate with respect to the director circle.

Given the point (α, β), (x', y') is determined as the intersection of the polar of (α, β) with regard to the director circle and the equilateral hyperbola which passes through the feet of the normals to S from (α, β).

If (x', y') lie on a fixed line parallel to an axis or passing through the centre of S, the locus of (α, β) is a conic. If (x', y') lie on the quartic
$$\frac{c^4x^2y^2}{a^4b^4} - (a^2 + b^2)\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right) = 0,$$

(α, β) lies on the director circle.

32. *To find the equation of the polar circle of the triangle formed by two tangents to a conic and their chord of contact.*

The coordinates of the centre are already known, and we find the absolute term by expressing that (x', y') is the pole of $\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1 = 0$, with regard to the circle; hence the equation sought is

$$\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)(x^2+y^2)-\frac{2x'}{a^2}\left(a^2+b^2+\frac{c^2y'^2}{b^2}\right)x-\frac{2y'}{b^2}\left(a^2+b^2-\frac{c^2x'^2}{a^2}\right)y$$
$$+a^2+b^2+\frac{b^2}{a^2}x'^2+\frac{a^2}{b^2}y'^2=0.$$

If r is the radius of the circle,

$$r^2=\frac{a^2b^2(x'^2+y'^2-a^2-b^2)\left(\dfrac{x'^2}{a^6}+\dfrac{y'^2}{b^6}-\dfrac{c^4x'^2y'^2}{a^6b^6}\right)}{\left(\dfrac{x'^2}{a^2}+\dfrac{y'^2}{b^2}\right)^2}.$$

The equation of the circle may be written

$$\left(a^2+b^2+\frac{c^2y'^2}{b^2}\right)\left(\frac{x-x'}{a^2}\right)^2+\left(a^2+b^2-\frac{c^2x'^2}{a^2}\right)\left(\frac{y-y'}{b^2}\right)^2$$
$$+(x'^2+y'^2-a^2-b^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=0,$$

which gives the equation of its chords of intersection with S, which meet in (x', y'). These chords never meet S in real points.

If $x'=$ a constant, the circle cuts orthogonally the fixed circle $x^2+y^2-\dfrac{a^2}{x'}x-b^2=0$, and, therefore, has double contact with a bicircular quartic. If (x', y') lie on the directrix, the quartic breaks up into two circles, which are imaginary for the ellipse but real for the hyperbola.

If (1) $a^2+b^2-\dfrac{c^2x'^2}{a^2}=0$, or (2) $a^2+b^2+\dfrac{c^2y'^2}{b'^2}=0$, the corresponding equations are (1) $x^2+y^2\pm\dfrac{2c}{a}\sqrt{(a^2+b^2)}\,x+a^2=0$, (2) $x^2+y^2\pm\dfrac{2c}{b}\sqrt{-(a^2+b^2)}\,y+b^2=0$, which represent fixed circles having double contact with S. These circles are such that S is its own reciprocal with regard to each of them; they are all imaginary for the ellipse, but the two latter are real for a hyperbola whose director circle is real.

33. From the equations of the circumscribed and polar circles of the triangles formed by two tangents to a conic and their chord of contact we can deduce the equation of the nine-point circle of the same triangle:—

$$2\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)(x^2+y^2)-\frac{xx'}{a^2}\left\{x'^2+\left(\frac{2a^2-b^2}{b^2}\right)y'^2+3a^2+b^2\right\}$$
$$-\frac{yy'}{b^2}\left\{y'^2+\left(\frac{2b^2-a^2}{a^2}\right)x'^2+3b^2+a^2\right\}+x'^2+y'^2+a^2+b^2=0.$$

34. *To find the locus of the vertex of a triangle formed by two tangents to a conic* $\left(S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$ *and their chord of contact, if the centre of the inscribed circle lie on* S.

If α and β are the tangents, and γ their chord of contact, S must be capable of being written in the form $\alpha\beta-k\gamma^2=0$. But $\alpha=\beta=\gamma$, for the centre of the inscribed circle, whence $k=1$. Substituting now for α, &c. $x\cos\alpha+y\sin\alpha-p$, &c. and equating the coefficient of xy to nothing, we see that the base and the internal bisector of the vertical angle must make equal angles with the axis.

Hence the locus is the confocal conic

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{c^2}{a^2+b^2}=0.$$

The base, in the same case, is normal to the equilateral hyperbola which passes through the feet of the normals to S from the vertex of the triangle.

35. *From a point (x', y') tangents are drawn to a conic $\left(S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0\right)$, to find the coordinates of the focus of the parabola having double contact with S at their points of contact.*

The equation of the parabola in tangential coordinates is $(x'\lambda + y'\mu + \nu)^2 + a^2\lambda^2 + b^2\mu^2 - \nu^2 = 0$; and if (x, y) is the focus, we have (Salmon's *Conics*, p. 228),

$$2x = \frac{x'(x'^2 + y'^2 + c^2)}{x'^2 + y'^2}, \quad 2y = \frac{y'(x'^2 + y'^2 - c^2)}{x'^2 + y'^2}.$$

If (x', y') lie on a line through the centre of S or a concentric circle, (x, y) lies on a confocal conic.

If (x', y') lie on a line, (x, y) lies on a nodal circular cubic which has its foci in common with S.

Given (x, y), (x', y') is determined thus: let a confocal hyperbola be described through (x, y); then the tangent to the hyperbola at (x, y) intersects the asymptotes of the hyperbola in the corresponding positions of (x', y'). The directrix of the parabola is $2x'x + 2y'y - x'^2 - y'^2 - a^2 - b^2 = 0$.

36. From a point (x', y') tangents are drawn to a conic; prove that the centre of the equilateral hyperbola, having double contact with the conic at their points of contact, is the inverse of (x', y') with regard to the director circle.

37. *To find the locus of the vertices of equilateral triangles self-conjugate with regard to a given conic.*

Let the conic be $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \equiv S = 0$, and let

$$x\cos\alpha + y\sin\alpha = 0, \quad x\cos\beta + y\sin\beta = 0,$$

be lines through the origin parallel to the two sides of the triangle meeting in the vertex (x, y). Then, expressing that these sides form a harmonic pencil with the tangents from (x, y), we have

$$\cos\alpha\cos\beta\,(y^2-b^2)+\sin\alpha\sin\beta\,(x^2-a^2)-xy\sin(\alpha+\beta)=0,$$

or $(x^2+y^2-a^2-b^2)\cos(\alpha-\beta)$

$$=(x^2-y^2-c^2)\cos(\alpha+\beta)+2xy\sin(\alpha+\beta),$$

now $\cos(\alpha-\beta)=\frac{1}{2}$, and $\alpha+\beta=2\omega$,

where $\cos\omega=\dfrac{px}{a^2}$, $\sin\omega=\dfrac{py}{b^2}$, hence

$$\tfrac{1}{2}(x^2+y^2-a^2-b^2)\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)=(x^2-y^2-c^2)\left(\frac{x^2}{a^4}-\frac{y^2}{b^4}\right)+\frac{4x^2y^2}{a^2b^2},$$

or $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2-\frac{3c^4x^2y^2}{a^4b^4}+(3b^2-a^2)\frac{x^2}{a^4}+(3a^2-b^2)\frac{y^2}{b^4}=0,$$

a quartic curve with a node at the origin.

If $a^2-3b^2=0$, the locus breaks up into two imaginary conics.

For the parabola $y^2-4ax=0$, the locus is

$$y^2(3x+7a)-4a^2(x-3a)=0.$$

38. *To find an expression for the radius of a circle circumscribing a triangle self-conjugate with regard to a conic.*

Double the area of the triangle formed by the lines

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0,\quad \frac{xx_2}{a^2}+\frac{yy_2}{b^2}-1=0,\quad \frac{xx_3}{a^2}+\frac{yy_3}{b^2}-1=0,$$

is equal to

$$\frac{a^6b^6}{(x_1y_2-y_1x_2)(y_2x_3-y_3x_2)(x_1y_3-y_1x_3)}\times\begin{vmatrix}\dfrac{x_1}{a^2}, & \dfrac{x_2}{a^2}, & \dfrac{x_3}{a^2}\\[2ex] \dfrac{y_1}{b^2}, & \dfrac{y_2}{b^2}, & \dfrac{y_3}{b^2}\\[2ex] 1, & 1, & 1\end{vmatrix}^2$$

$$=\frac{4a^2b^2\Delta^2}{\alpha\beta\gamma p_1p_2p_3},$$

where α, β, γ are the sides, and Δ the area of the triangle, and p_1, p_2, p_3 are the perpendiculars from the centre of the conic on the sides. But, if R be the radius of the circumscribing circle, $\alpha\beta\gamma = 4\Delta R$, therefore

$$2\Delta = \frac{4a^2b^2}{4\Delta R} \cdot \frac{\Delta^2}{p_1p_2p_3},$$

or

$$2R = \frac{a^2b^2}{p_1p_2p_3}.$$

In the same way we can find the expression for the radius of the circle circumscribing a triangle formed by two tangents and their chord of contact (Salmon's *Conics*, p. 230).

39. *Given three points on a conic and that a directrix touches a fixed conic, to find the locus of the corresponding focus.*

Let the tangential equation of the conic be

$$A\alpha^2 + B\beta^2 + C\gamma^2 + 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta = 0,$$

then the equation of the locus is, if ρ_1, ρ_2, ρ_3 are the distances of a variable point from the three fixed points,

$$A\rho_1^2 + B\rho_2^2 + C\rho_3^2 + 2F\rho_2\rho_3 + 2G\rho_1\rho_3 + 2H\rho_1\rho_2 = 0,$$

or, putting $A\rho_1^2 + B\rho_2^2 + C\rho_3^2 = 2S$,

$$\begin{aligned} S^4 - 2S^2(F^2\rho_2^2\rho_3^2 + G^2\rho_1^2\rho_3^2 + H^2\rho_1^2\rho_2^2) \pm 8FGH\rho_1^2\rho_2^2\rho_3^2 S \\ + F^4\rho_2^4\rho_3^4 + G^4\rho_1^4\rho_3^4 + H^4\rho_1^4\rho_2^4 \\ - 2\rho_1^2\rho_2^2\rho_3^2(G^2H^2\rho_1^2 + F^2H^2\rho_2^2 + F^2G^2\rho_3^2) = 0, \end{aligned}$$

which represents two curves of the eighth order having the circular points at infinity for quadruple points.

If two sides of the fixed triangle are conjugate with respect to the fixed conic, these two curves coincide.

If two pairs of sides are conjugate with respect to the fixed conic, the locus is a bicircular quartic of which the two corresponding vertices of the triangle are foci.

If the triangle is self-conjugate with regard to the conic, the locus is the director circle of the conic.

If a directrix of the variable conic pass through a fixed point, the locus of the corresponding focus is a bicircular quartic of which the three points are foci.

If this fixed point is the centre of one of the circles touching the sides of the triangle, the locus is the circle circumscribed to the triangle.

We shall get similar results if we substitute for the focus, "the centre of a circle of given radius having double contact with the conic," and for the directrix "the chord of contact of this circle" (Salmon's *Conics*, p. 230).

Let the chord of contact pass through the centre of a circle touching the sides of the triangle, then (Salmon's *Conics*, p. 364) $\alpha \sin A \pm \beta \sin B \pm \gamma \sin C = 0$, therefore

$$\sin A \sqrt{(\rho_1^2 - k^2)} \pm \sin B \sqrt{(\rho_2^2 - k^2)} \pm \sin C \sqrt{(\rho_3^2 - k^2)} = 0,$$

where k is the radius of the circle having double contact with the conic; hence, by Dr. Casey's equation (Salmon's *Conics*, p. 114), we see that the circle having double contact with the conic touches the circumscribing circle of the triangle.

40. Given three points on a conic, if a focus lie on a fixed circle, prove that the corresponding directrix touches a curve of the fourth class, of which the centres of the circles touching the sides of the triangle formed by the points are double points.

41. *Given a self-conjugate triangle with regard to a conic, if a directrix pass through a fixed point, to find the locus of the corresponding focus.*

If p, p' are the perpendiculars from two points conjugate with respect to a conic on its directrix, S the square of the tangent from the corresponding focus to the circle described on the points as diameter, and e the eccentricity of the conic, we have $S = e^2pp'$.

Hence, if the directrix pass through the fixed point $l\alpha + m\beta + n\gamma = 0$, the corresponding focus lies on the bicircular quartic

$$lS_2S_3 + mS_3S_1 + nS_1S_2 = 0.$$

This bicircular quartic circumscribes the triangle, passes through the feet of the perpendiculars, and has four foci on the polar circle.

If the directrix be parallel to a fixed line, the locus is a circular cubic, of which the intersection of perpendiculars and vertices of the triangle are centres of inversion.

If the directrix touch a fixed conic inscribed in the triangle, the locus is the director circle of this conic.

42. A conic U is inscribed in a triangle self-conjugate with regard to a conic V; prove that the circles, having double contact with V, whose chords of contact touch U, cut orthogonally the director circle of U.

43. *Given three tangents to a conic, and that a directrix passes through a fixed point, to find the locus of the corresponding focus.*

If ρ_1 be the distance of the focus from a vertex A of the triangle formed by the tangents, ϕ_1 the angle subtended at the focus by the side opposite A, α the perpendicular from A on the directrix, and e the eccentricity of the conic, we have (Salmon's *Conics*, p. 177) $e\alpha = \rho_1 \cos\phi_1 = \dfrac{\rho_1 S_1}{\rho_2\rho_3}$, where S_1 is equal to the square of the tangent drawn from the focus to the circle described on the side opposite A as diameter. Hence, if the directrix pass through the fixed point whose tangential equation is $l\alpha + m\beta + n\gamma = 0$, the focus lies on $l\rho_1^2 S_1 + m\rho_2^2 S_2 + n\rho_3^2 S_3 = 0$, which represents a bicircular quartic circumscribing the triangle.

When the fixed point is the intersection of the perpendiculars of the triangle, the quartic becomes the product of the circumscribing and polar circles of the triangle. The former circle belongs to the case when the conic is a parabola. If the directrix be parallel to a given line, the locus is a circular cubic, as may be also proved thus: let x, y, z be the perpendiculars from the focus on the sides of the triangle, λ, μ, ν the angles between the sides and the given line; then $x^2+2cx\cos\lambda-b^2=0$, $y^2+2cy\cos\mu-b^2=0$, $z^2+2cz\cos\nu-b^2=0$; and, eliminating b and c, the locus is

$$x\cos\lambda\,(y^2-z^2)+y\cos\mu\,(z^2-x^2)+z\cos\nu\,(x^2-y^2)=0,$$

a circular cubic, of which the centres of the circles touching the sides are centres of inversion.

44. *Given a self-conjugate triangle with regard to a conic, and that half the length of the least axis is equal to the radius of the polar circle, to find the envelope of the major axis.*

If two points are conjugate with respect to a conic $S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, we have

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1=0,$$

or

$$x_1x_2+y_1y_2=a^2-\frac{c^2}{b^2}\,y_1y_2,$$

which may be written $S_3=a^2-\frac{e^2}{b^2}p_1p_2$, where S_3 is the square of the tangent from the origin to the circle described on the points as diameter, and p_1, p_2 are the perpendiculars from the points on the major axis.

But if A, B, C are the angles and Δ the area of the triangle,

$$S_1 \tan A + S_2 \tan B + S_3 \tan C = 2\Delta + t^2 \tan A \tan B \tan C,$$

where t is the tangent drawn to the circumscribing circle of the triangle. Now $t^2 = a^2 + b^2$, and $r^2 = -2\Delta \cot A \cot B \cot C$, where r is the radius of the polar circle. Hence

$$2\Delta\left(1 - \frac{b^2}{r^2}\right) + \frac{c^2}{b^2}(\beta\gamma \tan A + \gamma\alpha \tan B + \alpha\beta \tan C) = 0;$$

and the equation of the envelope is

$$\beta\gamma \tan A + \gamma\alpha \tan B + \alpha\beta \tan C = 0,$$

which represents a conic, inscribed in the triangle, and concentric with the circumscribing circle.

If the latus-rectum of the variable conic be equal to the radius of the polar circle, the envelope of the major axis is a conic confocal with the conic just found.

45. Given a triangle inscribed in a conic, if the radical axis of the circumscribing circle and a circle S, having double contact with the conic, pass through the centre of an inscribed circle, prove that the chord of contact of S touches a conic, having the given triangle for a self-conjugate triangle, and concentric with the same inscribed circle.

46. Given a self-conjugate triangle with regard to a conic, if the radical axis of the polar circle of the triangle and a circle S, having double contact with the conic, pass through the centre of the circumscribing circle, the chord of contact of S touches a conic inscribed in the triangle.

If the radical axis pass through the centre of the nine-point circle, the chord of contact touches a conic inscribed in the triangle.

47. A circle S is inscribed in a triangle self-conjugate with regard to a conic $U \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, and touches a line; prove that the centre of S lies on the equilateral hyperbola which has double contact with U where it is met by the line.

If the line touch the confocal conic $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2+b^2}$, the circle S has its centre on one or other of the tangents to U where it is met by the line.

If circles inscribed in triangles, self-conjugate with regard to U, have their centres on a fixed line, show that they have double contact with a conic whose foci are the points where the fixed line meets U.

Conversely, if a circle have double contact with U (the chord of contact being parallel to the minor axis) show that it is inscribed in triangles self-conjugate with regard to the conic

$$b^2(x^2 - c^2) - a^2y^2 + 2fy = 0,$$

where f is an arbitrary constant.

48. *Given a triangle circumscribed to a conic and the length* (2*a*) *of the major axis, to find the locus of the foci.*

Let A, B, C be the angles of the triangle, and ρ_1, ρ_2, ρ_3 the distances of a point from its vertices; then $\rho_1 \sin A$, $\rho_2 \sin B$, $\rho_3 \sin C$ are the sides of the triangle formed by the feet of the perpendiculars from the point on the sides of the given triangle, and a is the radius of the circle passing through the feet of the perpendiculars. Hence

$$a^2 (2\rho_1^2\rho_2^2 \sin^2 A \sin^2 B + 2\rho_2^2\rho_3^2 \sin^2 B \sin^2 C$$
$$+ 2\rho_3^2\rho_1^2 \sin^2 C \sin^2 A - \rho_1^4 \sin^4 A - \rho_2^4 \sin^4 B - \rho_3^4 \sin^4 C)$$
$$= \rho_1^2\rho_2^2\rho_3^2 \sin^2 A \sin^2 B \sin^2 C$$

from the expression for the radius of the circumscribing circle of a triangle in terms of the sides.

Now we may infer from Ptolemy's theorem that the expression between the brackets is equal to

$$4 \sin^2 A \sin^2 B \sin^2 C t^4,$$

where t is the length of the tangent to the circumscribing circle.

Hence $\rho_1\rho_2\rho_3 = 2at^2$, a curve of the sixth order. We have also $\rho_1'\rho_2'\rho_3' = 2at'^2$ for the other focus; but $t^2t'^2 = 4b^2R^2$, where R is the radius of the circumscribing circle, from the invariant relation between the conic and the circumscribing circle of a circumscribed triangle; therefore

$$R^2 = \frac{\rho_1\rho_2\rho_3\rho_1'\rho_2'\rho_3'}{16a^2b^2}.$$

49. The equation of the circle described on a chord of a conic as diameter is

$$(a^2l^2 + b^2m^2)(x^2 + y^2) - 2a^2lx - 2b^2my + a^2 + b^2 - a^2b^2(l^2 + m^2) = 0,$$

where $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \equiv U = 0$, and $lx + my - 1 = 0$, are the equations of the conic and chord respectively.

This equation may be written

$$(l^2 + m^2)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + \frac{c^2}{a^2b^2}(lx + my - 1)\left(lx - my - \frac{a^2 + b^2}{c^2}\right) = 0,$$

showing that the circle meets the conic again on the line $lx - my - \frac{(a^2 + b^2)}{c^2} = 0.$

Hence, if the circle touch the curve, the chord touches

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{c^4}{(a^2 + b^2)^2}.$$

If the ratio of the lengths of an internal and external common tangent of the director circle, and the circle described on a chord as diameter, be given, the chord touches a confocal conic; for, if m be the ratio, and a' the semi-axis major of the conic touched by the chord, we can show that

$$a^2 - a'^2 = \frac{a^2b^2}{a^2+b^2}\left(\frac{1-m}{1+m}\right)^2.$$

If the circle described on the chord $lx + my - 1 = 0$ as diameter cut orthogonally the fixed circle

$$S \equiv x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = 0,$$

the chord touches the conic

$$a^2(k^2-b^2)\,l^2 + b^2(k^2-a^2)\,m^2 - 2a^2\alpha l - 2b^2\beta m + a^2 + b^2 = 0.$$

If $\dfrac{a^2\alpha^2}{k^2-b^2} + \dfrac{b^2\beta^2}{k^2-a^2} = a^2 + b^2$, the chord passes through one or other of two fixed points; and S is then the polar circle of the triangle formed by the line joining the points and the tangents to U where it is met by this line.

The circles described on parallel chords of a conic as diameters have double contact with a concentric conic.

50. For the equilateral hyperbola $x^2 - y^2 - a^2 \equiv U = 0$, the circle described on the chord $lx + my - 1 = 0$ as diameter, coincides with the polar circle of the triangle formed by the tangents to U from the point $x = \dfrac{1}{l}$, $y = \dfrac{1}{m}$, and their chord of contact.

Prove that the circles described on conjugate or rectangular chords of an equilateral hyperbola as diameters cut orthogonally.

51. *To show that the algebraic sum of the reciprocals of the common tangents of a circle and a conic is equal to zero.*

Let $x \cos w + y \sin w - \sqrt{\{(a^2 \cos^2 w + b^2 \sin^2 w)\}} = 0$ be a tangent of the conic, and let p be the perpendicular on it from the point (x', y'); then $\frac{dw}{dp}$ is the reciprocal of the common tangent of the conic and a circle of radius p, whose centre is (x', y').

Hence, for the four common tangents,

$$\Sigma \frac{1}{t} = \frac{d}{dp} \Sigma w = 0, \text{ as we shall show.}$$

Putting $e^{iw} = z$ in

$$x' \cos w + y' \sin w - p - \sqrt{(a^2 \cos^2 w + b^2 \sin^2 w)} = 0,$$

we obtain

$$\{(x' - iy') z^2 - 2pz + x' + iy'\}^2 - \{c^2 z^4 + 2(a^2 + b^2) z^2 + c^2\} = 0,$$

from the absolute term of which equation we deduce, if $\tan 2\phi = \frac{2x'y'}{x'^2 - y'^2 - c^2}$, $\Sigma w = 4\phi$. Σw is therefore independent of p.

If the circle touch the conic, we have

$$\frac{1}{t_1} \pm \frac{1}{t_2} = \frac{2\rho \cot \theta}{(\rho - r)^2},$$

where r is the radius of the circle, ρ the radius of curvature of the conic at the point of contact, and θ' the angle which the diameter of the conic at the same point makes with the curve. We can show in a similar manner that a circle meets a conic at angles, the sum of whose cotangents is equal to zero.

If a circle touching a conic meet the curve again at angles α and β, we have $\cot \alpha + \cot \beta = \frac{2r^2 \cot \theta}{(\rho - r)^2}$, where θ has the same meaning as before.

52. *A triangle is circumscribed to an ellipse S, and inscribed in a confocal ellipse; to show that the osculating circles of S at the points of contact of the sides touch the fourth common tangent of S and the inscribed circle.*

Using tangential coordinates, the equation of S, referred to the triangle, is $\mu\nu \sin^2\frac{1}{2}A + \nu\lambda \sin^2\frac{1}{2}B + \lambda\mu \sin^2\frac{1}{2}C = 0$; and $\Omega - 4S = 0$, represents the confocal conic circumscribed to the triangle where A, B, C are the angles of the triangle, and

$$\Omega \equiv \lambda^2 + \mu^2 + \nu^2 - 2\mu\nu \cos A - 2\nu\lambda \cos B - 2\lambda\mu \cos C.$$

Then $S - (\lambda \sin^2\frac{1}{2}B + \mu \sin^2\frac{1}{2}A)(l\lambda + m\mu) = 0$, represents a conic having contact of the second order with S on the side opposite ν; and expressing that this conic passes through the points represented by $\Omega = 0$, we obtain the equation of the circle osculating S on the side opposite ν in the form

$$\Omega - \left\{\lambda\left(\frac{2 \sin^2\frac{1}{2}B}{\sin^2\frac{1}{2}C} - 1\right) + \mu\left(\frac{2 \sin^2\frac{1}{2}A}{\sin^2\frac{1}{2}C} - 1\right) + \nu\right\}^2 = 0.$$

But this equation is satisfied by

$$\lambda = \frac{1}{\cos B - \cos C}, \quad \mu = \frac{1}{\cos C - \cos A}, \quad \nu = \frac{1}{\cos A - \cos B},$$

which are the coordinates of the fourth common tangent of S and the inscribed circle. The three osculating circles have, therefore, this line for a common tangent.

53. *Given four tangents to a conic, to find the locus of the centre of a circle of given radius having double contact with the curve.*

Let $l\alpha + m\beta + n\gamma + p\delta = 0$ be an identical relation connecting the four tangents α, β, γ, δ, and let r be the given radius; then the equation of the locus is

$$l\sqrt{(\alpha^2 - r^2)} + m\sqrt{(\beta^2 - r^2)} + n\sqrt{(\gamma^2 - r^2)} + p\sqrt{(\delta^2 - r^2)} = 0,$$

which represents a curve of the sixth order passing through the circular points at infinity, and the points at infinity on the diagonals of the quadrilateral formed by the tangents.

Putting $r = \delta$, we see that the locus of the points where the normal at the point of contact of one of the tangents meets the axes of the conic, is a cubic passing through the points where δ is met by α, β, γ.

Given three tangents to a parabola, the locus of the centre of a circle of given radius, having double contact with the curve, is a unicursal curve of the fifth order.

54. *Given four points on a conic, to find the locus of the centre of a circle of given radius having double contact with the curve.*

Let ρ_1, ρ_2, ρ_3, ρ_4 be the distances of a point from the four points, l, m, n, p the areas of the four triangles formed by the points, and r the given radius; then the equation of the locus is

$$l\sqrt{(\rho_1^2 - r^2)} + m\sqrt{(\rho_2^2 - r^2)} + n\sqrt{(\rho_3^2 - r^2)} + p\sqrt{(\rho_4^2 - r^2)} = 0,$$

which represents a curve of the sixth order. (Salmon's *Conics*, p. 206, Ex. 10).

If the four points lie on a circle, the locus breaks up into two circular cubics whose foci are situated on a concentric circle.

If the four points form a parallelogram, the locus reduces to a curve of the fourth order.

Putting $r = \rho_4$, we see that the locus of the points where the normal at one of the given points meets the axes of the conic is a conic; but when the given points lie on a circle, the locus consists of two lines passing through the centre of the circle.

55. If ρ_1, ρ_2 are the distances of a variable point from the foci of a conic, and if we put $\rho_1' = 2a\cos^2\frac{1}{2}\theta$, $\rho_2' = 2a\sin^2\frac{1}{2}\theta$, for a point on the curve, $\frac{\rho_1^2}{\cos^2\frac{1}{2}\theta} + \frac{\rho_2^2}{\sin^2\frac{1}{2}\theta} - 4a^2 = 0$ will represent a circle having double contact with the conic. This equation will become, if we put

$$\rho_1 + \rho_2 = 2\mu,\ \rho_1 - \rho_2 = 2\nu,\ \mu^2 + \nu^2 - 2\mu\nu\cos\theta - a^2\sin^2\theta = 0.$$

Differentiating the equation of the circle and eliminating θ, we obtain for the differential equation of the system of circles

$$\frac{d\mu}{\sqrt{(a^2 - \mu^2)}} \pm \frac{d\nu}{\sqrt{(a^2 - \nu^2)}} = 0.$$

Hence, two circles of the system make equal angles with the confocal conics which pass through their points of intersection.

To find the angle between the two circles that pass through a point. Transforming the equation

$$\mu^2 + \nu^2 - 2\mu\nu\cos\theta - a^2\sin^2\theta = 0$$

to Cartesian coordinates by means of the relations

$$\mu^2 + \nu^2 = x^2 + y^2 + c^2,\ \mu\nu = cx,$$

we find if r be the radius of the circle, and x' the abscissa of its centre $r = b\sin\theta$, $x' = c\cos\theta$. Hence, if d is the distance between the centres of two circles,

$$d = -2c\sin\tfrac{1}{2}(\theta - \theta')\sin\tfrac{1}{2}(\theta + \theta'),$$

and $d^2 = r^2 + r'^2 - 2rr'\cos\phi$, where ϕ is the angle sought. But solving for the intersection of the circles, we obtain $\mu = a\cos\frac{1}{2}(\theta - \theta')$, $\nu = a\cos\frac{1}{2}(\theta + \theta')$; therefore

$$\tan\tfrac{1}{2}\phi = \sqrt{\left\{\frac{(a^2 - \mu^2)(c^2 - \nu^2)}{(\mu^2 - c^2)(a^2 - \nu^2)}\right\}},$$

and

$$\sin\tfrac{1}{2}\phi = \frac{1}{b}\sqrt{\left\{\frac{(a^2 - \mu^2)(c^2 - \nu^2)}{(\mu^2 - \nu^2)}\right\}}.$$

If the circles cut at right angles, the locus of their intersection is the concentric conic passing through the foci $x^2 + \left(\frac{2a^2 - b^2}{b^2}\right) y^2 = c^2$. If the circles cut at a constant angle, the locus of their intersection is a curve of the fourth order.

56. If t be the length of an external common tangent of two circles of the system,

$$t = \sqrt{\{d^2 - (r - r')^2\}} = 2 \sin \tfrac{1}{2} (\theta - \theta') \sqrt{\{a^2 \sin^2 \tfrac{1}{2} (\theta + \theta') - b^2\}}$$
$$= \frac{2}{a} \sqrt{\{(a^2 - \mu^2)(c^2 - \nu^2)\}}.$$

If ψ be the angle which the same tangent makes with the axis $\sin \psi = \frac{r - r'}{d} = -\frac{b}{c} \cot \frac{1}{2} (\theta + \theta')$, and for an external common tangent $\sin \psi' = -\frac{b}{c} \cot \frac{1}{2} (\theta - \theta')$.

Hence, one or other of the common tangents is parallel to a fixed line when the intersection of the circles lies on a confocal conic.

57. *If n circles, having double contact with a conic, form, with a single point of intersection of each, a polygon $n - 1$ of whose vertices move along confocal conics, the n^{th} vertex will also move along a confocal conic.*

This follows at once from the expressions for μ, ν in terms of θ, θ'.

For a triangle the semi-axes μ_1, μ_2, μ_3 of the confocal ellipses are connected by the relation

$$a^3 - a(\mu_1^2 + \mu_2^2 + \mu_3^2) + 2\mu_1\mu_2\mu_3 = 0.$$

58. If d be the distance between the centres of two circles

$$d = \frac{2c}{a^2} \sqrt{\{(a^2 - \mu^2)(a^2 - \nu^2)\}} = \frac{2bc}{a} \sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}$$

in Cartesian coordinates.

Hence, the theorem in the last example is true, if we substitute for confocal conics "concentric, similar, and similarly-situated conics."

59. If A, B are the base angles of the triangle formed by the centres of two circles and a point of intersection, we have

$$\tan \tfrac{1}{2}A \tan \tfrac{1}{2}B = \frac{r + r' - d}{r + r' + d} = \frac{b\mu + c\sqrt{(a^2 - \mu^2)}}{b\mu - c\sqrt{(a^2 - \mu^2)}} = \text{a constant},$$

if the intersection of the circles lies on a confocal ellipse. And $\dfrac{\tan \frac{1}{2}A}{\tan \frac{1}{2}B}$ is constant if the intersection of the circles lies on a confocal hyperbola. If the difference of A and B is given, the locus of the intersection of the circles is an equilateral hyperbola passing through the foci.

60. The centres of similitude of two circles having double contact with a conic are harmonically conjugate with the foci.

We have $\quad x = \dfrac{x_1 r_2 + r_1 x_2}{r_1 + r_2} = \dfrac{c \cos \frac{1}{2}(\theta - \theta')}{\cos \frac{1}{2}(\theta + \theta')}$;

and $\quad x' = \dfrac{c \cos \frac{1}{2}(\theta + \theta')}{\cos \frac{1}{2}(\theta - \theta')}$;

therefore $xx' = c^2$.

Steiner

see Gundelfinger

For the system of circles which have their centres on the minor axis of the conic, the circle described on the line joining the centres of similitude as diameter passes through the foci.

61. If tangents be drawn from the foci to circles having double contact with a conic, show that the product of the sines of the angles they make with the axis is constant.

62. *The square of the distance from a focus of a point of intersection of two circles having double contact with a conic*

is equal to the product of the distances from the same focus of their points of contact. The equations of the circles may be written in Cartesian coordinates

$$x^2 + y^2 - 2e^2x_1x + e^2x_1^2 - b^2 = 0, \quad x^2 + y^2 - 2e^2x_2x + e^2x_2^2 - b^2 = 0,$$

where x_1, x_2 are the abscissæ of the points of contact. Hence, for a point of intersection, we have

$$x^2 + y^2 - b^2 = e^2x_1x_2, \quad 2x = x_1 + x_2;$$

therefore $$y^2 + (x \pm c)^2 = (a \pm ex_1)(a \pm ex_2).$$

63. If two circles having double contact with a conic intersect in the points $(x', \pm y')$, the equation of the circle passing through their points of contact may be written

$$x^2 + y^2 - 2e^2x'x + x'^2 + y'^2 - 2b^2 \equiv S = 0.$$

If S be fixed, and the conics form a confocal system, the locus of the points $(x', \pm y')$ is a circle cutting S orthogonally.

If the points $(x', \pm y')$ are fixed, and the conics form a confocal system, S has double contact with the Cartesian oval

$$(x^2 + y^2 + x'^2 + y'^2 + 2c^2)^2 - 16c^2x'x = 0.$$

If the radius of S is given, the locus of $(x', \pm y')$ is the conic

$$a^4y^2 + b^2(2a^2 - b^2)x^2 = a^4(2b^2 - r^2).$$

If the tangents to S from $(x', \pm y')$ contain a constant angle, the polars of $(x', \pm y')$ with regard to the given conic are touched by a confocal.

64. The envelope of the polar of a fixed point with regard to the circles having double contact with a conic is a parabola.

For the system of circles which have their centres on the minor axis of the conic, the envelope is

$$(y - y')^2 + \frac{4a^2b^2}{c^2}\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1\right) = 0,$$

(x', y') being the fixed point, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ the equation of the conic. Putting $y' = 0$, $x' = c$, we see that the feet of the perpendiculars from a focus on its polars with regard to this system of circles lie on the corresponding directrix.

65. Tangents parallel to a given line are drawn to the circles having double contact with a conic; show that their points of contact lie on a concentric conic passing through the foci.

66. The radical axes of a fixed circle, and the system of circles having double contact with a conic, touch a parabola which passes through the points of intersection of the conic and the fixed circle. We have two parabolas for the two systems of circles, the axis of the parabola being, in each case, parallel to the chords of contact of the circles.

Hence, if circles having double contact with a conic be described to touch a fixed circle, the tangents at the points of contact are touched by the corresponding parabola.

67. *To find the locus of the centres of similitude of a fixed circle, and the system of circles having double contact with a conic.*

If α, β are the perpendiculars from the foci of the conic on a line, $\alpha \tan\frac{1}{2}\theta + \beta \cot\frac{1}{2}\theta - 2b \equiv \Sigma = 0$ will be a tangential equation of a circle having double contact with the conic $\alpha\beta - b^2 = 0$; and if $\gamma^2 - r^2 \equiv \Sigma' = 0$ be the equation of the fixed circle, $\alpha \tan\frac{1}{2}\theta + \beta \cot\frac{1}{2}\theta \pm \frac{2b}{r}\gamma = 0$ will be the equation of a centre of similitude of Σ and Σ'. The envelope of this equation with respect to θ gives the locus required, viz.

$\alpha\beta - \frac{b^2}{r^2}\gamma^2 = 0$, which represents a conic touching the common tangents of Σ' and the given conic.

This conic passes through the points on Σ where it is touched by circles of the system $\Sigma' = 0$.

68. Taking two circles

$$\alpha \tan\tfrac{1}{2}\theta + \beta \cot\tfrac{1}{2}\theta - 2b = 0, \quad \alpha \tan\tfrac{1}{2}\theta' + \beta \cot\tfrac{1}{2}\theta' - 2b = 0,$$

having double contact with the conic $\alpha\beta - b^2 = 0$, we have for their common tangents

$$b^2 - \alpha\beta = (\alpha + \beta)^2 \tan^2\tfrac{1}{2}(\theta - \theta').$$

Hence, when their common tangents touch a concentric, similar and similarly situated conic, the intersection of the circles lies on a confocal conic; for the angle θ is that used in Ex. 55.

For the system of circles which have their centres on the minor axis, if the tangents to the conic from the intersection of the circles contain a constant angle, their common tangents will touch a confocal conic.

69. *If a right line touch two circles having double contact with a conic, its points of contact with them lie on the same concentric, similar and similarly situated conic.*

Let $x \cos w + y \sin w - p = 0$, $y^2 + (x - c\cos\theta)^2 - b^2 \sin^2\theta = 0$ be the equations of the line and one of the circles respectively, and let (x, y) be the coordinates of their point of contact; then

$$x = c\cos\theta + b\sin\theta\cos w, \quad y = b\sin\theta\sin w,$$

$$p = b\sin\theta + c\cos\theta\cos w;$$

hence $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{a^2}\{p^2 + (c^2\cos^2\theta - b^2\sin^2\theta)\sin^2 w\}$$

$$+ \sin^2\theta\sin^2 w = \frac{p^2 + c^2\sin^2 w}{a^2} = \frac{a'^2}{a^2},$$

where a' is half the major axis of the confocal conic touched by the line. The conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{a'^2}{a^2}$, therefore, passes through the points where the line is touched by both the circles.

Putting $a'=c$, we see that the tangents drawn to the circles from a focus have their points of contact on the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=e^2$.

70. *To find the distance between the centres of the two circles which touch the line* $x\cos w+y\sin w-p=0$. From the equation $p-b\sin\theta-c\cos\theta\cos w=0$, or

$$\cos^2\theta\,(a^2\cos^2 w+b^2\sin^2 w)-2cp\cos w\cos\theta+p^2-b^2=0,$$

we find $$\cos\theta-\cos\theta'=\frac{2b\sqrt{\{(a^2\cos^2 w+b^2\sin^2 w-p^2)\}}}{a^2\cos^2 w+b^2\sin^2 w}.$$

Now $d=c(\cos\theta-\cos\theta')$, and if δ be the intercept of the line on the conic

$$\delta=\frac{2ab\sqrt{\{(a^2\cos^2 w+b^2\sin^2 w-p^2)\}}}{a^2\cos^2 w+b^2\sin^2 w};$$

therefore $d=e\delta$.

71. *To find the condition that four circles, having double contact with a conic, should be all touched by the same circle.*

If the circles

$$(x-c\cos\theta)^2+y^2-b^2\sin^2\theta=0,\quad (x-\alpha)^2+(y-\beta)^2-r^2=0,$$

touch one another, we have

$$(r\pm b\sin\theta)^2=(\alpha-c\cos\theta)^2+\beta^2.$$

Now this equation also expresses that the point whose coordinates are $c\cos\theta$, $ib\sin\theta$, lies on the circle

$$(x-\alpha)^2+(y-ir)^2+\beta^2=0.$$

Hence, from the known condition that four points on a conic should lie on a circle, we have $\theta_1+\theta_2+\theta_3+\theta_4=0$.

When this condition is satisfied, a confocal conic passing through the intersection of one pair of circles will also pass through the intersection of the remaining pair (see Ex. 55).

If four circles having double contact with a parabola be all touched by the same fifth circle, the algebraic sum of their radii is equal to zero.

72. Three pairs of osculating circles of a conic can be described to touch a circle which has double contact with the conic.

