Symbolic Integration

Lecture 14

The problem of integration in finite terms

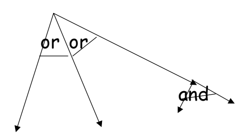
1. The Freshman Calculus Problem
2. The Artificial Intelligence Problem
3. The Algebraic (Rational Function) Problem
4. The Decision Problem (Liouville-Risch...)
5. The Computer Algebra System problem.
6. Extensions to definite integration, numerical integration, approximation by Taylor series, Laurent series, asymptotic series.

The Freshman Calculus Problem. Find the flaws.

1. Humans are intelligent
2. MIT freshman are especially intelligent humans.
3. Freshman study integral calculus (this is in 1960's before Advanced Placement).
4. Therefore solving freshman calculus problems requires intelligence.
5. If a computer can solve freshman calculus problems, then it is intelligent (An example of Artificial Intelligence).
6. To solve these problems, imitate freshman students.
7. Future work: do the rest of math, and the rest of AI.

Freshman Calc → AI: James Slagle's approach

- AND-OR trees for evaluation of difficulty of a path (do we abandon this route or plow on?)



- Game playing: could be any complex task.

Slagle's program SAINT Symbolic Automatic INTegrator

- ELINST: elementary instance expression pattern matching (in assembler).
- Simplification program (in assembler).
- Used lisp prefix expressions (* x (sin x))
- First major lisp program (in Lisp 1.5).
- Took about a minute per problem (about the same time as a student!) running uncompiled on an IBM 7090 (32k word x 36 bits/word).
- Could not do rational function integration (out of space.)

The Algebraic (Rational Function) Problem

- Many attempts to do this right, e.g. Theses of R.Tobey, E. Horowitz, and with extensions, M. Rothstein, B. Trager.
- Main idea is to take a rational expression

q(x)/r(x), express it as a polynomial + "proper" fraction P+Q/R with deg(Q)<= deg(R).

Integrate P, a polynomial, trivially.

Expand Q/R in partial fractions.

Partial fractions "trivialized"

- From Q/R Find all the complex roots of R,

$a_0, \ldots, a_n$.

Express Q/R as $\sum c_i/(x-a_i)$ which integrates to $\sum c_i \log(x-a_i)$.

Can we always find $\{a_i\}$? Complex roots can be found numerically. Can we find c_i?

It turns out that we can expand the summation and solve everywhere... with a big problem..

an example of the general cubic...

$$\frac{c_3}{x-a_3}+\frac{c_2}{x-a_2}+\frac{c_1}{x-a_1} =$$

$$\frac{(c_3+c_2+c_1)\,x^2+((-a_2-a_1)\,c_3+(-c_2-c_1)\,a_3-a_1\,c_2-c_1\,a_2)\,x+a_1\,a_2\,c_3+(a_1\,c_2+c_1\,a_2)\,a_3}{x^3+(-a_3-a_2-a_1)\,x^2+((a_2+a_1)\,a_3+a_1\,a_2)\,x-a_1\,a_2\,a_3}$$

What's the problem?

- If the denominator R= $s_3*x^3+s_2*x^2+s_1*x+s_0$, then the each of the roots a_i look like this:

$$\left(-\frac{\sqrt{3}\,i}{2}-\frac{1}{2}\right)\sqrt[3]{\frac{\sqrt{27\,s_0^2\,s_3^2+\left(4\,s_1^3-18\,s_0\,s_1\,s_2\right)\,s_3+4\,s_0\,s_2^3-s_1^2\,s_2^2}}{6\,\sqrt{3}\,s_3^2}-\frac{27\,s_0\,s_3^2-9\,s_1\,s_2\,s_3+2\,s_2^3}{54\,s_3^3}}$$

$$-\frac{\left(\frac{\sqrt{3}\,i}{2}-\frac{1}{2}\right)\left(3\,s_1\,s_3-s_2^2\right)}{9\,s_3^2\,\sqrt[3]{\frac{\sqrt{27\,s_0^2\,s_3^2+(4\,s_1^3-18\,s_0\,s_1\,s_2)\,s_3+4\,s_0\,s_2^3-s_1^2\,s_2^2}}{6\,\sqrt{3}\,s_3^2}-\frac{27\,s_0\,s_3^2-9\,s_1\,s_2\,s_3+2\,s_2^3}{54\,s_3^3}}}-\frac{s_2}{3\,s_3}$$

The answer is unmanageable even if we can compute it.

- For the cubic, each c_i looks something like this

$$-\left(\frac{-s_0-s_1\,a_1-s_2\,{a_1}^2}{(a_1-a_2)\,(a_1-a_3)}\right)$$

- and so it is ugly. Often enough these expressions can be simplified, and so the search has been to find a minimal algebraic extension in which the c_i can be expressed. We DON'T need to see the a_i, but can use facts like $a_1a_2a_3$ is constant term of polynomial R; we have neat forms for all symmetric functions of roots.

