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Heights of Subvarieties over M-Fields

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Introduction

An important tool in diophantine geometry are heights. The height of a point measures the arithmetic complexity of its coordinates. Let us briefly recall its definition. Let P be a point of projective space with rational coordinates. We may assume that the coordinates $x_0, \ldots, x_n$ are entire and have no common divisor. Then the height of P is defined by $h(P) := \max_{j=0,\ldots,n} |x_j|$. Easily, this is generalized to points with coordinates in the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. For a projective variety X over a number field, we use an embedding (or more generally a morphism) φ into some projective space to define the height $h_\varphi := h \circ \varphi$ on $X(\bar{\mathbb{Q}})$. The basic question is how h_φ depends on the choice of φ. This is answered by the following theorem of Weil [We]: The height h_φ depends only on the isomorphy class of $\varphi^*\mathcal{O}_{\mathbb{P}^n}(1)$ up to bounded functions. Any element of the Picard group of X may be written as the difference of two elements of the form above. Therefore we get a homomorphism from the Picard group to the real functions on $X(\bar{\mathbb{Q}})$ modulo bounded functions.

Let Y be a subvariety of projective space. Then there is a canonical multi-homogeneous polynomial associated to Y called the Chow form (cf. Example 1.2 below). Nesterenko and Philippon have defined the height of Y as the height of the coefficient vector of the Chow form (cf. [Ph]). Using arithmetic intersection theory [GS], Faltings [Fa] defined equivalently the height of Y as an arithmetic degree of Y. This idea leads to many important properties of heights of subvarieties by translating the corresponding properties of the usual degree. As for points. we can define the height of a subvariety of any projective variety. In [Gu], Weil's theorem was generalized to heights of subvarieties. It says that the height h_φ on subvarieties depends only on the isomorphy class of $\varphi^*\mathcal{O}_{\mathbb{P}^n}(1)$ up to functions bounded by a multiple of the degree. Using Tate's limit argument, one can easily deduce Néron–Tate heights for subvarieties of an abelian variety (cf. [Gu]).

The theory of heights of points works not only in the number field case. We may use any ground field with a set of absolute values satisfying the product formula. Moreover, Vojta pointed out that in Nevanlinna theory, the characteristic function is an analogue of the height of points and the first

main theorem in Nevanlinna theory corresponds to Weil's theorem (cf. [Vo], [La2]). This leads to far reaching results and conjectures for number fields and in Nevanlinna theory.

Since Faltings' definition of the height of a subvariety uses arithmetic intersection theory, Weil's theorem for subvarieties was proved in the number field case. Here, we omit this restriction and we prove the theorem over very general ground fields. We do not assume that the product formula is satisfied. In particular, we get a generalization of the first main theorem in Nevanlinna theory.

To explain that, let K be the field of meromorphic functions on $\mathbb{C}$. For any $R > 0$, the closed disc M_R of radius R induces a set of absolute values on K. To be more precise, assume that v is in the interior of the disc, then the order in v induces a discrete valuation on K. If v is on the boundary and $f \in K$, then we define $|f|_v := |f(v)|$. For fixed f, this is only defined up to finitely many v. Strictly speaking, $|\ |_v$ is not an absolute value, but the axioms are true if we skip a null set on the boundary. Let us fix a holomorphic map P from $\mathbb{C}$ into a complex projective variety X. Then P may be viewed as a K-valued point of X and a similar construction as in the number field case leads to a global height $h_R(P)$. To be more precise, let L be a line bundle on X with hermitian metric $\|\ \|$ and non-zero meromorphic sections s. Then the local height of P in an element v of the boundary is $-\log \|s(P(v))\|$. If v is not on the boundary of M_R, then we have a canonical local height given by the order of $s \circ P$ in v. Integrating the local heights with respect to a suitable measure on M_R, we get $h_R(P)$. The first main theorem says that $h_R(P)$ depends only on the isomorphy class of L up to $O(1)$ with respect to R and fixed P. Using our general theory, we can replace P by any subvariety of X defined over the algebraic closure of K. In particular, it applies to any holomorphic map from a finite covering of $\mathbb{C}$ to X. For a discussion of the second main theorem in Nevanlinna theory of coverings, the reader is refered to a paper of Cherry [Ch].