If circles be described to have double contact with the conic at the points of contact of each pair of these osculating circles, they will intersect on the confocal conic

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - 4c^2} = \tfrac{1}{4}.$$

73. *To find the equations of the circles which touch three given circles having double contact with a conic*

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0\right).$$

From the equation of the circle passing through three points on the conic $\frac{x^2}{c^2} - \frac{y^2}{b^2} = 1$, we obtain (see Ex. 1), if the equation of the circle sought is

$$x^2 + y^2 - 2Ax - 2By + C = 0,$$

$$A = \frac{a^2}{c} \cos\tfrac{1}{2}(\theta_1 + \theta_2) \cos\tfrac{1}{2}(\theta_2 + \theta_3) \cos\tfrac{1}{2}(\theta_3 + \theta_1),$$

$$B = \pm \frac{1}{bc} \sqrt{\{c^2 - a^2 \cos^2\tfrac{1}{2}(\theta_1 + \theta_2)\}\{c^2 - a^2 \cos^2\tfrac{1}{2}(\theta_2 + \theta_3)\} \times \{c^2 - a^2 \cos^2\tfrac{1}{2}(\theta_3 + \theta_1)\}},$$

$$C = \tfrac{1}{2}a^2 \{\cos(\theta_1 + \theta_2) + \cos(\theta_2 + \theta_3) + \cos(\theta_3 + \theta_1)\} - \tfrac{1}{2}(a^2 - 2b^2).$$

Also, if R is the radius of this circle,

$$R=\frac{a^2}{b}\sin\tfrac{1}{2}(\theta_1+\theta_2)\sin\tfrac{1}{2}(\theta_2+\theta_3)\sin\tfrac{1}{2}(\theta_3+\theta_1).$$

The equations of three other pairs of circles are obtained by altering the signs of two of the angles θ_1, θ_2, θ_3.

74. *To find the angle between a tangent to a conic and a circle having double contact with the conic.*

Let the tangent be

$$x\cos w+y\sin w=p=\sqrt{(a^2\cos^2 w+b^2\sin^2 w)},$$

and the circle $x^2+(y-\beta)^2=r^2=a^2\left(1+\frac{\beta^2}{c^2}\right)$. Now, if θ be the angle required, $r\cos\theta=\beta\sin w-p$, and $r\sin\theta=c\sin w+\frac{\beta p}{c}$.

Let $p=a\sin\alpha$, $c\sin w=a\cos\alpha$, $\beta=c\cot\phi$, and $r=\frac{a}{\sin\phi}$, then $\theta=\alpha-\phi$.

Hence, the tangents to a conic meet two fixed circles, having double contact with the conic, at angles whose sum or difference is constant.

Also a variable circle having double contact with a conic meets two fixed tangents to the conic at angles whose sum or difference is constant.

75. *To express the same angle in terms of the coordinates (x, y) of the intersection of the tangent and circle.*

We find $\cos\theta=\frac{a+ex}{\rho}$, or $=\frac{a-ex}{\rho'}$, where ρ, ρ' are the distances of (x, y) from the foci; θ is therefore equal to half the angle which the points of contact of tangents from (x, y) subtend at a focus (Salmon's *Conics*, Art. 121).

This result may be obtained by means of elliptic coordinates; for the differential equation of the tangents to the conic $\mu = a$, is

$$\frac{d\mu}{\sqrt{(\mu^2 - a^2)(\mu^2 - c^2)}} \pm \frac{d\nu}{\sqrt{(a^2 - \nu^2)(c^2 - \nu^2)}} = 0.$$

(Williamson's *Integral Calculus*, p. 249, Ex. 32), and that of the circles is

$$\frac{\mu d\mu}{\sqrt{(\mu^2 - a^2)(\mu^2 - c^2)}} \pm \frac{\nu d\nu}{\sqrt{(a^2 - \nu^2)(c^2 - \nu^2)}} = 0,$$

as can be seen by integrating and transforming to Cartesian coordinates. But when two curves are represented by the equations in elliptic coordinates,

$$Pd\mu + Qd\nu = 0, \quad P'd\mu + Q'd\nu = 0,$$

the angle θ between them is given by

$$\tan\theta = \frac{\{PQ' - P'Q)\sqrt{(\mu^2 - c^2)(c^2 - \nu^2)}}{PP'(\mu^2 - c^2) + QQ'(c^2 - \nu^2)};$$

whence, in the present case,

$$\tan\theta = \frac{\sqrt{(\mu^2 - a^2)(a^2 - \nu)}}{(a^2 \pm \mu\nu)}, \text{ and } \cos\theta = \frac{a^2 \pm \mu\nu}{a(\mu \pm \nu)}.$$

Hence it may be deduced, that the tangents drawn to a conic from the centre of a circle having double contact with it meet the circle on the directrices.

76. *Two circles are described through a point* (x, y) *to have double contact with a conic; to find the angle subtended by their centres at a focus.*

A circle having double contact with the conic being written

$$x^2 + y^2 - 2\beta y - a^2 - \frac{c^2}{b^2}\beta^2 = 0,$$

we have for the two circles of the system which pass through (x, y)

$$\beta_1\beta_2 = \frac{c^2}{b^2}(a^2 - x^2 - y^2), \quad \beta_1 - \beta_2 = \frac{2ac}{b}\sqrt{S},$$

where
$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1;$$

but
$$\tan\phi = \frac{c(\beta_1 - \beta_2)}{c^2 + \beta_1\beta_2}, \quad = \frac{2ab\sqrt{S}}{x^2 + y^2 - a^2 - b^2},$$

or, the angle sought is equal to the angle between the tangents drawn to the curve from (x, y) (Salmon's *Conics*, Art. 169, Ex. 3).

77. *To find the envelope of the tangents to the circles having double contact with a conic at the points where these circles are intersected by a fixed tangent to the conic.*

Writing the circle

$$x^2 + y^2 - 2\beta y - a^2 - \frac{b^2}{c^2}\beta^2 \equiv S = 0,$$

and the fixed tangent

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi - 1 = 0,$$

we have for a point (x', y') of their intersection

$$\frac{x'}{a} = \frac{\cos\phi \pm \left(e + \frac{b\beta}{ac}\sin\phi\right)}{1 \pm e\cos\phi}, \quad \frac{y'}{b} = \frac{\sin\phi \mp \frac{b\beta}{ac}\cos\phi}{1 \pm e\cos\phi}.$$

The tangent to S at (x', y') is, then,

$$ax\left\{\cos\phi \pm \left(e + \frac{b\beta}{ac}\sin\phi\right)\right\} + b(y - \beta)\left\{\sin\phi \mp \frac{b\beta}{ac}\cos\phi\right\}$$
$$- \left(\frac{b^2}{c^2}\beta^2 + \beta y + a^2\right)(1 \pm e\cos\phi) = 0,$$

which touches, when β varies, one or other of two parabolas.

78. Through the centre of a circle having double contact with a conic tangents are drawn to a confocal conic; show that they intersect the circle on two chords of intersection of the conics.

79. Let $S \equiv (x-\alpha)^2 + y^2 - r^2 = 0$ be a circle having double contact with the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, and

$$S' \equiv x^2 + (y-\beta)^2 - r'^2 = 0,$$

a circle having double contact with the confocal conic

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} - 1 = 0,$$

then $$\frac{r^2}{b^2} + \frac{\alpha^2}{c^2} = 1, \quad \frac{r'^2}{a'^2} - \frac{\beta^2}{c^2} = 1;$$

therefore $$\frac{r'^2}{a^2} - \frac{r^2}{b^2} = \frac{\alpha^2 + \beta^2}{c^2} = \frac{r^2 + r'^2 - 2rr' \cos\phi}{c^2},$$

where ϕ is the angle at which S and S' intersect.

Hence, when ϕ is given, the ratio of r to r' is constant. Putting $r = nr'$, we have

$$b^2\alpha^2 + n^2a'^2\beta^2 = c^2(b^2 - n^2a'^2),$$

showing that the line joining the centres of S and S' is normal to

$$\frac{x^2}{b^2} + \frac{y^2}{n^2a'^2} = \frac{c^2}{b^2 - n^2a'^2},$$

which represents a conic confocal with the given conics.

If (x, y) be a centre of similitude of S and S', we have, in the same case,

$$b^2x^2 + a'^2y^2 = \frac{c^2(b^2 - n^2a'^2)}{(n+1)^2};$$

when, therefore, S and S' touch, the point of contact lies on

$$\frac{x^2}{a'^2} + \frac{y^2}{b^2} = 1.$$

80. If, in the preceding example, S and S' cut orthogonally, show that they intersect on two chords of intersection of the given conics.

81. *Two circles having double contact with a conic are described to touch a tangent to a confocal conic; to show that the angle subtended by their centres at a focus is constant.*

Expressing that the circle

$$x^2 + (y - \beta)^2 - r^2 = 0$$

has double contact with the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

and touches the line

$$x \cos w + y \sin w - p = 0,$$

we have

$$\frac{r^2}{a^2} - \frac{\beta^2}{c^2} = 1, \quad \beta \sin w - p = r.$$

Eliminating β, we obtain from the equation in r,

$$r_1 r_2 = \frac{p^2 + c^2 \sin^2 w}{1 - e^2 \sin^2 w};$$

and eliminating r from the equation in β,

$$\beta_1 \beta_2 = \frac{e^2 (a^2 - p^2)}{1 - e^2 \sin^2 w}.$$

But, if ϕ be the angle sought,

$$r_1 r_2 \cos \phi = \beta_1 \beta_2 + c^2;$$

therefore

$$\cos\phi = \frac{e^2(2a^2 - p^2 - c^2\sin^2 w)}{p^2 + c^2\sin^2 w} = c^2\left(\frac{2}{a'^2} - \frac{1}{a^2}\right),$$

where a' is the transverse semi-axis of the confocal conic touching the line.

As a particular case we have :—The circles drawn through the foci to touch a variable tangent to a conic cut each other under a constant angle.

82. *To find the orthogonal trajectory of the system of circles having double contact with a conic.*

The differential equation of the system of circles being

$$\frac{\mu d\mu}{\sqrt{(\mu^2 - a^2)(\mu^2 - c^2)}} \pm \frac{\nu d\nu}{\sqrt{(a^2 - \nu^2)(c^2 - \nu^2)}} = 0,$$

that of the orthogonal trajectory will be

$$\frac{d\mu}{\mu}\sqrt{\left(\frac{\mu^2 - a^2}{\mu^2 - c^2}\right)} \mp \frac{d\nu}{\nu}\sqrt{\left(\frac{a^2 - \nu^2}{c^2 - \nu^2}\right)} = 0.$$

Taking the lower sign and integrating, we obtain

$$\log\left\{\frac{a}{\mu}\sqrt{(\mu^2 - c^2)} + \frac{c}{\mu}\sqrt{(\mu^2 - a^2)}\right\}\left\{\frac{a}{\nu}\sqrt{(c^2 - \nu^2)} + \frac{c}{\nu}\sqrt{(a^2 - \nu^2)}\right\}$$

$$- e\log\{\sqrt{(\mu^2 - a^2)} + \sqrt{(\mu^2 - c^2)}\}\{\sqrt{(a^2 - \nu^2)} + \sqrt{(c^2 - \nu^2)}\}$$

$$= \text{a constant}.$$

If, then, the eccentricity of the conic be the ratio of two integers, the orthogonal trajectory will be algebraic.

For the system of circles which have their centres on the major axis, the orthogonal trajectory is transcendental when the curve is an ellipse and algebraic when the curve is an hyperbola whose eccentricity $= \dfrac{m}{\sqrt{(m^2 - n^2)}}$, where m and n are integers.

83. *A conic has double contact with two fixed circles, the circles belonging to different systems; to show that its eccentricity is given.*

If the circles

$$(x-\alpha)^2+y^2-r^2=0, \quad x^2+(y-\beta)^2-r'^2=0,$$

have double contact with the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

we have $$\frac{\alpha^2}{c^2}+\frac{r^2}{b^2}=1, \quad \frac{r'^2}{a^2}-\frac{\beta^2}{c^2}=1;$$

therefore $$d^2=e^2r'^2-\frac{e^2}{1-e^2}r^2,$$

where d is the distance between the centres of the circles.

Hence it can be shown that the locus of the foci of this system of conics consists of two pairs of circles concentric with the given circles.

84. *To show that the asymptotes, in the same case, pass through fixed points.*

Let A, B be the centres of the given circles, and C the centre of the conic, then ACB is a right angle, and the asymptotes are inclined to CA and CB at constant angles; they, therefore, pass through fixed points on the circle described on AB as diameter.

85. If two circles are connected by a certain relation, an infinite number of equilateral hyperbolæ can be described to have double contact with them.

86. *A conic has double contact with two fixed circles, the circles belonging to different systems; to find the envelope of its directrices.*

The conic

$$x^2 + y^2 - 2ax + k^2 - e^2 (x \cos \alpha + y \sin \alpha)^2 = 0,$$

has double contact with the circles

$$x^2 + y^2 - 2ax + k^2 = 0, \quad (1 - e^2)(x^2 + y^2) - 2ax + k^2 = 0,$$

the origin being a limiting point of the circles.

Now $\qquad x \cos \alpha + y \sin \alpha - p = 0$

will be a directrix of the conic when

$$x^2 + y^2 - 2ax + k^2 + e^2p^2 - 2e^2p (x \cos \alpha + y \sin \alpha) = 0$$

represents a point; which condition gives

$$e^2 (1 - e^2) p^2 - 2e^2ap \cos \alpha + k^2 - a^2 = 0,$$

showing that the directrix touches a conic having the origin for focus.

87. *To find the envelope of the director circles of the same system of conics.*

The equation of the conic being the same as in the last example, the equation of the director circle is

$$(1-e^2)(x^2+y^2) - 2ax(1-e^2\sin^2\alpha) - 2e^2ay\sin\alpha\cos\alpha + k^2(2-e^2) - a^2 = 0.$$

The envelope is, therefore, the Cartesian oval

$$\{(1 - e^2)(x^2 + y^2) - (2 - e^2) ax + k^2 (2 - e^2) - a^2\}^2 - e^4a^2 (x^2 + y^2) = 0.$$

88. Let S be a variable circle having double contact with a hyperbola, and S_1, S_2, two fixed circles of the other system having double contact with the hyperbola; if t_1 be a common tangent of S and S_1, and t_2 of S and S_2, shew that $t_1 - t_2 =$ a constant.

89. *Given three circles with their centres on a line, there is one conic which has double contact with them.*

If S_1, S_2, S_3 are the equations of the circles in Cartesian coordinates, l, m, n the distances between their centres,

$$l\sqrt{S_1} + m\sqrt{S_2} + n\sqrt{S_3} = 0,$$

represents the conic which has double contact with the circles

If Σ_1, Σ_2, Σ_3 are the squares of the intercepts of a line on the circles, the tangential equation of the conic may be written $l\sqrt{\Sigma_1} + m\sqrt{\Sigma_2} + n\sqrt{\Sigma_3} = 0$.

90. If in the preceding example, the line joining the centres of the circles is the major axis of the conic, the eccentricity (e) of the conic is given by

$$\frac{e^2}{1-e^2} = \frac{lmn}{lr_1^2 + mr_2^2 + nr_3^2},$$

where r_1, r_2, r_3 are the radii of the circles.

If the same line is the minor axis of the conic,

$$e^2 = \frac{lmn}{lr_1^2 + mr_2^2 + nr_3^2}.$$

In the first case the foci of the conic are the double points of the involution determined by the centres of similitude of the circles (see Ex. 60).

In the second case the foci of the conic are the points at which the centres of similitude of each pair of circles subtend a right angle.

91. A conic has double contact with two fixed circles, the circles belonging to the same system; shew that its asymptotes touch a parabola of which the middle point of the centres of the circles is the focus.

92. The envelope of the director circle of the same conic is a Cartesian oval, of which the middle point of the centres of the circles is the double focus, and the centres of similitude of the circles are single foci.

93. *To find the locus of the points through which the conics, drawn to have double contact with two fixed circles, cut orthogonally.*

When the circles belong to different systems, the locus is found to be a bicircular quartic having the centres of the circles for double foci.

When the circles belong to the same system, the locus is the circle described on the line joining the centres of similitude of the circles as diameter.

94. A variable circle has double contact with a conic; shew that the tangents, drawn through one of the points of contact to a fixed confocal conic, intercept on the circle segments of given length.

95. PP', QQ' are chords of a conic, parallel to an axis of the curve; if circles through PP' intersect circles through QQ' on the curve, shew that the distance between their centres is constant.

96. *If pairs of circles, having their centres on an axis and cutting each other orthogonally, be described through a fixed point on a conic, the circles passing through the variable points where they meet the curve again have a common radical axis.*

The circle passing through two points x_1, x_2 on the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

may be written

$$x^2+y^2-e^2(x_1+x_2)x+e^2x_1x_2-b^2\equiv S=0.$$

Now, α being the abscissa of the fixed point, the circles

$$x^2+y^2-e^2(\alpha+x_1)x+e^2\alpha x_1-b^2=0,$$

$$x^2+y^2-e^2(\alpha+x_2)x+e^2\alpha x_2-b^2=0,$$

cut orthogonally, when

$$e^4(\alpha+x_1)(\alpha+x_2)-2e^2\alpha(x_1+x_2)+4b^2=0,$$

subject to which condition S passes through the points determined by

$$x=\left(\frac{2}{e^2}-1\right)\alpha,\quad x^2+y^2=\left(1+\frac{4}{e^2}\right)b^2+e^2\alpha^2.$$

These points lie on the circle

$$x^2+y^2-\frac{2a^2}{\alpha}x+a^2+c^2-e^2\alpha^2=0,$$

which represents the circle cutting the conic at right angles at the points where it is met by $x=\alpha$.

97. If r be the radius and x' the abscissa of the centre of the circle

$$x^2+y^2-\frac{2a^2}{\alpha}x+a^2+c^2-e^2\alpha^2=0,$$

we have

$$r^2=\frac{(x'^2-a)(x'^2-c^2)}{x'^2}.$$

This equation is not altered by interchanging a and c; whence we infer that the circles cutting the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

orthogonally at points on a perpendicular to the transverse axis, and the circles related to the conic

$$\frac{x^2}{c^2}-\frac{y^2}{b^2}-1=0,$$

in a similar manner, belong to the same system algebraically.

98. The envelope of the system of circles

$$x^2+y^2-\frac{2a^2}{\alpha}x+a^2-c^2+e^2\alpha^2=0,$$

is
$$(x^2+y^2+a^2+c^2)^3-27a^2c^2x^2=0,$$
or in elliptic coordinates
$$(\mu^2+\nu^2+a^2)^3-27a^2\mu^2\nu^2=0,$$
which may be written
$$\mu^{\frac{2}{3}}+\nu^{\frac{2}{3}}+a^{\frac{2}{3}}=0.$$

99. The length of the tangent drawn from a focus to a circle cutting a conic orthogonally at points P, Q, PQ being parallel to the transverse axis, is equal to the semi-diameter parallel to the tangent at P. Also, the angle between the tangents from a focus to the circle equals $\pi-2\phi$, where ϕ is the eccentric angle at P.

100. If the circles
$$x^2+y^2-\frac{2a^2}{\alpha}x+a^2+c^2-e^2\alpha^2=0,$$
$$x^2+y^2-\frac{2b^2}{\beta}y+b^2-c^2+\frac{c^2}{b^2}\beta^2=0,$$
cut each other orthogonally, the line joining their centres touches the conic
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{c^2}{a^2+b^2}.$$

101. *To describe through a point on a conic circles cutting the curve twice at right angles.*

Eliminating y between the equations
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$
$$x^2+y^2-\frac{2a^2}{\alpha}x+a^2+c^2-e^2\alpha^2=0,$$
and dividing by $x-\alpha$, we obtain
$$e^2(\alpha^2+x\alpha)-2a^2=0.$$

Hence two circles may be described, and the circles passing through the variable points where they meet the curve again passes through fixed points on the minor axis.

102. *If normals be drawn from a point to a conic, and the line joining the feet of two normals pass through a fixed point, to shew that the line joining the feet of the other two normals touches a parabola.*

If one chord of intersection of

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

and

$$c^2xy + b^2y'x - a^2x'y = 0,$$

(*Salmon's Conics*, Art. 181, Ex. 1) is

$$lx + my + n = 0,$$

the other must be

$$\frac{b^2x}{l} + \frac{a^2y}{m} - \frac{a^2b^2}{n} = 0.$$

Hence, if one chord pass through the fixed point (x', y'), the other touches the parabola

$$\sqrt{\left(-\frac{xx'}{a^2}\right)} + \sqrt{\left(-\frac{yy'}{b^2}\right)} = 1.$$

The locus of the intersection of normals, at the extremities of a chord which touches the parabola

$$\sqrt{(\lambda x)} + \sqrt{(\mu y)} = 1$$

is a curve of the third order (Salmon's *Conics*, Art. 370, Ex.).

103. A triangle being inscribed in a conic

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

and circumscribed to a conic

$$S' \equiv \frac{x^2}{a'^2} + \frac{y^2}{b'^2} - 1 = 0,$$

the normals to S at the vertices of the triangle pass through a point; to find the locus of the point.

Writing down the conditions that the sides of the triangle (*Salmon's Conics*, Art. 231, Ex. 2) should touch S', and eliminating a' and b', we obtain a relation which expresses that the normals pass through a point.

Forming the invariants of S' and

$$S'' \equiv 2\,(c^2xy + b^2y'x - a^2x'y),$$

we have, since S'' circumscribes a triangle circumscribed about S',

$$\frac{a'^2}{a^4}x'^2 + \frac{b'^2}{b^4}\,y'^2 = c^2.$$

104. If, in the preceding example,

$$a' = \frac{a^3}{c^2},\ b' = \frac{b^3}{c^2},$$

relations which agree with

$$\frac{a'}{a} - \frac{b'}{b} = 1,$$

the invariant relation between S and S', the normals at the vertices of the triangle intersect on the curve.

The locus of the centre of the circumscribing circle is, in this case (see Ex. 12),

$$a^6x^2 + b^6y^2 = \tfrac{1}{4}a^4b^4;$$

and the envelope of the circumscribing circle is the bicircular quartic

$$(x^2 + y^2 - a^2 - b^2)^2 - a^4b^4\left(\frac{x^2}{a^6} + \frac{y^2}{b^6}\right) = 0.$$

105. *To find the locus of the centroid of the same triangle.*

Eliminating y between

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,\ c^2xy+b^2y'x-a^2x'y=0,$$

we obtain

$$c^4x^4-2a^2c^2x'x^3+a^2(a^2x'^2+b^2y'^2-c^4)\,x^2+\&c.=0.$$

Hence
$$x_1+x_2+x_3+x'=\frac{2a^2}{c^2}\,x',$$

since x' is one of the roots of the equation in x; therefore $3x=\frac{(a^2+b^2)}{c^2}\,x'$, and similarly $3y=-\frac{(a^2+b^2)}{c^2}\,y'$, whence the equation of the locus is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{9}\,\frac{(a^2+b^2)^2}{c^4}.$$

To find the area of the same triangle.

We have
$$2\Delta=\begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix},$$

therefore, squaring

$$4\Delta^2=\begin{vmatrix} \Sigma x^2, & \Sigma xy, & \Sigma x \\ \Sigma xy, & \Sigma y^2, & \Sigma y \\ \Sigma x, & \Sigma y, & 3 \end{vmatrix}.$$

But we have

$$\Sigma x=\frac{(a^2+b^2)}{c^2}\,x',\quad \Sigma y=-\frac{(a^2+b^2)}{c^2}\,y';$$

we also find
$$\Sigma x^2=\frac{2a^2}{c^4}\,\{(a^2+2c^2)\,x'^2-b^2y'^2+c^4\},$$

$$\Sigma y^2=\frac{2b^2}{c^4}\,\{(b^2-2c^2)\,y'^2-a^2x'^2+c^4\},$$

$$\Sigma xy=-\frac{(a^2+b^2)^2}{c^4}\,x'y'.$$

Hence, substituting and reducing by means of the equation

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1,$$

we obtain

$$\Delta^2 = \frac{a^2b^2}{c^4}\left\{(a^2 - 2b^2)^3 \frac{y'^2}{b^4} - (2a^2 - b^2)^3 \frac{x'^2}{a^4}\right\}.$$

106. From the point where a normal to the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ touches the evolute, two other normals are drawn to the curve; show that the line joining their feet is always normal to the conic $a^2x^2 + b^2y^2 = \frac{a^4b^4}{c^4}$.

107. Normals are drawn from a point (x, y) to the parabola $y^2 - 4mx = 0$; if Δ is the area of the triangle formed by their feet, show that

$$\Delta^2 = 4m(x - 2m)^3 - 27m^2y^2.$$

108. Normals to the parabola $y^2 - 4mx = 0$, include a constant angle $= \tan^{-1}t$; show that the locus of their intersection is

$$\{t^3(y^2 + 3m^2 - mx) + t(2m^2 - x^2 + mx)\}^2$$
$$= m(1 + t^2)^2\{4(x - 2m)^3 - 27my^2\}.$$

109. *To draw a normal to an equilateral hyperbola from a point on the curve.*

The curve referred to the asymptotes being written $2xy - a^2 = 0$, the hyperbola which passes through the feet of the normals from (x', y') is $x^2 - y^2 - x'x + y'y = 0$. We have, therefore, to determine x

$$4x'x^4 - 4x'^2x^3 + a^4x - a^4x' = 0,$$

which gives, after dividing by $x - x'$, $x^3 = -\frac{a^4}{4x'}$.

110. The anharmonic ratio of the pencil, which joins any point on the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ to the feet of the normals drawn to the curve from (x, y), is given by the equation

$$\frac{S^3}{T^2}=-\frac{(a^2x^2+b^2y^2-c^4)^3}{a^2b^2c^4x^2y^2}.$$

111. *A conic circumscribes a triangle so that the normals at the vertices pass through a point; to find the locus of its centre.*

The condition that the normals should pass through a point can be written $\cot\vartheta_1+\cot\vartheta_2+\cot\vartheta_3=0$, where $\vartheta_1, \vartheta_2, \vartheta_3$ are the angles which the sides of the triangle make with the diameters bisecting them. But, α, β, γ being the perpendiculars on the sides of the triangle,

$$\cot\vartheta_1=\frac{\beta\sin B-\gamma\sin C+\alpha\sin(B-C)}{2\alpha\sin B\sin C},$$

and similar expressions for ϑ_2, ϑ_3; thus, we have for the equation of the locus

$$\frac{\alpha}{\sin A}(\beta^2-\gamma^2)+\frac{\beta}{\sin B}(\gamma^2-\alpha^2)+\frac{\gamma}{\sin C}(\alpha^2-\beta^2)=0.$$

This cubic is also the locus of foci of conics inscribed in the triangle, whose axis major passes through the centroid.

112. *To find the condition that the normals at six points on a conic should be all touched by a conic.*

Expressing that the normal whose equation is

$$\frac{ax}{\cos\phi}-\frac{by}{\sin\phi}-c^2=0,$$

touches a conic given by the general equation, we obtain an equation of the eighth degree in $\tan\frac{1}{2}\phi$, between the roots of which we find three relations by eliminating the constants in the equation of the conic. Eliminating from these relations two of the roots, we have the condition required

$$PQ - RS = 0,$$

where $P = \Sigma \cos\frac{1}{2}(\phi_1 + \phi_2 + \phi_3 - \phi_4 - \phi_5 - \phi_6)$,

$$Q = \Sigma \sin(\phi_1 + \phi_2 + \phi_3 + \phi_4), \quad R = \Sigma \cos(s - \phi_1),$$

$$S = \sin 2s + \Sigma \sin(\phi_1 + \phi_2),$$

$2s$ being equal to $\Sigma\phi$.

If the normals at six points on the parabola $y^2 - px = 0$ are all touched by a conic, we have $\Sigma y = 0$.

113. *To find the locus of the poles of lines making a constant angle with the conic* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$.

The equation of one of the lines may be written

$$x(b\cos\phi + ma\sin\phi) + y(a\sin\phi - mb\cos\phi) - (ab + mc^2\sin\phi\cos\phi) = 0,$$

where m is the tangent of the given angle; comparing this with $\dfrac{xx'}{a^2} + \dfrac{yy'}{b^2} - 1 = 0$, we have

$$\frac{x'}{a^2} = \frac{b\cos\phi + ma\sin\phi}{ab + mc^2\sin\phi\cos\phi}, \quad \frac{y'}{b^2} = \frac{a\sin\phi - mb\cos\phi}{ab + mc^2\sin\phi\cos\phi};$$

hence, eliminating ϕ, the equation of the locus is

$$m^2\left(\frac{c^4x^2y^2}{a^6b^6} - \frac{x^2}{a^6} - \frac{y^2}{b^6}\right) - \frac{2mc^2xy}{a^4b^4}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + \frac{1}{a^2b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = 0,$$

a quartic having a node at the origin and two nodes at infinity.

114. *To find the condition that three lines making an angle* $\tan^{-1} m$ *with* $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$, *at points whose eccentric angles are* α, β, γ *should meet in a point.*

The points at which lines making a constant angle with S pass through (x', y') are determined as the points of intersection of S with the hyperbola

$$S' \equiv m\,(c^2xy + b^2y'x - a^2x'y) - a^2b^2\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1\right).$$

Hence, we must have $S' + kS = \lambda PQ$, where

$$P = \frac{x}{a}\cos\tfrac{1}{2}(\alpha+\beta) + \frac{y}{b}\sin\tfrac{1}{2}(\alpha+\beta) - \cos\tfrac{1}{2}(\alpha-\beta),$$

$$Q = \frac{x}{a}\cos\tfrac{1}{2}(\gamma+\delta) + \frac{y}{b}\sin\tfrac{1}{2}(\gamma+\delta) - \cos\tfrac{1}{2}(\gamma-\delta);$$

and equating the coefficients of x^2, y^2, xy and the absolute terms in this identity, we obtain

$$k = \lambda\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\gamma+\delta) = \lambda\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\gamma+\delta),$$

$$mc^2ab = \lambda\sin\tfrac{1}{2}(\alpha+\beta+\gamma+\delta),$$

$$a^2b^2 - k = \lambda\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\gamma-\delta);$$

hence,
$$\alpha+\beta+\gamma+\delta = \pi,$$

and
$$\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = \frac{2ab}{mc^2}.$$

115. If a triangle be inscribed in $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ and circumscribed about a concentric conic, three lines at the vertices of the triangle making a constant angle $\tan^{-1} m$ with S will pass through a point.

If the conic touched by the sides is

$$S'' \equiv a'x^2 + b'y^2 + 2h'xy - 1,\quad m = \frac{a^2b^2}{c^2}\left(\frac{h'^2 - a'b'}{h'}\right),$$

and the locus of the points through which the lines pass is the conic

$$a'a^4(y+mx)^2+b'b^4(x-my)^2-2h'a^2b^2(y+mx)(x-my)-m^2c^4=0.$$

116. Lines making a constant angle with a conic at the vertices of an inscribed triangle pass through a point on the curve; show that the locus of the centroid of the triangle is a concentric conic.

117. If three lines making a given angle with the parabola $y^2-4ax=0$ at the points y_1, y_2, y_3 pass through a point, $y_1+y_2+y_3=\frac{2a}{m}$, where m is the tangent of the given angle. Now

$$y_1+y_2+y_3+y_4=0,$$

is the condition that four points should lie on a circle; hence the circle passing through the three points passes through the fixed point on the curve $y=-\frac{2a}{m}$.

118. The locus of the intersection of lines making a constant angle with a conic at the extremities of a chord which passes through a fixed point, is a curve of the third degree (Salmon's *Conics*, Art. 370, Ex.) If the fixed point is on the diameter which cuts the curve at the given angle, the locus reduces to a conic, as the diameter in this case is part of the locus. The locus also reduces to a conic if the fixed point be at infinity.

119. From the point of intersection of two lines making a constant angle with the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, at the extremities of a chord which passes through a fixed point

(x', y'), the two other lines are drawn which make the same angle with the curve; show that the chord joining the feet of the latter pair of lines touches the parabola whose tangential equation is

$$a^2b^2(a^2\lambda^2 + b^2\mu^2) + mc^2(b^2x'\mu\nu + a^2y'\nu\lambda - a^2b^2\lambda\mu) = 0.$$

120. A conic circumscribes a fixed triangle, so that lines making a given angle ϑ with the curve at the vertices pass through a point; the locus of its centre, referred to the triangle, is

$$\frac{\alpha}{\sin A}(\beta^2 - \gamma^2) + \frac{\beta}{\sin B}(\gamma^2 - \alpha^2) + \frac{\gamma}{\sin C}(\alpha^2 - \beta^2) + 2\cot\vartheta\,\alpha\beta\gamma = 0,$$

a cubic passing through the vertices and middle points of sides.

121. If two lines be drawn through the point y' on the parabola $y^2 - 4ax = 0$ to meet the curve again at the angle $\tan^{-1} m$, the equation of the line joining their feet is

$$4max - (2a - my')\,y + 2a(4ma + y') = 0,$$

which, when y' varies, passes through the fixed point

$$y = -\frac{2a}{m}, \quad x = -\frac{a}{m^2}(1 + 2m^2).$$

The locus of this point for different values of m, is the parabola $y^2 + 4a(x + 2a) = 0$.

122. Eight points on a conic lie on a bicircular quartic; show that the sum of their eccentric angles $= 0$ or $2m\pi$.

123. Six points whose eccentric angles are ϕ, &c., are taken on the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0;$$

show that the middle points of the centres of the ten pairs of circles which pass through them lie on the line

$$ax \sin s - by \cos s - \tfrac{1}{8}c^2 (P \sin s + Q \cos s) = 0,$$

where $\quad 2s = \Sigma\phi, \quad P = \Sigma \cos\phi, \quad Q = \Sigma \sin\phi.$

124. Four points whose eccentric angles are ϕ_1, &c., being taken on the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

the equation of the equilateral hyperbola passing through them is

$$2ab \cos s\,(x^2 - y^2) + 2\,(a^2 + b^2) \sin s xy - b\,(a^2 + b^2)\,Px$$
$$- a\,(a^2 + b^2)\,Qy + (a^2 + b^2)(R - \cos s) = 0,$$

where $\quad 2s = \Sigma\phi, \quad P = \Sigma \cos(s - \phi), \quad Q = \Sigma \sin(s - \phi),$

$$R = \Sigma \cos\tfrac{1}{2}(\phi_1 + \phi_2 - \phi_3 - \phi_4).$$

125. Four tangents to the conic $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$, being drawn at the points whose eccentric angles are ϕ_1, &c., the tangential equation of the parabola which touches them is

$$a^2 (R + \cos s)\,\lambda^2 + b^2 (R - \cos s)\,\mu^2 + 2ab \sin s\lambda\mu$$
$$+ aP\nu\lambda + bQ\mu\nu = 0,$$

where P, Q and R have the same meaning as in the preceding example.

126. Given five points, it is possible to find an equilateral hyperbola such that the centre of the circle passing through any three of the points is the pole, with regard to the hyperbola, of the line bisecting at right angles the line joining the remaining two points.

The five points being taken on the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

the asymptotes of the hyperbola are parallel to the axes, and the coordinates of its centre are given by the equations

$$x=\frac{c^2}{4a}(P+\cos 2s), \quad y=-\frac{c^2}{4b}(Q-\sin 2s).$$

127. *A system of conics has double contact with a fixed circle and a fixed conic; to find the locus of the foci.*

Let the fixed conic be

$$\Sigma \equiv a^2\lambda^2+b^2\mu^2-1=0,$$

and the circle

$$S\equiv(\alpha\lambda+\beta\mu-1)^2-r^2(\lambda^2+\mu^2)=0,$$

in tangential coordinates; then, if h is a root of the equation

$$\frac{\alpha^2}{a^2-h^2}+\frac{\beta^2}{b^2-h^2}+\frac{r^2}{h^2}-1=0,$$

$$\Sigma+\frac{h^2}{r^2}S=EF,$$

where E and F are the extremities of a diagonal of the quadrilateral formed by the common tangents of Σ and S, and

$$\vartheta^2E^2+2\vartheta\left(\Sigma-\frac{h^2}{r^2}S\right)+F^2=0,$$

represents a conic having double contact with Σ and S. The latter equation may be written

$$(\vartheta E+F)^2-4\vartheta\frac{h^2}{r^2}(\alpha\lambda+\beta\mu-1)^2+4\vartheta h^2(\lambda^2+\mu^2)=0,$$

showing that the points

$$\vartheta E+F\pm\frac{2h}{r}\sqrt{\vartheta}(\alpha\lambda+\beta\mu-1)=0,$$

are foci. Hence, taking the discriminant with respect to ϑ, the locus of the foci is

$$EF - \frac{h^2}{r^2}(\alpha\lambda + \beta\mu - 1)^2 = 0, \text{ or } \Sigma - h^2(\lambda^2 + \mu^2) = 0,$$

a conic confocal with Σ and passing through the points E and F.

This conic gives the locus of the foci, when the major axis of the variable conic passes through the centre of S. When the minor axis of the variable conic passes through the centre of S, the foci are the anti-points (Salmon's *Higher Plane Curves*, Art. 139) of two points on a conic which are collinear with a fixed point. To find the locus in this case, let the conic be

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

and the fixed point (α, β); forming then the equation of the chords of intersection of S and $(x - x')^2 + (y - y')^2 = 0$ (Salmon's *Conics*, Art. 370, Ex.), and, expressing that this equation is satisfied for the point (α, β), we have the locus required

$$\{(x-\alpha)^2 + (y-\beta)^2\}^2 + \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1\right)(x^2 + y^2 - a^2 - b^2)$$

$$\times \{(x-\alpha)^2 + (y-\beta)^2\} - a^2b^2\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1\right)^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = 0,$$

a bicircular quartic having its foci in common with S, which it cuts at right angles at the points of contact of tangents from (α, β), and having a node at the foot of the perpendicular from (α, β) on its polar with regard to S.

Hence the complete locus for the three systems of variable conics consists of three confocal conics and three nodal bicircular quartics.

128. A system of conics passes through two fixed points and has double contact with a fixed circle; show that the locus of the foci consists of two confocal conics and two bicircular quartics.

129. A system of conics touches two fixed lines and has double contact with a fixed circle; show that the locus of the foci consists of two pairs of lines and two circles.

130. A system of conics passes through a fixed point, touches a fixed line, and has double contact with a fixed circle; show that the locus of the foci consists of two conics and two bicircular quartics.

131. A system of conics has double contact with a fixed circle and touches two fixed conics having double contact with the circle; show that the locus of the foci consists of eight conics and eight bicircular quartics (see Salmon's *Conics*, Art. 387, Ex. 1).

132. A system of conics touches two fixed parallel lines and has double contact with a fixed conic; show that the envelope of the asymptotes consists of two conics, concentric, similar and similarly situated with the fixed conic.

133. *A system of conics has double contact with two fixed confocal conics; to find the locus of the foci.*

Let $\Sigma \equiv a^2\lambda^2 + b^2\mu^2 - 1 = 0$, $\Sigma' \equiv a'^2\lambda^2 + b'^2\mu^2 - 1 = 0$,

where $$a'^2 - a^2 = b'^2 - b^2 = h^2,$$

be the equations of the fixed conics in tangential coordinates; then the equation of the system which have their centres on the axis major of Σ and Σ' may be written

$$\vartheta^2 h^2 (c\lambda + 1)^2 + 2\vartheta (b'^2\Sigma + b^2\Sigma') + h^2 (c\lambda - 1)^2 = 0;$$

and the equations for determining the foci give

$$C(x^2-y^2)-2Gx+A-B=0, \quad y(Cx-G)=0,$$

where $$C=\frac{1}{b^2b'^2}\{h^2(\vartheta^2+1)-2(b^2+b'^2)\vartheta\},$$

$$G=\frac{ch^2}{b^2b'^2}(\vartheta^2-1), \quad A-B=\frac{c^2}{b^2b'^2}\{h^2(\vartheta^2+1)+2(b^2+b'^2)\vartheta\}.$$

Hence, besides the axis of x, the locus consists of the two circles

$$x^2+y^2\pm c\left(\frac{b^2+b'^2}{bb'}\right)y-c^2=0.$$

The equation of the system which is concentric with Σ and Σ' may be written

$$h^2\cos\vartheta(\lambda^2-\mu^2)+2h^2\sin\vartheta\lambda\mu+\Sigma+\Sigma'=0,$$

the equations for determining the foci giving in this case

$$x^2-y^2-c^2-h^2\cos\vartheta=0, \quad 2xy-h^2\sin\vartheta=0;$$

hence the locus is the oval of Cassini

$$(x^2+y^2)^2-2c^2(x^2-y^2)+c^4-h^4=0.$$

The locus of the intersection of rectangular tangents to Σ and Σ' is the director circle of every conic of this system; and, if $h^2=-(a^2+b^2)$, all the conics of the system are equilateral hyperbolas.