For a general quartic, it gets worse.

- And for order 5 or more there is in general no formula "in radicals" for the roots. That is, we can't compute the a_i in general for symbolic polynomials
- So we can find approximate numerical roots in $\mathcal{C}$ if the problem has a denominator solely in $\mathcal{Q}$[x], all bets are off "symbolically".
- The solution is "RootSum(f(x),poly,x)" expressions where we have $\sum f(a_i)$ where $\{a_i\}$ are the roots of the polynomial poly.

This trick works for numeric denominators too

$$\int \frac{1}{x^3+4\,x^2+7\,x+9}\,dx =$$

$$\mathrm{root\_sum}\left(x\,\log\left(-5430\,x^2+1748\,x+192\right),\,543\,x^3-5\,x-1,\,x\right)$$

Rothstein (etal), find the c_i easier

- Let B(x) be square free and of degree >= A(x),
- then $\int (A/B)dx = c_i \log(v_i)$
- where c_i, are the distinct roots of the polynomial R(c)=Resultant(x,A(x)-cB'(x),B(x)) and $v_i = \gcd(A(x)-c_iB'(x),B(x))$ for i = 1,...n.

- We need Square-free computation and Resultant wrt x

Squarefree isn't a problem

- Let the problem be $\int A(x)/B(x)^n$ with deg(A)<=deg(B).
- Since B itself is square free, its GCD with B'=dB/dx is 1. Thus we can compute c,e such that c*B+e*B'=1.
- Now multiply through by A to get cAB+eB'A=A, and substitute in the original $\int A(x)/B(x)^n$ to get $\int (cAB+eB'A)/B(x)^n$. Now dividing through we get $\int cA/B^{n-1} + \int eA^* (B'/B^n)$ where the second term can be integrated by parts to lower the power of B^n in the denominator.

(reminder: $\int u\, dv = uv - \int v\, du$. Let u =eA, dv=$B'/B^n$, $v=1/((1-n)B^{n-1})$, du=(eA)', so $\int v\, du$ comes out as $\langle\text{something}\rangle/B^{n-1} + \int \langle\text{something}\rangle/B^{n-1}$. Iterate until $B^n = B^1$.)

Resultant isn't a problem

- Well, actually it is something we'd prefer not to do since it takes a while, but we have algorithms for it.

- Conclusion: we can do rational function integration pretty well. But it took us into the mid-1980s to do it right.

The decision problem

- Liouville [1809-1882] During the period (1833-1841) presented a theory of integration; proved elliptic integrals cannot have elementary expressions.
- Various other writers advanced the subject in late 1800's
- J. Ritt (1948) *Integration in Finite Terms* Columbia Univ. Press
- M. Rosenlicht (AMM. 1972) *Integration in Finite Terms*

- R. Risch (1968) [unreadable]
- B. Trager, J. Davenport, M.Bronstein [implementation]

Risch's result

Theorem: Let $\mathcal{K}$ be a differential field and f be from $\mathcal{K}$. Then an elementary extension of the field $\mathcal{K}$ which has the same field of constants as $\mathcal{K}$ and contains an element g such that g' = f exists if and only if there exist constants $c_1..c_n$ from $\mathcal{K}$ and functions $u_0,..,u_n$ from $\mathcal{K}$ such that

$f = u_0' + \sum_{i=1,\ldots,n} c_i(u_i'/u_i)$

or

$g = \int f dx = u_0 + \sum_{i=1..n} c_i \log(u_i)$

note: this allows additional logs in the answer, but that's all. The structure of the integral is specified.

What's a Differential Field?

- Equip a field with an additional operator D which satisfies identities parallel to the rules for differentiation of functions. These structures obviously include various fields of functions (e.g. the field of rational functions in one variable over the real field).
- The goal is to provide a setting to answer concretely such questions as, "Does this function have an elementary antiderivative?"

Formally, start with a field F and a derivation map

- Derivation is a map of F into itself
  a $\mapsto$ a' such that
  (a+b)' = a' + b'
  (ab)'=a'b+ab'

CONSEQUENTLY
  $(a/b)' = (a'b-ab')/b^2$ if a,b, $\in$ F and $b \neq 0$
  $(a^n)' = na^{n-1}a'$ for all integers n
  1' =0; the constants F are a subfield of F

Also

- If *a, b* are in a differential field F with *a* being nonzero, we call *a* an exponential of *b* or *b* a logarithm of *a* if *b' = a'/a.*
- For algebraic extensions of F, there is a theorem that says that F is of characteristic zero and K is an algebraic extension field of F then the derivation on F can be extended uniquely to K.