The paper is organized in the following way. In the first section, we introduce local heights of complete varieties X over K. Here, the field K is assumed to be complete with respect to a given absolute value. The local heights are well-known in the archimedean case and also for discrete valuations. In the first case, the local height of X is given by the $*$-product of Green forms [GS] for hermitian line bundles. If the absolute value is discrete, then the local height of X is an intersection number of line bundles on a model of X over the valuation ring. It will be shown in a subsequent paper that one can define a similar intersection number if the non-archimedean absolute value is not discrete. To include this case, we use an axiomatic approach to the theory of local heights for subvarieties. To get estimates for change of metrics, we have to impose a positivity condition on the hermitian line bundles. In the archimedean case, this means semi-positive curvature and for discrete valuations, we use base-point-free line bundles on the models. Similarly as in measure theory, we can canonically complete a given theory of local heights. The resulting theory is closed under uniform limits of metrics. It has the advantage that the canonical local heights for subvarieties of an abelian variety are contained in the theory. This was pointed out by Zhang in [Zh] using similar ideas as Néron in the classical case.

In the second section, we introduce the notion of M-fields. This is a field K together with a measure space M. Moreover, a family $|\ |_v$, $v \in M$, is given which may be viewed as absolute values on K. The difference with the usual absolute values is, that for given arguments, they are defined and satisfy the axioms only for almost all $v \in M$. It is assumed that for non-zero $f \in K$, $\log |f|_v$ is integrable on M. The prototype of an M-field is the field of meromorphic functions on $\mathbb{C}$ with measure space M_R. It will be proved that the algebraic closures of $\mathbb{Q}$ and of the field of meromorphic functions on $\mathbb{C}$ are canonically M-fields. At the end of the section, M-bounded subsets are introduced. The standard example for an M-bounded subset is the set of rational points in a complete variety. If all members of the family $|\ |_v$, $v \in M$, are indeed absolute values, then this concept is a straightforward generalization of the usual theory of bounded sets (cf. [La1]). We have simply to replace the usual bounds C_v, which have to be zero up to finitely many places v, by an integrable bound C_v. If not all $|\ |_v$ are absolute values, then we have to deal with some technical problems. The idea is to reduce to the case above by passing to a sufficiently large, finitely generated subfield K' of K.

The third section deals with global heights of subvarieties over an M-field K. Let $L_0, \dots, L_t$ be base-point free line bundles on a complete K-variety X with suitable metrics and non-zero meromorphic sections $s_0, \dots, s_t$. Then the global height of a t-dimensional subvariety Y of X with respect to these data is the integral over M of the corresponding local heights. This does not necessarily exist, but using morphisms to multi-projective space, we deduce that there are suitable metrics such that the global height is defined for all subvarieties Y intersecting $\mathrm{div}(s_0), \dots, \mathrm{div}(s_t)$ properly. In Theorem 3.13, we give the generalization of Weil's theorem to this situation. Since we do not assume that the product formula is satisfied, the global height depends on the choice of $s_0, \dots, s_t$. However, we prove a formula which describes the dependence on the meromorphic sections in terms of the defect of product formula. If K is a number field, then the global heights agree with the absolute version of Faltings' definition. Finally, the generalization of the first main theorem in Nevanlinna theory is given.

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1. Local Heights

Let K be a field with a complete absolute value $|\ |$. It has a unique continuation to an absolute value on the algebraic closure $\bar{K}$. In this paper, a K-variety is an irreducible and reduced scheme X of finite type and separated over K.

1.1) First, we recall basic concepts about metrics and boundedness. For the latter, the basic reference is [La1]. The reader may also consult the end of section 2.

Let L be a line bundle on the complete K-variety X. Recall that a metric $\|\ \|$ on L is a family $(\|\ \|_x)_{x \in X(\bar{K})}$ where the non-negative function $\|\ \|_x$ on the fibre L_x is mapped to the absolute value $|\ |$ under a suitable $\bar{K}$-isomorphism $L_x \tilde{\to} \bar{K}$. Note that we have made no continuity assumptions. As usual, we define the tensor product and pull back of metrized line bundles.

The metric $\|\ \|$ on L is called bounded if for any affine open subset U of X such that L is trivial on U the following is satisfied: For any $s \in L(U)$ without zeros in $U(\bar{K})$, the function $\log \|s\|$ is bounded on bounded subsets of $U(\bar{K})$. (Here, U may be viewed as a closed subscheme of an affine space and it makes sense to speak about bounded subsets of $U(\bar{K})$.) In the terminology of [La1], such a metric should be called locally bounded but since X is complete, locally bounded and globally bounded are the same. In particular, if L is trivial on X and s is a global section without zeros, then $\log \|s\|$ is a bounded function on $X(\bar{K})$.