134. The differential equation in elliptic coordinates

$$\frac{d\mu}{\sqrt{\{\mu^2-a^2)(a'^2-\mu^2)\}}}\pm\frac{d\nu}{\sqrt{\{(\nu^2-a^2)(a'^2-\nu^2)\}}}=0,$$

represents a system of conics having double contact with the confocal conics $\nu=a$, $\mu=a'$; for the integral of the equation may be written in either of the forms

$$A\mu\nu+B\sqrt{\{(\mu^2-a^2)(\nu^2-a^2)\}}+C=0,$$

$$A'\mu\nu+B'\sqrt{\{(a'^2-\mu^2)(a'^2-\nu^2)\}}+C'=0,$$

where A, B, &c. are constants, and $\mu\nu = cx$,

$$\sqrt{\{(\mu^2 - a^2)(\nu^2 - a^2)\}} = ab\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)},$$

$$\sqrt{\{(a'^2 - \mu^2)(a'^2 - \nu^2)\}} = a'b'\sqrt{\left(1 - \frac{x^2}{a'^2} - \frac{y^2}{b'^2}\right)}.$$

This system of conics have their centres on the axis major of the given conics. The differential equation of the system having their centres on the axis minor is

$$\frac{\mu d\mu}{\sqrt{\{(\mu^2 - c^2)(\mu^2 - a^2)(a'^2 - \mu^2)\}}} \pm \frac{\nu d\nu}{\sqrt{\{(c^2 - \nu^2)(a^2 - \nu^2)(a'^2 - \nu^2)\}}} = 0.$$

135. The differential equation of the system which is concentric with the given conics is

$$\frac{d\mu}{\sqrt{(\{\mu^2 - c^2)(\mu^2 - a^2)(a'^2 - \mu^2)\}}} \pm \frac{d\nu}{\sqrt{\{(c^2 - \nu^2)(a^2 - \nu^2)(a'^2 - \nu^2)\}}} = 0.$$

If two of this system cut at right angles, their intersection lies on the bicircular quartic

$$(x^2 + y^2)^2 - (a^2 + a'^2)x^2 - (b^2 + b'^2)y^2 + a^2a'^2 + b^2b'^2 = 0.$$

136. Two conics of the same system are described through a point to have double contact with two confocal conics; show that they make equal angles with the conics confocal with the given ones which pass through the point.

137. *To find the differential equation in elliptic coordinates of the evolutes of a system of confocal conics.*

The evolute of $\mu = a$, is

$$(a^2\mu^2\nu^2)^{\frac{1}{3}} + \{b^2(\mu^2 - c^2)(c^2 - \nu^2)\}^{\frac{1}{3}} - c^2 = 0.$$

Now this is the relation which exists between μ, ν and a, when it is possible to determine λ, so that the expression

$$x(x - c^2) + \lambda(x - \mu^2)(x - \nu^2)(x - a^2)$$

may be a perfect cube. Hence, if

$$(x-\mu^2)(x-\nu^2)(x-a^2)=\phi(x),$$

we have $$\lambda\phi(x)=(px+q)^3-x(x-c^2).$$

Differentiating this identity, and substituting μ^2, ν^2 and a^2 for x in succession, we obtain

$$\frac{\lambda d(\mu^2)}{\{\mu^2(\mu^2-c^2)\}^{\frac{2}{3}}}+\frac{3(\mu^2dp+dq)}{\phi'(\mu^2)}=0,$$

$$\frac{\lambda d(\nu^2)}{\{\nu^2(c^2-\nu^2)\}^{\frac{2}{3}}}+\frac{3(\nu^2dp+dq)}{\phi'(\nu^2)}=0,$$

$$\frac{3(a^2dp+dq)}{\phi'(a^2)}=0;$$

therefore, since

$$\Sigma\frac{\mu^2}{\phi'(\mu^2)}=\Sigma\frac{1}{\phi'(\mu^2)}=0,$$

$$\frac{\mu d\mu}{\{\mu^2(\mu^2-c^2)\}^{\frac{2}{3}}}+\frac{\nu d\nu}{\{\nu^2(c^2-\nu^2)\}^{\frac{2}{3}}}=0.$$

It follows from this equation that only one real curve of the system passes through a point.

138. If t, p, n be the tangents drawn from the centre of an equilateral hyperbola to the circumscribing, polar, and nine-point circles of a triangle whose area is Δ, and t', p', n' be the corresponding values for the reciprocal triangle, show that

$$\frac{t^2}{\Delta}=\frac{t'^2}{\Delta'},\quad n^2=\frac{\Delta}{2\Delta'}p'^2,\quad n'^2=\frac{\Delta'}{2\Delta}p^2.$$

139. *To find the condition that a conic S_1 should circumscribe a triangle whose reciprocal with regard to S_3 is self-conjugate with regard to S_2.*

We must have $\Theta = 0$, between S_1 and the reciprocal of S_2 with regard to S_3; we find thus $\Theta_{133}\Theta_{233} = \Delta\Theta_{123}$, where Δ is the discriminant of S_3 and Θ_{133}, &c. are the coefficients of the several powers of l, m, n in the discriminant of $lS_1 + mS_2 + nS_3$.

If
$$S_1 \equiv x^2 + y^2 - 2\alpha_1 x - 2\beta_1 y + k_1^2,$$
$$S_2 \equiv x^2 + y^2 - 2\alpha_2 x - 2\beta_2 y + k_2^2,$$
$$S_3 \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1,$$
we have
$$k_1^2 k_2^2 - 2(a^2\alpha_1\alpha_2 + b^2\beta_1\beta_2) + a^4 + b^4 = 0.$$

Hence, when S_3 and S_1 are fixed, S_2 cuts orthogonally the circle
$$k_1^2(x^2 + y^2) - 2a^2\alpha_1 x - 2b^2\beta_1 y + a^4 + b^4 = 0.$$

140. Show that the intersection of the perpendiculars of a triangle formed by three tangents to an equilateral hyperbola and the centre of the circle passing through the points of contact of the tangents are conjugate with respect to the curve.

141. If a conic S_1 touch the sides of a triangle whose reciprocal with regard to S_3 is self-conjugate with regard to S_2, show that $\Theta_{322}\Theta_{311} = \Phi$, where Φ is the invariant which corresponds in tangential coordinates to Θ_{123}.

142. If a conic S_1 circumscribe a triangle whose reciprocal with regard to S_3 is inscribed in S_2, show that
$$(\Theta_{133}\Theta_{233} - \Delta\Theta_{123})^2 = 4\Delta^2(\Theta_{311}\Theta_{322} - \Phi).$$

143. If a conic U be such that an inscribed and a self-conjugate triangle are reciprocal with regard to a conic V, show that $\Theta^2 = 2\Delta\Theta'$.

If
$$U \equiv (x-\alpha)^2 + (y-\beta)^2 - r^2,$$
$$V \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1,$$
we have $(\alpha^2 + \beta^2 - r^2)^2 - 2(a^2\alpha^2 + b^2\beta^2) + a^4 + b^4 = 0.$

144. If a conic U be such that a circumscribed and a self-conjugate triangle are reciprocal with regard to a conic V, show that $\Theta'^2 = 2\Delta'\Theta$.

145. If a conic U circumscribe two triangles which are reciprocal with regard to a conic V, we have
$$\Theta^3 - 4\Delta\Theta\Theta' + 8\Delta'\Delta^2 = 0,$$
which is also the condition that U should circumscribe quadrilaterals circumscribed about V. If U be a circle and V an equilateral hyperbola, the areas of the triangles are equal.

146. If two triangles, reciprocal with regard to a conic U, be such that their centroids are conjugate with respect to the curve, show that a conic circumscribing either triangle, so that the tangent at each vertex shall be parallel to the opposite side, will pass through the centre of U.

147. Let lines drawn from the centre of a conic to the vertices of a triangle whose area is Δ meet the sides of the triangle in L, M, N; if the area of the triangle L, M, N is equal to A, and A', Δ' be corresponding values for the reciprocal triangle, show that
$$\frac{A}{\Delta} = \frac{A'}{\Delta'}.$$

148. A system of conics passes through the four points determined by the equations in rectangular coordinates

$$ax^2 + by^2 + 2gx + 2fy \equiv U = 0,$$
$$x^2 + y^2 - k^2 \equiv V = 0;$$

show that the locus of the vertices consists of the two cubics

$$(a - b)\, xy^2 + g\,(y^2 - x^2) - 2fxy - k^2\,(ax + g) = 0,$$
$$(a - b)\, x^2y + f\,(y^2 - x^2) + 2gxy + k^2\,(by + f) = 0.$$

These two cubics pass through the points of intersection of U and V and the vertices of the common self-conjugate triangle of U and V, cutting each other at right angles at these seven points.

149. Let the tangent to an hyperbola at a point P meet the asymptotes in A, B; if perpendiculars to the asymptotes at A, B meet the normal at P in A', B', show that the middle point of A', B' is the centre of curvature at P.

150. If six osculating circles of the conic $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$ be described to cut orthogonally the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

show that their centres lie on the conic

$$(2gx + 2fy + c)^2 - (a^2 + b^2)\,(2gx + 2fy + c)$$
$$- 3\,(a^2x^2 + b^2y^2) + a^4 + b^4 - a^2b^2 = 0.$$

151. If the osculating circle of a conic cut the director circle at an angle ϑ, show that $\cos\vartheta = \dfrac{3p}{2k}$, where p is the perpendicular from the centre on the tangent at the point of contact, and k is the radius of the director circle.

152. A, B, C are three points forming a triangle inscribed in a conic, so that the tangent at each vertex is parallel to the opposite side; P is any point on the curve. If PA, PB, PC meet the opposite sides of the triangle in L, M, N, show that the area of the triangle LMN is double that of the triangle ABC.

153. Two conics U and V are inscribed in the same quadrilateral, the focus of U being the centre of V; show that the points of contact of V with the sides of the quadrilateral lie on a circle.

154. If the two conics described through a point P to touch four fixed lines cut orthogonally, show that P lies on the circular cubic, which is the locus of the foci of conics touching the lines.

155. If F be the focus of a parabola passing through three points A, B, C, show that a circle, touching the lines which bisect FA, FB, FC at right angles, passes through the centre of the circle circumscribing the triangle ABC.

156. If two circles touch the sides of triangles self-conjugate with regard to a conic S, show that their centres of similitude are conjugate with respect to S.

157. Two conics are described through P to touch the lines A, B, C, D, and two to touch A, B, C, E. If the tangents to the conics at P have a constant anharmonic ratio, the locus of P is a conic touching D and E.

158. *To find the locus of the intersection of rectangular tangents to the conics whose equations to rectangular axes are*

$$ax^2 + by^2 + 2gx + 2fy + c \equiv S = 0,$$
$$bx^2 + ay^2 - 1 \equiv S' = 0.$$

If ϑ be the angle which a tangent makes with the axis, we have, from the equations of the pairs of tangents which pass through (x, y),

$$S\ (a\cos^2\vartheta + b\sin^2\vartheta) = \{(ax+g)\cos\vartheta + (by+f)\sin\vartheta\}^2,$$
$$S'\ (a\cos^2\vartheta + b\sin^2\vartheta) = (bx\sin\vartheta - ay\cos\vartheta)^2;$$

whence, by division, if $\dfrac{S}{S'} = \mu^2$, $\tan\vartheta = \dfrac{\mu ay + ax + g}{\mu bx - by - f}$.

Substituting this value of ϑ in the equation of the second pair of tangents, we obtain

$$S'\{a(\mu bx - by - f)^2 + b(\mu ay + ax + g)\} = \{ab(x^2+y^2) + bgx + afy\}^2;$$

or, restoring the value of μ,

$$abS(1+S') + S'(abS + af^2 + bg^2 - abc)$$
$$+ 2ab\sqrt{(SS')}\{(a-b)xy + gy - fx\} = \{ab(x^2+y^2) + bgx + afy\}^2.$$

Now $\{ab(x^2+y^2) + bgx + afy\}^2$

$$\equiv (abS + af^2 + bg^2 - abc)(1+S') - ab\{(a-b)xy + gy - fx\}^2$$

identically; therefore we have

$$\{(a-b)xy + gy - fx\}^2 + 2\sqrt{(SS')}\{(a-b)xy + gy - fx\} + SS'$$
$$= \frac{af^2 + bg^2 - abc}{ab},$$

or $$\left\{(a-b)xy + gy - fx \pm \sqrt{\left(\frac{af^2 + bg^2 - abc}{ab}\right)}\right\}^2 = SS',$$

which represents a pair of bicircular quartics, each having quartic contact with the two given conics.

159. If the tangents to the conic $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$ from (x, y) form a harmonic pencil with perpendiculars to the tangents to the conic $\dfrac{x^2}{a'^2} + \dfrac{y^2}{b'^2} - 1 = 0$ from the same point, show that the locus of (x, y) is the bicircular quartic

$$(x^2+y^2)^2 - (a^2+a'^2)x^2 - (b^2+b'^2)y^2 + a^2a'^2 + b^2b'^2 = 0.$$

160. If the tangents from a point P to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

contain an angle α, show that the tangents from P to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{4a^2b^2+(a^2-b^2)^2\sin^2\alpha}{(a^4-b^4)\sin^2\alpha}$$

contain an angle β, where

$$\cos\beta=\left(\frac{a^2-b^2}{a^2+b^2}\right)\cos\alpha.$$

161. A conic passes through four fixed points; prove that the locus of the pole of a given triangle with respect to the conic (Salmon's *Conics*, Art. 375) is an unicursal quartic, of which the vertices of the given triangle are nodes.

162. A conic touches four fixed lines; prove that the locus of the pole of a given triangle with respect to the conic is a conic circumscribing the given triangle.

163. A triangle is inscribed in a given conic so that the polar of a fixed point with respect to the triangle is a fixed line; show that the triangle is circumscribed about a fixed conic.

164. If Σ be the sum of the squares of the lengths of the six chords of intersection of the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, and the circle $(x-\alpha)^2+(y-\beta)^2-r^2=0$, show that

$$\Sigma=16\left\{r^2-(a^2-b^2)^2\left(\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}\right)\right\}.$$

For the parabola $y^2-px=0$, $\Sigma=4(\delta^2-p^2)$, where δ is the intercept of the circle on the axis.

165. A conic passes through four fixed points; show that the right line which passes through the middle points of the diagonals of the quadrilateral formed by drawing tangents at these points touches a conic. Show also that the right line, which passes through the intersections of the perpendiculars of the four triangles formed by the same tangents, passes through a fixed point.

166. *Given four concyclic points on a conic, to show that perpendiculars at these points to the focal radii vectores are all touched by the same circle.*

Eliminating r between the polar equation of the conic $r = \dfrac{l}{1 + e\cos\vartheta}$, and the circle $r^2 - 2r(\alpha\cos\vartheta + \beta\sin\vartheta) + k^2 = 0$, we have

$$l^2 - 2l(1 + e\cos\vartheta)(\alpha\cos\vartheta + \beta\sin\vartheta) + k^2(1 + e\cos\vartheta)^2 = 0,$$

and expressing that the line

$$x\cos\vartheta + y\sin\vartheta = r = \frac{l}{1 + e\cos\vartheta}$$

touches the circle $(x - \alpha')^2 + (y - \beta')^2 - r'^2 = 0$, we have

$$(\alpha'\cos\vartheta + \beta'\sin\vartheta - r')(1 + e\cos\vartheta) - l = 0;$$

and these two conditions coincide, if

$$l\alpha' = 2l\alpha - ek^2, \quad \beta' = 2\beta, \quad lr' = k^2.$$

167. Show that the polar circle of the triangle formed by three tangents to an equilateral hyperbola touches the nine-point circle of the triangle formed by the points of contact of the tangents at the centre of the curve.

168. A conic passes through two fixed points and touches two fixed lines; show that the envelope of the director circle is one or other of two nodal bicircular quartics, the node common to each being the intersection of the fixed lines.

169. Two circles are described through two points A, B on a conic to touch the curve elsewhere; show that the angle between the circles is equal to double the angle which their points of contact subtend at A or B.

170. A parabola passes through three fixed points; show that the locus of the intersection of the perpendiculars of the triangle formed by drawing tangents at these points is a curve of the fourth class.

171. A conic circumscribes a triangle so that two of the vertices are at the extremities of conjugate diameters; show that the locus of its centre is a conic with regard to which the given triangle is self-conjugate.

172. A conic passes through three fixed points A, B, C; if the diameter of the conic parallel to AB be given in length, show that the locus of its centre is a conic, whose asymptotes are parallel to AC, BC, and with regard to which C is the pole of AB.

173. If from any point of the quartic curve

$$2(x^2+y^2)\left(1-\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-(a^2-b^2)\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=0,$$

tangents be drawn to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0,$$

show that the sum of the squares of their lengths is equal to a^2-b^2.

174. Given a vertex and two points on a conic, show that the locus of its centre is a cubic, of which the given vertex is a node.

175. If a chord of an equilateral hyperbola subtend a right angle at a fixed point, show that the locus of the middle point of the chord is a curve of the third order.

176. A point P moves along a right line; show that the locus of the foot of the perpendicular from P on its polar with regard to a conic is a circular cubic passing through the foci of the conic.

177. Show that the line

$$(a^2 - b^2)(lx - my) = a^2 + b^2 + 2ab \cot\phi \sqrt{(a^2l^2 + b^2m^2 - 1)}$$

passes through the two points on the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

at which the chord $lx + my = 1$ subtends the angle ϕ.

178. A circle passes through the centre of the conic

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0;$$

show that the product of the perpendiculars from the centre of S on a pair of chords of intersection of S and the circle is equal to $\dfrac{a^2b^2}{a^2 - b^2}$.

179. A circle touches two fixed tangents to a conic; show that a pair of its chords of intersection with the conic are parallel to given lines.

180. A circle passes through two fixed points on a conic; show that the extremities of one of the diagonals of the quadrilateral formed by the common tangents of the circle and the conic lie on a given confocal conic.

181. A circle Σ cuts the circle $S \equiv x^2 + y^2 - k^2 = 0$ orthogonally, and touches the conic

$$V \equiv ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0;$$

show that the equation of the reciprocal polar of the locus of the centre of Σ is the trinodal quartic

$$(k^2 V - cS)^2 + 4k^2 (gx + fy + c)^2 S = 0.$$

182. Three fixed points A, B, C are taken on an hyperbola so that their centroid lies on the curve; if lines be drawn bisecting PA, PB, PC at right angles, where P is a variable point on the hyperbola, the centroid of the triangle so formed will lie on a line which passes through the centre of the circle circumscribing the triangle A, B, C.

183. The centre of the circle passing through a point P, and the point of contact of tangents from P to the conic

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

lies on the director circle of S; show that the locus of P is the bicircular quartic

$$(x^2 + y^2)^2 - 2c^2 (x^2 - y^2) + c^4 - 4(a^2 + b^2)(x^2 + y^2) = 0,$$

where $c^2 = a^2 - b^2$.

184. Let A, B, C be three points on a conic U, and Σ a circle having double contact with U at points on a parallel to its minor axis. Let t, t' be the lengths of the direct and transverse common tangents of Σ and the circle circumscribing the triangle A, B, C, and let p_1, p_2, p_3, p_4 be the perpendiculars from the centres of the four circles which touch AB, BC, CA on the chord of contact of Σ and U. Show that $t^2t'^2 = e^2 p_1 p_2 p_3 p_4$, where e is the eccentricity of U.

185. S, H are the given foci of a conic U, and S', H' of a conic V; if U and V are similar, show that their common tangents envelope a conic which touches the lines SS', SH', $S'H$, HH'.

186. The pairs of tangents from a point P to two concentric equilateral hyperbolas contain equal angles; show that the locus of P is an equilateral hyperbola concentric with the given ones.

187. The perpendicular to a chord of a conic at its middle point passes through a fixed point O on one of the axes of the curve; show that the chord touches a parabola, of which O is the focus.

188. From any point of the circle $x^2+y^2=a^2+b^2+h^2$ pairs of tangents are drawn to the confocal conics

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0, \quad \frac{x^2}{a^2+h^2}+\frac{y^2}{b^2+h^2}-1=0;$$

show that the difference of the squares of the reciprocals of their lengths is the same for each pair.

189. A triangle is circumscribed about a conic so that the lines, which are drawn from each vertex to the point of contact of the opposite side with the circle escribed to that side, intersect on the curve; show that the circle inscribed in the triangle formed by the middle points of the sides passes through the centre of the conic.

190. Through the points on the conic $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$, whose eccentric angles are α, β, $-\alpha$, $-\beta$, a circle is described; if ϑ be the angle subtended by α, β at the centre of the circle, show that $\tan\frac{1}{2}\vartheta=\dfrac{a}{b}\tan\frac{1}{2}(\alpha-\beta)$.

191. If from points on the line $\lambda x + \mu y + \nu = 0$, normals be drawn to the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, show that the poles of the chords joining their extremities form a quadrilateral inscribed in the cubic

$$\frac{\lambda x}{a^2}\left(1 - \frac{y^2}{b^2}\right) - \frac{\mu y}{b^2}\left(1 - \frac{x^2}{a^2}\right) + \frac{\nu}{a^2 - b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 0.$$

192. If the foot of the perpendicular from a point P on its polar with regard to the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ move along the right line $\lambda x + \mu y + \nu = 0$, show that the locus of P is the cubic

$$\frac{\lambda x}{b^2}\{b^4 + (a^2 - b^2)\, y^2\} + \frac{\mu y}{a^2}\{a^4 - (a^2 - b^2)\, x^2\} + \nu a^2 b^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right) = 0.$$

193. Two tangents to a conic, whose foci are given, pass through fixed points on the axis minor, show that the locus of their intersection is a circle passing through the foci.

If the tangents, instead of passing through fixed points, are parallel to fixed lines, the locus is an equilateral hyperbola passing through the foci.

194. Given two similar parabolas, show that a right line, which cuts off from them areas which are in a constant ratio to one another, touches a parabola similar to the given ones.

195. A variable tangent to a conic whose point of contact is P meets a concentric, similar, and similarly situated conic in A, B; show that the lines joining A, B to the points where the normal at P meets the axes of the conic are inclined to AB at constant angles.

196. Show that tangents to the parabolas

$$S \equiv y^2 + ax + c = 0, \quad S' \equiv x^2 + by + c' = 0,$$

intersect at right angles on

$$(2abxy - ab^2x - a^2by - cb^2 - c'a^2)^2 - 4a^2b^2SS' = 0,$$

a nodal circular cubic having triple contact with S and S'.

197. Show that the polar equation of the evolute of a parabola, referred to the focus, may be written

$$\left(\frac{m}{\rho}\right)^{\frac{1}{3}} = (\cos\tfrac{1}{2}\theta)^{\frac{2}{3}} - (\sin\tfrac{1}{2}\theta)^{\frac{2}{3}}.$$

If the evolute cut a confocal parabola at an angle ϕ, show that $\cot^3\phi = \cot\frac{1}{2}\theta$.

198. A triangle is inscribed in the parabola

$$U \equiv y^2 - px = 0,$$

and circumscribed about the parabola

$$(\alpha x + \beta y)^2 - 4p\beta^2 x + \gamma\alpha y - p\beta\gamma = 0,$$

show that the normals to U at the vertices of the triangle pass through a point, and show that the locus of this point is a right line.

199. Tangents to a parabola intersect on a parabola having the same focus and axis; show that the product or ratio of the tangents of the halves of the angles which they make with the axis is constant.

200. Three circles osculate the parabola $y^2 - px = 0$ at the points y_1, y_2, y_3; show that the equation of the circle which cuts them orthogonally is

$$(y_1y_2 + y_2y_3 + y_3y_1)(x^2 + y^2) - \tfrac{3}{4}(y_1 + y_2)(y_2 + y_3)(y_3 + y_1)\,y$$

$$- \frac{1}{p}\{y_1^2y_2^2 + y_2^2y_3^2 + y_3^2y_1^2 + y_1y_2y_3(y_1 + y_2 + y_3)\}\,x$$

$$+ \tfrac{1}{2}\{y_1^2y_2^2 + y_2^2y_3^2 + y_3^2y_1^2 + y_1y_2y_3(y_1 + y_2 + y_3)\} + \frac{3y_1^2y_2^2y_3^2}{p^2} = 0.$$

If four osculating circles be cut orthogonally by the same circle, show that $\Sigma \dfrac{1}{y} = 0$.

201. Show that there are a real pair of lines passing through the points of intersection of the point circle

$$(x - x')^2 + (y - y')^2 = 0,$$

and the conic $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

whose equations are

$$L \equiv \frac{\nu x}{a} + \sqrt{(c^2 - \nu^2)}\frac{y}{b} - \frac{1}{c}\{a\mu + b\sqrt{(\mu^2 - c^2)}\} = 0,$$

$$M \equiv \frac{\nu x}{a} - \sqrt{(c^2 - \nu^2)}\frac{y}{b} - \frac{1}{c}\{a\mu - b\sqrt{(\mu^2 - c^2)}\} = 0,$$

where μ, ν are the elliptic coordinates of (x', y'). Show also that

$$(x - x')^2 + (y - y')^2 = (a^2 - \nu^2)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + LM.$$

If $$\frac{x}{a}\cos\tfrac{1}{2}(\alpha + \beta) + \frac{y}{b}\sin\tfrac{1}{2}(\alpha + \beta) - \cos\tfrac{1}{2}(\alpha - \beta) = 0$$

be the equation of a chord of the conic, show that the elliptic coordinates of the antipoints (Salmon's *Curves*, Art. 139) of its extremities are given by

$$\mu = c\cos\{\theta \pm \tfrac{1}{2}(\alpha - \beta)\}, \quad \nu = c\cos\tfrac{1}{2}(\alpha + \beta),$$

where $\cos\theta = \dfrac{a}{c}$.

202. Given five lines

$$\alpha \equiv x\cos\alpha + y\sin\alpha - p = 0, \text{ \&c.},$$

touching the parallel curve of a conic, the locus of the centre of the conic is

$$\begin{vmatrix} \alpha^2, & \beta^2, & \gamma^2, & \delta^2, & \varepsilon^2 \\ \alpha, & \beta, & \gamma. & \delta, & \varepsilon \\ \cos 2\alpha, & \cos 2\beta, & \cos 2\gamma, & \cos 2\delta, & \cos 2\varepsilon \\ \sin 2\alpha, & \sin 2\beta, & \sin 2\gamma, & \sin 2\delta, & \sin 2\varepsilon \\ 1, & 1, & 1, & 1, & 1 \end{vmatrix} = 0,$$

which represents a curve of the second order, as terms of the third order vanish identically.

203. A conic whose foci are F, F' is inscribed in a triangle; if F lie on the polar circle of the triangle, show that an equilateral hyperbola can be described having F' for centre and passing through the feet of the perpendiculars from F' on the sides.

If F lie on the inscribed circle, show that a parabola can be described having F' for focus and passing through the feet of the perpendiculars from F' on the sides.

204. If P be the principal parameter of the parabola osculating a curve at the point (x, y), show that

$$P = \frac{54q^5}{(9q^4 - 6prq^2 + r^2 + p^2r^2)^{\frac{3}{2}}},$$

where $$p = \frac{dy}{dx}, \quad q = \frac{d^2y}{dx^2}, \quad \&c.$$

205. If x', y' be the coordinates of the centre of the equilateral hyperbola osculating a curve at the point (x, y), show that

$$x' = x + \frac{3qr(1+p^2)}{(9q^4 - 6prq^2 + r^2 + p^2r^2)},$$

$$y' = y + \frac{3q(1+p^2)(pr - 3q^2)}{9q^4 - 6prq^2 + r^2 + p^2r^2}.$$

206. If ρ be the radius of curvature of the locus of the centre of the conic osculating a curve at the point (x, y), show that

$$\rho = \frac{(9q^4 - 6prq^2 + r^2 + p^2r^2)^{\frac{3}{2}}(9q^2t - 45qrs + 40r^3)}{q(3qs - 5r^2)^3}.$$

207. A triangle is circumscribed about the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

and inscribed in the circle $(x-\alpha)^2 + (y-\beta)^2 - r^2 = 0$; if s be half the sum of the angles subtended by the foci of the conic at the vertices of the triangle, show that

$$\alpha^2 + \beta^2 - r^2 - c^2 = 2ar\cos s, \quad c\beta = ar\sin s.$$

208. A circle circumscribing a triangle circumscribed about a conic meets the conic in A, B, C; show that the algebraic sum of the diameters parallel to AB, BC, CA is equal to zero.

209. A triangle is inscribed in the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, the semi-diameters parallel to the sides of which are b_1, b_2, b_3; if the circumscribing circle of the triangle cut the conic at an angle ϕ, show that

$$ab\cot\phi = \pm\frac{b_2^2b_3^2\sqrt{\{(a^2-b_1^2)\}}(b_1^2-b^2)}{(b_1^2-b_2^2)(b_1^2-b_3^2)} \pm \frac{b_3^2b_1^2\sqrt{(a^2-b_2^2)}(b_2^2-b^2)}{(b_2^2-b_1^2)(b_2^2-b_3^2)}$$
$$\pm\frac{b_1^2b_2^2\sqrt{(a^2-b_3^2)}(b^2-b_3^2)}{(b_3^2-b_1^2)(b_3^2-b_2^2)},$$

the variation of signs corresponding to the four values of ϕ.

210. A triangle is inscribed in a parabola, the sides of which make angles α, β, γ with the axis of the curve; if the

circumscribing circle of the triangle cut the parabola at an angle ϕ, show that

$$\cot\phi = \frac{\sin^2\alpha + \sin^2\beta + \sin^2\gamma - \sin^2(\alpha+\beta+\gamma)}{2\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta)},$$

the three other values of ϕ being obtained by changing the signs of α, β, γ.

211. If s be the sum of the angles at which the circle $x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = 0$ cuts the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, show that

$$\tan s = \frac{8a^2b^2\alpha\beta}{c^2(a^2+b^2+k^2)^2 - 4(a^4\alpha^2 - b^4\beta^2)}.$$

If the same circle meet the parabola $y^2 - 4mx = 0$, show that

$$\tan s = \frac{4m\beta}{k^2 + 2m\alpha - 3m^2}.$$

212. Show that the product of the lengths of the common tangents of the circle $(x-\alpha)^2 + (y-\beta)^2 - r^2 = 0$ and the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ is equal to

$$\frac{(h_1^2 - h_2^2)^2(h_2^2 - h_3^2)^2(h_3^2 - h_3^2)^2}{c^4r^4 - 2c^2r^2(a^2\beta^2 - b^2\alpha^2) + (a^2\beta^2 + b^2\alpha^2)^2},$$

where h_1^2, h_2^2, h_3^2 are the roots of the equation

$$\frac{\alpha^2}{a^2 - h^2} + \frac{\beta^2}{b^2 - h^2} + \frac{r^2}{h^2} - 1 = 0.$$

213. Show that the products of the lengths of the common tangents of the circle $(x-\alpha)^2 + (y-\beta)^2 - r^2 = 0$ and the parabola $y^2 - px = 0$ is equal to

$$\tfrac{1}{4}p^4(k_1 - k_2)^2(k_2 - k_3)^2(k_3 - k_1)^2,$$

where k_1, k_2, k_3 are the roots of the equation

$$(k+1)(4m^2k^2+4mak+r^2)-\beta^2k=0.$$

214. If λ be determined by the equation

$$\frac{a^2b^2}{\beta^2-\lambda a^2}-\frac{b^2a^2}{\beta^2-\lambda(a^2-b^2)}+\frac{r^2}{\lambda}-a^2=0,$$

show that

$$(x^2+y^2-2ax+a^2+\beta^2-r^2)^2-4\beta^2y^2+4\lambda(b^2x^2+a^2y^2-a^2b^2)=0$$

breaks up into two circles.

215. P, P' are the points of contact of a common tangent of two conics; if C be the centre of one of the conics and A the area of the triangle CPP', show that, taking the four common tangents, $\Sigma\,\dfrac{1}{A}=0$.

II.—Examples and Problems on Cubics.

216. If a triangle be inscribed in a cubic so that the sides meet the curve again in three points on a line, then there are four pairs of conics such that each pair will touch all the tangents drawn to the curve from the vertices of the triangle.

The cubic referred to such a triangle can be written

$$ax(y^2+z^2)+by(z^2+x^2)+cz(x^2+y^2)+2dxyz=0,$$

and the tangents drawn to the curve from xy are then

$$\{c(x^2+y^2)+2dxy\}^2-4xy(ax+by)(ay+bx)=0,$$

or

$$(x^2+y^2+2t_1xy)(x^2+y^2+2t_2xy)=0,$$

where t_1, t_2 are the roots of the equation

$$c^2t^2-2(cd-ab)t+d^2-a^2-b^2=0.$$

Now three pairs of lines, whose equations are

$$y^2+z^2+2fyz=0,\quad z^2+x^2+2gzx=0,\quad x^2+y^2+2hxy=0,$$

are all touched by the conic whose tangential equation is

$$\lambda^2+\mu^2+\nu^2-2f\mu\nu-2g\nu\lambda-2h\lambda\mu=0;$$

hence we see that the twelve tangents to the cubic are all touched by the pair of conics

$$U-2R\left(\frac{\mu\nu}{a^2}\pm\frac{\nu\lambda}{b^2}\pm\frac{\lambda\mu}{c^2}\right)=0,\quad U+2R\left(\frac{\mu\nu}{a^2}\pm\frac{\nu\lambda}{b^2}\pm\frac{\lambda\mu}{c^2}\right)=0,$$

where

$$U\equiv\lambda^2+\mu^2+\nu^2-2\left(\frac{ad-bc}{a^2}\right)\mu\nu-2\left(\frac{bd-ca}{b^2}\right)\nu\lambda-2\left(\frac{cd-ab}{c^2}\right)\lambda\mu,$$

$$R^2\equiv a^2b^2+b^2c^2+c^2a^2-2abcd,$$

four such pairs being obtained by the variations of sign.

If $a+b+c+d=0$, the curve has a double point at $x=y=z$, and the tangents drawn to the curve from the vertices of the triangle are all touched by the conic

$$(\lambda+\mu+\nu)^2+4(ab+bc+ca)\left(\frac{\mu\nu}{a^2}+\frac{\nu\lambda}{b^2}+\frac{\lambda\mu}{c^2}\right)=0.$$

217. If a triangle be inscribed in a cubic so that the tangents at the vertices pass through the same point on the curve, then four pairs of conics can be found such that each pair will touch all the tangents drawn to the curve from the vertices of the triangle.

The curve, referred to such a triangle, can be written

$$ax(y^2-z^2)+by(z^2-x^2)+cz(x^2-y^2)=0,$$

and the tangents drawn to the curve from xy are then

$$x^2+y^2+2t_1xy=0,\quad x^2+y^2+2t_2xy=0,$$

where t_1, t_2 are the roots of the equation See Wols. I.

$$c^2t^2+2abt+a^2+b^2-c^2=0;$$

thus we see that the twelve tangents are all touched by the pair of conics

$$U+V=0,\quad U-V=0,$$

where

$$U\equiv\lambda^2+\mu^2+\nu^2+\frac{2bc}{a^2}\mu\nu+\frac{2ca}{b^2}\nu\lambda+\frac{2ab}{c^2}\lambda\mu=0,$$

$$V\equiv 2\sqrt{\{(a^2-b^2)(a^2-c^2)\}}\frac{\mu\nu}{a^2}\pm 2\sqrt{\{(b^2-a^2)(b^2-c^2)\}}\frac{\nu\lambda}{b^2}$$

$$\pm 2\sqrt{\{(c^2-a^2)(c^2-b^2)\}}\frac{\lambda\mu}{c^2},$$

four such pairs being obtained by the variations of sign.

218. If S and T are the invariants of the cubic

$$U\equiv ax(y^2+z^2)+by(z^2+x^2)+cz(x^2+y^2)+2dxyz=0;$$

show that $S=3B-A^2,\quad T=4A(2A^2-9B),$

$$T^2+64S^3=432B^2(4B-A^2),$$

where $A\equiv a^2+b^2+c^2-d^2,\quad B\equiv a^2b^2+b^2c^2+c^2a^2-2abcd.$

Also show that when B vanishes the curve breaks up into a line and a conic, and that when $A^2=4B$ the curve has a node at one of the points $x^2=y^2=z^2$.

219. If x, y, z be a point on the cubic U, the points x, y, z; $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, are corresponding points on the curve. For corresponding points on the Hessian of the cubic

$$B(bcx^3+cay^3+abz^3)-(bcx+cay+abz)^3=0$$

p. 149 are easily seen to be connected by the relations

$$xx'=yy'=zz'=\frac{1}{B}(bcx+cay+abz)(bcx'+cay'+abz'),$$

and the Hessian of this cubic is the cubic U.

220. *The circle passing through the feet of the perpendiculars from a point P on the sides of a given triangle cuts orthogonally a fixed circle; to find the locus of P.*

The equation of the circle passing through the feet of the perpendiculars from α, β, γ on the sides of the triangle of reference is given in Salmon's *Conics*, Art. 132, Ex. 7. Expressing that this circle cuts orthogonally the circle S which we write

$$S\equiv(l\alpha+m\beta+n\gamma)(\alpha\sin A+\beta\sin B+\gamma\sin C)$$
$$+k(\alpha^2\sin A\cos A+\beta^2\sin B\cos B+\gamma^2\sin C\cos C)=0,$$

we obtain the equation of the locus:

$$U\equiv(m\sin C+n\sin B)\alpha(\beta^2+\gamma^2)+(n\sin A+l\sin C)\beta(\gamma^2+\alpha^2)$$
$$+(l\sin B+m\sin A)\gamma(\alpha^2+\beta^2)$$
$$+2(l\sin A+m\sin B+n\sin C+k\sin A\sin B\sin C)\alpha\beta\gamma=0,$$

which represents a cubic circumscribing the triangle and meeting the sides again in three collinear points. From what we have seen the points α, β, γ; $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are corresponding points on the locus, and hence by writing the equation of U in the form

$$(l\alpha + m\beta + n\gamma)(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C)$$
$$+ (l\beta\gamma + m\gamma\alpha + n\alpha\beta)(\alpha \sin A + \beta \sin B + \gamma \sin C)$$
$$+ 2k \sin A \sin B \sin C\, \alpha\beta\gamma = 0,$$

it appears that corresponding points are conjugate with respect to the circle

$$(l\alpha + m\beta + n\gamma)(\alpha \sin A + \beta \sin B + \gamma \sin C)$$
$$+ \tfrac{1}{2}k(\alpha^2 \sin A \cos A + \beta^2 \sin B \cos B + \gamma^2 \sin C \cos C) = 0.$$

It can be seen otherwise that corresponding points on a cubic are conjugate with respect to a fixed circle; for corresponding points on the Hessian of a given cubic are conjugate with respect to all the polar conics, and one of the polar conics is, in general, a circle.