We must also deal with algebraic pieces

- Algebraic extensions z where p(z)=0 in F
- Monomial extensions exp() and log(). Independent extensions are required e.g. $\log(x)$ and $\log(x^2)$ don't cut it since $(\log(x^2))-(2\log(x))$ is a constant (perhaps 0)

examples

- sometimes we prefer arctan (if a<0 below), sometimes logs.

$$\int \frac{1}{x^2+a}\,dx = \left\{ \frac{\arctan\left(\frac{x}{\sqrt{a}}\right)}{\sqrt{a}}, \frac{\log\left(\frac{x-\sqrt{-a}}{x+\sqrt{-a}}\right)}{2\sqrt{-a}} \right\}$$

- Risch converts $\int x^n \sin x$ to exponentials

$$-\frac{\int \left(i\,e^{2\,i\,x}-i\right)\,e^{n\,\log x - i\,x}\,dx}{2}$$

- and then can't do it..

Extensions of the Risch algorithm

- Allow certain functions beyond log, exp such that the form of the integral still holds. For example, $\mathrm{erf}(x) = (2/\pi)\int_0^x \exp(-t^2)dt$ has the appropriate structure.

$$\mathrm{risch}\left(x^2\,\mathrm{erf}\,x, x\right) = \frac{\sqrt{\pi}\,x^3\,\mathrm{erf}\,x + \left(x^2+1\right)\,e^{-x^2}}{3\,\sqrt{\pi}}$$

The Risch Algorithm: two steps back.

- There is a fallacy in claiming the Risch "algorithm" is an algorithm at all: it depends, for solution of subproblems, on heuristics to tell if certain expressions are equivalent to zero.
- We have gone over Richardson's arguments to show that the zero-equivalence problem over a class of expressions much smaller than that of interest for integration is recursively unsolvable.
- But that is not the problem with most implementations of the Risch algorithm. They *mostly have not been programmed completely*, because, even assuming you can solve the zero-equivalence problem, the procedure is *hard to program.*

The Risch Algorithm: answers are not ncessarily what you expect

- It is not nearly as useful as you might think, because it returns algebraic antiderivatives whose validity may be on a set of measure zero. Work by D. Jeffrey A. Rich (among others) on removing gratuitous discontinuities, is helpful.
- The Risch algorithm may also, in the vast majority of problems, simply say, after an impressive pause, nope. can't do it. There is no reasonable complexity analysis for the process, which is probably why certain authors ignore it.

The Risch Algorithm: answering wrong problem

- Also, most people are interested in definite, not indefinite integrals, at least once they've finished with Freshman calculus.
- And in cases where approximate solutions are easily obtained for the corresponding definite integral, the **Risch** algorithm may grind on for a long while and then say there is no closed form. Or if there is a closed form, if it is to be ultimately evaluated numerically, the closed form may be less useful than the quadrature formula!

Nevertheless, the Risch Algorithm fascinates us

- The theory of algebraic integration, and its corresponding history is interesting.
- The "solution" to calculus problems can probably be presented this way.
- It's a good advertising slogan.
- The definite integration problem is also a classic in the applied mathematics literature:
- It would seem that vast tables (20,000 entries) could be replaced with a computer program. (Actually Risch alg. is almost irrelevant here)

It also eludes us.

It is in some sense at the pinnacle of CAS algorithms: it uses more mechanisms of a computer algebra system than nearly any other program.

- Rational function manipulation (GCD, partial fractions, factoring)
- Simplification in a differential field (algebraic, exponential/log, complex extensions)
- Solving certain ODEs
- J. Moses, J. Davenport, B. Trager, D. Lazard, R. Rioboo, A. Norman, G. Cherry, M. Bronstein,

Go over examples of Risch in Moses' paper

- Detailed understanding of Risch's original presentation is a challenge.

How should one write an integration program? (Moses)

- Quick solution to easy problems
- A collection of methods and transformations
- Radical methods
- (less known, table lookup)
- consider approximate or numerical approaches

  expand integrand in taylor series, orthogonal polynomials, fourier series, or asymptotic series or even approximate as rational functions.

  (for rational functions), approximate the roots of the denominator and do partial fraction expansion

  do the whole task by quadrature (a well-studied area) treating the integrand as a "black box" capable only of returning a value at a point. [difficulties with infinite range or nasty behaviors.

Losing numerically

- Try integrating x*sin(x) between 0 and 5000.
- Numerically, it's tough.
- Symbolically it is –5000 cos (5000)+sin(5000)

Losing symbolically

- Sometimes the answer is possible but ugly. Consider ∫1/(z^{64}+1) dz integrated to

$$F(z) = \frac{1}{32}\sum_{k=1}^{16} c_k \,\mathrm{arctanh}\left(\frac{2c_k}{z+1/z}\right) - s_k \arctan\left(\frac{2s_k}{z-1/z}\right)$$

where $c_k := \cos((2k-1)\pi/64)$
and $s_k := \sin((2k-1)\pi/64)$.

- Computer algebra systems get forms that are worse than this (try them!) and often contribute other monstrosities more easily integrated numerically