LOGIC & AUTOMATA — ASSIGNMENT 2

Due: 10 February, 3pm, Room AT 2.13 or AT 2.17

Note: marks do **not** reflect difficulty.

1. (*7 marks*) Recall that wMSO stands for weak MSO, i.e. MSO with quantification over *finite* sets. Prove that there is an algorithm that converts MSO sentences over $\langle \mathbb{N}, succ \rangle$ into equivalent wMSO sentences. Here *succ* stands for the successor relation $\{(i, i+1) \mid i \in \mathbb{N}\}$.

2. (*8 marks*) Now we deal with MSO sentences over $\langle \mathbb{N}, succ \rangle$ of the following form
$$\exists X_1 \ldots \exists X_k \bigvee_i \alpha_i, \qquad (1)$$
where each α_i is an atomic formula, or a negation of an atomic formula. We assume the version of MSO that does not have first-order quantification but instead uses predicates $Sng(X)$, $X \subseteq Y$, and $succ(X, Y)$, as we did when converting S1S formulae into automata.

   Prove that there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ such that each MSO formula of the form (1) of size n can be converted into an equivalent wMSO formula of size at most $n^{p(n)}$.

3. (*7 marks*) Consider S2S formulae $\phi(x)$, i.e. MSO formulae over $\langle \{0,1\}^*, succ_0, succ_1 \rangle$ with one free first-order variable. Prove that for each such formula, the set $\{s \mid \phi(s) \text{ is true}\} \subseteq \{0,1\}^*$ is regular.

For the next 2 problems, we define a class of *prefix-recognizable* rewrite systems (and graphs) which are used as an abstraction of several scenarios of verification of systems with an infinite state-space.

Fix a finite alphabet Σ. Let Δ be a set of rules of the form
$$L \ \rightarrow \ L', \qquad L, L' \subseteq \Sigma^* \ \text{ are regular}$$

We write $u \ \rightarrow_\Delta \ v$ iff there are strings x, y, z such that

(a) $u = xy$

(b) $v = xz$

(c) $y \in L$ and $z \in L'$ for some $L \rightarrow L'$ in Δ.

We write $u \rightarrow^*_\Delta v$ if $u = v$ or if there is a sequence
$$u \rightarrow_\Delta v_0 \rightarrow_\Delta v_1 \rightarrow_\Delta \ldots \rightarrow_\Delta v_k \rightarrow_\Delta v$$
(i.e. for the reflexive-transitive closure of $\rightarrow_\Delta$) Finally, for each set S, define
$$post_\Delta(S) \ = \ \{s' \mid \exists s \in S : \ s \rightarrow^*_\Delta s'\} \ \subseteq \ \Sigma^*$$

In the following problems, you are expected to use Rabin's tree theorem and decidability of S2S.

4. (*6 marks*) Prove that if S is regular, then so is $post_\Delta(S)$.
   Hint: use the result of the previous problem (it is ok to use it even if you did not solve the previous problem).

5. (*4 marks*) Consider the structure $\langle \{0,1\}^*, \rightarrow^*_\Delta \rangle$. Prove that its MSO theory is decidable.