221. *To reduce the equation of a cubic to the form*

$$U \equiv l\alpha(\beta^2 + \gamma^2) + m\beta(\gamma^2 + \alpha^2) + n\gamma(\alpha^2 + \beta^2) + 2p\alpha\beta\gamma = 0.$$

Let V be the cubic which has U for its Hessian (see Ex. 219), then the polar conic with respect to V of the point corresponding to $\alpha\beta$ is $\alpha^2 - \beta^2 = 0$, which pair of lines, being at right angles to one another, form a harmonic pencil with the lines drawn from $\alpha\beta$ to the circular points at infinity. Expressing, then, that the line joining two fixed points x_1, y_1, z_1; x_2, y_2, z_2 is cut harmonically by the polar conic of a point (x, y, z) with respect to $V (\equiv x^3 + y^3 + z^3 + 6m\,xyz)$, we get the equation of a line

$$\{x_1x_2 + m(y_1z_2 + z_1y_2)\}x + \{y_1y_2 + m(z_1x_2 + x_1z_2)\}y$$
$$+ \{z_1z_2 + m(x_1y_2 + y_1x_2)\}z = 0,$$

which meets U in the three points corresponding to the vertices of the triangle $\alpha\beta\gamma$. Since there are three systems of corresponding points, it follows that a cubic can be reduced to the required form in three ways. When

$$l\cos A + m\cos B + n\cos C - p = 0,$$

the cubic is circular and has its double focus on itself, and then the curve can be reduced to this form in an infinite number of ways.

222. *To find the locus of the middle point of corresponding points on a cubic.*

From Ex. 220 we see that a pair of corresponding points are foci of a conic inscribed in the triangle α, β, γ. Now, if α, β, γ be the coördinates of the centre of this conic, and a half the length of its major axis, we can deduce, by means of the expression for the perpendicular from the centre on a tangent in terms of the angles it makes with the axis, the relation

$$\sin A\sqrt{(a^2-\alpha^2)} + \sin B\sqrt{(a^2-\beta^2)} + \sin C\sqrt{(a^2-\gamma^2)} = 0;$$

but $a^2 = S =$ the square of the tangent drawn from α, β, γ to the circle S; hence we have for the equation of the locus

$$\sin A\sqrt{(S-\alpha^2)} + \sin B\sqrt{(S-\beta^2)} + \sin C\sqrt{(S-\gamma^2)} = 0,$$

which, being satisfied by the line at infinity, represents a curve of the third order.

We can obtain the equation of the locus otherwise. For, as can be easily seen, the line joining two poles of a line with respect to a cubic V meets the Hessian in a pair of corresponding points, and the two poles are harmonically conjugate with these points. Hence we have to find the locus of the poles of a line one of whose poles is at infinity. Writing V in the form $\alpha x^3 + \beta y^3 + \gamma z^3 + \delta v^3 = 0$, where $x + y + z + v = 0$, and v is the line at infinity, the given cubic (the Hessian of V) is $\frac{1}{\alpha x} + \frac{1}{\beta y} + \frac{1}{\gamma z} + \frac{1}{\delta v} = 0$, and the locus is

$$\sqrt{\{\beta\gamma(\alpha x^2 - \delta v^2)\}} + \sqrt{\{\gamma\alpha(\beta y^2 - \delta v^2)\}} + \sqrt{\{\alpha\beta(\gamma z^2 - \delta v^2)\}} = 0,$$

which, being divisible by v, represents a curve of the third order. When the given cubic is circular, the locus is circular and has two foci in common with the Cayleyan of V; and if it have also its double focus on itself, the locus is a right line.

223. A conic is inscribed in a given triangle; if the foci are conjugate with respect to a fixed equilateral hyperbola, show that they lie on a circular cubic which has its double focus on itself.

224. Show that

$$a(y-z)^2 + b(z-x)^2 + c(x-y)^2 = 0$$

represents the nodal tangents, and

$$(2bc + ca + ab)\,x + (2ca + ab + bc)\,y + (2ab + bc + ca)\,z = 0$$

the line of inflexions of the cubic

$$ax(y-z)^2 + by(z-x)^2 + cz(x-y)^2 = 0.$$

225. If the tangents at the vertices of a triangle inscribed in a cubic pass through a point, the cubic, referred to the triangle, can be written

$$U \equiv ax(y^2 - z^2) + by(z^2 - x^2) + cz(x^2 - y^2) + 2dxyz = 0,$$

from which it appears that the lines, joining the vertices to the points in which the opposite sides meet the curve again, pass through a point.

If S and T are the invariants of U, and

$$s \equiv a^2 + b^2 + c^2 + d^2, \quad q \equiv a^2b^2 + b^2c^2 + c^2a^2, \quad r \equiv a^2b^2c^2,$$

$$S = 12\,(s^2 - 3q), \quad T = -4\,(2s^3 - 9qs + 27r).$$

If U has a cusp, $qs - 9r = 0$, and $a^{-\frac{2}{3}} + b^{-\frac{2}{3}} + c^{-\frac{2}{3}} = 0$.

226. A triangle is inscribed in a cubic so that the tangent at each vertex passes through the point where the opposite side meets the curve again; the cubic referred to the triangle can be written

$$U \equiv x\,(ay^2 + bz^2) + y\,(az^2 + bx^2) + z\,(ax^2 + by^2) + 2cxyz = 0.$$

If $3\,(a + b) + 2c = 0$, the curve has a node at the point $x = y = z$, and then

$$x^2 + y^2 + z^2 - xy - yz - zx = 0$$

represents the nodal tangents, and $x + y + z = 0$ the line of inflexions.

If H be the Hessian of U,

$$H \equiv c^2U + (2abc - a^3 - b^3)\,(x^3 + y^3 + z^3 - 3xyz),$$

from which it is evident that $x^3 + y^3 + z^3 - 3xyz = 0$ must represent the sides of a canonical triangle. Hence each vertex is the pole of the opposite side with regard to one of the canonical triangles.

Thus, for the envelope of such triangles inscribed in the curve, U being written

$$x^3 + y^3 + z^3 + 6mxyz = 0,$$

we have the equation in tangential coordinates

$$\Sigma \equiv \beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3 + 6m\alpha^2\beta^2\gamma^2 = 0;$$

and, since there are four canonical triangles, the complete envelope consists of four curves of the sixth class.

By writing the tangential equation of the cubic in the form

$$(\alpha^3 + \beta^3 + \gamma^3 - 12m^2\alpha\beta\gamma)^2 - 4\,(1 + 8m^3)\,\Sigma = 0,$$

we see that Σ touches the cubic in eighteen points.

For the nodal cubic the envelope is a curve of the fourth class.

If a side AB of one of such triangles meet the curve again in C', the point of contact of AB with its envelope is the harmonic conjugate of C' with respect to A, B. If (x', y', z') be the coordinates of the vertex opposite AB, the equation of AB will be $\frac{x}{x'} + \frac{y}{y'} + \frac{z}{z'} \equiv L = 0$; and, differentiating this equation, subject to the condition

$$x'^3 + y'^3 + z'^3 + 6mx'y'z' = 0,$$

we find for the coordinates (x_1, y_1, z_1) of the point of contact

$$x_1 = x'(p+q), \quad y_1 = y'(p\vartheta^2 + q\vartheta), \quad z_1 = z'(p\vartheta + q\vartheta^2),$$

where $p \equiv x'^3 + \vartheta y'^3 + \vartheta^2 z'^3$, $q \equiv x'^3 + \vartheta^2 y'^3 + \vartheta z'^3$, $\vartheta^3 \equiv 1$.

But $C'(x_2, y_2, z_2)$, being the tangential of (x', y', z'), has for its coordinates

$$x_2 = x'(p-q), \quad y_2 = y'(p\vartheta^2 - q\vartheta), \quad z_2 = z'(p\vartheta - q\vartheta^2);$$

hence (x_1, y_1, z_1) and (x_2, y_2, z_2) divide harmonically the line joining the points $x', \vartheta^2 y', \vartheta z'$; $x', \vartheta y', \vartheta^2 z'$. Now these points lie on the line L and also on the curve; they are, therefore, the points A, B.

227. A triangle is inscribed in a circular cubic so that the tangent at each vertex passes through the point where the opposite side meets the curve again; show that the osculating circles at the vertices pass through the same point on the curve.

228. Show that the curve

$$2(ax + by + cz)^3 - (a + b + c)(ax + by + cz)^2(x + y + z) \\ + (a + b + c)^3 xyz = 0$$

has a node at the point $x = y = z$, and that

$$(a^2 - b^2 - c^2 + 4bc)(y - z)^2 + (b^2 - c^2 - a^2 + 4ca)(z - x)^2 + (c^2 - a^2 - b^2 + 4ab)(x - y)^2 = 0$$

represents the nodal tangents.

If $\sqrt{a} + \sqrt{b} + \sqrt{c} = 0$, the node becomes a cusp, and

$$(b - c)x + (c - a)y + (a - b)z = 0$$

represents the cuspidal tangent.

229. Show that the invariant S vanishes for the cubic

$$(x + y + z)^3 - 24xyz = 0,$$

and the invariant T for either of the cubics

$$(x + y + z)^3 - 6(3 \pm \sqrt{3})xyz = 0.$$

230. A variable cubic is inscribed in two fixed triangles so that the points of contact of each of the triangles lie on a line; to show that the point of intersection of these lines is one or other of six fixed points.

If ABC, $A'B'C'$ be the two triangles, the cubic can be written in either of the forms

$$ABC - D^2F = 0, \quad A'B'C' - D'^2F' = 0.$$

Hence we can deduce the identical relation

$$ABC + \lambda A'B'C' = D^2F + \lambda D'^2F',$$

from which it is evident that the point DD' (the node of the cubic $D^2F + \lambda D'^2F' = 0$) is one of the six critic centres of the triangles ABC, $A'B'C'$ (Salmon's *Higher Plane Curves*, Art. 192).

Also F and F', the satellites of D and D', touch and intersect on one or other of six nodal cubics.

Taking one of the critic centres for origin and writing

$$ABC + \lambda A'B'C' = D^2F + \lambda D'^2F' = x^3 + y^3 + 6xyz,$$

we may assume

$$D = y + \theta x, \quad -4\theta^3 F = (\theta^4 - 2\theta)\, x + (1 - 2\theta^3)\, y - 6\theta^2 z,$$

and then

$$(y + \theta x)^2 \{(\theta^4 - 2\theta)\, x + (1 - 2\theta^3)\, y - 6\theta^2 z\} + 4\theta^3 ABC = 0$$

will represent one of the variable cubics inscribed in the two triangles. Since the parameter θ occurs in the sixth degree, and there are six systems corresponding to the six critic centres, it follows that thirty-six of the cubics can be drawn through a given point.

231. Show that the polar conic S of a point P with respect to a cubic U meets the Hessian in six points, at which the tangents to S touch the Cayleyan, and the points corresponding to which are the points of contact of the tangents from P.

232. A triangle is formed by three points on a cubic, and another by three corresponding points of the same system; if the triangles are homologous, show that each of them meets the curve again in three points on a line.

233. If, in the preceding example, U be the conic with regard to which the triangles are reciprocal, show that the tangents to U from corresponding vertices of the triangles intersect on the cubic, and that if O be any point on the cubic, and P, P' the points of contact of the tangents from O to U, O, P, P' lie on a conic passing through the vertices of one of the triangles.

234. *A conic U has triple contact with a cubic; to show that the six points on the cubic, at which the tangents touch U, lie on a conic.*

If x, y, z be the tangents at the points of contact, the cubic may be written

$$(ax + by + cz)(x^2 + y^2 + z^2 - 2yz - 2zx - 2xy) + xyz = 0,$$

where $$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy \equiv U;$$

and then the locus of the poles with respect to the cubic of the tangents to U is found to be

$$xyz(x + y + z) + (ab + bc + ca) U^2 - 8(ax + by + cz)^2 U$$
$$- 12xyz(ax + by + cz) + \{(b + c) yz + (c + a) zx + (a + b) xy\} U = 0.$$

Now this locus evidently touches the cubic at the three points of contact with U, and meets it again in the six points at which the tangents touch U; therefore, putting

$$xyz = -(ax + by + cz) U$$

from the equation of the cubic, and dividing by U, we obtain

$$4(ax + by + cz)^2 + (ab + bc + ca) U - (ax^2 + by^2 + cz^2) = 0,$$

which represents a conic passing through the six points on the cubic at which the tangents touch U.

235. I is an inflexion of a cubic, and A, B are two points on a line passing through I; if a radius vector through I meet the curve in P, Q, show that the locus of the intersection of PA and QB is a cubic.

Hence show, by letting A, B become the circular points at infinity, that the cubics, whose equations to rectangular axes are

$$x(ax^2 + by^2 + 2hxy + 2gx - bk^2) - 1 = 0,$$

$$b^2y\{h^2x^2 + (h^2 + b^2 - ab) y^2 + 2bhxy + 2ghy + k^2h^2\} - h^3 = 0,$$

cut each other at right angles at three points on the line $hx + by = 0$.

236. Show that a line which meets the cubic

$$(x+y+z)^3 - kxyz = 0$$

and the line $x+y+z=0$ in four points whose invariant S vanishes, also meets the lines x, y, z, $x+y+z$ in four points whose S vanishes.

237. Show that a line which meets the cubics

$$(x+y+z)^3 \pm kxyz = 0$$

in six points in involution, is divided harmonically by the lines x, y, z, $x+y+z$.

238. If two points are such, that the polar of each with respect to a cubic U passes through the other, show that their coordinates (x, y, z), (x', y', z') are connected by the relations

$$\frac{xx' + m(yz' + zy')}{yz' - zy'} = \frac{yy' + m(zx' + xz')}{zx' - xz'} = \frac{zz' + m(xy' + yx')}{xy' - yx'}$$

$$= \sqrt{\left(\frac{H}{U}\right)} = \sqrt{\left(\frac{H'}{U'}\right)},$$

where $\quad U \equiv x^3 + y^3 + z^3 + 6mxyz,$

$\quad H \equiv -m^2(x^3 + y^3 + z^3) + (1 + 2m^3)\,xyz.$

239. Show that a point and the intersection of its polars with respect to a cubic U and its Hessian H lie on the same cubic of the system $U + \lambda H = 0$.

240. Two conics are described through four points on a cubic, the tangents at which pass through the same point O on the curve; show that the line joining any point on the cubic to the intersection of its polars with regard to the conics passes through O.

241. Lines drawn from a point P to three fixed points A, B, C intersect three fixed lines respectively in three points which form a triangle of given area; show that the locus of P is a cubic passing through A, B, C.

242. Lines drawn from a point P to three fixed points A, B, C intersect the lines BC, CA, AB in three points which form a triangle homologous with a given triangle; show that the locus of P is a cubic passing through A, B, C.

243. The points of contact of the tangents drawn from a point P to a conic U lie on a conic which passes through four fixed points; show that the locus of P is a cubic which passes through the vertices of the quadrilateral formed by the polars of the fixed points with respect to U.

244. *A conic S meets the Hessian of a cubic U in three pairs of corresponding points; to show that S belongs to one of two systems of conics which have a common Jacobian.*

Writing the conic

$$S \equiv (a, b, c, f, g, h)(x, y, z)^2 = 0,$$

and the cubic

$$U \equiv x^3 + y^3 + z^3 + 6mxyz = 0,$$

if we form the invariant Θ of S and the polar conic of (x, y, z) with respect to U, we obtain

$$\Theta = a(yz - m^2x^2) + b(zx - m^2y^2) + c(xy - m^2z^2)$$
$$+ 2f(m^2yz - mx^2) + 2g(m^2zx - my^2) + 2h(m^2xy - mz^2).$$

Now, by considering the case when the polar conic breaks up into right lines, we see that the conic $\Theta = 0$ passes through the six points on the Hessian corresponding to its points of intersection with S. Hence, when S passes through three pairs of corresponding points on the Hessian, it must coincide with Θ, and, therefore, be of the form

$$a(\lambda x^2 + yz) + b(\lambda y^2 + zx) + c(\lambda z^2 + xy),$$

where $\lambda = m^2 \pm \sqrt{(m^4 - m)}$, which belongs to the system having the common Jacobian

$$\lambda(x^3 + y^3 + z^3) - (1 + 4\lambda^3)\,xyz = 0.$$

245. If the discriminant of the covariant conic Θ vanishes, a conic S will meet the Hessian at the vertices of two triangles, such that each meets the curve again in three points on a line, and these two lines are the factors of Θ.

Hence, a triangle being inscribed in a cubic so that the sides meet the curve again in three points on a line, if the circumscribing circle S cut orthogonally a fixed circle, the locus of the centre of S is one or other of three cubics.

246. Given the six tangents drawn to a cubic from a point of the Hessian H, to find the absolute invariant of the curve.

The cubic being written in the form

$$U \equiv ax^3 + by^3 + cz^3 + dv^3 = 0,$$

where $x + y + z + v = 0$, the point xy is on H, and the tangents from this point to the cubic are

$$(\sqrt{c} + \sqrt{d})^2 (ax^3 + by^3) - cd\,(x + y)^3 \equiv \phi = 0,$$

$$(\sqrt{c} - \sqrt{d})^2 (ax^3 + by^3) - cd\,(x + y)^3 \equiv \phi' = 0,$$

which form, therefore, two sets of three satisfying the invariant relation $Q = 0$ (Salmon's *Higher Algebra*, Art. 199) with one another. If we write the discriminant of $\phi + k\phi'$ (which has a square factor when Q vanishes)

$$(k - \alpha)(k - \beta)(k - \gamma)^2,$$

we find
$$\frac{T^2}{64S^3} = -\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2,$$

where
$$S = -abcd,$$

$$T = b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 + a^2b^2c^2$$
$$- 2abcd\,(ab + ac + ad + bc + cd + db).$$

The tangents to the Cayleyan from xy are also given in the same case, for the Jacobian of ϕ and ϕ' is

$$(x+y)^2\,(ax^2 - by^2) = 0,$$

and $x+y=0$, $ax^2 - by^2 = 0$ are the three tangents to the Cayleyan from xy.

The Hessians of ϕ and ϕ' are the tangents from xy to H.

247. *To find the length of the segment* (t) *which the tangent at any point of a cubic intercepts on the curve.*

Suppose the curve, referred to the tangent and normal at the point as axes of coordinates, to be written

$$U \equiv (x\cos\theta_1 + y\sin\theta_1)\,(x\cos\theta_2 + y\sin\theta_2)\,(x\cos\theta_3 + \sin\theta_3)$$
$$+ ax^2 + by^2 + 2hxy + 2fy = 0,$$

where θ_1, θ_2, θ_3 are the angles which the normal makes with the asymptotes; then putting $y=0$, we have

$$t = \frac{-a}{\cos\theta_1\cos\theta_2\cos\theta_3}.$$

Now, at the origin,

$$\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2 = 4f^2,$$

and $a\rho = -f$, where ρ is the radius of curvature; therefore, since the value of $\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2$ is independent of the axes to which the curve is referred, we have referred to any axes,

$$t = \frac{\sqrt{\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2\right\}}}{2\rho \cos\theta_1 \cos\theta_2 \cos\theta_3}.$$

For the circular cubic

$$U \equiv (x\cos\theta + y\sin\theta)(x^2 + y^2) + ax^2 + by^2 + 2hxy + 2fy = 0,$$

we have $$t = \frac{\sqrt{\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2\right\}}}{2\rho\cos\theta} = \frac{\sqrt{(r_1 r_2 r_3 r_4)}}{2\rho\cos\theta},$$

where r_1, r_2, r_3, r_4 are the distances of the origin from four concyclic foci; for, expressing the condition that $x + iy - p = 0$ should touch the curve, we find

$$p^4 + \&c. - 4f^2(\cos 2\theta + i\sin 2\theta) = 0,$$

whence $$p_1 p_2 p_3 p_4 = -4f^2(\cos 2\theta + i\sin 2\theta),$$

and therefore $r_1 r_2 r_3 r_4 = 4f^2$.

For the curve whose polar equation is

$$r^3\cos 3\theta + 3br^2 + a^3 = 0,$$

we have $$t = \tfrac{3}{2}r^2\sqrt{\left\{\frac{r^4 - 8b^2r^2 - 4ba^3}{r^6 - (a^3 + 3br^2)^2}\right\}},$$

and for the curve $r^3\cos 3\theta = a^3$, $t = \frac{3}{2}\frac{r^4}{\sqrt{(r^6 - a^6)}}$, which is a minimum when $r = a\sqrt[3]{2}$. For the curve $r\cos 3\theta = a$,

$$t = \tfrac{1}{2}r\sqrt{\left(\frac{9r^2 - 8a^2}{r^2 - a^2}\right)},$$

which is a minimum when $r = \dfrac{2a}{\sqrt{3}}$.

For the curves whose equations in rectangular coordinates are

$$x^3 - 3ay^2 = 0,\ x^3 - 3a^2y = 0,\ xy^2 - a^3 = 0,$$

we find

$$t = \tfrac{3}{8}x\sqrt{\left(\frac{3x}{a}+4\right)},\quad \frac{6x}{a^2}\sqrt{(x^4+a^4)},\quad \frac{3}{2}\sqrt{\left(4x^2+\frac{a^3}{x}\right)},$$

respectively.

For the cissoid $(a-x)y^2 - x^3 = 0$,

$$t = \tfrac{3}{2}\frac{ax}{\sqrt{\{(a-x)(4a-3x)\}}}.$$

248. A, A'; B, B' are two pairs of points on a cubic, such that the lines AA', BB' intersect on the curve; if P be a variable point on the cubic, and PA, PA' intercept a segment d, and PB, PB' a segment d' on one of the asymptotes, show that $\frac{\lambda}{d}+\frac{\mu}{d'}=1$, where λ and μ are constants.

249. *To show that a line* (δ) *meets a circular cubic at angles whose sum is equal to that which* δ *makes with its satellite.*

If ε be the satellite of δ, and α, β, γ the tangents to the cubic where it is met by δ, the curve may be written

$$\alpha\beta\gamma - k\delta^2\varepsilon = 0.$$

If we write now $\alpha \equiv x\cos\alpha + y\sin\alpha - p$, &c., where x, y are rectangular Cartesian coordinates, we have for the conditions that the curve should be circular $k=1$, $\alpha+\beta+\gamma = 2\delta+\varepsilon$, the latter of which equations gives the result stated above. Since for a non-singular cubic the coordinates of the satellite involve the coordinates of the line in the fourth degree, it follows that a line which meets the curve at angles whose sum is constant touches a curve of the fifth class.

If a line be written $\alpha x+\beta y+\gamma z=0$, its satellite with regard to the cubic

$$x^3+y^3+z^3+6mxyz \equiv U = 0$$

is

$$(\alpha^4 - 2\alpha\beta^3 - 2\alpha\gamma^3 - 6m\beta^2\gamma^2)\,x + (\beta^4 - 2\beta\gamma^3 - 2\beta\alpha^3 - 6m\gamma^2\alpha^2)\,y$$
$$+ (\gamma^4 - 2\gamma\alpha^3 - 2\gamma\beta^3 - 6m\alpha^2\beta^2)\,z = 0;$$

and, therefore, if the line intersect its satellite on $z = 0$, we have, after dividing by $\alpha^3 - \beta^3$, $\alpha\beta + 2m\gamma^2 = 0$. Hence, x, y being rectangular Cartesian coordinates, a tangent to the conic

$$4a\,(y^2 - 3x^2) + 3b = 0$$

meets the circular cubic

$$x\,(x^2 + y^2) + a\,(y^2 - 3x^2) + b = 0$$

at angles, whose sum $= 0$.

For a circular cubic with a node the envelope is a curve of the third class.

For a circular cubic with a cusp the envelope is a parabola, and if the sum of the angles $= 0$, the envelope touches the cuspidal and inflexional tangents. If s be the sum of the angles at which the line $lx + my - 1 = 0$ meets the cissoid $(a - x)\,y^2 - x^3 = 0$, we have

$$\tan s = \frac{3m\,(4al - 1)}{4a\,(2m^2 - l^2) + 3l}.$$

250. Show that, in a circular cubic with a node, the angle between two tangents which intersect on the curve is equal to double the angle subtended by their points of contact at the node.

251. If two conics be described through four points on a circular cubic, the angle between the two chords in which the conics meet the curve again, is equal to double the angle between the axes of the conics.

252. If a, a'; b, b'; c, c' be three pairs of corresponding points on a circular cubic, show that $ab'.bc'.ca' = ac'.ba'.cb'$,

where ab' denotes the length of the line joining the points a, b', &c.

253. If any point of a circular cubic be joined to four concyclic foci of the curve, the perpendiculars erected at the foci to the joining lines are all touched by the same circle. For any point of the cubic is a focus of a conic passing through the four foci (Salmon's *Conics*, Art. 228, Ex. 10), from which, by Ex. 166, the truth of the theorem becomes evident.

254. If perpendiculars be dropped from any point of a circular cubic on the sides of a quadrilateral formed by four concyclic foci, four lines joining their feet will form a quadrilateral circumscribed about a circle. This theorem is an interpetation of the equation $\dfrac{\rho+\rho''}{a+c}=\dfrac{\rho'+\rho'''}{b+d}$ in Salmon's *Higher Plane Curves*, Art. 279.

255. If any point P of a circular cubic be joined to four concyclic foci, the tangent to the curve at P is divided in a constant anharmonic ratio by the perpendiculars erected at the foci to the joining lines.

From the equation of the cubic

$$(b+c)\,\rho_1+(a-b)\,\rho_3=(a+c)\,\rho_2,$$

(Salmon's *Higher Plane Curves*, Art. 279}, we have, by differentiation,

$$(b+c)\frac{\rho_1}{PN_1}+(a-b)\,\frac{\rho_3}{PN_3}=(a+c)\,\frac{\rho_2}{PN_2},$$

where N_1, &c. are the points in which the tangent meets the perpendiculars; whence, eliminating ρ_2,

$$\frac{\rho_1}{\rho_3}=\left(\frac{a-b}{b+c}\right)\frac{PN_1.N_2N_3}{PN_3.N_1N_2}.$$

Also, from the equation

$$(c-d)\,\rho_1 + (a+d)\,\rho_3 = (a+c)\,\rho_4,$$

in the same way we have

$$\frac{\rho_1}{\rho_3} = \left(\frac{a+d}{c-d}\right)\frac{PN_1 . N_3N_4}{PN_3 . N_1N_4};$$

therefore, equating the values of $\frac{\rho_1}{\rho_3}$, we obtain

$$\frac{N_1N_2 . N_3N_4}{N_2N_3 . N_1N_4} = \frac{(a-b)\,(c-d)}{(b+c)\,(a+d)}.$$

To show that one of the conics described through P to touch the four perpendiculars is touched by the cubic at P.

A circle Σ cutting orthogonally the circle

$$J \equiv x^2 + y^2 - k^2 = 0,$$

and having its centre on the parabola

$$F \equiv (y-\beta)^2 - 4m\,(x-\alpha),$$

can be written

$$\Sigma \equiv x^2 + y^2 - 2\,(\alpha + m\mu^2)\,x - 2\,(\beta + 2m\mu)\,y + k^2 = 0,$$

and the envelope of Σ is a circular cubic, of which the points of intersection of F and J are foci. Hence, the envelope of the polars of $P\,(x', y')$ with regard to Σ, which will evidently touch the four perpendiculars, is

$$(x+x')\,\{xx' + yy' - \alpha\,(x+x') - \beta\,(y+y') + k^2\} + m\,(y+y')^2 = 0;$$

but this conic, as can be easily seen, touches the cubic at x', y'.

256. Show that the tangent at any point P of a circular cubic is divided in a constant anharmonic ratio by the polars of P with regard to four fixed circles having double contact with the cubic, and show that this anharmonic ratio is equal to that of the pencil joining any point on the parabola F to the centres of the four circles.

257. Two circles have double contact with a circular cubic, and a third circle is described through their points of contact; if P be any point on the curve, show that the tangent at P is divided harmonically at P, and the points where it meets the polars of P with respect to the three circles.

258. If a variable point P of a circular cubic be joined to three centres of inversion A, B, C, and lines be drawn bisecting PA, PB, PC at right angles, the intersection of the perpendiculars of the triangle so formed will lie on a parallel to the asymptote through the centre of the circle passing through A, B, C.

This line is the directrix of the focal parabola corresponding to the fourth centre of inversion.

259. If a variable point P of a circular cubic be joined to the points denoted by the letters S, U, V in the figure in Salmon's *Higher Plane Curves*, Art. 278, and lines be drawn bisecting PS, PU, PV at right angles, the centre of the circle circumscribing the triangle so formed will lie on a parallel to the asymptote through the centre of the circle passing through S, U, V.

260. The point U (Salmon's *Curves*, Art. 278) is that point on the infinite branch of the curve from which the tangents to the oval contain a maximum angle.

Since any conic meets a cubic, so that three chords of intersection meet the cubic again in three points on a line, it follows that a circle meets a circular cubic, so that two finite chords of intersection meet the cubic again in two points which lie on a parallel to the asymptote. But the chord of contact of the tangents from U to the oval is parallel to the asymptote (Salmon's *Curves*, Art. 150); hence a circle can be described touching the curve at U and passing through the points of contact of the tangents from U to the oval, which shows that the angle between the tangents is a maximum.

261. If γ be the chord of curvature through a centre of inversion at any point P of a circular cubic, show that $\gamma = \frac{t^2 \sin\theta}{\delta}$, where t is the length of the tangent from P to the circle of inversion, θ is the angle which the chord makes with the asymptote, and δ is the projection on the asymptote of the segment of the tangent at P intercepted by the curve.

262. A circular cubic having its double focus on itself passes through AA', BB', CC' the extremities of the three diagonals of a quadrilateral; if P be any point on the curve, the relation holds

$$MN.PA.PA' \pm NL.PB.PB' \pm LM.PC.PC' = 0,$$

where L, M, N are the middle points of the diagonals.

For any point P on this cubic is the focus of a conic inscribed in the quadrilateral, and therefore the feet of the perpendiculars from P on the sides lie on a circle (the auxiliary circle of the conic); Ptolemy's theorem then furnishes the relation given above.

263. A circular cubic has its double focus on itself; if a line through the double focus meet the curve again in A, B, show that the circle described on AB as diameter passes through two fixed points.

264. Show that the circular cubics, whose equations in rectangular coordinates are

$$lx(x^2+y^2+c^2) + my(x^2+y^2-c^2) + n(x^2+y^2) = 0,$$
$$l'x(x^2+y^2+c^2) + m'y(x^2+y^2-c^2) + n'(x^2+y^2) = 0,$$

cut each other at an angle equal to that between their asymptotes.

265. Points on the cubic

$$y(x^2+y^2-c^2)-a(x^2+y^2)=0,$$

connected by the relations

$$xx'-yy'-c^2=0, \quad xy'+yx'=0,$$

are corresponding points; if d be the distance between such a pair of points, and δ the distance of their middle point from the origin, shew that

$$d^4=16\{a^2c^2+(\delta^2-c^2)^2\}.$$

From this expression, it follows that the minimum value of d is $2\sqrt{(ac)}$ if $2c>a$, but if $2c<a$, the minimum value is $\sqrt{(a^2+4c^2)}$.

266. The parabola

$$(x^2-y^2-c^2)\cos 2\phi+2xy\sin 2\phi$$
$$-(x^2+y^2-2\alpha x-2\beta y+k^2)\equiv U=0$$

has triple contact with the circular cubic

$$\{y^2+(x-c)^2\}\{y^2+(x+c)^2\}-(x^2+y^2-2\alpha x-2\beta y+k^2)^2=0,$$

of which the points $x=\pm c$, $y=0$ are evidently foci, and since the cubic can be reduced to this form in six ways, it follows that there are six systems of parabolas having triple contact with the curve.

Writing U in the form

$$2(x\sin\phi-y\cos\phi)^2=2\alpha x+2\beta y-(k^2+c^2\cos 2\phi),$$

we have, differentiating with regard to ϕ,

$$2(x\sin\phi-y\cos\phi)(x\cos\phi+y\sin\phi)=c^2\sin 2\phi,$$

whence, by division,

$$(x\cos\phi+y\sin\phi)\{2\alpha x+2\beta y-(k^2+c^2\cos 2\phi)\}$$
$$-c^2\sin 2\phi(x\sin\phi-y\cos\phi)=0,$$

which represents a third conic passing through the three points of contact. Combining these three equations so as to obtain the equation of a circle, we find

$$\cos\phi\,\{\alpha\,(x^2+y^2+c^2)-(k^2+c^2)\,x\}+\sin\phi\,\{\beta\,(x^2+y^2-c^2)-(k^2-c^2)\,y\}$$
$$+\,(\alpha\cos\phi+\beta\sin\phi)\,(x^2+y^2-2\alpha x-2\beta y+k^2)=0,$$

which is the equation of the circle passing through the three points of contact. This circle, therefore, passes through two fixed points.

The nine-point circle of the triangle formed by the points of contact passes through a fixed point (the middle point of the line joining the foci); for an equilateral hyperbola having this point for centre passes through the points of contact. Writing U in the form

$$2\,\{(x-\tfrac{1}{2}\alpha)\sin\phi-(y-\tfrac{1}{2}\beta)\cos\phi\}^2$$
$$=2\,(\alpha\cos\phi+\beta\sin\phi)\,(x\cos\phi+y\sin\phi)$$
$$+\tfrac{1}{2}\,(\alpha\sin\phi-\beta\cos\phi)^2-(k^2+c^2\cos 2\phi),$$

we sec that the axis passes through the fixed point $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$, and that the tangent at the vertex touches the parabola

$$(2\alpha y+2\beta x-\alpha\beta)^2-4\,(2\alpha x-c^2-k^2-\tfrac{1}{2}\beta^2)(2\beta y-k^2+c^2-\tfrac{1}{2}\alpha^2)=0.$$

If p be the principal parameter $p=\alpha\cos\phi+\beta\sin\phi$; hence we see that the directrix touches a parabola, and this parabola, by considering two consecutive curves of the system, is seen to be the locus of the intersection of the perpendiculars of the triangle formed by the tangents at the points of contact. Also the locus of the focus is a circular cubic with a node at the point $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$, and this cubic is touched by the circle circumscribing the triangle formed by the tangents at the points of contact.

267. Two parabolas of the same system described through a point P to have triple contact with a circular cubic cut

each other orthogonally at P; show that the locus of P is a bicircular quartic.

268. If t_1, t_2, t_3, t_4 be the lengths of the tangents drawn from any point of the curve to a circular cubic, show that

$$\frac{(t_1^2 - t_2^2)(t_3^2 - t_4^2)}{(t_1^2 - t_3^2)(t_2^2 - t_4^2)}$$

is equal to the anharmonic ratio of the tangents taken in the proper order.

269. A, B, C, D are four points on a circular cubic, and A', B', C', D' are the centres of the four circles circumscribing the triangles ABC, &c.; if P be the point where the asymptote meets the curve, and F the double focus, show that the anharmonic ratio of the pencil PA, PB, PC, PD is equal to that of the pencil FA', FB', FC', FD'.

270. If the tangent at a point P of a circular cubic meet the curve again in A, and the asymptote in B, show that $AB = 2PN$, where N is the foot of the perpendicular from the double focus on the tangent.

271. Show that any circle meets a circular cubic at angles the sum of whose cotangents $= 0$.

If a line meet the curve at angles α, β, γ, and the asymptote at an angle δ, show that

$$\cot\alpha + \cot\beta + \cot\gamma = \cot\delta.$$

272. A circle S cuts orthogonally the circle

$$x^2 + y^2 - 2x'x - 2y'y + k^2 = 0,$$

and passes through the points where parallels to the asymptote meet the circular cubic

$$(x + a)(x^2 + y^2) + lx + my + n = 0;$$

show that the locus of the centre of S is the conic

$$2(my' + n - la)y^2 + 2m(x' + a)xy + m^2x - m(k^2 + l)y = 0.$$

273. A series of circles having the origin for centre meets the circular cubic

$$(lx + my)(x^2 + y^2) + ax^2 + by^2 + 2gx + 2fy + c = 0;$$

show that the locus of the centres of the quadrangle formed by the points of intersection is the quartic

$$(ly - mx)^2(gx + fy + c) + \{(a - b)xy + gy - fx\}$$
$$\times \{lm(x^2 + y^2) - (l^2 - m^2)xy + 2max - 2lby + mg - lf\} = 0.$$

274. Show that the locus of the centre of the circle passing through the three centres of the same quadrangle is the cubic

$$(lx + my + a + b)^2\{2bgx + 2afy + c(a + b)\}$$
$$- 2(lx + my + a + b)(gx + fy + c)(lbx + may + 2ab)$$
$$= \{(lf - mg)x + f(a + b) - mc\}^2 + \{(lf - mg)y - g(a + b) + lc\}^2.$$

275. Four lines

$$x\cos\theta_1 + y\sin\theta_1 - p_1 = 0, \text{ \&c.,}$$

are tangents to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0;$$

if $\Sigma d\theta = 0$,

$$\Sigma\frac{\cos\theta d\theta}{\sqrt{(a^2\cos^2\theta + b^2\sin^2\theta)}} = 0, \quad \Sigma\frac{\sin\theta d\theta}{\sqrt{(a^2\cos^2\theta + b^2\sin^2\theta)}} = 0,$$

show that they form a quadrilateral inscribed in a fixed circular cubic.

✓276. AA', BB' are two fixed chords of a circular cubic which intersect on the curve; if P be a variable point on the curve, and α, β the angles subtended by AA', BB' at P, show that $\lambda \cot\alpha + \mu \cot\beta = 1$, where λ and μ are constants.

277. Let a point P be taken on the line bisecting at right angles the line joining two poiuts F_1, F_2, then if ρ_1, ρ_2, ρ be the distances of a point from F_1, F_2, P, respectively, the equation of a circular cubic having P for a node, and F_1, F_2 for foci, is $\rho_1 + \rho_2 = 2\rho$. Transforming to elliptic coordinates, F_1, F_2 being the foci of the system of conics, this equation becomes

$$\sqrt{(c^2 - \nu^2)}\,\{\mu + \sqrt{(\mu^2 - c^2)}\} = c\beta,$$

where β is the distance of P from the middle point of F_1, F_2.

If we take a point Q on the line joining F_1, F_2, the equation of a circular cubic having Q for a node, and F_1, F_2 for foci is in elliptic coordinates

$$\nu\,\{\mu + \sqrt{(\mu^2 - c^2)}\} = c\alpha,$$

where α is the distance of Q from the middle point of F_1, F_2.

Three poiuts of intersection of these two cubics lie on a line perpendicular to PQ, and the remaining points of intersection lie on the point circle $(x - \alpha)^2 + (y - \beta)^2 = 0$. At one of the points on the line the two curves cut each other at right angles, for their differential equations are

$$\frac{d\mu}{\sqrt{(\mu^2 - c^2)}} \mp \frac{\nu d\nu}{\sqrt{(c^2 - \nu^2)}} = 0,\quad \frac{d\mu}{\sqrt{(\mu^2 - c^2)}} \pm \frac{d\nu}{\nu} = 0,$$

and at the two other points they cut each other at an angle $= \cos^{-1}\dfrac{2\alpha\beta}{c^2}$, where $c = \frac{1}{2}F_1F_2$. If δ be the distance between the two latter points,

$$\delta = \sqrt{\left\{(\alpha^2 + \beta^2)\left(\frac{c^4}{4\alpha^2\beta^2} - 1\right)\right\}}.$$

278. Show that, in the preceding example, the Cartesian

coordinates x, y of a point can be expressed rationally in terms of α and β as follows:

$$2x = \frac{\alpha(\alpha^2+\beta^2+c^2)}{\alpha^2+\beta^2}, \quad 2y = \frac{\beta(\alpha^2+\beta^2-c^2)}{\alpha^2+\beta^2}.$$

279. Two nodal circular cubics having their foci in common meet each other in three points on a line and four points on a circle; show that, at one of the points on the line, they intersect at an angle equal to that between their asymptotes. If this angle is right, show that the radius of the circle of intersection vanishes.

A nodal circular cubic being written

$$\{(x-\alpha)^2+(y-\beta)^2-r^2\}^2-\rho^2\rho'^2=0,$$

where ρ, ρ' are the distances of x, y from the points $\pm c$, 0, show that

$$(\alpha^2+\beta^2-2r^2)(\alpha^2+\beta^2)=2c^2(\alpha^2-\beta^2),$$

and that
$$\frac{\alpha(r^2+c^2)}{\alpha^2+\beta^2}, \quad \frac{\beta(r^2-c^2)}{\alpha^2+\beta^2},$$

are the coordinates of the node.

280. The equation of a nodal cubic, referred to the triangle formed by the inflexional tangents, may be written $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}=0$, or in tangential coordinates $\lambda^{-\frac{1}{2}}+\mu^{-\frac{1}{2}}+\nu^{-\frac{1}{2}}=0$. If we combine with the latter equation the equation of a point $x\lambda+y\mu+z\nu=0$, we get a biquadratic which determines the tangents drawn from (x, y, z) to the curve.

Let us consider the cubics

$$f=au^3+bv^3+cw^3, \quad f'=a'u^3+b'v^3+c'w^3,$$

where $u+v+w=0$. Then the discriminant of $f+kf'$ is seen to be

$$(a+ka')^{-\frac{1}{2}}+(b+kb')^{-\frac{1}{2}}+(c+kc')^{-\frac{1}{2}}=0.$$

Let us suppose that this equation in k coincides with the biquadratic found above; we must have, then,

$$\lambda = a + ka', \quad \mu = b + kb', \quad \nu = c + kc';$$

and, since $x\lambda + y\mu + z\nu = 0$, identically,

$$\frac{x}{(bc')} = \frac{y}{(ca')} = \frac{z}{(ab')}.$$

Now the invariants, S and T, of the equation in k are expressed in terms of the combinants P and Q, of the cubics f and f', thus (Salmon's *Higher Algebra*, Art. 204),

$$S = 3P(P^3 - 24Q),$$

$$T = -(P^6 - 36P^3Q + 216Q^2).$$

But $$P = (ab') + (bc') + (ca'),$$

$$Q = (ab')(bc')(ca');$$

hence $$P = x + y + z, \quad Q = xyz,$$

and $$S = 3(x + y + z)\{(x + y + z)^3 - 24xyz\},$$

$$T = -\{(x + y + z)^6 - 36xyz(x + y + z)^3 + 216x^2y^2z^2\};$$

whence $$\frac{S^3}{T^2} = \frac{27\lambda(\lambda - 24)^3}{(\lambda^2 - 36\lambda + 216)^2},$$

where $$(x + y + z)^3 = \lambda xyz.$$

Let us calculate the invariants of the cubic

$$(x + y + z)^3 - \lambda xyz = 0.$$

We find $S' = 3\lambda^3(\lambda - 24)$, $T' = -\lambda^4(\lambda^2 - 36\lambda + 216)$.

Hence we infer, that the absolute invariant $\dfrac{S^3}{T^2}$ of the tangents drawn from any point to the nodal cubic

$$(x + y + z)^3 - 27xyz = 0$$

is equal to the absolute invariant $\dfrac{S'^3}{T'^2}$ of the cubic of the system $(x + y + z)^3 - \lambda xyz = 0$ which passes through the point.

281. Using trilinear coordinates and writing the cubic

$$(l\alpha + m\beta + n\gamma)^3 - 27lmn\alpha\beta\gamma = 0,$$

if a cubic of the system

$$(l\alpha + m\beta + n\gamma)^3 - \lambda\alpha\beta\gamma = 0$$

pass through the circular points at infinity, the foci of the curve will lie, by fours, on four circles (Salmon's *Higher Plane Curves*, Art. 168). This condition is equivalent to making the line of the inflexions meet the curve at angles whose sum $= 0$. For writing

$$\alpha = X\cos\alpha + Y\sin\alpha - p_1, \text{ \&c.},$$

$$l\alpha + m\beta + n\gamma = k\delta = k(X\cos\delta + Y\sin\delta - p),$$

where X, Y are rectangular Cartesian coordinates, we have for the circular points $\alpha : \beta : \gamma : \delta = e^{\pm i\alpha} : e^{\pm i\beta} : e^{\pm i\gamma} : e^{\pm i\delta}$; and, therefore, when $\alpha\beta\gamma - \lambda'\delta^3 = 0$ passes through the two circular points $\lambda' = 1$, $\alpha + \beta + \gamma = 3\delta$, the latter of which equations gives the condition stated above.

The curve, in this case, has all its foci in common with two circular cubics.

282. A nodal cubic, being referred to the triangle formed by the nodal tangents and the line of the inflexions, can be written $x^3 + y^3 + 6xyz = 0$. Eliminating z between the equations of the curve and the polar conic of (x', y', z')

$$U \equiv x'(x^2 + 2yz) + y'(y^2 + 2zx) + 2z'xy,$$

we obtain

$$y'x^4 - 2x'x^3y - 6z'x^2y^2 - 2y'xy^3 + x'y^4 = 0.$$

Multiplying this equation by $x'x^2 + y'y^2$, it becomes

$$x'y'(x^6 + y^6 - 4x^3y^3) + (x'^2 - 6y'z')x^2y^4 + (y'^2 - 6z'x')y^2x^4$$
$$- 2x'^2x^5y - 2y'^2xy^5 = 0.$$

But, from the equation of the curve,

$$x^6 + y^6 - 4x^3y^3 = 6x^2y^2(6z^2 - xy),$$

$$x^5y = -x^2y^2(y^2 + 6zx), \quad xy^5 = -x^2y^2(x^2 + 6yz);$$

hence, substituting and dividing by $3x^2y^2$, we have

$$(y'^2 - 2z'x')x^2 + (x'^2 - 2y'z')y^2 + 12x'y'z^2$$
$$+ 4y'^2yz + 4x'^2zx - 2x'y'xy = 0,$$

which represents a conic passing through the points of contact of tangents from (x', y', z') to the curve.

If we call this conic V, $V + \lambda U = 0$ represents any conic passing through the points of contact of the tangents.

Hence, the locus of points, from which the tangents have their points of contact on a conic passing through two fixed points, is a cubic $U_1V_2 - V_1U_2 = 0$. By taking for the fixed points the circular points at infinity, we have the locus of points whence the tangents have their points of contact on a circle.

Putting $\lambda = -2z'$ in the equation of the conic $V + \lambda U = 0$, we obtain the equation of the conic of the system which passes through (x', y', z'):

$$(y'^2 - 4z'x')x^2 + (x'^2 - 4y'z')y^2 + 12x'y'z^2$$
$$+ 4(y'^2 - z'x')yz + 4(x'^2 - y'z')zx - 2(x'y' + 2z'^2)xy = 0.$$

The tangent to this conic at (x', y', z') is

$$z'(x'^2 + 2y'z')x + z'(y'^2 + 2z'x')y - (x'^3 + y'^3 + 4x'y'z')z = 0,$$

which coincides with the tangent to the cubic

$$z'^3(x^3 + y^3 + 6xyz) - (x'^3 + y'^3 + 6x'y'z')z^3 = 0$$

at the same point, as it ought (see Salmon's *Curves*, Art. 169).

283. The discriminant of $V + \lambda U$ is found to be, after dividing by the Hessian $(\equiv x^3 + y^3 - 2xyz)$,

$$\lambda^3 - 6z'\lambda^2 + 4(x'^3 + y'^3 + 6x'y'z') = 0.$$

By means of this result we can find the locus of the intersection of the tangents at the extremities of a chord which passes through a fixed point. Forming the equation of the chords of intersection of U and V, and expressing this equation is satisfied by the coordinates of the fixed point, we obtain the equation of the locus

$$W^3 + 6zPW^2 - 4(x^3 + y^3 + 6xyz)P^3 = 0,$$

where $$W = (y'^2 + 4z'x')x^2 + (x'^2 + 4y'z')y^2 - 2y'^2yz - 2x'^2zx + 2(6z'^2 - x'y')xy,$$

$$P = (x'^2 + 2y'z')x + (y'^2 + 2z'x')y + 2x'y'z,$$

(x, y, z) being now the coordinates of a point on the locus, and (x', y', z') of the fixed point.

When the fixed point is on the curve,

$$W = -\frac{1}{x'y'}(xx'^2 + yy'^2)P,$$

and the locus, after having been divided by P^3, becomes

$$4x'^3y'^3(x^3 + y^3) + (xx'^2 + yy'^2)^3 - 6x'y'(xx'^2 - yy'^2)^2 z = 0,$$

which represents a cubic with xy for a cusp.

284. If tangents be drawn from (x', y', z') to the cubic $x^3 + y^3 + 6xyz = 0$, show that the anharmonic ratios of the lines joining the node to their points of contact is given by the equation

$$\frac{T^2}{S^3} = \frac{(u' - 4z'^3)^2}{432z'^6},$$

where $$u' \equiv x'^3 + y'^3 + 6x'y'z'.$$

Show that the anharmonic ratio of the lines joining the node to the points where the tangents meet the curve again is given by the equation

$$\frac{T'^2}{S'^3} = \frac{(u'^2 - 36z'^3u' + 216z'^6)^2}{1728z'^3(9z'^3 - u')^3}.$$

285. Show that the conic we have called W (Ex. 283) passes through the points where the tangents from (x', y', z') meet the cubic again.

286. A line, which meets the cubic $x^3 + y^3 - 3xyz = 0$ and the line $y - mx = 0$, so that the invariant S of the four points of intersection vanishes, is a tangent to

$$\alpha^2 - \beta\gamma + m(\alpha\beta - \gamma^2) + \beta^2 - \gamma\alpha = 0,$$

which represents a conic touching the three inflexional tangents and the curve.

If the cubic be written $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0$, and the line

$$lx + my + nz = 0,$$

where $l + m + n = 0$, the conic is

$$\left(\frac{x}{l}\right)^{\frac{1}{2}} + \left(\frac{y}{m}\right)^{\frac{1}{2}} + \left(\frac{z}{n}\right)^{\frac{1}{2}} = 0.$$

287. *Given the foci of a nodal cubic, to determine the curve.*

Or, in other words, to inscribe a nodal cubic in two pencils of four lines each. Suppose the cubic to be written in the form $\theta x = f_1$, $\theta y = f_2$, $\theta z = f_3$, where f_1, f_2, f_3 are binary quantics of the third degree in a parameter λ, and xz, yz are the vertices of the pencils. Since we obtain the equation of the tangents from xz to the curve by equating to zero the discriminant of $xf_2 - yf_1$, it follows that if the coefficients of f_1 and f_2 be given, these tangents will be given. If we write now

$$f_2 = a\lambda^3 + 3b\lambda^2 + 3c\lambda + d,\ f_3 = a'\lambda^3 + 3b'\lambda^2 + 3c'\lambda + d',$$

we determine a', b', c', d', by expressing that the tangents from yz to the curve are given, or that the coefficients of y^4, &c. in the discriminant of $yf_3 - zf_2$ are given. Hence the quantities ΔD, $\Delta^2 D$, $\Delta^3 D$, $\Delta^4 D$, where

$$\Delta = a' \frac{d}{da} + b' \frac{d}{db} + c' \frac{d}{dc} + d' \frac{d}{dd},$$

and D is the discriminant of f_2, must be given. These equations being of the first, second, third, and fourth degrees respectively, give twenty-four values of a', b', c', d'. Hence there are twenty-four curves satisfying the given conditions.

288. Let U denote the nodal cubic

$$(l\alpha + m\beta + n\gamma)^3 - 27 lmn\alpha\beta\gamma = 0;$$

if
$$\frac{l}{\cos^2 \frac{1}{2}A} + \frac{m}{\cos^2 \frac{1}{2}B} + \frac{n}{\cos^2 \frac{1}{2}C} = 0,$$

where A, B, C are the angles of the triangle of reference, U touches the inscribed circle and has a focus on the circumcircle at the point

$$\frac{\sin A \cos^2 \frac{1}{2}A}{l}, \quad \frac{\sin B \cos^2 \frac{1}{2}B}{m}, \quad \frac{\sin C \cos^2 \frac{1}{2}C}{n}.$$

For it can be easily seen that

$$\cos \tfrac{1}{2}A\, (\alpha)^{-\frac{1}{2}} + \cos \tfrac{1}{2}B\, (\beta)^{-\frac{1}{2}} + \cos \tfrac{1}{2}C\, (\gamma)^{-\frac{1}{2}} = 0$$

represents a tricuspidal quartic passing through the circular points at infinity. Reciprocating this equation (see Salmon's *Conics*, Art. 311), and identifying the result with the tangential equation of U, we have, since the origin is a focus,

$$\frac{\sin \theta_1 \sin^2 \frac{1}{2}\theta_1}{l\alpha'} = \frac{\sin \theta_2 \sin^2 \frac{1}{2}\theta_2}{m\beta'} = \frac{\sin \theta_3 \sin^2 \frac{1}{2}\theta_3}{n\gamma'},$$

where θ_1, θ_2, θ_3 are the angles the sides of the triangle subtend at a focus. But for a point on the circumscribing circle

$$\theta_1 = \pi - A, \text{ \&c., and } \frac{a}{\alpha'} + \frac{b}{\beta'} + \frac{c}{\gamma'} = 0;$$

therefore
$$\frac{l}{\cos^2 \frac{1}{2}A} + \frac{m}{\cos^2 \frac{1}{2}B} + \frac{n}{\cos^2 \frac{1}{2}C} = 0.$$

289. Writing $x=\theta$, $y=\theta^2$, $z=1+\theta^3$ for a point on the cubic $x^3+y^3-xyz=0$, if $a\theta\theta'+b(\theta+\theta')+c=0$, the chord joining the points θ, θ' touches a conic having triple contact with the curve. For the equation of the chord is

$$(\theta^2\theta'^2-\theta-\theta')\,x+(1-\theta^2\theta'-\theta'^2\theta)\,y+\theta\theta'z=0,$$

whence the coordinates of the chord may be written

$$\lambda=bt^2+at+c,\quad \mu=at^2+ct+b,\quad \nu=bt,$$

where $t=\theta\theta'$, showing that the envelope of the chord is a conic. If we substitute the coordinates of the chord in the tangential equation of the curve

$$27\nu^4-18\lambda\mu\nu^2+4(\lambda^3+\mu^3)\,\nu-\lambda^2\mu^2=0,$$

we must get a result proportional to

$$(\theta-\theta')^2(\theta-\theta'')^2(\theta'-\theta'')^2,$$

where $\theta''=-\dfrac{1}{\theta\theta'}$ is the parameter of the point where the chord meets the curve again. Hence it appears that the conic has triple contact with the cubic at the points

$$b\theta^3+c\theta^2-a\theta-b=0,$$

and that the points of contact of the common tangents are

$$a\theta^2+2b\theta+c=0.$$

From the latter equation it can be seen that the problem "To describe a conic having triple contact with a nodal cubic to touch two given tangents to the curve" admits of a single solution.

If two lines meet the cubic in the points α, β, γ; α', β', γ', respectively, and if

$$a\alpha\alpha'+b(\alpha+\alpha')+c=0,\quad a\beta\beta'+b(\beta+\beta')+c=0,$$
$$a\gamma\gamma'+b(\gamma+\gamma')+c=0,$$

in which case the chords $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ are tangents to a conic having triple contact with the curve, then the point of intersection of the lines is fixed. For eliminating γ, γ', α', β' between these equations and $\alpha\beta\gamma = \alpha'\beta'\gamma' = -1$, we obtain

$$b(a^2 - bc)(\alpha^2\beta^2 - \alpha - \beta) + b(c^2 - ab)(1 - \alpha^2\beta - \beta^2\alpha) + (2b^3 - a^3 - c^3)\,\alpha\beta = 0,$$

which shows that the line $\alpha\beta$, and of course also the line $\alpha'\beta'$, passes through the point

$$x = b(a^2 - bc),\quad y = b(c^2 - ab),\quad z = 2b^3 - a^3 - c^3.$$

In the same way, putting $y = \theta x$ in the equation of the cuspidal cubic $y^3 - x^2z = 0$, if

$$a\theta\theta' + b(\theta + \theta') + c = 0,$$

the chord $\theta\theta'$ is enveloped by a conic having double contact with the curve and touching the inflexional tangent. Also, if two lines meet the curve in the points α, β, γ; α', β', γ', and if $a\alpha\alpha' + b(\alpha + \alpha') + c = 0$, &c., the point of intersection of the lines is the fixed point

$$x = -3a^2b,\quad y = a(ac + 2b^2),\quad z = 3b^2c.$$

By reciprocation we see that, if

$$a\theta\theta' + b(\theta + \theta') + c = 0,$$

the tangents at θ, θ' intersect on a conic having double contact with the curve and passing through the cusp.

290. If the tangent at a point α of the cubic

$$x^3 + y^3 - xyz = 0$$

meet the curve again in α', we have $\alpha^2\alpha' = -1$; hence we see that the line

$$(1 + \alpha^4\beta^2 + \beta^4\alpha^2)\,x + (\alpha^4\beta^4 + \alpha^2 + \beta^2)\,y + \alpha^2\beta^2z = 0$$

is the *satellite* of

$$(\alpha^2\beta^2 - \alpha - \beta)\,x + (1 - \alpha^2\beta - \beta^2\alpha)\,y + \alpha\beta z = 0.$$

Writing the latter line

$$\lambda x + \mu y + \nu z = 0,$$

the satellite will be, then,

$$(\mu^2 - 2\nu\lambda)\,x + (\lambda^2 - 2\mu\nu)\,y + \nu^2z = 0.$$

Hence if the satellite pass through the fixed point x', y', z', the four corresponding lines will form a quadrilateral inscribed in the cubic and circumscribed about the conic

$$y'(\lambda^2 - 2\mu\nu) + x'(\mu^2 - 2\nu\lambda) + z'\nu^2 = 0.$$

By considering the case when the satellite is a tangent to the curve, we see that this conic touches the eight tangents drawn to the curve from the points of contact of tangents from x', y', z'.

If a line pass through the fixed point x', y', z', the satellite touches the conic

$$(2z'x' - y'^2)x^2 + (2y'z' - x'^2)y^2 + y'^2yz + x'^2zx + (z'^2 + 2x'y')xy = 0.$$

This conic, it can easily be seen, touches the tangents to the cubic at the points where the tangents from x', y', z' meet the curve again.

391 If the curve be written $(x+y+z)^3 - 27xyz = 0$, the satellite of $\lambda x + \mu y + \nu z = 0$ is

$$(\lambda^2 - 2\lambda\mu - 2\lambda\nu + 6\mu\nu)x + (\mu^2 - 2\mu\lambda - 2\mu\nu + 6\nu\lambda)y$$
$$+ (\nu^2 - 2\nu\lambda - 2\nu\mu + 6\lambda\mu)z = 0.$$

291. If we put $y = \theta x$ in the equation of the cubic

$$x^3 + y^3 + 6xyz = 0,$$

the condition that the chord θ_1, θ_2, should pass through the fixed point x', y', z' is

$$(\theta_1^2\theta_2^2 - \theta_1 - \theta_2)x' + (1 - \theta_1^2\theta_2 - \theta_2^2\theta_1)y' - 6\theta_1\theta_2 z' = 0,$$

which may be written

$$A_2\theta_1^2 + B_2\theta_1 + C_2 = A_1\theta_2^2 + B_1\theta_2 + C_1 = 0,$$

where A_1, &c. are functions of θ_1, and A_2, &c. of θ_2. Hence, differentiating, we have

$$(A_2\theta_1 + B_2)\,d\theta_1 + (A_1\theta_2 + B_1)\,d\theta_2 = 0;$$

therefore $\sqrt{(B_2^2 - 4A_2C_2)}\, d\theta_1 + \sqrt{(B_1^2 - 4A_1C_1)}\, d\theta_2 = 0,$

or
$$\frac{d\theta_1}{\sqrt{f(\theta_1)}} + \frac{d\theta_2}{\sqrt{f(\theta_2)}} = 0,$$
where
$$f(\theta) = y'^2\theta^4 + 4\,(3y'z' + x'^2)\,\theta^3 + 6\,(6z'^2 - x'y')\theta^3 + 4\,(3z'x' + y'^2)\,\theta + x'^2.$$

In a similar manner we find for the cuspidal cubic
$$y^3 - x^2z = 0, \quad \frac{d\theta_1}{\sqrt{f(\theta_1)}} + \frac{d\theta_2}{\sqrt{f(\theta_2)}} = 0,$$
where $\qquad f(\theta) = (y' - \theta x')\,(x'\theta^3 + 3y'\theta^2 - 4z').$

292. A triangle being inscribed in the cuspidal cubic defined by the equations $y = \theta x$, $z = \theta^3 x$, so that the tangents at the vertices pass through the point x, y, z, the equations
$$L - \theta_2\theta_3 M = 0, \quad L - \theta_3\theta_1 M = 0, \quad L - \theta_1\theta_2 M = 0,$$
where
$$L = 2\theta_1\theta_2\theta_3 x - (\theta_1 + \theta_2 + \theta_3)^2 y + z, \qquad M = (\theta_1 + \theta_2 + \theta_3)\,x + 3y,$$
represent the lines joining the vertices to the points in which the opposite sides meet the curve again. These lines, therefore (see Ex. 225), pass through a point, the coordinates of which are, since θ_1, θ_2, θ_3 are the roots of $2x\theta^3 - 3y\theta^2 + z = 0$,
$$x' = 8x^3, \quad y' = -4x^2y, \quad z' = 8zx^2 - 9y^3.$$

If we are given the point x', y', z', we have
$$x = x'^3, \quad y = -2x'^2y', \quad z = x'^2z' - 9y'^3,$$
and
$$\frac{8zx^2}{y^3} + \frac{z'x'^2}{y'^3} = 9.$$
Hence if one of the points lie on a locus V, the other will lie on a curve having the same deficiency as V (see Salmon's *Curves*, Art. 364).

293. If a triangle be inscribed in the cubic $k^2y = x^3$ so that the tangents at the vertices pass through a point, the

axis of y will be a tangent to the conic touching the sides of the triangle at their middle points.

Let $x+y+z=L$, $ax+by+cz=M$, $a^3x+b^3y+c^3z=N$, where x, y, z are the sides of the triangle, then

$$L^2N-M^3=x\{(a-b)^2(a+2b)y^2+(a-c)^2(a+2c)z^2\}$$
$$+y\{(b-c)^2(b+2c)z^2+(b-a)^2(b+2a)x^2\}$$
$$+z\{(c-a)^2(c+2a)x^2+(c-b)^2(c+2b)y^2\}+2xyz(a^3+b^3+c^3-3abc).$$

If we seek now the condition that the tangents to this cubic at the vertices of the triangle should pass through a point, we find $ab+bc+ca=0$; and when this relation is satisfied the line M touches $\sqrt{x}+\sqrt{y}+\sqrt{z}=0$. But if $x+y+z=0$ is the line at infinity, this conic touches the sides of the triangle at their middle points. In the same case the lines joining the vertices to the points in which the opposite sides meet the curve again intersect on the conic $xy+yz+zx=0$.

294. Let three points θ_1, θ_2, θ_3 be taken on the cubic $y=\theta x$, $z=\theta^3x$, such that $\theta_1+\theta_2+\theta_3=0$,

$$4(\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1)^3+27k\,\theta_1^2\theta_2^2\theta_3^2=0,$$

then the tangents at these points form a triangle inscribed in the cubic $y^3-kx^2z\equiv V=0$. The points of contact of the sides of this triangle lie on a line which touches $ky^3-x^2z=0$, and the tangents to V at the vertices of the triangle pass through a point which lies on $y^3-k^2x^2z=0$.

295. Show that the locus of the intersection of rectangular tangents to the cubic $(ax+by)^2-x^3=0$ is a parabola having double contact with the curve; and, reciprocally, that the envelope of a chord which subtends a right angle at the cusp is a conic having double contact with the curve and passing through the cusp.

296. The cubic whose equation in rectangular coordinates is $xy^2 - 4a^3 = 0$ has three foci at the points

$$x = -3a,\ y = 0;\quad x = \tfrac{3}{2}a,\ y = \pm\tfrac{3}{2}a\sqrt{3}.$$

The locus of the intersection of rectangular tangents to the same cubic is $x^2 + y^2 - 3ax = 0$, a circle having double contact with the curve at points on the line $x = 2a$.

A chord of the curve, which subtends a right angle at the origin, touches the parabola

$$x^2 + y^2 - (x - 2a\sqrt[3]{4})^2 = 0.$$

297. If the coefficient of xy be absent in the equation of a conic, show that it meets the cubic $y^3 - x^2z = 0$ in six points where the tangents to the cubic are touched by a conic.

298. The equation of the conic, osculating the cubic $xy^2 - z^3 = 0$ at the point where $z = \theta y$, is

$$x^2 - 5\theta^6y^2 - 45\theta^4z^2 + 40\theta^3xy + 24\theta^5yz - 15\theta^2zx = 0.$$

Hence six conics of the system can be drawn through a point, and the tangents at the points of contact are all touched by a conic.

Also we can show that the locus of the centres of hyperbolas osculating the cubic $3y = x^3$ is $125x^3 + 192y = 0$.

299. Show that the anharmonic ratio of the tangents from x, y, z to the cubic $y^3 - x^2z$, and (1) the line from the same point to yz is given by the equation $\dfrac{27T^2}{S^3} = k$, (2) the line to zx, $\dfrac{27T^2}{S^3} = (1-2k)^2$, (3) the line to xy,

$$\frac{27T^2}{S^3} = \frac{(8k^2 - 36k + 27)^2}{(9-8k)^3},$$

where
$$\frac{x^2z}{y^3} = k.$$

Hence show that, if the angles of the triangle of reference are connected by the relation $2C - A = \pi$, the cubic $\beta^3 - k\alpha^2\gamma = 0$ has three foci on the circumscribing circle.

Show also that the circle circumscribing the triangle formed by three foci of the cubic $\gamma\alpha^2 + 2\beta^3 \cos(2C - A) = 0$ passes through $\alpha\gamma$.

300. The tangent at any point of the nodal cubic

$$(y + kz)(y + 4kz)^2 - x^2 z = 0$$

is one of the fourth harmonics to the three tangents from the point to $y^3 - x^2 z = 0$.

Writing $x = \theta y$ in $y^3 - x^2 z$, the coordinates of the intersection of the tangents at θ, θ' are

$$x = \tfrac{1}{2}\theta\theta'(\theta + \theta'), \quad y = \tfrac{1}{3}(\theta^2 + \theta'^2 + \theta\theta'), \quad z = 1.$$

Hence if $\theta - \theta' = c$, we have $x = \frac{1}{2}\theta\theta'(\theta + \theta')$, $cy = \frac{1}{3}(\theta^3 - \theta'^3)$, $c^3 z = (\theta - \theta')^3$; whence, eliminating $\dfrac{\theta}{\theta'}$,

$$(y + kz)(y + 4kz)^2 - x^2 z = 0, \text{ where } k = -12c^2.$$

Now the tangent to this curve is easily seen to meet z on the line $2(\theta + \theta')x - (\theta^2 + \theta'^2 + 4\theta\theta')y = 0$, and the tangents to $y^3 - x^2 z = 0$ meet z on the lines

$$2x - 3\theta y = 0, \quad 2x - 3\theta' y = 0, \quad 2x + 3(\theta + \theta')y = 0;$$

and these four lines form a harmonic system.

The tangents to the three curves of the system

$$(y + kz)(y + 4kz)^2 - x^2 z = 0,$$

which pass through a point, are evidently the three fourth harmonics.

301. The polar equation of the cissoid, referred to its double focus, can be written $\tan\frac{1}{2}\theta = \left\{\dfrac{r - a}{r + a}\right\}^{\frac{3}{2}}$.

If p be the perpendicular on the tangent, and ρ the radius of curvature,

$$p^2 = \frac{9a^2(r^2-a^2)}{r^2+15a^2}, \quad \rho = \frac{(r^2-a^2)^{\frac{1}{2}}(r^2+15a^2)^{\frac{3}{2}}}{48a^3}.$$

302. The cubic, whose equation in rectangular coordinates is

$$(b^2+c^2)(y+b)\,x^2 - (y^2+c^2)\,\{(a^2-b^2-c^2)\,y + b\,(a^2+b^2+c^2)\} = 0,$$

has two foci on itself, viz. $x = \pm a$, $y = b$.

The lines joining these foci to the points $x = \pm c$, $y = 0$ are tangents to the curve.

303. The triangle of reference being equilateral, the equation

$$\alpha\,(\beta^2+\gamma^2) + \beta\,(\gamma^2+\alpha^2) + \gamma\,(\alpha^2+\beta^2) - \alpha\beta\gamma = 0$$

represents a cubic of which the vertices of the triangle are foci. Transforming this equation to rectangular axes by writing

$$\alpha = \tfrac{1}{2}(x + y\sqrt{3}) - a, \quad \beta = \tfrac{1}{2}(x - y\sqrt{3}) - a, \quad \gamma = -(a+x),$$

we find $$x^3 - 3xy^2 - \tfrac{3}{4}a\,(x^2+y^2) - 5a^3 = 0.$$

304. The cubic whose equation to rectangular axes is

$$x^3 - p\,(x^2+y^2) + qx - r = 0,$$

being the envelope of the circle

$$p\,(x^2+y^2) - (q+3\mu^2)\,x + r + 2\mu^3 = 0,$$

has four foci determined by the equation

$$27\,(px^2-r)^2 - 4\,(2px-q)^3 = 0.$$

If $q^2 = 4pr$, two of the foci coincide at the point $2px = q$, and the square of the distance of any point of the curve from this focus is in a constant ratio to the cube of its distance from a fixed line.

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305. A cubic is such that two asymptotes meet on the curve; if A, B be two fixed and P a variable point on the curve, show that $\frac{\lambda}{d}+\frac{\mu}{d'}=1$, where d, d' are the segments intercepted on the two asymptotes by PA, PB, and λ and μ are constants.

306. Two asymptotes of a cubic are at right angles to one another; if a perpendicular to the other asymptote, at the point where it meets the curve, intersect the cubic again in A, B, show that the locus of the centre of the nine-point circle of the triangle PAB is a hyperbola, P being a variable point on the curve.

307. If the cubic whose polar equation is $r^3\cos 3\theta=a^3$ be inscribed in a triangle so that the points of contact lie on a line, the pole of the curve is one of the points of contact of the nine-point circle with the circles touching the sides.

The polar conic of the line $ax+by+cz=0$ with respect to the cubic $x^3+y^3+z^3=0$ is

$$a^2yz+b^2zx+c^2xy=0;$$

it therefore circumscribes the triangle xyz. Now let x, y pass through the circular points and z be the line at infinity, when the cubic becomes $r^3\cos 3\theta=a^3$. Then, since the polar conic of a line touches the tangents to the curve where it is met by the line, it follows that one of the circles touching these tangents passes through the pole.

Again, when the cubic is written in the form

$$ABC-D^2F=0,$$

where A, B, C are the tangents to the curve at its points of intersection with D, it is evident that the polar conic of the point DF circumscribes the triangle ABC. But the polar conic of any point with regard to the curve $r^3 \cos 3\theta = a^3$ is an equilateral hyperbola having the pole for centre. Hence the nine-point circle of the triangle ABC passes through the pole of the curve.

We can show that the nine-point circle passes through the pole otherwise thus: If a conic Σ have ABC for a self-conjugate triangle and touch D where it is met by F, the result of substituting differential symbols in Σ and operating on $ABC - D^2F$ vanishes; but such a conic for the cubic $x^3 + y^3 + z^3 = 0$ must be of the form

$$f\mu\nu + g\nu\lambda + h\lambda\mu = 0,$$

or for the curve $r^3 \cos 3\theta = a^3$ must be a parabola having the pole for focus. Now the nine-point circle of a triangle self-conjugate with regard to a parabola passes through the focus; therefore &c.

308. The locus of the poles with regard to the cubic

$$U \equiv x^3 + y^3 + z^3 = 0$$

of the tangents to the conic

$$\Sigma \equiv (a\lambda + b\mu + c\nu)^2 - \lambda\mu = 0$$

breaks up into the factors

$$ax^2 + by^2 + cz^2 \pm xy = 0,$$

and these two conics intersect U at the points where its tangents touch Σ. If z is the line at infinity and x, y pass through the circular points, the polar equation of the curve may be written $\rho^3 \cos 3\theta = a^3$, and we see that if six tangents to the curve be touched by the circle

$$(x - x')^2 + (y - y')^2 - r^2 = 0,$$

the points of contact will lie on the conic

$$(r + x')\, x^2 + (r - x')\, y^2 - 2y'xy - a^3 = 0.$$

When the conic and the circle touch one another, it is evident that they will both touch the curve. Thus we find the condition that the circle should touch the cubic by equating to zero the discriminant of the equation

$$r^2\lambda^3 - \{x'^3 - 3x'y'^2 - a^3 + r(x'^2 + y'^2) - 2r^3\}\lambda^2$$
$$+ \{2a^3r + (x'^2 + y'^2 - r^2)^2\}\lambda - a^3(x'^2 + y'^2 - r^2) = 0.$$

When this equation has three equal roots the circle osculates the cubic.

309. If the normal at a point P of the curve

$$x^3 - 3xy^2 - a^3 = 0$$

meet one of the lines $y^3 - 3x^2y = 0$ in Q, show that

$$(\delta - 2r)(\delta + r)^2 = a^3,$$

where $r = PQ$, and δ is the distance of Q from the origin.

If Q_1, Q_2, Q_3 be the points corresponding to the three lines, show that

$$\frac{1}{PQ_1} + \frac{1}{PQ_2} + \frac{1}{PQ_3} = 0.$$

310. Show that the six lines represented by the equation

$$x^{\frac{3}{2}} + y^{\frac{3}{2}} + (ax + by)^{\frac{3}{2}} = 0$$

are tangents to both the cubics

$$(cx - bz)^3 - (cy + az)^3 + z^3 = 0,$$

$$x^3 + y^3 + z^3 - 3xyz - (ax + by)^3 = 0.$$

F. ... ~ Rat Behandlung

der Böhm Gesells. der Wiss. 1873
Schoute Sitzber der Wien Ac. der Wiss, Bd 88 1884.

III. *Examples and Problems on Bicircular Quartics.*

311. To find the points on a bicircular quartic from which the tangents have their points of contact on a conic.

Let us write the quartic

$$U \equiv x^2y^2 + z^2(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0,$$

where z is the line at infinity and xz, yz are the circular points, and let V be the polar cubic of a point (x', y', z') on U. Now if six points of intersection of a quartic and a cubic lie on a conic, the remaining six points of intersection must also. Thus when the points of contact of the six tangents lie on a conic, the remaining points of intersection of U and V must also; but these latter points are the point x', y', z' and the nodes xz, yz, each counted twice over. Hence a conic can be described so as to touch V at each of these points, or a circle whose centre is the double focus of V must touch the curve at x', y', z'.

We may express this condition by substituting the coordinates of the double focus of V in the equation of the normal at x', y', z'. Now the tangents to V at xz, yz are

$$xx' + bzz' = 0, \quad yy' + azz' = 0,$$

and the intersection of these tangents is the double focus; also the normal to U at x', y', z' is

$$(xz' - zx')\frac{dU'}{dx'} - (yz' - zy')\frac{dU'}{dy'} = 0;$$

hence we have, after dividing by z and dropping the accents,

$$xy(fy - gx) + hz(ax^2 - by^2) + z^2(afx - bgy) = 0,$$

which represents a circular cubic intersecting the quartic in eight points satisfying the given condition.

Weyr's th. on pts of contact of tan from pt on line
Schoute's is the 6 ~~pts~~ poles of a line for m

If $f=g=h=0$, the quartic becomes an ellipse of Cassini, and the cubic vanishes identically. Hence we see that the points of contact of the tangents from every point of this curve lie on a conic. Equation of this conic?

312. The equation of the quartic being the same as in the preceding example, show that the locus of a point, whose polar cubic has its double focus on itself, is the circular cubic

$$hz(ax^2+by^2-2abz^2)+3z^2(afx+bgy)$$
$$-xy(gx+fy)+2(ab-c)xyz=0.$$

If P be an arbitrary point, and P' the double focus of the polar cubic of P, show that OP and OP' are equally inclined to FF', and that $OP.OP'=OF^2$, where F, F' are the double foci of the quartic, and O is the middle point of FF'.

313. The equation of the quartic being the same as before, the locus of the double foci of the polar cubics of every point on the curve is the bicircular quartic

$$cx^2y^2-2xyz(afx+bgy)+abz^2(ax^2+by^2+abz^2+2hxy)=0.$$

This locus is identical with the given quartic, if $f=g=0$, $c=ab$. Hence the double focus of the polar cubic of any point on the quartic

$$(x^2+bz^2)(y^2+az^2)+2hxyz^2=0$$

lies on the curve.

314. A bicircular quartic has a finite node; to find the locus of the points from which the tangents have their points of contact on a conic.

Let us write the quartic

$$U\equiv x^2y^2+y^2z^2+z^2x^2+2xyz(ax+by+cz),$$

where xz, yz are the circular points and xy is the finite node, and let V be the polar cubic of x', y', z'.

Then since the points of contact of the tangents lie on a conic, another conic can be described through the vertices of the triangle xyz to touch V at these points, or the normal to V at xy must pass through the double focus of V. This condition gives

$$(a-bc)\,x\,(y^2-z^2)+(b-ca)\,y\,(z^2-x^2)+(c-ab)\,z\,(x^2-y^2)=0,$$

which represents a circular cubic having its double focus on itself.

It is evident that this cubic intersects the quartic in points such that the anharmonic ratio of the tangents from them is a maximum or a minimum.

315. If, in the preceding example, $a=b=c=0$, the quartic becomes a lemniscate and the points of contact of the tangents from any point to the curve lie on a conic.

We proceed to find the equation of this conic. If we form the contravariant σ (Salmon's *Curves*, Art. 292) of the quartic

$$U\equiv x^2y^2+y^2z^2+z^2x^2,$$

we find
$$\sigma=(\alpha^2+\beta^2+\gamma^2)^2.$$

For other cases of σ breaking up see pp. 167, 162, 187.

Now, if we put

$$\alpha=yz'-zy',\quad \beta=zx'-xz',\quad \gamma=xy'-yx'$$

in σ, the result must be proportional to the invariant S of the biquadratic in k obtained by putting $x+kx'$, &c. for x, &c. in $U=0$. Thus we obtain the identity

$$12\,(U'U-PV)+Q^2=\{(yz'-zy')^2+(zx'-xz')^2+(xy'-yx')^2\}^2,$$

where

$$V\equiv x'x\,(y^2+z^2)+y'y\,(z^2+x^2)+z'z\,(x^2+y^2),$$

$$P\equiv x'\,(y'^2+z'^2)\,x+y'\,(z'^2+x'^2)\,y+z'\,(x'^2+y'^2)\,z,$$

$$Q\equiv (y'^2+z'^2)\,x^2+(z'^2+x'^2)\,y^2+(x'^2+y'^2)\,z^2$$
$$+4y'z'yz+4z'x'zx+4x'y'xy.$$

See 349 and note at end.

Hence, putting $U = V = 0$, we see that the points of contact of the tangents from x', y', z' to the curve lie on the conic

$$(y'^2 + z'^2)\, x^2 + (z'^2 + x'^2)\, y^2 + (x'^2 + y'^2)\, z^2 + y'z'yz + z'x'zx + x'y'xy = 0,$$

the factor $\qquad y'z'yz + z'x'zx + x'y'xy$

being rejected as irrelevant.

The discriminant of this conic is proportional to

$$(x'^2 + y'^2 + z'^2)\,(x'^2y'^2 + y'^2z'^2 + z'^2x'^2).$$

Taking the case when $x'^2 + y'^2 + z'^2 = 0$, the conic breaks up into the lines

$$xx' + \theta yy' + \theta^2 zz' = 0, \quad xx' + \theta^2 yy' + \theta zz' = 0,$$

where θ is a cube root of unity. These two lines touch the conics

$$x^2 + \theta^2 y^2 + \theta z^2 = 0, \quad x^2 + \theta y^2 + \theta^2 z^2 = 0,$$

and intersect each other at the point $\dfrac{1}{x'}, \dfrac{1}{y'}, \dfrac{1}{z'}$ on U.

If we write the lemniscate in polar coordinates

$$r^2 = 2c^2 \cos 2\theta,$$

we see that if tangents be drawn to the curve from any point of the equilateral hyperbola $2r^2 \cos 2\theta = c^2$, their points of contact will lie on two lines which intersect on the curve and touch the equilateral hyperbolas

$$2r^2 \cos\left(2\theta \pm \frac{2\pi}{3}\right) = c^2.$$

316. From the identity in the preceding example we see that the polar line of any point (x', y', z') meets the curve at its points of intersection with the conics

$$x'yz + y'zx + z'xy = 0, \quad y'z'yz + z'x'zx + x'y'xy = 0.$$

Hence the tangent at x', y', z' meets the curve again at its intersection with

$$x'yz + y'zx + z'xy = 0.$$

t

Th [illegible] the $x_0 + \omega x_1 + \omega^2 x_2$. The oth.
3 lines ~~touch~~ at pts on a line of (x^3), and lines of curve
~~in pt t~~ where ~~t meet line~~ meet on (x^3) at pt of contact.

317. If a point lie on one of the conics

$$x^2 + \theta y^2 + \theta^2 z^2 = 0, \quad x^2 + \theta^2 y^2 + \theta z^2 = 0,$$

show that the invariant S of its polar cubic with regard to $x^2y^2 + y^2z^2 + z^2x^2 = 0$ vanishes.

If a point lie on $x^2 + y^2 + z^2 = 0$, show that the invariant T of its polar cubic vanishes (see Ex. 218).

318. Four points on the quartic $x^2y^2 + y^2z^2 + z^2x^2 = 0$, lie on the line $\alpha x + \beta y + \gamma z = 0$; the tangents at these points meet the curve again in eight points lying on the conic

$$\alpha^4 x^2 + \beta^4 y^2 + \gamma^4 z^2 - 2(\alpha^2 + \beta^2 + \gamma^2)(\beta\gamma yz + \gamma\alpha zx + \alpha\beta xy)$$
$$+ 8\alpha\beta\gamma(\alpha yz + \beta zx + \gamma xy) = 0.$$

The discriminant of this conic is

$$\alpha^2\beta^2\gamma^2 \{(\alpha^2 + \beta^2 + \gamma^2)^3 - 27\alpha^2\beta^2\gamma^2\} = \alpha^2\beta^2\gamma^2$$

multiplied by the tangential equation of the curve, as it ought to be.

319. *To find an expression for the angle of aberrancy* δ (Salmon's *Curves*, Art. 407) *at any point of a bicircular quartic.*

Taking the tangent and normal at the point as axes of coordinates, the equation of the curve may be written

$$U \equiv (x^2 + y^2)^2 + (lx + my)(x^2 + y^2) + ax^2 + by^2 + 2hxy + 2fy = 0;$$

and then the conic

$$a'x^2 + b'y^2 + 2h'xy + 2f'y = 0$$

is easily seen to have four point contact with the curve at the origin, if $a' = a^2$, $h' = ha - lf$, $f' = af$.

The line drawn from the origin to the centre of this conic is $a^2x + (ha - lf)y = 0$; whence $\tan\delta = \dfrac{lf - ha}{a^2}$. Now if we express the condition that the circle $x^2+y^2-2ry=0$ should touch the curve again, we obtain

$$(ar+f)(4r^3+2mr^2+br+f)-r^2(lr+h)^2=0;$$

whence
$$r_1r_2r_3r_4=\frac{f^2}{4a-l^2},$$

and
$$(\rho-r_1)(\rho-r_2)(\rho-r_3)(\rho-r_4)=\frac{f^2(lf-ha)^2}{a^4(l^2-4a)},$$

where r_1, r_2, r_3, r_4 are the radii of the four circles which may be described through the origin to have double contact with the curve, and $\rho=-\dfrac{f}{a}$ is the radius of curvature at the origin. Hence

$$\tan^2\delta=-\left(\frac{\rho}{r_1}-1\right)\left(\frac{\rho}{r_2}-1\right)\left(\frac{\rho}{r_3}-1\right)\left(\frac{\rho}{r_4}-1\right).$$

Now if the quartic be considered as the envelope of a circle whose centre moves along the conic

$$F\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

and which cuts orthogonally the circle

$$J\equiv x^2+y^2-2\alpha x-2\beta y+k^2=0,$$

we obtain the equation of the curve in the form

$$(x^2+y^2-k^2)^2-4a^2(x-\alpha)^2-4b^2(y-\beta)^2=0,$$

or $S^2-PQ=0$, showing that the points of contact of the double tangents PQ lie on the circle S. Since the quartic can be generated thus in four ways, it follows that S is one of four concentric circles. If we write now

$$(x^2+y^2)^2+(lx+my)(x^2+y^2)+ax^2+by^2+2hxy+2fy$$
$$\equiv\{x^2+y^2+\tfrac{1}{2}(lx+my)+\lambda\}^2-PQ,$$

we must have

$$S=x^2+y^2+\tfrac{1}{2}(lx+my)+\lambda,$$
$$PQ=\lambda^2+\lambda(lx+my)+2\lambda(x^2+y^2)+\tfrac{1}{4}(lx+my)^2$$
$$-(ax^2+by^2+2hxy+2fy).$$

But if $x^2+y^2-2ry=0$ touch the curve again, we know that it must cut J orthogonally, or, if $J\equiv x^2+y^2-2x'x-2y'y+c'$, we must have $c'=2ry'$. Now J cuts S orthogonally, therefore $\frac{1}{2}(lx'+my')+c'+\lambda=0$, and the point PQ is the centre of J, whence $\lambda^2+\frac{1}{2}\lambda(lx'+my')=fy'$. Eliminating $lx'+my'$ and c', we obtain $-2r\lambda=f$. Now f is the value of

$$\tfrac{1}{2}\sqrt{\left\{\left(\frac{dU}{dx}\right)^2+\left(\frac{dU}{dy}\right)^2\right\}}$$

at the origin, and this function is independent of the axes to which the curve is referred; also $\lambda=\rho^2-k^2$, where ρ is the distance of the origin from the centre of F, and k is the radius of S. Hence

$$r_1=\tfrac{1}{4}\frac{\sqrt{\left\{\left(\frac{dU}{dx}\right)^2+\left(\frac{dU}{dy}\right)^2\right\}}}{\rho^2-k_1^2},$$

$$r_2=\tfrac{1}{4}\frac{\sqrt{\left\{\left(\frac{dU}{dx}\right)^2+\left(\frac{dU}{dy}\right)^2\right\}}}{\rho^2-k_2^2},$$

$$r_3=\&c.$$

We can express $\left(\frac{dU}{dx}\right)^2+\left(\frac{dU}{dy}\right)^2$ in terms of the distances $\rho_1, \rho_2, \rho_3, \rho_4$ of the origin from four concyclic foci. Forming the condition that $x+iy-p=0$ should touch the curve, we get

$$\{m^2-l^2+4(a-b)+2i(lm-4h)\}p^4+\ldots+4f^2=0;$$

therefore

$$p_1p_2p_3p_4=\frac{4f^2}{m^2-l^2+4(a-b)+2i(lm-4h)},$$

and $$\rho_1^2\rho_2^2\rho_3^2\rho_4^2=\frac{16f^4}{(m^2-l^2+4a-4b)^2+4(lm-4h)^2}.$$

Now from the expression for PQ given above we can deduce that

$$(m^2 - l^2 + 4a - 4b)^2 + 4(lm - 4h)^2 = 16\delta^4,$$

where δ is the distance between the double foci of the curve.

Hence we have $f = \delta\sqrt{(\rho_1\rho_2\rho_3\rho_4)}$, and

$$\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2 = 4\delta^2(\rho_1\rho_2\rho_3\rho_4).$$

320. Putting $y = 0$ in the equation

$$U \equiv (x^2 + y^2)^2 + (lx + my)(x^2 + y^2) + ax^2 + by^2 + 2hxy + 2fy = 0,$$

we get, after dividing by x^2, $x^2 + lx + a = 0$, whence

$$d^2 = l^2 - 4a = -\frac{16(\rho^2 - k_1^2)(\rho^2 - k_2^2)(\rho^2 - k_3^2)(\rho^2 - k_4^2)}{\delta^2\rho_1\rho_2\rho_3\rho_4},$$

where d is the length of the segment which the tangent intercepts on the curve.

For the central bicircular quartic

$$(x^2 + y^2 + k^2)^2 - 4(a^2x^2 + b^2y^2) = 0,$$

we find

$$d^2 = -\frac{4(\rho^4 - k^4)(\rho^2 + k^2 - 2a^2)(\rho^2 + k^2 - 2b^2)}{(a^2 + b^2 - k^2)(\rho^4 + k^4) - 2(k^4 - k^2a^2 - k^2b^2 + 2a^2b^2)\rho^2},$$

where ρ is the distance of the point on the curve from the centre. For the ellipse of Cassini

$$(x^2 + y^2)^2 - 2c^2(x^2 - y^2) + k^4 = 0, \quad d = 2\frac{\sqrt{\{(r^4 - k^4)(c^4 - r^4)\}}}{r\sqrt{(c^4 - k^4)}}.$$

When $k = 0$, this curve becomes a lemniscate, and

$$d = \frac{2r}{c^2}\sqrt{(c^4 - r^4)},$$

which is a maximum, when $c^4 = 3r^4$.

For the Cartesian oval, since the distance between the double foci must vanish, we have

$$l^2 - m^2 - 4(a-b) = 0, \quad lm - 4h = 0;$$

and if the curve be written in the form

$$(x^2+y^2-k^2)^2 - a^3(x-m) = 0, \quad d^2 = 8(k^2-r^2),$$

where r is the distance of the point on the curve from the triple focus.

321. When the origin is a point of inflexion on the quartic

$$U \equiv (x^2+y^2)^2 + (lx+my)(x^2+y^2) + ax^2 + by^2 + 2hxy + 2fy = 0,$$

a vanishes, and then $d^2 = l^2$. But $l^2 = 16q^2$, where q is the length of the perpendicular from the centre of the focal conic on the normal, or when the curve is written

$$(x^2+y^2-k^2)^2 - 4a^2(x-\alpha)^2 - 4b^2(y-\beta)^2 = 0,$$

$$q = \frac{8\{(a^2-b^2)xy + b^2\beta x - a^2\alpha y\}}{\sqrt{\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2\right\}}}.$$

Hence, the points of inflexion, when the curve is written in the latter form, must satisfy the equation

$$(\rho^2-k_1^2)(\rho^2-k_2^2)(\rho^2-k_3^2)(\rho^2-k_4^2) + 16\{(a^2-b^2)xy + b^2\beta x - a^2\alpha y\}^2 = 0,$$

which may be combined with the equation of the curve so as to give a quartic passing through the circular points at infinity.

For the Cartesian oval

$$(x^2+y^2-k^2)^2 - a^3(x-m) = 0,$$

the points of inflexion lie on the circular cubic

not on conic

$$16k^2(x-m)(x^2+y^2-k^2) + a^3(9x^2+3y^2-24mx+16m^2-k^2) = 0.$$

322. The area which the tangent at a point P of a bicircular quartic cuts off from the curve is a maximum or a minimum, show that the normal to the curve at P passes through the centre of the focal conic.

323. If A, B be the points in which the tangent at a point P of the lemniscate $r^2 = 2c^2 \cos 2\theta$ meets the curve again, and S be the area which the chord AB cuts off from the curve, we have evidently

$$dS = \tfrac{1}{2} (PA^2 - PB^2)\, d\phi,$$

where ϕ is the angle which the tangent makes with a fixed line. But (see *Ex.* 320)

$$PA + PB = 4q = 4 \frac{dp}{d\phi},$$

where p is the perpendicular from the node on the tangent, and

$$PA - PB = -\frac{2r}{c^2} \sqrt{(c^4 - r^4)},$$

(*Ex.* 320); therefore

$$dS = -\frac{4r}{c^2} \sqrt{(c^4 - r^4)}\, dp = -\frac{6r^3 dr}{c^4} (c^4 - r^4)^{\frac{1}{2}},$$

since $r^3 = 2c^2 p$ for the lemniscate. Hence, integrating, we have $S = \dfrac{(c^4 - r^4)^{\frac{3}{2}}}{c^4}$, adding no constant, as S must vanish with the segment AB.

324. A tangent to the lemniscate $r^2 = 2c^2 \cos 2\theta$ meets the curve again in A, B; the locus of the middle point of AB is $r^2 = 2c^2 \cos \frac{1}{5} (2\theta)$.

If ϕ be the angle which AB subtends at the node,

$$\cos \phi = \frac{r^2}{c^2}.$$

325. *To find the locus of the centre of gravity of an arc of the lemniscate which is of given length.*

Let $r^2 = a^2 \cos 2\theta$ be the polar equation of the curve, then, if ds be the element of the arc, $ds = \dfrac{a d\theta}{\sqrt{(\cos 2\theta)}}$; and, if x, y be the rectangular coordinates of the centre of gravity of an arc of length l,

$$x = \frac{\int x ds}{\int ds} = \frac{a^2}{l}\int_{\theta_2}^{\theta_1} \cos\theta\, d\theta = \frac{a^2}{l}(\sin\theta_1 - \sin\theta_2),$$

and $y = \dfrac{a^2}{l}(\cos\theta_2 - \cos\theta_1)$; therefore

$$\sin^2\tfrac{1}{2}(\theta_1 - \theta_2) = \frac{l^2}{4a^4}(x^2 + y^2), \quad \tan\tfrac{1}{2}(\theta_1 + \theta_2) = \frac{y}{x}.$$

Now, by the theory of elliptic functions, when

$$\int_0^{\theta_1} \frac{d\theta}{\sqrt{(\cos 2\theta)}} - \int_0^{\theta_2} \frac{d\theta}{\sqrt{(\cos 2\theta)}} = \text{a constant},$$

we have $$m\cos\theta_1\cos\theta_2 + n\sin\theta_1\sin\theta_2 = 1,$$

where m and n are constants connected by the relation $m^2 + n^2 = 2$. Hence, the equation of the locus is

$$(m+n)(x^2+y^2)^2 = \frac{4a^4}{l^2}\{(m-1)x^2 + (n-1)y^2\}.$$

326. A variable lemniscate $r^2 = a\cos 2\theta + b\sin 2\theta$ touches the Cassinian oval $r^4 - 2c^2r^2\cos 2\theta - k^4 = 0$; to show that it cuts off a constant area from the Cassinian

$$r^4 - 2c^2r^2\cos 2\theta - k'^4 = 0,$$

where $k' < k$.

Let S be the area cut off by the lemniscate from the curve $r^2 = \phi(\theta)$, then

$$S = \tfrac{1}{2}\int_{\theta_2}^{\theta_1} \{a\cos 2\theta + b\sin 2\theta - \phi(\theta)\}\, d\theta,$$

where θ_1, θ_2 are two roots of the equation

$$a\cos 2\theta + b\sin 2\theta - \phi(\theta) = 0.$$

Hence, if S remain constant while a and b vary, we must have

$$\int_{\theta_2}^{\theta_1} (da\cos 2\theta + db\sin 2\theta)\, d\theta = 0,$$

as the terms outside the sign of integration vanish by the condition given above. Performing the integration and dividing by $\sin(\theta_1 - \theta_2)$, we obtain

$$da \cos(\theta_1 + \theta_2) + db \sin(\theta_1 + \theta_2) = 0.$$

But it is evident that if we seek the point of contact of the lemniscate with its envelope, we have

$$da \cos 2\theta + db \sin 2\theta = 0;$$

hence $2\theta = \theta_1 + \theta_2$, or the radius vector to the point of contact of the lemniscate with its envelope bisects the angle between the radii vectores to the two points of intersection with the curve $r^2 = \phi(\theta)$.

Let us now seek the intersection of the lemniscate with the Cassinian oval

$$r^4 - 2c^2r^2 \cos 2\theta - k'^4 = 0.$$

We find

$$(a + b \tan 2\theta)^2 - 2c^2 (a + b \tan 2\theta) - k'^4 (1 + \tan^2\theta) = 0,$$

whence $$\tan 2(\theta_1 + \theta_2) = \frac{2b(c^2 - a)}{b^2 - a^2 + 2c^2a},$$

which being independent of k', the truth of the theorem becomes evident.

327. If r be the radius vector and p the perpendicular from the origin on the tangent, to find the relation between p and r for the quartic

$$(x^2 + y^2 + k^2)^2 - 4(a^2x^2 + b^2y^2) = 0.$$

The conic $\theta^2 + \theta(x^2 + y^2 + k^2) + a^2x^2 + b^2y^2 = 0$

touches the quartic, and the points of contact lie on the circle

$$x^2 + y^2 + k^2 + 2\theta = 0.$$

Now when we are given a conic

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1,$$

we have
$$\frac{1}{p^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} - \frac{r^2}{\alpha^2\beta^2};$$

hence, in this case,
$$\frac{1}{p^2} = -\frac{(a^2+b^2+2\theta)}{\theta(\theta+k^2)} - \frac{(a^2+\theta)(b^2+\theta)r^2}{\theta^2(\theta+k^2)^2};$$

but from the circle $2\theta = -(k^2+r^2)$;

therefore
$$\frac{k^2}{p^2} = \frac{4a^2b^2}{(r^2+k^2)^2} - \frac{4a'^2b'^2}{(r^2-k^2)^2},$$

where
$$a'^2 = a^2 - k^2, \quad b'^2 = b^2 - k^2.$$

In a similar manner we can find a relation between p and r for the Cartesian oval
$$(x^2+y^2-2ax+k^2)^2 - 4a^2(x^2+y^2) = 0,$$

where the origin is a focus, by considering the curve as the envelope of the circle
$$\mu^2(x^2+y^2) + \mu(x^2+y^2-2ax+k^2) + a^2 = 0.$$

We have from this circle
$$r^2 = \frac{a^2+\mu k^2}{\mu(\mu+1)} + 2p\sqrt{\left\{\frac{a^2}{(\mu+1)^2} - \frac{(a^2+\mu k^2)}{\mu(\mu+1)}\right\}};$$

and, since the points of contact lie on $\mu^2(x^2+y^2) - a^2 = 0$, we have $\mu = \frac{a}{r}$; hence
$$2p = \frac{r^2-k^2}{\sqrt{\left\{a^2-(a+r)\left(a+\frac{k^2}{r}\right)\right\}}}.$$

We can also find the relation between p and r for the Cartesian oval when the origin is the triple focus, by considering the curve as the envelope of the circle
$$\mu^2 + 2\mu(x^2+y^2-k^2) + a^3(x-m) = 0.$$

328. F, F' are the double foci of the quartic

$$(x^2+y^2+k^2)^2-4(a^2x^2+b^2y^2)=0;$$

if the normal at a point O of the curve meet the axis of y in P, and a parallel through O to FP meet the axis of x in Q, then

$$OQ=\frac{a}{c}\sqrt{(a^2-k^2)}.$$

If the normal at x', y' meet the axis of y in 0, β, we have

$$\beta=\frac{2c^2y'}{x'^2+y'^2+k^2-2a^2}=\frac{c^2y'}{\sqrt{(a^2a'^2-c^2y'^2)}},$$

since the equation of the curve may be written

$$(x^2+y^2+k^2-2a^2)^2+4c^2y^2=4a^2a'^2,$$

where $$a'^2=a^2-k^2.$$

Hence, if $PFF'=\theta$, $cy'=aa'\sin\theta$, and $OQ=\dfrac{aa'}{c}$.

Also $$FQ=x'-c+\frac{2aa'}{c}\cos\theta=\frac{1}{2c}(d^2-OF^2),$$

where $$a^2+b^2-k^2=d^2;$$

and since the curve can be written

$$(OF^2-d^2)(OF'^2-d^2)=4b^2(b^2-k^2),$$

we have

$$FQ.F'Q'=\frac{b^2}{c^2}(b^2-k^2).$$

The circle

$$(x^2+y^2-a^2-a'^2)\cos\tfrac{1}{2}(\theta_1+\theta_2)$$
$$+2cy\sin\tfrac{1}{2}(\theta_1+\theta_2)-2aa'\cos\tfrac{1}{2}(\theta_1-\theta_2)=0$$

evidently passes through the two points θ_1, θ_2 on the curve. Hence the line bisecting at right angles the chord θ_1, θ_2

meets the axis of y at the point $y' = -c \tan\frac{1}{2}(\theta_1 + \theta_2)$, from which it readily follows that any line meets the curve so that the sum of the angles θ_1, &c. is equal to 0 or $2m\pi$.

329. The quartic

$$(x^2 + y^2 + k^2)^2 - 4(a^2x^2 + b^2y^2) = 0,$$

being written in the form

$$\{y^2 + (x - c^2) - d^2\}\{y^2 + (x + c)^2 - d^2\} = \frac{b^2}{c^2}(b^2 - k^2),$$

where $\qquad d^2 = a^2 + b^2 - k^2, \quad c^2 = a^2 - b^2,$

the circle S having double contact with the curve, where

$$S \equiv x^2 + y^2 - 2ax\cos\phi - 2by\sin\phi + k^2 = 0,$$

cuts off constant arcs from the circles

$$y^2 + (x \pm c)^2 - d^2 = 0.$$

For the chord of intersection of S and $y^2 + (x - c)^2 - d^2 = 0$ is $(a\cos\phi - c)x + b\sin\phi y - b^2 = 0$, and the perpendicular on this line from c, 0 is equal to a.

330. If ρ, ρ' be the distances of a point P on the same curve from the double foci, we have

$$(\rho^2 - d^2)(\rho'^2 - d^2) = \frac{b^2}{c^2}(b^2 - k^2),$$

and, therefore,

$$\frac{\frac{\rho d\rho}{ds}}{\rho^2 - d^2} + \frac{\frac{\rho' d\rho'}{ds}}{\rho'^2 - d^2} = 0.$$

Now, if the tangent at P meet the polars of P with regard to the circles $\rho^2 - d^2 = 0$, $\rho'^2 - d^2 = 0$ in A, B, we have

$$PA = \frac{\rho^2 - d^2}{\frac{\rho' d\rho'}{ds}}, \quad PB = \frac{\rho'^2 - d^2}{\frac{\rho d\rho}{ds}},$$

from which it follows that the portion of the tangent intercepted between these two lines is bisected at the point of contact.

331. Let

$$S \equiv x^2 + y^2 - 2ax\cos\phi - 2by\sin\phi + k^2 = 0$$

be a circle having double contact with the quartic

$$(x^2 + y^2 + k^2) - 4(a^2x^2 + b^2y^2) = 0,$$

then if we form the discriminant of $S + \lambda PQ$, where

$$PQ \equiv (a^2 - k^2)\,x^2 + (b^2 - k^2)\,y^2 = 0$$

represents two double tangents of the curve, we obtain

$$\frac{a^2\cos^2\phi}{1+\lambda(a^2-k^2)} + \frac{b^2\sin^2\phi}{1+\lambda(b^2-k^2)} - k^2 = 0,$$

or $$(1-\lambda k^2)(\lambda a'^2b'^2 + a'^2\cos^2\phi + b'^2\sin^2\phi) = 0,$$

if we write

$$a^2 - k^2 = a'^2, \quad b^2 - k^2 = b'^2.$$

Taking the value $\lambda = \frac{1}{k^2}$, we see that two chords of intersection of S and PQ are

$$(ax - k^2\cos\phi)^2 + (by - k^2\sin\phi)^2 = 0.$$

These two lines are parallel to the double tangents

$$a^2x^2 + b^2y^2 = 0,$$

and intersect on the conic $a^2x^2 + b^2y^2 = k^4$. If we take the value

$$\lambda = -\frac{(a'^2\cos^2\phi + b'^2\sin^2\phi)}{a'^2b'^2},$$

the two chords of intersection are

$$\frac{x\cos\phi}{b'} \pm \frac{y\sin\phi}{a'} + \left(\frac{ab' \mp ba'}{c^2}\right) = 0.$$

If 2θ be the angle which one of these chords subtends at the centre of S, we find

$$\cos\theta = \frac{aa' \mp bb'}{c^2}.$$

Since one of these chords touches the conic

$$a'^2x^2 + b'^2y^2 = \frac{a'^2b'^2}{c^4}(ab' \mp ba')^2,$$

whose asymptotes are $a'^2x^2 + b'^2y^2 = 0$, we see that the quartic can be generated as the envelope of a circle described through the points where a tangent to an hyperbola meets the asymptotes, so that the angle subtended by these points at the circumference is given.

332. If we write

$$U \equiv (x^2 + y^2 + k^2)^2 - 4(a^2x^2 + b^2y^2),$$

$$S \equiv x^2 + y^2 - 2\alpha x - 2\beta y + t^2, \quad S' \equiv x^2 + y^2 + 2\alpha x + 2\beta y + t^2,$$

we have

$$SS' - U \equiv t^4 - k^4 + 2(t^2 - k^2)(x^2 + y^2) - 4(\alpha x + \beta y)^2 + 4(a^2x^2 + b^2y^2),$$

from which it follows that any circle S meets U in four points which lie on a concentric conic. Hence, being given four concyclic points on a bicircular quartic with a centre, the locus of the centre coincides with the locus of the centres of conics through the same points, viz: the equilateral hyperbola which passes through the middle points of all the lines joining the given points. This quartic is determined by six conditions, and when we are given four points, the curve will still contain two arbitrary parameters. Thus the quartic may have a node or become an ellipse of Cassini, and the locus of the centre of either of these curves, when we are given four concyclic points, is the equilateral hyperbola determined above.

If we write

$$V \equiv ax^2 + by^2 + 2gx + 2fy + c,$$
$$S \equiv x^2 + y^2 - k^2,$$

it can be easily seen that the equation

$$S\left\{\left(x + \frac{2g}{a+\theta}\right)^2 + \left(y + \frac{2f}{b+\theta}\right)^2 - k^2 + \lambda\theta\right\} + \lambda V = 0,$$

where θ and λ are arbitrary parameters, represents any central bicircular quartic passing through the intersection of S and V. The coordinates of the centre are $\frac{-g}{a+\theta}, \frac{-f}{b+\theta}$. If the curve is an ellipse of Cassini U, the centre must satisfy the equation

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} = 0;$$

and then if $a+b=0$, or V is an equilateral hyperbola, we get $\lambda\theta = 2k^2$.

Thus the equation

$$S\left\{\left(x + \frac{2g}{\theta+a}\right)^2 + \left(y + \frac{2f}{\theta-a}\right)^2 + k^2\right\} + \frac{2k^2}{\theta} V = 0$$

represents a system of Cassinian ovals passing through the intersection of S and V.

If we compare this equation with

$$\{(x-x')^2 + (y-y')^2\}\{(x-x'')^2 + (y-y'')^2\} - d^4 = 0,$$

we have, to determine the double foci

$$x^2 - y^2 + \frac{2gx}{\theta+a} - \frac{2fy}{\theta-a} + \frac{ak^2}{\theta} = 0, \quad xy + \frac{gy}{\theta+a} + \frac{fx}{\theta-a} = 0;$$

whence, eliminating θ, we obtain

$$(axy + gy - fx)(x^2 + y^2) + k^2axy = 0.$$

333. If the conic

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

pass through four concyclic points on the bicircular quartic

$$(x^2 + y^2)^2 - 2\alpha x^2 - 2\beta y^2 + k^4 = 0,$$

the conic $$\frac{(gh - af)}{f} x^2 + \frac{(fh - bg)}{g} y^2 + \frac{fg - ch}{h} = 0$$

has quartic contact with the curve. This readily follows from the fact that if U be a conic having quartic contact with the curve, any conic having double contact with U meets the curve in eight points which lie on two circles.

334. A circle of given radius passes through a fixed point F, and an equal circle through a fixed point F'; if the sum or difference of the arcs FP, $F'P$ be given, where P is a point of intersection of the circles, show that the locus of P is a central bicircular quartic of which F, F' are foci.

335. Show that the quartics

$$(x^2 + y^2 + k^2)^2 - 4d^2xy = 0, \quad (x^2 + y^2 - k^2)^2 - 4a^2(x^2 - y^2) = 0,$$

cut each other orthogonally.

336. If S_1, S_2, S_3, S_4 be four bicircular quartics having five points in common, the equation

$$l_1S_1 + l_2S_2 + l_3S_3 + l_4S_4 = 0$$

represents any bicircular quartic passing through these five points. Comparing this equation with $SS' - L = 0$, where S, S' are circles whose centres are the double foci, and L is a line, we determine the double foci by the equations

$$4(x_1x_2 - y_1y_2)\,\Sigma l = \Sigma l\,(a - b), \quad 2(x_1y_2 + y_1x_2)\,\Sigma l = \Sigma lh,$$

$$(x_1 + x_2)\,\Sigma l = 2\Sigma l\alpha, \qquad (y_1 + y_2)\,\Sigma l = 2\Sigma l\beta,$$

where

$$S_1 \equiv (x^2+y^2)^2 - 4(\alpha_1 x + \beta_1 y)(x^2+y^2) + a_1 x^2 + b_1 y^2$$
$$+ 2h_1 xy + 2g_1 x + 2f_1 y + c_1,$$

$S_2 \equiv$ &c.

Hence, eliminating l_1, l_2, l_3, l_4, we obtain a result of the form

$$A(x_1x_2 - y_1y_2) + H(x_1y_2 + y_1x_2) + G(x_1+x_2) + F(y_1+y_2) + C = 0,$$

which shows that the double foci are conjugate with respect to the fixed equilateral hyperbola

$$A(x^2-y^2) + 2Hxy + 2Gx + 2Fy + C = 0.$$

When the double foci coincide, the quartic becomes a Cartesian oval; and thus we see that this equilateral hyperbola is the locus of the triple foci of all the Cartesian ovals passing through the five points.

For a bicircular quartic U with a centre, the centre, in addition to the equations, $x\Sigma l = \Sigma l\alpha$, $y\Sigma l = \Sigma l\beta$, satiffies $\dfrac{dU}{dx}=0$, $\dfrac{dU}{dy}=0$. We find thus that the locus of the centres of such quartics which pass through five fixed points is a curve of the fourth order passing through the circular points at infinity.

In the same way the equation $l_1S_1 + l_2S_2 + l_3S_3 = 0$, where S_1, S_2, S_3 have six points in common, represents a system of bicircular quartics passing through six fixed points. If one of the quartics reduce to a conic, the six points will lie on a conic, and the centres of the focal conics of the system will lie on a right line. When the quartic breaks up into two circles, the centre of the focal conic is the middle point of the line joining their centres; hence we obtain the theorem of Ex. 123.

337. Writing the equation of a bicircular quartic in the form $SS' - L = 0$, we see that any circle Σ meets the quartic at four points of its intersection with the conic

$$(S - \Sigma)(S' - \Sigma) - L = 0.$$

Now the asymptotes of this conic are parallel to $S - \Sigma = 0$, $S' - \Sigma = 0$, and these lines are perpendicular to the lines joining the centre of the circle to the double foci. Hence, since the chords of intersection of a conic and a circle are equally inclined to an axis of the conic, we see that a pair of chords of intersection of a circle and a bicircular quartic are equally inclined to the bisectors of the angle between the lines drawn from the centre of the circle to the double foci.

338. To show that a line meets a bicircular quartic at angles the sum of whose cotangents is equal to zero.

If we take an arbitrary point for origin and draw any line through the origin to meet the curve, it is evident that the continued product of the four radii vectores is constant. Hence, differentiating, since $\dfrac{dr}{r\,d\theta} = \cot\phi$, we have $\Sigma \cot\phi = 0$.

We can arrive at this result otherwise thus: Let us write the curve

$$(x^2 + y^2)^2 + (lx + my)(x^2 + y^2) + ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0,$$

then if the line is the axis of x, and $f(x)$ is the result of putting $y = 0$ in the equation of the curve, the equation of the tangent at the point $(x_1, 0)$ is

$$(x - x_1) f'(x_1) + y\,(mx_1^2 + 2hx_1 + 2f) = 0;$$

whence $$\cot\theta_1 = -\frac{(mx_1^2 + 2hx_1 + 2f)}{f'(x_1)},$$

and, therefore, $$\Sigma \cot\theta = 0.$$

We have also $\Sigma x \cot\theta = -m = 4y'$, where y' is the ordinate of the centre of the focal conics, from which it follows that, if G be the centroid of the four points in which the normals intersect a perpendicular to the line, a parallel to the line through G will pass through the centre of the focal conic.

If the line is a tangent to the curve, the sum of two cotangents is replaced by $2\cot\delta$, where δ is the angle which the axis of aberrancy makes with the curve.

Hence, if the tangent at a point P of the quartic meet the curve again in A, B, and if the line joining the middle point of AB to the intersection of the tangents at A and B meet the perpendicular from the centres of the focal conic on the tangent in C, then the axis of aberrancy at P passes through C.

By inversion we see that a circle meets a bicircular quartic at angles the sum of whose cotangents is equal to zero.

339. *To draw through a point on a bicircular quartic a circle to meet the curve again at the vertices of an equilateral triangle.*

Taking the point on the curve as origin and the axes passing through the circular points at infinity, the equation of the curve may be written

$$x^2y^2 + xy\,(lx + my) + ax^2 + by^2 + 2hxy + 2gx + 2fy = 0,$$

and $xy - \alpha y - \beta x = 0$ represents a circle through the origin whose centre is α, β. Forming the equation of the lines which join the origin to the points where the circle meets the curve again we have

$$(\alpha y + \beta x)^3 + (\alpha y + \beta x)^2\,(lx + my) \\ + (\alpha y + \beta x)\,(ax^2 + by^2 + 2hxy) + 2xy\,(gx + fy) = 0.$$

Now if the coefficients of x^2y and xy^2 in this equation vanish, these three lines will be parallel to the sides of an equilateral triangle; thus we have

$$3\alpha^2\beta + l\alpha^2 + 2m\alpha\beta + b\beta + 2h\alpha + 2f = 0,$$

$$3\alpha\beta^2 + m\beta^2 + 2l\alpha\beta + a\alpha + 2h\beta + 2g = 0.$$

Multiplying the first equation by β and the second by α, and subtracting, we get

$$m\alpha\beta^2 - l\alpha^2\beta + b\beta^2 - a\alpha^2 + 2f\beta - 2g\alpha = 0;$$

and, combining these three equations, we have

$$(l^2 - 3a)\,\alpha^2 - (m^2 - 3b)\,\beta^2 + (2hl - ma - 6g)\,\alpha$$
$$- (2hm - lb - 6f)\,\beta + 2\,(lf - mg) = 0,$$

which represents an equilateral hyperbola having five points in common with the two cubics. Thus we see that five circles can be drawn to satisfy the given conditions, and that their centres lie on an equilateral hyperbola.

340. *A rectangle is inscribed in a bicircular quartic; to find the locus of its centroid.*

Taking the origin at the centre of the focal conic and the axes passing through the circular points at infinity, the curve may be written

$$x^2y^2 + ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0.$$

Let $xy - \alpha y - \beta x + k^2 = 0$ be the equation of the circle circumscribing the rectangle; then, eliminating y between the equations of the circle and quartic, we get

$$(\beta^2 + a)\,x^4 - 2\,(k^2\beta + a\alpha - h\beta - g)\,x^3 + \&c. = 0;$$

hence, $\qquad 2x'\,(\beta^2 + a) = k^2\beta + a\alpha - h\beta - g,$

and, similarly, $2y'\,(\alpha^2 + b) = k^2\alpha + b\beta - h\alpha - f,$

where x', y' are the coordinates of the centroid of the rectangle.

But the centroid of the rectangle coincides with the centre of the circle; therefore

$$\beta(k^2 - 2\alpha\beta) = a\alpha + h\beta + g, \quad \alpha(k^2 - 2\alpha\beta) = h\alpha + b\beta + f,$$

whence, eliminating k^2, we have

$$a\alpha^2 - b\beta^2 + g\alpha - f\beta = 0$$

an equilateral hyperbola which passes through the feet of the normals which can be drawn to the curve from the origin.

341. By exactly the same method which we used in Ex. 255, we can show that, if any point P of a bicircular quartic be joined to four concyclic foci, the tangent to the curve at P is divided in a constant anharmonic ratio by the perpendiculars erected to the joining lines at the foci. Also, if

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

be a focal conic, and

$$x^2 + y^2 - 2\alpha x - 2\beta y + t^2 = 0$$

the corresponding Jacobian circle of the quartic, a circle having double contact with the curve may be written

$$\Sigma \equiv x^2 + y^2 - 2a(x - \alpha)\cos\phi - 2b(y - \beta)\sin\phi - t^2 = 0;$$

and the envelope of the polars of $P(x', y')$ with regard to the circles Σ is the conic

$$(xx' + yy' - t^2)^2 = a^2(x + x' - 2\alpha)^2 + b^2(y + y' - 2\beta)^2,$$

which, by the mode of generation, evidently touches the four perpendiculars, and also, as can be seen from its equation, touches the curve at P. Now, if a point P be joined to four points on a circle and perpendiculars be erected at these points to the joining lines, we know that these perpendiculars are tangents to a conic of which P is a focus. But the two conics, described through any point P of the locus of the foci of inscribed conics to touch the sides of a quadrilateral, cut each other at right angles at P; in fact, the tangents to these conics are harmonically conjugate with every pair of tangents drawn from P to an inscribed conic (Salmon's *Conics*, Art. 344); and since, when P is a focus of an inscribed conic, one of these pairs passes through the circular points at infinity, it follows that the tangents are at right angles to one another. The two conics evidently correspond to the two confocal quartics which pass through P, and we thus have a proof that two such quartics cut orthogonally.

We can also show that the tangent at any point P of a bicircular quartic is divided in a constant anharmonic ratio by the polars of P with respect to four fixed circles of the same system having double contact with the curve. This ratio, it is not difficult to see, is equal to that of the pencil joining any point on the focal conic to the centres of the four circles.

342. To find the tangential equation of a conic touching a bicircular quartic at four points on a circle.

If
$$F_1 \equiv \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - 1 = 0$$
be a focal conic and
$$J_1 \equiv x^2 + y^2 - 2x_1x - 2y_1y + t_1^2 = 0,$$
the corresponding Jacobian circle of the quartic, the equation of the curve is
$$(x^2 + y^2 - t_1^2)^2 - 4a_1^2 (x - x_1)^2 - 4b_1^2 (y - y_1)^2 = 0.$$

Now we know that the quartic can be generated in this manner in four ways, and that the four conics F_1, F_2, &c. are confocal; hence, equating the terms in two forms of the equation of the curve, we obtain the following cubic in a^2
$$\frac{x_1^2}{a^2} + \frac{y_1^2}{a^2 - c^2} - \frac{k_1^2}{a^2 - a_1^2} = 1$$

(where k_1 is the radius of F_1) to determine the semiaxes a_2, a_3, a_4 of the three other focal conics. From this equation we obtain

$$k_1^2 = \frac{(a_1^2 - a_2^2)(a_1^2 - a_3^2)(a_1^2 - a_4^2)}{a_1^2 b_1^2},$$

and, by symmetry, we have similar values for k_2^2, k_3^2, k_4^2; also

$$t_1^2 = a_2^2 + a_3^2 + a_4^2 - a_1^2 - c^2.$$

Now the equation

$$\theta^2 + \theta(x^2 + y^2 - t_1^2) + a_1^2(x - x_1)^2 + b_1^2(y - y_1)^2 \equiv V = 0$$

represents a conic touching the quartic at four points on the circle $x^2 + y^2 + 2\theta - t_1^2 = 0$, and if we form the condition that the line $x \cos\omega + y \sin\omega - p = 0$ should touch V, and arrange according to powers of θ, we have

$$\theta^3 + \ldots + a_1^2 b_1^2 (x_1 \cos\omega + y_1 \sin\omega - p)^2 = 0;$$

whence $\theta_1\theta_2\theta_3 = -a_1^2 b_1^2 p_1^2$, where θ_1, θ_2, θ_3 are the parameters of three conics of the system which touch the line, and p_1 is the perpendicular from the centre of J_1 on the line. Let us put $\theta = \lambda^2 - a_1^2$, and let r be the radius of the circle through the contact of V, then

$$r^2 = t_1^2 - 2\theta = a_1^2 + a_2^2 + a_3^2 + a_4^2 - c^2 - 2\lambda^2,$$

and
$$p_1^2 = \frac{(a_1^2 - \lambda_1^2)(a_1^2 - \lambda_2^2)(a_1^2 - \lambda_3^2)}{a_1^2 b_1^2},$$

and by symmetry there are similar values for p_2, p_3, p_4.

Eliminating λ_2 and λ_3 between these values of p_1, &c., we obtain

$$\frac{p_1^2}{k_1^2(\lambda^2 - a_1^2)} + \frac{p_2^2}{k_2^2(\lambda^2 - a_2^2)} + \frac{p_3^2}{k_3^2(\lambda^2 - a_3^2)} + \frac{p_4^2}{k_4^2(\lambda^2 - a_4^2)} = 0,$$

which may be regarded as the tangential equation of V.

The discriminant of this equation with regard to λ^2 will evidently give the tangential equation of the quartic.

343. The three conics of the system in the preceding example which can be described to touch an inflexional tangent coincide; hence, these tangents are given by the equations

$$p_1^2 = \frac{(a_1^2 - \lambda^2)^3}{a_1^{2}b_1^2}, \; p_2^2 = \frac{(a_2^2 - \lambda^2)^3}{a_2^{2}b_2^2}, \; p_3^2 = \frac{(a_3^2 - \lambda^2)^3}{a_3^{2}b_3^2}, \; p_4^2 = \frac{(a_4^2 - \lambda^2)^3}{a_4^{2}b_4^2},$$

and since, as can be easily seen, $\Sigma \frac{p}{k^2} = 0$, identically, λ^2 is given by the equation

$$\frac{(a_1^2 - \lambda^2)^{\frac{3}{2}}}{a_1b_1k_1^2} + \frac{(a_2^2 - \lambda^2)^{\frac{3}{2}}}{a_2b_2k_2^2} + \frac{(a_3^2 - \lambda^2)^{\frac{3}{2}}}{a_2b_2k_2^2} + \frac{(a_4^2 - \lambda^2)^{\frac{3}{2}}}{a_4b_4k_4^2} = 0,$$

which when cleared of radicals is of the twelfth degree, as it ought to be.

If we eliminate λ^2 between the values of p_1, p_2, p_3, we obtain

$$(a_2^2 - a_3^2)(a_1b_1p_1)^{\frac{2}{3}} + (a_3^2 - a_1^2)(a_2b_2p_2)^{\frac{2}{3}} + (a_1^2 - a_2^2)(a_3b_3p_3)^{\frac{2}{3}} = 0;$$

hence, the twelve inflexional tangents are touched by this curve, which is a projected form of the lemniscate, and, therefore, of the sixth class and fourth degree. There are of course four such curves corresponding to the four centres of inversion.

If the tangential equation of V be written

$$A\lambda^6 + B\lambda^4 + C\lambda^2 + D = 0,$$

the equation

$$l(B^2 - 3AC) + m(BC - 9AD) + n(C^2 - 3BD) = 0$$

will represent a system of curves of the fourth class touching the twelve inflexional tangents.

344. If the conic

$$\theta^2 + \theta(x^2 + y^2 - t_1^2) + a_1^2(x - \alpha_1)^2 + b_1^2(y - \beta_1)^2 = 0$$

be referred to its axes, show that it can be written

$$\lambda^2 x^2 + (\lambda^2 - c^2)\, y^2 + \frac{(\lambda^2 - a_1^2)(\lambda^2 - a_2^2)(\lambda^2 - a_3^2)(\lambda^2 - a_4^2)}{\lambda^2 (\lambda^2 - c^2)} = 0,$$

where λ, a_1, &c. have the same meaning as before.

345. If the two conics of the system

$$\theta^2 + \theta (x^2 + y^2 - t^2) + a^2 (x - \alpha)^2 + b^2 (y - \beta)^2 = 0,$$

which pass through a point P, cut each other orthogonally at P, show that the locus of P is the circular cubic

$$(a^2 \alpha x + b^2 \beta y - a^2 \alpha^2 - b^2 \beta^2)(x^2 + y^2)$$
$$- t^2 a^2 x (x - \alpha) - t^2 b^2 y (y - \beta) - a^4 (x - \alpha)^2 - b^4 (y - \beta)^2 = 0.$$

346. Show that the locus of the poles of a fixed line with regard to the same system of conics is a nodal cubic passing through the four centres of inversion.

347. Show that the locus of the vertices of the same system of conics consists of the two quartics

$$x^2 \{a^2 (x - \alpha)^2 + b^2 (y - \beta)^2\} - a^2 x (x - \alpha)(x^2 + y^2 - t^2) + a^4 (x - \alpha)^2 = 0,$$
$$y^2 \{a^2 (x - \alpha)^2 + b^2 (y - \beta)^2\} - b^2 y (y - \beta)(x^2 + y^2 - t^2) + b^4 (y - \beta)^2 = 0.$$

Show that these two curves cut each other at right angles at the four centres of inversion.

348. If $V = \theta^2 + \theta (x^2 + y^2 - t^2) + a^2 (x - \alpha)^2 + b^2 (y - \beta)^2$,

$$S = x^2 + y^2 - t^2 + 2\theta,$$

the quartic which is the envelope of V can be written $S^2 - 4V = 0$.

Now let us transfer the origin by parallel axes to a point on V, and transform to polar coordinates, then, if we consider the radius vector at the origin which touches V, the four points where this radius vector meets the quartic will be given by the equation

$$(\rho^2 + M\rho + N)^2 - 4\{(a^2+\theta)\cos^2\phi + (b^2+\theta)\sin^2\phi\}\rho^2 = 0,$$

where $\rho^2 + M\rho + N$ is the result of transforming S, and ϕ is the angle which the radius vector makes with the axis of x. Hence, if α, β be the roots of the equation

$$\rho^2 + M\rho + N + 2\surd\{(a^2+\theta)\cos^2\phi + (b^2+\theta)\sin^2\phi\}\rho = 0,$$

and γ, δ those of the equation

$$\rho^2 + M\rho + N - 2\surd\{(a^2+\theta)\cos^2\phi + (b^2+\theta)\sin^2\phi\}\rho = 0,$$

we have

$$(\alpha+\beta-\gamma-\delta)^2 = 16\{(a^2+\theta)\cos^2\phi + (b^2+\theta)\sin^2\phi\}.$$

Now if we consider the three conics which can be described to touch a given line, we shall evidently obtain the equations

$$(\alpha+\beta-\gamma-\delta)^2 = 16(\theta_1 + a^2\cos^2\phi + b^2\sin^2\phi),$$
$$(\alpha+\gamma-\beta-\delta)^2 = 16(\theta_2 + a^2\cos^2\phi + b^2\sin^2\phi),$$
$$(\alpha+\delta-\beta-\gamma)^2 = 16(\theta_3 + a^2\cos^2\phi + b^2\sin^2\phi),$$

where θ_1, θ_2, θ_3 are the parameters of the conics. Hence, we have

$$(\alpha-\delta)(\beta-\gamma) = 4(\theta_1-\theta_2),$$
$$(\beta-\delta)(\gamma-\alpha) = 4(\theta_3-\theta_1), \quad (\gamma-\delta)(\alpha-\beta) = 4(\theta_2-\theta_3),$$

and
$$\Sigma(\alpha-\delta)^2(\beta-\gamma)^2 = 16\Sigma(\theta_1-\theta_2)^2,$$
$$\Sigma(\alpha-\delta)^2(\beta-\gamma)^2(\alpha-\beta)(\delta-\gamma)$$
$$= 64(\theta_2+\theta_3-2\theta_1)(\theta_1+\theta_3-2\theta_2)(\theta_1+\theta_2-2\theta_3).$$

Thus if a line meet the quartic in a, b, c, d so that $\Sigma ad^2.bc^2$, or $\Sigma ad^2.bc^2.ab.cd$, is a constant, its envelope will be a curve of the fourth or sixth class, obtained by putting H or G of the cubic for λ^2 in Ex. 342 equal to a constant.

349. If a line meet the Cassinian oval

$$(x^2+y^2)^2 - 2c^2(x^2-y^2) + k^4 = 0$$

in a, b, c, d, so that $\Sigma ad^2 . bc^2 = d^4$, its envelope will consist of the confocal conics

$$\frac{x^2}{\mu_1^2} + \frac{y^2}{\mu_1^2 - c^2} = 1, \quad \frac{x^2}{\mu_2^2} + \frac{y^2}{\mu_2^2 - c^2} = 1,$$

where μ_1^2, μ_2^2 are the roots of the equation

$$\mu^4 - c^2\mu^2 + \tfrac{1}{4}(c^4 + 3k^4) - \tfrac{1}{32}d^4 = 0.$$

If d vanish the line is divided equi-anharmonically by the quartic; thus we see that if the curve be written

$$x^2y^2 + y^2z^2 + z^2x^2 - k^2z^4 = 0,$$

the contravariant σ breaks up into the factors

$$\alpha^2 + \beta^2 + \gamma^2 \pm 2k\sqrt{3}\alpha\beta = 0.$$

350. If a line is divided equi-anharmonically by the quartic

$$(x^2 + y^2)^2 - 4(a^2x^2 + b^2y^2) = 0,$$

we can show that it touches the curve parallel to the conic

$$\frac{x^2}{3a^2 - b^2} + \frac{y^2}{3b^2 - a^2} = \tfrac{1}{4}$$

at a distance equal to $\frac{1}{2}\sqrt{\{3(a^2 + b^2)\}}$.

If $a^2 = 3b^2$, we see that for the curve

$$(x^2 + y^2)^2 - 4b^2(y^2 + 3x^2) = 0$$

the contravariant σ is the product of the two circles

$$x^2 + y^2 \pm 2b\sqrt{2}x - b^2 = 0.$$

351. Referring to Ex. 348 we see that the absolute terms of the two quadratics, which determine the intersection of a tangent to the conic

$$U \equiv \theta^2 + \theta(x^2 + y^2 - t^2) + a^2(x - \alpha)^2 + b^2(y - \beta)^2 = 0$$

with the quartic, are equal; hence $oa.ob = oc.od$, where o is the point of contact of the tangent. If U be fixed, we can find the locus of the middle points of ab and cd. We have

$$\alpha + \beta - \gamma - \delta = \tfrac{1}{4}\sqrt{\{(a^2 + \theta)\cos^2\phi + (b^2 + \theta)\sin^2\phi\}};$$

but if p, be the perpendicular from the centre of the focal conic on the line bisecting at right angles ab or cd, it can be easily seen that $p = \pm \frac{1}{4}(\alpha + \beta - \gamma - \delta)$; therefore

$$p^2 = (a^2 + \theta)\cos^2\phi + (b^2 + \theta)\sin^2\phi,$$

from which it follows that the lines bisecting ab and cd at right angles are tangents to the conic $\dfrac{x^2}{a^2+\theta} + \dfrac{y^2}{b^2+\theta} = 1$. Now this conic and U are of the same form as the two conics in Ex. 158, and the locus of the middle points of ab and cd is evidently the locus of the intersection of rectangular tangents to these two conics. Thus we see that the locus breaks up into two bicircular quartics.

In the case of the central bicircular quartic we can arrive at this result directly. For the perpendicular from the origin on the tangent to the conic

$$\theta^2 + \theta(x^2 + y^2 + k^2) + a^2x^2 + b^2y^2 = 0$$

is proportional to

$$\sqrt{\{(a^2 + \theta)\cos^2\phi + (b^2 + \theta)\sin^2\phi\}},$$

and is, therefore, in a constant ratio to the perpendicular from the origin on the line bisecting ab or cd at right angles. The middle points of ab and cd are, then, the points where lines drawn from the centre of a conic meet tangents to the conic at a given angle, and this locus evidently consists of two bicircular quartics having the origin for a node.

352. Show that the locus of the foot of the perpendicular from the origin on a line cutting off two equal intercepts from the quartic

$$(x^2 + y^2 - t^2)^2 - 4a^2(x - \alpha)^2 - 4b^2(y - \beta)^2 = 0$$

is the equilateral hyperbola $\dfrac{a^2\alpha}{x} - \dfrac{b^2\beta}{y} - c^2 = 0.$

353. Show that the common tangents of the conics

$$\theta^2 + \theta\,(x^2 + y^2 - t^2) + a^2\,(x - \alpha)^2 + b^2\,(y - \beta)^2 = 0,$$

$$\frac{x^2}{t^2 - b^2 - 3\theta} + \frac{y^2}{t^2 - a^2 - 3\theta} - 1 = 0,$$

are divided harmonically by the quartic whose equation is given in the preceding example.

354. If the quartic

$$(x^2 + y^2 - t^2)^2 - 4a^2\,(x - \alpha)^2 - 4b^2\,(y - \beta)^2 = 0$$

consist of two ovals, one wholly inside the other, two tangents to the conic

$$V \equiv \theta^2 + \theta\,(x^2 + y^2 - t^2) + a^2\,(x - \alpha)^2 + b^2\,(y - \beta)^2 = 0$$

will cut off from the space between the two ovals areas whose difference can be expressed by means of logarithmic and circular functions, the conic V being supposed to lie wholly within the inner oval.

Referring to Ex. 348, we have

$$\alpha^2 + \gamma^2 - \beta^2 - \delta^2 = (\alpha + \gamma - \beta - \delta)\,(\alpha + \beta + \gamma + \delta)$$

$$= 16q\,\sqrt{\{(a^2 + \theta)\cos^2\phi + (b^2 + \theta)\sin^2\phi\}},$$

where q is the perpendicular from the origin on the normal to V at the point of contact. But if dS, dS' be elements of the areas cut off from the space between the ovals by a tangent to V, we have

$$dS = \tfrac{1}{2}\,(\alpha^2 - \beta^2)\,d\phi, \quad dS' = \tfrac{1}{2}\,(\delta^2 - \gamma^2)\,d\phi;$$

therefore $$dS - dS' = \tfrac{1}{2}(\alpha^2 + \gamma^2 - \beta^2 - \delta^2)\,d\phi$$

$$= 8q\sqrt{\{(a^2 + \theta)\cos^2\phi + (b^2 + \theta)\sin^2\phi\}}\,d\phi$$

$$= 8\,\sqrt{\{(a^2 + \theta)\cos^2\phi + (b^2 + \theta)\sin^2\phi\}}\,dp,$$

where p is the perpendicular from the origin on the tangent. Now, writing V in the form

$$\frac{(x - x')^2}{b^2 + \theta} + \frac{(y - y')^2}{a^2 + \theta} = \mu^2,$$

we find

$$p = x' \sin\phi - y' \cos\phi + \mu \sqrt{\{(a^2+\theta)\cos^2\phi + (b^2+\theta)\sin^2\phi\}};$$

hence

$$S - S' = 8x' \int_{\phi_2}^{\phi_1} u \cos\phi \, d\phi + 8y' \int_{\phi_2}^{\phi_1} u \sin\phi \, d\phi + 4\mu \, (u_1^2 - u_2^2),$$

where $\qquad u^2 = (a^2+\theta)\cos^2\phi + (b^2+\theta)\sin^2\phi.$

The points α, δ are supposed to lie on the outer oval, and β, γ on the inner, and the point of contact with V is such that $\alpha\gamma = \beta\delta$; the tangents also are supposed to intersect within the inner oval.

For the central bicircular quartic x' and y' vanish, and the difference of the areas is algebraic.

355. If the equation of a Cartesian oval be written

$$(x^2+y^2-k^2)^2 - a^3(x-m) = 0,$$

the equation

$$S \equiv \mu^2 + 2\mu\,(x^2+y^2-k^2) + a^3(x-m) = 0$$

represents one of the circles having double contact with the curve at points on a perpendicular to the axis. The equation of the curve may then be written $(x^2+y^2-k^2+\mu)^2 = S$, and by the method which we adopted in Ex. 348, we can show that, if a line meet the curve in a, b, c, d, we shall have

$$\alpha+\beta-\gamma-\delta = 2\sqrt{(2\mu_1)}, \quad \alpha+\gamma-\beta-\delta = 2\sqrt{(2\mu_2)},$$
$$\alpha+\delta-\beta-\gamma = 2\sqrt{(2\mu_3)},$$

where μ_1, μ_2, μ_3 are the parameters of the three circles of the system which touch the line, and α, &c. are the distances of a, &c. from a point on the line.

Now if r be the radius and x' the abscissa of the centre of S, we find

$$x'r^2 = (x'-d_1)(x'-d_2)(x'-d_3),$$

where d_1, d_2, d_3 are the distances of the three collinear foci from the origin; and from this relation we see that, if S touch the line

$$x\cos\omega + y\sin\omega - p = 0,$$

we shall have

$$x'(p - x'\cos\omega)^2 - (x' - d_1)(x' - d_2)(x' - d_3) = 0.$$

Let x_1, x_2, x_3 be the roots of this equation, then since

$$x_1 = \frac{a^3}{4\mu_1}, \quad x_2 = \frac{a^3}{4\mu_2}, \quad x_3 = \frac{a^3}{4\mu_3},$$

we have $$x_1 = \frac{2a^3}{(\alpha + \beta - \gamma - \delta)^2}, \quad x_2 = \&c.$$

Hence, being given the sum or difference of the intercepts of a line on a Cartesian, the envelope of the line is a circle having double contact with the curve.

Putting $d_1 + d_2 + d_3 = p_1$, &c. the cubic in x' can be written

$$x'^3 \sin^2\omega - x'^2(p_1 - 2p\cos\omega) + x'(p_2 - p^2) - p_3 = 0,$$

from which we see that a line touching a circle about the double focus as centre meets the curve so that

$$\Sigma(\alpha + \beta - \gamma - \delta)^2 = \text{a constant},$$

and that a line parallel to a given one meets the curve so that

$$(\alpha + \beta - \gamma - \delta)(\alpha + \gamma - \beta - \delta)(\alpha + \delta - \beta - \gamma)$$

is given.

Substituting d_1, d_2, d_3 successively, for x' in the identity

$$(x' - d_1)(x' - d_2)(x' - d_3) - x'(p - x'\cos\omega)^2$$
$$\equiv \sin^2\omega\,(x' - x_1)(x' - x_2)(x' - x_3),$$

and eliminating p and ω, we obtain

$$\Sigma(d_2 - d_3)\sqrt{\left\{\frac{(d_1 - x_1)(d_1 - x_2)(d_1 - x_3)}{d_1}\right\}} = 0,$$

which gives the identical relation connecting the intercepts of a line on the curve.

If $p_2^2 = 3p_1p_3$, in which case the curve consists of a single oval, we can show that any line drawn through the double focus is divided equi-anharmonically by the curve.

356. If two tangents be drawn to the circle S in the same manner as in Ex. 354, the difference of the areas intercepted on the space between the ovals of the Cartesian will be algebraic.

357. If a bicircular quartic meet a conic, the sum of the eccentric angles of the eight points of intersection is equal to zero or 2π, see Ex. 122. If, then, the conic touch the quartic four times, the sum of the eccentric angles of the points of contact is equal to zero or π. In the former case the points of contact are concyclic, and the system of conics is that considered in Ex. 342. In the latter case the points of contact lie on an equilateral hyperbola whose asymptotes are parallel to the axes of the conic, see Ex. 124. If we write the quartic

$$S^2 - m^2\rho^2\rho'^2 = 0,$$

where

$$S \equiv x^2 + y^2 - 2\alpha x - 2\beta y + k^2, \quad \rho^2 \equiv y^2 + (x-c)^2, \quad \rho'^2 \equiv y^2 + (x+c)^2,$$

the equation

$$U \equiv S - m\cos\phi\,(x^2 - y^2 - c^2) - 2mxy\sin\phi = 0$$

will represent one of the latter system of conics. The points ρ, ρ' are evidently foci of the quartic, and since there are six pairs of foci, there thus appear to be six systems of these conics.

If θ be the angle between the asymptotes of U, we have $\tan\theta = \sqrt{(m^2-1)}$, and $m = \sec\theta$, from which we see that the eccentricity of U is given.

358. We find for the equation of the director circle of U

$$(1-m^2)(x^2+y^2) - 2(\alpha + m\alpha\cos\phi + m\beta\sin\phi)x$$
$$-2(\beta + m\alpha\sin\phi - m\beta\cos\phi)y + 2(k^2 + mc^2\cos\phi) - \alpha^2 - \beta^2 = 0,$$

from which we see that the centre of U lies on the circle

$$(x-\alpha)^2 + (y-\beta)^2 - m^2(x^2+y^2) = 0.$$

We also see that the director circle cuts orthogonally a fixed circle whose centre is $\dfrac{c^2\alpha}{\alpha^2+\beta^2}, \dfrac{-c^2\beta}{\alpha^2+\beta^2}$, and that it, therefore, has double contact with a Cartesian oval.

359. Putting $m = \sec\theta$, we may write U

$$(\cos\theta - \cos\phi)x^2 + (\cos\theta + \cos\phi)y^2 - 2\sin\phi xy - 2\cos\theta(\alpha x + \beta y)$$
$$+ k^2\cos\theta + c^2\cos\phi = 0.$$

The equations of the asymptotes of U are then found to be

$$y + \alpha\cot\theta - \tan\tfrac{1}{2}(\phi-\theta)(x - \beta\cot\theta) = 0,$$
$$y - \alpha\cot\theta - \tan\tfrac{1}{2}(\phi+\theta)(x + \beta\cot\theta) = 0;$$

they, therefore, pass through the fixed points $\beta\cot\theta$, $-\alpha\cot\theta$; $-\beta\cot\theta$, $\alpha\cot\theta$, respectively.

From the equations of the asymptotes we easily find the equations of the axes of U, viz.

$$y - \frac{\beta}{1+\sec\theta} - \tan\tfrac{1}{2}\phi\left(x - \frac{\alpha}{1+\sec\theta}\right) = 0,$$
$$y - \frac{\beta}{1-\sec\theta} + \cot\tfrac{1}{2}\phi\left(x - \frac{\alpha}{1-\sec\theta}\right) = 0;$$

they, therefore, pass through fixed points. This might be seen at once from the fact that the asymptotes include a constant angle and pass through fixed points on the circle on which they intersect; for it can be seen that the points $\mp\beta\cot\theta$, $\pm\alpha\cot\theta$ lie on the circle

$$(x-\alpha)^2+(y-\beta)^2-m^2(x^2+y^2)=0 \text{ (Ex. 358).}$$

360. Writing U in the form

$$(1-m)X^2+(1+m)Y^2-G=0,$$

where $$X=\left(x-\frac{\alpha}{1-m}\right)\cos\tfrac{1}{2}\phi+\left(y-\frac{\beta}{1-m}\right)\sin\tfrac{1}{2}\phi,$$

$$Y=\left(x-\frac{\alpha}{1+m}\right)\sin\tfrac{1}{2}\phi-\left(y-\frac{\beta}{1+m}\right)\cos\tfrac{1}{2}\phi,$$

$$G=\frac{\alpha^2(1+m\cos\phi)+\beta^2(1-m\cos\phi)+2m\alpha\beta\sin\phi}{1-m^2}$$
$$-(k^2+mc^2\cos\phi),$$

the foci will evidently be given by the equations

$$Y=0,\ X^2=\frac{2mG}{1-m^2},\ \text{ or } X=0,\ Y^2=\frac{2mG}{m^2-1},$$

from which we see that the locus of the foci consists of two nodal bicircular quartics, the node in each case being the point through which the corresponding axis of U passes.

The equations of the tangents at the vertices of U being

$$(1-m)X^2-G=0,\ (1+m)Y^2-G=0,$$

these lines touch a pair of conics, and the vertices themselves lie on a pair of nodal bicircular quartics.

361. A circle is described through the centres of the quadrilateral formed by the points of contact of U; show that the locus of the centre of the circle is

$$(2\alpha x+2\beta y+\alpha^2+\beta^2-2k^2)^2=4m^2\{(\alpha x-\beta y-c^2)^2+(\alpha y+\beta x)^2\},$$

a conic of which the radical centre of the director circles of U is a focus.

362. To find the equation of a Cartesian oval passing through four points on a circle.

Let $$S \equiv x^2 + y^2 - k^2 = 0$$

be the equation in rectangular coordinates of the circle, and let $\alpha^2 - \beta = 0$, where

$$\alpha \equiv lx + my, \quad \beta \equiv px + qy + r$$

denote one of the parabolas whose intersection with S determines the four points, then

$$\theta^2 (\alpha^2 - \beta) + 2 (\theta\alpha + \lambda) S + S^2 = 0,$$

where θ and λ are arbitrary parameters, represents a Cartesian oval passing through the four points; for this equation may be written

$$(S + \theta\alpha + \lambda)^2 - (\lambda^2 + \theta^2\beta + 2\theta\lambda\alpha) = 0,$$

showing that the curve is a Cartesian oval, of which

$$\lambda^2 + \theta^2\beta + 2\theta\lambda\alpha = 0$$

is the double tangent, and the centre of the circle

$$S + \theta\alpha + \lambda = 0$$

the triple focus.

Since two parabolas can be described through the four points, it follows that there are two systems of Cartesian ovals passing through the points.

From the equation of the double tangent we see that it is always a tangent to the parabola $\alpha^2 - \beta = 0$, and from the equation $S + \theta\alpha + \lambda = 0$ we see that the triple focus lies on the perpendicular to α at the centre of S.

For the equation of the axis of symmetry of the curve we find

$$2\theta (qx - py) + 4\lambda (mx - ly) + (lq - mp) \theta^2 = 0,$$

whence, if the curve pass through another fixed point, the axis will touch a conic.

The equation of a circle Σ, having its centre on the axis of symmetry, and having double contact with the curve, is easily seen to be

$$\mu^2 + 2\mu(S + \theta\alpha + \lambda) + \lambda^2 + \theta^2\beta + 2\theta\lambda\alpha = 0,$$

or $$(\lambda + \mu)^2 + 2\theta(\lambda + \mu)\alpha + \theta^2\beta + 2\mu S = 0;$$

from which it appears that the radical axis

$$(\lambda + \mu)^2 + 2\theta(\lambda + \mu)\alpha + \theta^2\beta = 0$$

of Σ and S touches the parabola $\alpha^2 - \beta = 0$.

Hence, when the radius of Σ is given, its centre lies on a given circular cubic; for, expressing that

$$2x'x + 2y'y - (x'^2 + y'^2 + k^2 - r^2) = 0,$$

the radical axis of S and

$$\Sigma \equiv (x - x')^2 + (y - y')^2 - r^2)$$

touches the parabola $\alpha^2 - \beta = 0$, we obtain a relation of the form

$$Ax^2 + By^2 + 2Hxy + (Gx + Fy)(x^2 + y^2 + k^2 - r^2) = 0.$$

Putting r equal to the distance of x, y from one of the given points, we see that the normal to the curve at one of these points meets the axis on two fixed lines passing through the centre of S. When r vanishes we obtain the cubic locus of the three collinear foci.

363. Writing a Cartesian oval in the form

$$V \equiv x^2y^2 + z^2(cz^2 + 2fyz + 2gzx + 2hxy) = 0,$$

where x, y, pass through the circular points, and z is the line at infinity, we can verify the identity

$$(gx' + fy')V - \tfrac{1}{2}\left(x'\frac{dV}{dx} + y'\frac{dV}{dy}\right)L \equiv (xy + hz^2)$$
$$\times\{(gx' + fy')(xy + hz^2) - (x'y + y'x)L\},$$

where $$L = 2gx + 2fy + (c - h^2)z.$$

Hence the points of contact of parallel tangents lie on a conic which passes through four fixed points. By writing V in the form

$$(xy + hz^2)^2 + z^3 \{2gx + 2fy + (c - h^2) z\} = 0,$$

we see that two of these points are the points of contact of the double tangent.

If we seek the locus of points from which the tangents to V have their points of contact on a conic by the method of Ex. 311, we obtain the product of the line at infinity by the axis of the curve.

364. A triangle is formed by two foci of a Cartesian and a variable point on the curve; show that the locus of the centre of the inscribed circle is a circular cubic passing through the same foci.

365. Four points A, B, C, D are taken on a Cartesian oval, of which O is a focus; if

$$P \equiv OA.BCD - OB.CDA + OC.ABD - OD.ABC,$$

$$Q \equiv OA^2.BCD - OB^2.CDA + OC^2.ABD - OD^2.ABC,$$

where ABC is the area of the triangle ABC, &c. show that the ratio of P to Q is of an absolute constant of the curve.

366. Using the notation of the preceding example, if O be the triple focus of the Cartesian, and

$$R \equiv OA^4.BCD - OB^4.CDA + OC^4.ABD - OD^4.ABC,$$

show that the ratio of Q to R is an absolute constant of the curve.

367. A circle passing through the double foci and a point P on an ellipse of Cassini meets the normal at P in Q; show that the locus of Q is the inverse of the curve with respect to the circle described on the double foci as diameter.

368. If δ be the angle of aberrancy at any point of an ellipse of Cassini, show that $\cot\delta = \frac{\rho^2}{r^2}\sin\theta\cos\theta$, where ρ is the radius of curvature, r the central radius vector and θ the angle which the central radius vector makes with the curve.

369. Given four points on an oval of Cassini to find the locus of the centre.

The polar equation of the curve referred to its centre

$$r^4 - 2c^2r^2\cos 2\theta + c^4 - k^4 = 0$$

is satisfied by assuming

$$r^2\cos 2\theta = c^2 + k^2\cos 2\phi, \quad r^2\sin 2\theta = k^2\sin 2\phi;$$

hence, if ρ_{12} denote the central radius vector to the middle point of the chord (12), we have

$$(12)\,\rho_{12} = \tfrac{1}{2}\sqrt{\{r_1^4 + r_2^4 - 2r_1^2r_2^2\cos 2(\theta_1 - \theta_2)\}} = k^2\sin(\phi_1 - \phi_2),$$

from which, by means of the identity

$$\sin(\phi_1 - \phi_2)\sin(\phi_3 - \phi_4) + \sin(\phi_2 - \phi_3)\sin(\phi_1 - \phi_4) + \sin(\phi_3 - \phi_1)\sin(\phi_2 - \phi_4) = 0,$$

we deduce

$$(12)(34)\,\rho_{12}\rho_{34} + (23)(14)\,\rho_{23}\rho_{14} + (31)(24)\,\rho_{31}\rho_{24} = 0.$$

Now this relation may also be written

$$BC.BC'.PA.PA' \pm CA.CA'.PB.PB' \pm AB.AB'.PC.PC' = 0,$$

where A, B, C are three fixed points on the curve, A', B', C' the points diametrically opposite A, B, C, and P a variable point on the curve.

But the middle points of the sides and diagonals of a quadrangle form two triangles such as ABC, $A'B'C'$, the common point of bisection of AA', &c. being the centroid, and the chords (12), (34), &c. are equal to double the lines $AB.AB'$, &c. respectively. Thus we see that the required locus is an oval of Cassini passing through the middle points of the six lines joining the given points.

When the four points lie on a circle, we have seen that the locus is an equilateral hyperbola (see Ex. 332).

370. A circle S, cutting orthogonally the circle

$$J \equiv x^2 + y^2 - a^2 + b^2 = 0,$$

meets one of the ovals of the Cassinian

$$r^4 - 2(a^2 + b^2) r^2 \cos 2\theta + (a^2 - b^2)^2 = 0$$

in A, B, A', B', the points A', B' being inverse to A, B, respectively, with regard to J; if the difference of the arcs AB, $A'B'$ be given, to show that the locus of the centre of S is a conic.

If O, O' be the points in which the radius vector $\theta = 0$ meets the oval, we can show that

$$\operatorname{arc} AO - \operatorname{arc} A'O' = \frac{2ab}{\sqrt{(a^2 + b^2)}} F(k, \phi),$$

where $\quad a \sin\phi = \sqrt{(a^2 + b^2)} \sin\theta, \quad k = \dfrac{a}{\sqrt{(a^2 + b^2)}}$

We have, then, when $\operatorname{arc} AB - \operatorname{arc} A'B'$ is given,

$$\cos\sigma = \cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2 \sqrt{(1 - k^2 \sin^2\sigma)},$$

where σ is a constant, subject to which condition it can be seen that the centre of S moves along a conic confocal with $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

If the sum of the arcs AB, $A'B'$ is given, the centre of S will move along a curve of the fourth order; for it can be shown that

$$\operatorname{arc} AO + \operatorname{arc} A'O' = \frac{2ab}{\sqrt{(a^2 + b^2)}} F(k', \phi'),$$

where $\quad b \sin\phi' = \sqrt{(a^2 + b^2)} \sin\theta, \quad k'^2 = 1 - k^2.$

371. If we take two points P, P', connected by the relations $rr' = c^2$, $\theta + \theta' = 0$, on the curve

$$r^4 - 2(2a^2 + c^2 \cos 2\theta) r^2 + c^4 = 0,$$

the distance between P and P' is equal to $2a$, and the locus of the middle point Q of PP' is the Cassinian oval

$$r^4 - 2c^2 \cos 2\theta r^2 + c^4 - a^4 = 0.$$

The normals to the curve at P and P' intersect at a point Q' on the Cassinian oval

$$(c^4 - a^4) r^4 - 2c^6 \cos 2\theta r^2 + c^8 = 0,$$

Q and Q' being connected by the same relations as P and P'.

372. To obtain an expression for the arc of the inverse of the parabola $y^2 = 4m(x - \alpha)$, the origin being the centre of inversion.

Putting $x = \alpha + m\mu^2$, $y = 2m\mu$, we have, for the arc s of the inverse curve,

$$ds = \frac{k^2 \sqrt{(dx^2 + dy^2)}}{x^2 + y^2} = \frac{2mk^2 \sqrt{(1 + \mu^2)}\, d\mu}{(\alpha + m\mu^2)^2 + 4m^2\mu^2}.$$

If, then, α be greater than $3m$, we find

$$s = \frac{k^2}{2\sqrt{\{m(\alpha + m)\}}} \left[\sqrt{\left(1 - \frac{m}{p}\right)} \log \left\{\frac{\sqrt{p} + z\sqrt{(p - m)}}{\sqrt{p} - z\sqrt{(p - m)}}\right\} - \sqrt{\left(1 - \frac{m}{q}\right)} \log \left\{\frac{\sqrt{q} + z\sqrt{(q - m)}}{\sqrt{q} - z\sqrt{(q - m)}}\right\}\right],$$

where

$$p = \alpha + 2m + 2\sqrt{\{m(\alpha + m)\}}, \quad q = \alpha + 2m - 2\sqrt{\{m(\alpha + m)\}},$$

$$z = \frac{\mu}{\sqrt{(1 + \mu^2)}}.$$

If $\alpha < 3m$ and $> -m$, the second member is replaced by a circular function, and when

$$\alpha = 3m, \quad s = \frac{k^2}{3m\sqrt{2}} \log \left(\frac{3 + 2z\sqrt{2}}{3 - 2z\sqrt{2}}\right)$$

The semi-perimeter of this curve will be bisected at the point where $z = \frac{3}{4}$.

When $\alpha + m > 0$, we can obtain the integral by assuming

$$m\mu^2 + \alpha + 2m = 2\sqrt{\{-m(\alpha + m)\}} \tan(\theta + \gamma),$$

and determining γ by the condition

$$\tan 2\gamma = -2\frac{(2\alpha + 3m)}{\alpha + 6m}\sqrt{\left(\frac{-m}{\alpha + m}\right)}.$$

(See Williamson's *Integral Calculus*, Art. 76).

373. A is a point on a fixed circle, and B a point on another fixed circle; if the line AB pass through a fixed point, show that the locus of the anti-points of A and B is a bicircular quartic.

374. Show that the bicircular quartics

$$\alpha x(x^2 + y^2 - c^2) + \beta y(x^2 + y^2 + c^2) + \gamma(x^2 + y^2)$$
$$+ (x^2 + y^2)^2 - 2c^2(x^2 - y^2) + c^4 = 0,$$

$$\alpha' x(x^2 + y^2 - c^2) + \beta' y(x^2 + y^2 + c^2) + \gamma'(x^2 + y^2)$$
$$+ (x^2 + y^2)^2 - 2c^2(x^2 - y^2) + c^4 = 0,$$

cut each other orthogonally, if

$$\alpha\alpha' + \beta\beta' = 2(\gamma + \gamma').$$

IV. *Miscellaneous Examples.*

375. To find the equation of the circle circumscribing the triangle formed by three tangents to a tri-cuspidal hypocycloid.

If we write the equation of a tangent to the curve in the form $x\cos\alpha + y\sin\alpha - b\cos 3\alpha = 0$ (*Salmon's Curves*, Art. 310, Ex. 5), the equation of the circle will be

$$\sin(\beta-\gamma)(x\cos\beta + y\sin\beta - b\cos 3\beta)(x\cos\gamma + y\sin\gamma - b\cos 3\gamma)$$
$$+\sin(\gamma-\alpha)(x\cos\gamma + y\sin\gamma - b\cos 3\gamma)(x\cos\alpha + y\sin\alpha - b\cos 3\alpha)$$
$$+\sin(\alpha-\beta)(x\cos\alpha + y\sin\alpha - b\cos 3\alpha)(x\cos\beta + y\sin\beta - b\cos 3\beta) = 0.$$

Multiplying out this equation, and dividing by

$$\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta),$$

we get

$$x^2 + y^2 - 2b(\cos 2\alpha + \cos 2\beta + \cos 2\gamma)x + 2b(\sin 2\alpha + \sin 2\beta + \sin 2\gamma)y$$
$$+ b^2\{8\cos(\beta-\gamma)\cos(\gamma-\alpha)\cos(\alpha-\beta) - 1 - 2\cos 2(\alpha+\beta+\gamma)\} = 0.$$

If R is the radius of this circle, we find $R = 2b\cos(\alpha+\beta+\gamma)$. Hence we see that when the tangents pass through a point, $\alpha+\beta+\gamma = \frac{1}{2}\pi$. When the normals at the points of contact pass through a point, $\alpha+\beta+\gamma = 0$, and then $R = 2b$.

376. To find the equation of the polar circle of the same triangle.

The equation of this circle is

$$\sin 2(\beta-\gamma)(x\cos\alpha + y\sin\alpha - b\cos 3\alpha)^2$$
$$+\sin 2(\gamma-\alpha)(x\cos\beta + y\sin\beta - b\cos 3\beta)^2$$
$$+\sin 2(\alpha-\beta)(x\cos\gamma + y\sin\gamma - b\cos 3\gamma)^2 = 0,$$

which becomes, when we multiply out and reduce,

$$x^2 + y^2 - 2b\{\cos 2(\beta+\gamma) + \cos 2(\gamma+\alpha) + \cos 2(\alpha+\beta)\}x$$
$$- 2b\{\sin 2(\beta+\gamma) + \sin 2(\gamma+\alpha) + \sin 2(\alpha+\beta)\}y$$
$$+ b^2\{1 - 8\cos(\beta-\gamma)\cos(\gamma-\alpha)\cos(\alpha-\beta)\cos 2(\alpha+\beta+\gamma)\}=0.$$

From the expressions for their coordinates, we see that the centres of the circles in this and the preceding example are equidistant from the origin; hence we infer that the line bisecting at right angles the line joining the centre of the circumscribing circle and the intersection of the perpendiculars of the triangle formed by three tangents to a tri-cuspidal hypocycloid passes through the intersection of the cuspidal tangents.

If α, β, γ be the equations of the sides of the triangle in trilinear coordinates, the equation of this line is

$$\alpha \sin 3A + \beta \sin 3B + \gamma \sin 3C = 0.$$

If we take four tangents, the four such lines, which correspond to the four triangles formed by the tangents, pass through the intersection of the cuspidal tangents. Thus we see that, for the four triangles which can be formed out of a quadrilateral, these lines in general pass through a point.

377. From the equations of the circumscribing and polar circles we can obtain the equation of the nine-point circle of the same triangle, viz.

$$x^2+y^2-2b\sin s\{\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)+\sin(\alpha+\beta-\gamma)\}x$$
$$+2b\sin s\{\cos(\beta+\gamma-\alpha)+\cos(\gamma+\alpha-\beta)+\cos(\alpha+\beta-\gamma)\}y$$
$$+b^2\{8\sin^2 s\cos(\beta-\gamma)\cos(\gamma-\alpha)\cos(\alpha-\beta)-\cos 2s\}=0,$$

where $s = \alpha+\beta+\gamma$.

If $s = 0$, or the normals at the points of contact pass through a point, the nine-point circle is the fixed circle $x^2 + y^2 - b^2 = 0$.

378. To find the equation of the circle inscribed in the same triangle.

Expressing that the tangent $x\cos\alpha + y\sin\alpha - b\cos 3\alpha = 0$ touches a circle whose centre is x', y', and radius r, we have

$$b(t^6+1) + 2rt^3 - (x'-iy')t^4 - (x'+iy')t^2 = 0,$$

where $e^{i\alpha} = t$. From this equation we get $\Sigma\cos\alpha = 0$, $\Sigma\sin\alpha = 0$, $\Sigma\alpha = 0$, $x' - iy' = -b\Sigma e^{i(\alpha+\beta)}$, $x' + iy' = -b\Sigma e^{-i(\alpha+\beta)}$, and from these relations, by eliminating three of the angles, we obtain

$$x' = b(P^2 - Q^2 - 2Q\sin s), \quad y' = 2bP(Q - \sin s),$$

where $P = \cos\alpha + \cos\beta + \cos\gamma$, $Q = \sin\alpha + \sin\beta + \sin\gamma$,

$$s = \alpha + \beta + \gamma.$$

We also get $r = b\cos s(P^2 + Q^2 - 1)$, and if k^2 be the absolute term in the equation of the circle,

$$k^2 = b^2\{(P^2 + Q^2 + 1)^2\sin^2 s + 2(P^2 + Q^2) + 4Q(Q^2 - 3P^2)\sin s - 1\}.$$

As the equation of a tangent is unaltered by increasing α by π, we shall evidently obtain the equations of the three escribed circles by increasing one of the angles by π.

379. If four tangents $x\cos\alpha + y\sin\alpha - b\cos 3\alpha = 0$, &c. of a tri-cuspidal hypocycloid are all touched by the same circle, we have $\Sigma\sin(\alpha + s) = 0$, or any one of the conditions obtained by interchanging α, β, &c. in either of the following:

$$\sin(\alpha+s) + \sin(\beta+s) - \sin(\gamma+s) - \sin(\delta+s) = 0,$$

$$\cos(\alpha+s) + \cos(\beta+s) + \cos(\gamma+s) - \cos(\delta+s) = 0,$$

where $$2s = \alpha + \beta + \gamma + \delta.$$

380. If the normal at a point P of a tri-cuspidal hypocycloid meet one of the axes of symmetry in Q, show that $x^3 = 27br^2$, where $PQ = r$, and x is the distance of Q from the corresponding cusp. If r_1, r_2, r_3 be the three values of r corresponding to the three axes of symmetry, show that

$$(r_1 - r_2)^2 + (r_2 - r_3)^2 + (r_3 - r_1)^2 = 162b^2, \quad r_1^{\frac{1}{3}} + r_2^{\frac{1}{3}} + r_3^{\frac{1}{3}} = 0.$$

381. If two tangents from a point P to a tri-cuspidal hypocycloid contain a constant angle, show that the locus of P is a tri-nodal quartic which touches the line at infinity at the circular points. Show also that the nodes of this quartic lie on the axes of symmetry of the curve and form an equilateral triangle.

382. A line cutting off a constant intercept between the lines $x^2 - y^2 = 0$ may be written $x \cos\omega + y \sin\omega - d\cos 2\omega = 0$, the envelope of which, as is well known, is the curve

$$(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2d^{\frac{2}{3}}.$$

If four tangents of this curve $x \cos\alpha + y \sin\alpha - d\cos 2\alpha = 0$, &c. are connected by the relation $\alpha + \beta + \gamma + \delta = 0$, it can be shown that they are all touched by the same circle.

It is not, however, necessary that this relation should hold if four tangents are touched by the same circle.

383. Show that three tangents from any point P of the circle $x^2 + y^2 - d^2 = 0$ to the curve, whose equation is given in
* the preceding example, are parallel to the sides of an equilateral triangle. Show also that the osculating circles at the points of contact of these tangents are touched by the fourth tangent from P to the curve.

384. Show that the equation of the circle inscribed in the triangle formed by the lines $x\cos\alpha + y\sin\alpha - d\cos 2\alpha = 0$, &c. is

$$(x - x')^2 + (y - y')^2 - r^2 = 0,$$

where

$$\begin{aligned} x' &= 4d \cos\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\gamma+\alpha)\cos\tfrac{1}{2}(\alpha+\beta), \\ y' &= -4d \sin\tfrac{1}{2}(\beta+\gamma)\sin\tfrac{1}{2}(\gamma+\alpha)\sin\tfrac{1}{2}(\alpha+\beta), \\ r &= -d\{\cos(\beta+\gamma) + \cos(\gamma+\alpha) + \cos(\alpha+\beta)\}. \end{aligned}$$

* Wols. London Math. Soc. IV p. 321.

385. Show that an infinite number of rigid equilateral triangles can be circumscribed about the curve which is the envelope of the line $x \cos\omega + y \sin\omega = k + d\cos 2\omega$. Show also that the locus of the centroids of the triangles is the circle $x^2 + y^2 = d^2$.

386. If a circle whose centre is x', y', and radius r touch the lines

$$x\cos\alpha + y\sin\alpha - (k + d\cos 2\alpha) = 0,$$
$$x\cos\beta + y\sin\beta - (k + d\cos 2\beta) = 0,$$

show that

$$\frac{x'^2}{\cos^2\frac{1}{2}(\alpha+\beta)} + \frac{y'^2}{\sin^2\frac{1}{2}(\alpha+\beta)} = 8d\{k \pm r - d\cos(\alpha+\beta)\}.$$

387. To show that the algebraic sum of the reciprocals of the common tangents of a circle and an arbitrary curve is equal to zero.

We may write the tangential equation of the curve

$$p^n + u_1 p^{n-1} + \ldots u_n = 0,$$

where u_r is a rational homogeneous function of $\cos\omega$, $\sin\omega$ of the r^{th} degree. Putting

$$\cos\omega = \frac{t^2+1}{2t}, \quad \sin\omega = \frac{t^2-1}{2it},$$

this equation becomes

$$t^n p^n + t^{n-1} p^{n-1}(a_0 + a_2 t^2) + \ldots + l_0 t^{2n} + l_2 t^{2n-2} + \ldots l_{2n} = 0,$$

from which we see that the product of all the values of t, corresponding to the same value of p, is independent of p;

therefore $\dfrac{d}{dp}\Sigma\omega = \Sigma\dfrac{1}{t} = 0.$

In a similar manner, by the use of polar coordinates, we can prove that the sum of the co-tangents of the angles at which a circle meets an arbitrary curve is equal to zero (see Exx. 51, 271, 338).

388. If t be the length of the tangent drawn from a point P to a curve, and ρ be the radius of curvature at the point of contact of the tangent, to show that, for all the tangents drawn from P to the curve, $\Sigma \frac{\rho}{t^3} = 0$.

If we write the equation of a tangent to a curve in the form $x + \mu y = \alpha$, it can be shown that $\frac{d^2\mu}{d\alpha^2} = \frac{\rho}{t^3}$, where t is the length of the tangent measured from the point of contact to the point α, 0. But the tangential equation of the curve is a rational function of the n^{th} degree in α, μ, from which it follows that the sum of the n values of μ corresponding to the same value of α, is of the form $a\alpha + b$. Hence we have

$$\Sigma \frac{\rho}{t^3} = \Sigma \frac{d^2\mu}{d\alpha^2} = 0.$$

389. If two curves be transformed by the substitutions of Ex. 35, show that the transformed curves cut each other under the same angles as the original curves.

390. If U is a conic, and x, y are lines, show that a common tangent of the curves

$$x^{2m} - \alpha y^{2n} U^{m-n} = 0, \quad y^{2m} - \beta x^{2n} U^{m-n} = 0$$

is divided harmonically at the points of contact and where it meets U.

391. Show that the four tangents to the tri-cuspidal
290 quartic

$$x^{-\frac{1}{2}} + y^{-\frac{1}{2}} + z^{-\frac{1}{2}} = 0,$$

at the points where it is met by the line

$$lx + my + nz = 0,$$

meet the curve again in eight points lying on the conic

$$lx^2 + my^2 + nz^2 + 2(3l - m - n)yz + 2(3m - n - l)zx + 2(3n - l - m)xy = 0.$$

392. Given three tangents of a tri-cuspidal quartic, show that a nodal cubic can be described having these lines for inflexional tangents and passing through the three cusps of the quartic.

393. Given three tangents and two of the cusps of a tri-cuspidal quartic, show that the locus of the intersection of the tangents at these cusps consists of nine right lines.

394. Show that the equation

$$\sqrt{(y^2+z^2+2ayz)}+\sqrt{(z^2+x^2+2bzx)}+\sqrt{(x^2+y^2+2cxy)}=0,$$

represents a quartic of which the vertices of the triangle of reference are nodes.

395. Let V be the polar cubic of a point P with respect to the tri-nodal quartic

$$x^2y^2+y^2z^2+z^2x^2+2\kappa xyz\,(ax+by+cz)=0,$$

then if the tangents to V at the nodes pass through a point, show that the locus of P is

$$(a+bc)\,x\,(y^2+z^2)+(b+ca)\,y\,(z^2+x^2)+(c+ab)\,z\,(x^2+y^2)$$
$$+\,2\,(1+abc)\,xyz=0.$$

396. The equation of a trinodal quartic being written in the same form as in the preceding example, show that

$$4\,\{(a-bc)\,\lambda+(b-ca)\,\mu+(c-ab)\,\nu\}^2$$
$$+\,3\,(1+2abc-a^2-b^2-c^2)\,\Sigma=0$$

is the tangential equation of the conic touching the six inflexional tangents, where

$$\Sigma\equiv\lambda^2+\mu^2+\nu^2-2a\mu\nu-2b\nu\lambda-2c\lambda\mu$$

is the conic touching the six nodal tangents (Salmon's *Curves*, Art. 286).

397. The points of contact of the six tangents from a point P to the quartic

$$x^4 + y^2z^2 + x^2y^2 + 2xy\,(ayz + bzx + cx^2) = 0,$$

(Salmon's *Curves*, Art. 289), lie on a conic; show, by the method of Ex. 311, that the locus of P is

$$2b\,(1 - a^2)\,x^2 + 2\,(1 - a^2)\,zx + (c - ab)\,yz + (a + bc - 2ab^2)\,xy = 0.$$

398. Show that the same locus for the oscnodal quartic

$$(yz + x^2)^2 + 2cxy\,(yz + x^2) + y^2\,(x^2 + y^2 + 2hxy + 2fyz) = 0,$$

is
$$(1 - c^2 - 2f)\,x + (h - cf)\,y = 0.$$

399. Show that the equations of the twenty-eight double tangents of the quartic

$$x^4 + y^4 + z^4 + 2fy^2z^2 + 2gz^2x^2 + 2hx^2y^2 = 0,$$

are
$$\begin{aligned}
&y^4 \sin^2\beta + x^4 \sin^2\alpha + 2x^2y^2 \sin\alpha \sin\beta \cos\gamma = 0,\\
&x^4 \sin^2\gamma + z^4 \sin^2\alpha + 2z^2x^2 \sin\gamma \sin\alpha \cos\beta = 0,\\
&z^4 \sin^2\beta + y^4 \sin^2\gamma + 2y^2z^2 \sin\beta \sin\gamma \cos\alpha = 0\,;\\
&x\sqrt{(\cot\tfrac{1}{2}\alpha)} \pm y\sqrt{(\cot\tfrac{1}{2}\beta)} \pm z\sqrt{(\cot\tfrac{1}{2}\gamma)} = 0,\\
&x\sqrt{(\tan\tfrac{1}{2}\alpha)} \pm y\sqrt{(\tan\tfrac{1}{2}\beta)} \pm z\sqrt{(-\cot\tfrac{1}{2}\gamma)} = 0,\\
&x\sqrt{(\tan\tfrac{1}{2}\alpha)} \pm y\sqrt{(-\cot\tfrac{1}{2}\beta)} \pm z\sqrt{(\tan\tfrac{1}{2}\gamma)} = 0,\\
&x\sqrt{(-\cot\tfrac{1}{2}\alpha)} \pm y\sqrt{(\tan\tfrac{1}{2}\beta)} \pm z\sqrt{(\tan\tfrac{1}{2}\gamma)} = 0,
\end{aligned}$$

where

$$f = \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma},\quad g = \frac{\cos\beta + \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma},\quad h = \frac{\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta}.$$

400. A circle S cuts orthogonally the circle $r = b$, and has its centre on the tri-cuspidal hypocycloid

$$r^4 + 8br^3\cos 3\theta + 18b^2r^2 = 27b^4\,;$$

show that the envelope of S consists of the two imaginary parabolas

$$(x \pm iy)^2 - b\,(x \mp iy) = 0.$$

401. Show that the curve

$$(ax+by+c)^2(x^2+y^2-k^2)-(x+a)^3=0$$

has six foci lying on a circle.

402. A triangle is inscribed in a conchoid of Nicomedes, so that the circumscribing circle passes through the node; show that the centre of the inscribed circle lies on the asymptote.

403. A line through the origin meets the conchoid

$$(x-b)^2(x^2+y^2)=a^2x^2$$

again in two points; show that the locus of the intersection of the normals at these points is the parobola $y^2+bx=0$.

404. If the centre of the nine-point circle of a triangle, whose base is fixed, move along a given conic; show that the locus of the vertex is a quartic curve of which the extremities of the base are nodes.

405. If the feet of the perpendiculars from a point P on five given lines lie on a Cartesian oval of which P is a focus, show that the locus of P is a quartic curve passing through all the points of intersection of the given lines.

406. Given four tangents to the curve parallel to a parabola, show that the locus of the focus is a nodal circular cubic passing through the centres of the circles inscribed in the four triangles formed by the given tangents.

407. If t be the length of the tangent drawn from a point P to the nine-point circle of the triangle formed by the feet of the perpendiculars from P on the sides of a given triangle, show that

$$t^2=\frac{\beta^2\gamma^2\sin 2A+\gamma^2\alpha^2\sin 2B+\alpha^2\beta^2\sin 2C}{2(\beta\gamma\sin A+\gamma\alpha\sin B+\alpha\beta\sin C)},$$

where α, β, γ are the trilinear coordinates of P with regard to the given triangle.

408. If Δ' be the area of the triangle formed by the lines bisecting at right angles the lines which join a point P to the vertices of a given triangle, show that

$$\Delta' = \frac{\Delta}{8R} \frac{(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C)^2}{\alpha\beta\gamma},$$

where Δ is the area, R the radius of the circumscribing circle, and α, β, γ the perpendiculars from P on the sides of the given triangle.

409. If δ be the angle of aberrancy at any point of the curve whose polar equation is $r^m = a^m \sin m\theta$, show that

$$\tan\delta = \tfrac{1}{3}\left(\frac{1-m}{1+m}\right)\cot\phi,$$

where ϕ is the angle which the radius vector makes with the curve.

410. A curve of the m^{th} degree, passing m times through each circular point, meets a conic; show that the sum of the eccentric angles of the $4m$ points of intersection is equal to zero.

411. Show that the foci of the curve

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$$

are the inverse points with respect to $x^2 + y^2 = k^2$ of the foci of the curve

$$\frac{x^n}{a'^n} + \frac{y^n}{b'^n} = 1,$$

if $$aa' = k^2, \quad bb' = k^2, \quad 2mn = m + n.$$

Show that this is also true if the axes are oblique, the circle of inversion being, in this case,

$$x^2 + y^2 + 2xy\cos\omega = k^2.$$

412. Show that the quartic

$$(xy + yz + zx)^2 = 16z^2xy$$

has a point of undulation at $x = y = 2z$, and show that the tangent at this point is $x + y - 4z = 0$.

413. If S and T are the invariants of the four tangents drawn from x, y, z to the quartic whose equation is given in the preceding example, show that

$$\frac{S^3}{27T^2} = \frac{16z^3(4z - x - y)}{(xy + yz + zx - 8z^2)^2}.$$

414. Show that the same quartic can be written in the form

$$16z(x-y)^3 = 27(y^2 + zx)^2,$$

and that the tangential equation of the curve is then

$$\beta^3(\beta + 4\alpha) - 16\alpha^3\gamma = 0.$$

Hence, show that the reciprocal of this curve is a quartic with a triple point at which all the tangents coincide.

415. Show that the contravariant σ of the quartic

$$x^4 + 3z^2(x^2 + y^2) = 0$$

breaks up into two conics whose equations in x, y, z coordinates are

$$x^2 - z^2 \pm 2yz = 0.$$

416. A, B are two variable points on a curve of which O is a fixed point; show that the loci of the middle point of the chord AB, corresponding to the conditions

(1) arc AB = a constant,

(2) arc AO + arc BO = a constant,

cut each other orthogonally. Show also that the tangents to the loci are parallel to the bisectors of the angles between the tangents to the curve at A, B.

417. Show that according as we consider u or v as a constant in the equations

$$x = a\cos(mu + mv) + ma\cos(u - v),$$
$$y = a\sin(mu + mv) + ma\sin(u - v),$$

we obtain two systems of epicycloids cutting each other orthogonally.

418. Show that the systems of curves whose equations in polar coordinates are

$$\left(r + \frac{k^2}{r}\right)^m (\cos\theta)^n = \alpha, \quad \left(r - \frac{k^2}{r}\right)^n (\sin\theta)^m = \beta,$$

cut each other orthogonally.

419. If r, r' are the sides PA, PB of a triangle formed by a variable point P and two fixed points A, B; θ, θ' the base angles of the same triangle, and x, y the coordinates of P with respect to the axes formed by the base AB and a perpendicular to it at its middle point, show that the system of curves

$$lr - mr' = \alpha + nx,$$

where α is variable, and l, m, n are constants, is cut orthogonally by the system $(\tan\frac{1}{2}\theta)^l(\tan\frac{1}{2}\theta')^m = \beta y^n$.

420. Show that the systems of curves

$$\frac{x^2y}{y-\beta} - \frac{y^3}{\beta} = c^2, \quad \frac{x^3}{\alpha} + \frac{xy^2}{\alpha - x} = c^2$$

cut orthogonally.

421. Show that the systems of curves

$$(x^2 + y^2)(x^2 - y^2 - k^2) = c_1 x, \quad y(y^2 + 3x^2 - k^2) = c_2(x^2 + y^2)$$

cut orthogonally.

422. Show that the cuspidal cubic, which is the envelope of the line $x + ty = m(1 + t^2) + at^3$, cuts orthogonally the tricuspidal quartic which is the envelope of the line

$$t(x + ty) = nt(1 + t^2) + a.$$

423. Show that, if we take different values of k, the curves of the fifth order, which are the envelopes of

$$tx + t^2 y = kt(1 + t^2) + m(1 + t^2)^2,$$

cut each other orthogonally.

This and the preceding example may be solved by means of the theorem that the curves which are the envelopes of the line $x \cos\omega + y \sin\omega - p = 0$, subject to the conditions

$$p \cos\omega + f(\omega) = c_1, \quad p \cos\omega - f(\omega + \tfrac{1}{2}\pi) = c_2,$$

cut orthogonally. This may be proved by differentiating the first equation, substituting $-p$, $\dfrac{dp}{d\omega}$ for $\dfrac{dp}{d\omega}$, p, and increasing ω by $\frac{1}{2}\pi$, when we find the result of differentiating the second equation.

424. To find the length of the inverse of the epicycloid with respect to the centre of the fixed circle.

We have (see Salmon's *Curves*, Art. 309) $p = b \cos n\omega$; but

$$ds = \frac{k^2\left(p + \dfrac{d^2p}{d\omega^2}\right)d\omega}{p^2 + \left(\dfrac{dp}{d\omega}\right)^2} = \frac{k^2}{nb}\,\frac{(1 - n^2)\,d(\sin n\omega)}{\cos^2 n\omega + n^2 \sin^2 n\omega}.$$

The arc, therefore, is a logarithm, or circular function according as n is greater or less than unity.

425. To find the length of the inverse of the epitrochoid with respect to the centre of the fixed circle.

We have, for the epitrochoid,

$$x = mb \sin\phi - d \sin m\phi, \quad y = mb \cos\phi - d \cos m\phi;$$

therefore

$$ds = k^2 \left\{1 + \frac{(m-1)(d^2 - mb^2)}{d^2 + m^2b^2 - 2mbd\cos\psi}\right\} \frac{d\psi}{\sqrt{(b^2 + d^2 - 2bd\cos\psi)}},$$

where $\psi = (m-1)\phi$.

The arc, therefore, depends on elliptic functions of the first and third kind.

426. If s be the length of the curve whose polar equation is $r^{\frac{2}{3}} = (\cos\theta)^{\frac{2}{3}} + (\sin\theta)^{\frac{2}{3}}$, show that

$$s = \sqrt{3}\,\{\tfrac{1}{3}\pi - \tan^{-1}(\sqrt{3}\cos 2\phi)\},$$

where $\tan^3\phi = \tan\theta$.

427. Show that the arc and area of the curve $r^3 \cos 3\theta = a^3$ are expressed by means of the same integrals as the area and arc of the curve $r^3 = a^3 \cos 3\theta$, respectively.

428. A circle passing through the origin and a fixed point on the curve $r^3 = a^3 \cos 3\theta$ meets the curve again in A, B; show that the middle point of the arc AB is fixed.

429. A tangent drawn from a point P to an epicycloid is of given length a; to find the arc of the locus of P.

If ρ be the radius of curvature of the epicycloid, and ω the angle which it makes with a fixed line, it may be shown that $d\sigma = \sqrt{(\rho^2 + a^2)}\,d\omega$, where σ is the arc of the locus; but $\rho = b\cos n\omega$; therefore,

$$d\sigma = \frac{1}{n}\sqrt{(a^2 + b^2\cos^2\phi)}\,d\phi,$$

putting $n\omega = \phi$. Thus we see that the arc required is always equal to that of an ellipse.

Notes and Solutions to some of the Problems.

7. By means of elliptic functions the equation of the circle inscribed in a circumscribed triangle can be written in a form similar to that of the circumscribing circle of an inscribed triangle. For if $u_1 = \int_0^{\phi_1} \frac{d\phi}{\sqrt{(1-e^2\sin^2\phi)}}$, &c. where ϕ_1, ϕ_2, ϕ_3 are the complements of the excentric angles of the points of contact of the tangents, it can be shown that

$$\nu_1 = c \operatorname{sn}\tfrac{1}{2}(u_2+u_3), \text{ \&c.}$$

(see Enneper's *Elliptische Functionen*, p. 501).

If, then, x', y' be the coordinates of the centre, and r the radius, we have

$$x' = \frac{c^2}{a} \operatorname{sn}\tfrac{1}{2}(u_2+u_3)\operatorname{sn}\tfrac{1}{2}(u_3+u_1)\operatorname{sn}\tfrac{1}{2}(u_1+u_2),$$

$$y' = -\frac{c^2}{b} \operatorname{cn}\tfrac{1}{2}(u_2+u_3)\operatorname{cn}\tfrac{1}{2}(u_3+u_1)\operatorname{cn}\tfrac{1}{2}(u_1+u_2),$$

$$r = \frac{a^2}{b} \operatorname{dn}\tfrac{1}{2}(u_2+u_3)\operatorname{dn}\tfrac{1}{2}(u_3+u_1)\operatorname{dn}\tfrac{1}{2}(u_1+u_2).$$

Comparing the equations of the circles in Ex. 7 and Ex. 1, we see that the points of contact of tangents to a conic, which are parallel to its chords of intersection with a circle, lie on confocal conics passing through the points of intersection of the common tangents of the conic and circle.

13. The locus of Ex. 12 can be written in the form

$$S' \equiv (C-\rho_1^2)(C-\rho_2^2) - 4b^2C = 0,$$

where ρ_1, ρ_2 are the distances of a point from the foci of S, from which it follows that a focus of S is an anti-point (Salmon's *Curves*, Art. 139) of two points of intersection of C on S'. But the points of intersection of C and S' (*Curves*, Art. 275) are foci of the quartic, and the anti-points of foci are also foci.

39. The equation $\sin A \sqrt{(\rho_1^2 - k^2)} + \&c. = 0$ represents two circles concentric with the circumscribing circle and situated at a distance $\pm k$ from it.

44. The point $\cos\lambda$, $\cos\mu$, $\cos\nu$ satisfies the equation of the cubic and also that of the line at infinity. But the tangents to the cubic at the points $1, \pm 1, \pm 1$ pass through $\cos\lambda$, $\cos\mu$, $\cos\nu$; these points are, therefore, the points of contact of the tangents parallel to the real asymptote (Salmon's *Curves*, Art. 278).

51. When the circle touches the conic, we have in the limit

$$t - t' = \tfrac{1}{2}\frac{ds^2}{\rho}\cot\theta, \text{ and } tt' = \tfrac{1}{4}(\rho - r)^2\frac{ds^2}{\rho^2}.$$

52. This proposition can be readily established by means of elliptic functions; for, by Chasles's theorem, the extremities of the diagonals of the quadrilateral, formed by the common tangents of a conic and a circle, lie on a confocal conic; hence, when four tangents are touched by a circle (see Note on Ex. 7), $u_1 + u_2 + u_3 + u_4 = 0$, or $4mK$.

Now three tangents coincide at the point of contact of an osculating circle; therefore, for the points of contact u_1, u_2, u_3 of osculating circles which touch the tangent u, we have $3u_1 + u = 0$, or $u_1 = -\tfrac{1}{3}u$, and $u_2 = -\tfrac{1}{3}u + \tfrac{4}{3}K$, $u_3 = -\tfrac{1}{3}u + \tfrac{8}{3}K$, from which it follows that the tangents u_1, u_2, u_3 are touched by a circle touching u. There are nine osculating circles touching a given tangent, but six of these are imaginary, corresponding to the imaginary periods of u.

53. Since there is only one circle of given radius having double contact with a parabola, it follows that the coordinates of its centre must be expressed rationally in terms of the coefficients of the curve. But given three tangents to a parabola, the coefficients are quadratic functions of a parameter. We thus see that the locus is unicursal.

122. Let $e^{i\phi} = t$, then

$$x = \tfrac{1}{2}a\left(t + \frac{1}{t}\right), \quad y = \tfrac{1}{2}\frac{b}{i}\left(t - \frac{1}{t}\right);$$

and substituting these values in the equation of the quartic we obtain $t^8 + \&c. \ldots + 1 = 0$, from which the result stated in the question follows.

126. Let $\rho_1^2 = (x - x_1)^2 + (y - y_1)^2$, $\rho_2^2 = \&c.$,
then the equation $\Sigma_5^1 l_i\rho_i^4 = 0$ will represent an equilateral hyperbola, if

$$\Sigma l_i = 0, \quad \Sigma l_i x_i = 0, \quad \Sigma l_i y_i = 0, \quad \Sigma l_i (x_i^2 + y_i^2) = 0.$$

Now this equation can be written $\Sigma_4^1 l_1\alpha_1^2 = 0$, where

$$\alpha_1 = \rho_1^2 - \rho_5^2, \quad \alpha_2 = \rho_2^2 - \rho_5^2, \ \&c.,$$

and $\Sigma_4^1 l_i\alpha_1 = 0$, identically. But when the curve is written in the latter form, the pole of $\alpha_4 = 0$ is found from the equations $\alpha_1 = \alpha_2 = \alpha_3$, therefore &c.

138. These relations are also true for the circle.

154. See Ex. 341.

155. Let ρ, r_1, r_2, r_3 be the distances of a point from F, A, B, C respectively, then the equation of the circle is

$$\sin\tfrac{1}{2}(\beta-\gamma)\sqrt{\left(\frac{\rho^2-r_1^2}{d_1}\right)}+\sin\tfrac{1}{2}(\gamma-\alpha)\sqrt{\left(\frac{\rho^2-r_2^2}{d_2}\right)}$$
$$+\sin\tfrac{1}{2}(\alpha-\beta)\sqrt{\left(\frac{\rho^2-r_3^2}{d_3}\right)}=0,$$

where $\alpha=$ angle AFO, &c., and $d_1=FA$, &c. But for the centre of the circle round ABC, $r_1=r_2=r_3$, and from the focal equation of the parabola $d_1=a\sec^2\frac{1}{2}\alpha$, &c.; therefore $\cos\frac{1}{2}\alpha\sin\frac{1}{2}(\beta-\gamma)+\cos\frac{1}{2}\beta\sin\frac{1}{2}(\gamma-\alpha)+\cos\frac{1}{2}\gamma\sin\frac{1}{2}(\alpha-\beta)=0$, which is satisfied identically. See also Ex. 166.

163. Let the conic be projected into a circle, and the fixed line to infinity, then the centroid of the projected triangle is fixed, subject to which condition it can be seen that the sides of the triangle touch a conic having the centre of the circle for focus.

165. By considering two consecutive curves of the system we see that the different loci of the centre, obtained according as the tangents or points as fixed, must touch one another. The line of intersections of perpendiculars is the radical axis of the director circles of conics touching four lines, and the chord of contact of the director circle, given four points on a conic, passes through a fixed point (see Ex. 18).

170. Considering two consecutive curves of the system, we see that the locus coincides with the envelope of the directrix.

182. If α, β, γ, ϕ are the eccentric angles of A, B, C, P respectively, it may be shown that the coordinates of the centroid of the variable triangle are given by the equations

$$x=\frac{c^2}{12a}\{2\Sigma\cos\alpha+3\cos\delta+\Sigma\cos(\alpha+\beta+\delta)\},$$

$$y=-\frac{c^2}{12b}\{2\Sigma\sin\alpha+3\sin\delta-\Sigma\sin(\alpha+\beta+\delta)\},$$

from which we see that the locus of P is a line if
$$\{\Sigma \cos(\alpha+\beta)\}^2 + \{\Sigma \sin(\alpha+\beta)\}^2 = 9,$$
the condition that the centroid of A, B, C should lie on the curve. The locus is in general a conic.

184. See Ex. 39.

207. Let α, β, γ be perpendiculars on the sides of the triangle, and let an imaginary focus satisfy $a\alpha'+b\beta'+c\gamma'=2\Delta$, then since for the imaginary foci $\alpha\alpha'=\beta\beta'=\gamma\gamma'=a^2$, and $a\beta\gamma+b\gamma\alpha+c\alpha\beta=-\frac{\Delta}{r}t^2$, we have $a^2t^2=-2r\alpha\beta\gamma$, where t is the length of the tangent drawn to the circle. But $\alpha=a\sqrt{\left(\frac{r_1}{s_1}\right)}$, where r_1, s_1 are the distances of the point of contact from the imaginary foci, and $\frac{r_1}{s_1}=e^{i\phi_1}$, where ϕ is the angle subtended by the same point at the real foci. Hence, since the sum of the angles subtended at the points of contact is equal to the sum of the angles subtended at the vertices of the triangle, we have $t^2=2are^{i\pi}$; therefore &c.

209. If ϕ_1, ϕ_2 are the angles of intersection at the points whose eccentric angles are α, β, we have
$$\cot\phi_1+\cot\phi_2 = 2\cot\theta\,\frac{\sin\psi_1\sin\psi_2}{\sin\phi_1\sin\phi_2},$$
where θ_1, ψ_1, ψ_2 are the angles which the chord $\alpha\beta$ makes with the diameter bisecting it and the curve respectively. But
$$\cot\theta = \frac{c^2}{2ab}\sin(\alpha+\beta),$$
and it can be shown, from the expressions for the coordinates of the centre of the circle in terms of eccentric angles, that
$$\sin\phi_1 = \frac{c^2}{ab}\,\frac{p_1}{r}\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\alpha-\gamma)\sin\tfrac{1}{2}(2\alpha+\beta+\gamma);$$

also $$b_1^2 - b_2^2 = -c^2 \sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\alpha+\beta+2\gamma),$$

and $$\sin\psi_1 \sin\psi_2 = \frac{p_1 p_2 \sin^2\tfrac{1}{2}(\alpha-\beta)}{b_3^2};$$

therefore, &c.

If α, β, γ are the angles which the sides of the triangle make with the tranverse axis, it can be shown that

$$\cot\phi = \frac{2(1-e^2) + e^2\{\sin^2\alpha + \sin^2\beta + \sin^2\gamma - \sin^2(\alpha+\beta+\gamma)\}}{2e^2 \sin(\alpha+\beta)\sin(\beta+\gamma)\sin(\gamma+\alpha)}.$$

211. The sum of the angles which the tangents to the circle make with the axis is seen to be zero. Eliminating then x, y from

$$x\cos\omega + y\sin\omega - \sqrt{(a^2\cos^2\omega + b^2\sin^2\omega)} = 0,$$

and the equations of the conic and circle, and putting $e^{i\omega} = t$, we find the required expression from the absolute term of the equation in t.

226. It is known that the sum of the arguments of three collinear points on a cubic is equal to a constant, the argument of a point being an elliptic function depending on the coordinates of the point. The following proof may be given of this theorem: Suppose the cubic, the point A on the curve being origin, to be written in the form

$$U \equiv ax^3 + \ldots fy = 0,$$

then, transforming to polar coordinates, if a line making an angle i_1 with the tangent at the origin meet the curve again in B, C, we have

$$AB.AC = \frac{P_1 \sin i_1}{Q},$$

where P_1 is the value of

$$\sqrt{\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2\right\}}$$

at A, and $Q = a\cos^3\theta + \&c.$, and, similarly,

$$BA.BC = \frac{P_2 \sin i_2}{Q}, \quad CA.CB = \frac{P_3 \sin i_3}{Q}.$$

Now let the line touch a curve in O, then we have, by infinitesimals

$$OAd\phi = \sin i_1 ds_1 = P_1 \sin i_1 du_1, \text{ if } ds = Pdu;$$

hence
$$\frac{du_1}{OA.BC} = \frac{du_2}{OB.CA} = \frac{du_3}{OC.AB},$$

and
$$d(u_1 + u_2 + u_3) = 0.$$

From
$$du = \frac{ds}{\sqrt{\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2\right\}}},$$

we have
$$du = \frac{dx}{\frac{dU}{dy}} = -\frac{dy}{\frac{dU}{dx}},$$

and by writing the curve in a particular form, as for instance,

$$y^2 = x(1-x)(1-k^2x),$$

it can be shown that u is an elliptic function. If we now inscribe a triangle in the curve so that the tangent at each vertex passes through the point where the opposite side meets the curve again, we must have the relations

$$u_1 - u_2 = \tfrac{1}{3}\omega, \quad u_2 - u_3 = \tfrac{1}{3}\omega, \quad u_3 - u_1 = -\tfrac{2}{3}\omega,$$

where
$$\omega \equiv 4mK + 2niK'.$$

Thus we see that there are four systems of triangles corresponding to the four distinct values of $\frac{1}{3}\omega$, viz.

$$\tfrac{1}{3}(4K), \quad \tfrac{1}{3}(2iK'), \quad \tfrac{1}{3}(4K) \pm \tfrac{1}{3}(2iK').$$

From the values given above for $du_1 : du_2$, we see that when $u_1 - u_2 =$ a constant, the line is divided harmonically at the three points on the curve and the point of contact with its envelope.

230. If we put $y = \theta x$ in the equation

$$x^3 + y^3 + 6xyz = 0,$$

$F = 0$ is the equation of the tangent, and corresponding points are connected by the relation $\theta + \theta' = 0$.

232. The lines joining corresponding vertices are tangents to the Cayleyan. Now if three tangents to the Cayleyan pass through a point, the three polar conics of which each tangent is a factor must have a common point; but this can only happen when the three points which give rise to the polar conics lie on a line. Hence the tangents at the vertices of one of the triangles meet the curve again in three points on a line (Salmon's *Curves*, Art. 180), in which case the points where the sides meet the curve again will also lie on a line.

235. If A, B are the circular points at infinity, the intersection of PA and QB is an anti-point of PQ. When P and Q coincide, the locus of the anti-points will cut the locus of P, Q orthogonally.

258. If ρ, r_1, r_2, r_3 denote the distances of a point from P, A, B, C respectively, we have, for the intersection of the perpendiculars of the variable triangle.

$$(\rho^2 - r_1^2)\frac{\cos\theta_1}{d_1} = (\rho^2 - r_2^2)\frac{\cos\theta_2}{d_2} = (\rho^2 - r_3^2)\frac{\cos\theta_3}{d_3},$$

where $$d_1 = PA, \quad \theta_1 = \angle BPC, \text{ \&c.}$$

Now, if $l + m + n = 0$, the equation

$$\frac{ld_1}{\cos\theta_1} + \frac{md_2}{\cos\theta_2} + \frac{nd_3}{\cos\theta_1} = 0, \text{ or } \frac{l}{S_1} + \frac{m}{S_2} + \frac{n}{S_3} = 0,$$

where S_1, S_2, S_3 are the circles described on the sides as diameters, represents a circular cubic of which A, B, C are centres of inversion. Hence

$$lr_1^2 + mr_2^2 + nr_3^2 = 0,$$

therefore, &c.

259. The equation

$$\frac{l}{d_1 \cos\theta_1} + \frac{m}{d_2 \cos\theta_2} + \frac{n}{d_3 \cos\theta_3} = 0,$$

where $l + m + n = 0$, represents a circular cubic of which the vertices of the triangle are the points S, U, V (see Ex. 43).

268. Taking the point on the curve as origin we may write the curve

$$y(x^2 + y^2) + \ldots + gx + fy = 0.$$

Transforming then to polar coordinates we have from the absolute term $t^2 = f + g \cot\theta$, from which we see that the equation whose roots are t_1^2, &c. is a homographic transformation of the equation which determines the direction of the tangents.

275. If α, β, γ, δ are the equations of the tangents, and $l\alpha + m\beta + n\gamma + p\delta = 0$, identically,

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} + \frac{p}{\delta} = 0$$

(Salmon's *Conics*, Art. 297, Ex. 15), represents a circular cubic passing through the foci and the extremities of the diagonals of an infinity of quadrilaterals, such as α, β, γ, δ, circumscribed about the conic. Now, if two tangents α, β to a conic make angles ϕ_1, ϕ_2 with the locus of their intersection, it may be shown that $\dfrac{du_1}{\sin\phi_1} = \dfrac{du_2}{\sin\phi_2}$, where u_1, u_2 have the same meaning as in the Note to Ex. 7. But the tangent to the cubic at $\alpha\beta$ is $\frac{\alpha}{l}+\frac{\beta}{m}=0$, whence $\frac{\sin\phi_1}{l}=\frac{\sin\phi_2}{m}$; therefore, $\frac{du_1}{l}=\frac{du_2}{m}=$, by symmetry, $\frac{du_3}{n}=\frac{du_4}{p}$. Now from the identity given above we have $\Sigma l\cos\theta=\Sigma l\sin\theta=\Sigma lp=0$, therefore, &c.

For the system of quadrilaterals inscribed in a cubic and circumscribed about a conic, see an article by Cayley, *Liouville's Journal*, tome x. p. 102.

291. This method of forming the differential equation was given by Euler (see Enneper's *Elliptische Functionen*, p. 132).

315. When the point is on the curve, the conic breaks up into the tangent at the point and a line on which the points of contact of the tangents from the point lie. This theorem was given by Dr. Casey.

343. The curve whose equation is $(a_2{}^2-a_3{}^2)(a_1b_1p_1)^{\frac{2}{3}}+\&c.=0$, can be shown to be the locus of the intersection of tangents at a pair of inverse points of the quartic (see Dr. Casey's *Memoir on Bicircular Quartics*).

370. These expressions for the arc of the Cassinian were given by Serret (see *Liouville's Journal*, tome VIII. p. 495).

428. The arc of the curve $r^3=a^3\cos 3\theta$ is the area of the inverse cubic $r^3\cos 3\theta=a^3$ (see then Note to Ex. 226).

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315 To apolar contravariant of
$\Sigma x^{2n} y^{2n} = 0$ is
$\sigma = (\Sigma \alpha^{2n})^2 = 0$.

Connect this two R. H's. with note on 315, thus
P is any point on a lemniscate whose centre is O, and P Q R is ~~an equi~~ a regular Δ whose centre is O. Then tangents where PQ, PR meet curve again meet at a point P_1 on ~~and~~ Q R, and the tangents where Q R meets curve meet at P.

N. B. Hypocycloid of class 4
$\Sigma \xi^2 \eta^2 = 0$ is reciprocal of lemniscate. $\therefore$ tangents where tangent at P meets curve meet on cusp circle