Let $\|\ \|, \|\ \|'$ be two metrics on the line bundle L. For any $x \in X(\bar{K})$, let s be a local section of L non-vanishing in x. Then

$$\|\ \|' / \|\ \| := \|s\|' / \|s\|$$

does not depend on the choice of s. Therefore it gives rise to a globally defined real function on $X(\bar{K})$. If the metrics are bounded, then the above shows that $\|\ \|' / \|\ \|$ is a bounded function on $X(\bar{K})$.

Example 1.2. On a projective space $\mathbb{P}^n$, we always fix a set of coordinates $x_0, \dots, x_n$ without mention. For any closed t-dimensional subvariety X of $\mathbb{P}^n_K$, we have the Chow form $F_X(\xi_0, \dots, \xi_t)$. It is a multi-homogeneous polynomial of multi-degree $(\deg X, \dots, \deg X)$ in the dual coordinates. If K is algebraically closed, then F_X is up to multiples determined by

$$\{F_X = 0\} = \left\{(\xi_0, \dots, \xi_t) \in (\check{\mathbb{P}}^n_K)^{t+1}; X \cap \xi_0 \cap \dots \cap \xi_t = \emptyset\right\}.$$

Here $\xi_0, \dots, \xi_t$ in the dual projective space $\check{\mathbb{P}}^n_K$ are viewed as hyperplanes in $\mathbb{P}^n_K$. In general, one can use base change to $\bar{K}$. The coefficients of F_X are in K.

Now we assume that the absolute value on K is non-archimedean with valuation ring K^o. We define a bounded metric on $O_{\mathbb{P}^n}(1)$ by

$$\|s(x)\| := |s(x)| \,/\, \max_j |x_j|$$

for any global section s of $O_{\mathbb{P}^n}(1)$. It is called the standard metric and the corresponding metrized line bundle is denoted by $\overline{O_{\mathbb{P}^n}(1)}$. The transformations of coordinates not changing this metric are induced by the group

$$G := \{A \in GL(n+1, K^o); |\det A| = 1\} .$$

Let $|F_X|$ be the maximum of the absolute values of the coefficients. Then $|F_X|$ is invariant under change of coordinates induced by G. We conclude that

$$-\log(|F_X(\xi_0, \ldots, \xi_t)| \,/\, |F_X|)$$

is invariant under G. It is called the local height with respect to the metrized line bundle $\overline{O_{\mathbb{P}^n}(1)}$ and the global sections $\xi_0, \ldots, \xi_t$. Note that it is a real number if and only if $\xi_0 \cap \cdots \cap \xi_t \cap X = \emptyset$.

Example 1.3. We still assume that the given absolute value on K is non-archimedean. Let $\mathbb{P} := \mathbb{P}^{n_0} \times \cdots \times \mathbb{P}^{n_t}$ be a multi-projective space (with a fixed set of coordinates on each factor). The pull back of $O_{\mathbb{P}^{n_j}}(1)$ under the j-th projection is denoted by $O_{\mathbb{P}}(e_j)$. Moreover, if we use the standard metric on $O_{\mathbb{P}^{n_j}}(1)$, we get a metrized line bundle $\overline{O_{\mathbb{P}}(e_j)}$.

Let X be a t-dimensional closed subvariety of P_K. The Chow form $F_X(\xi_0, \ldots, \xi_t)$ of X is a multi-homogeneous polynomial of degree d_i in the variables $\boldsymbol{\xi}_i = (\xi_{i0}, \ldots, \xi_{in_i})$. Here d_i is the degree of X with respect to the line bundles $(O_{\mathbb{P}}(e_j))_{j \in \{0,\ldots,t\}, j \neq i}$. If K is algebraically closed, then F_X is determined up to multiples by

$$\{F_X = 0\} = \{(\xi_0, \ldots, \xi_t) \in \check{\mathbb{P}}_K; X \cap \xi_0 \cap \cdots \cap \xi_t = \emptyset\} .$$

As in Example 2, the local height

$$-\log(|F_X(\xi_0, \ldots, \xi_t)| \,/\, |F_X|)$$

of X with respect to $\left(\overline{O(e_j)}, \xi_j \in \Gamma(\mathbb{P}, O(e_j))\right)_{j=0,\ldots,t}$ depends only on the metric and not on the specific choice of coordinates.

1.4) Next, we want to recall the $*$-product of Green currents. The main reference is [GS]. As we are interested in local heights with respect to line bundles, we may assume that one Green current belongs to a Cartier divisor.

Let X be a compact complex manifold and let Y be a closed subvariety of X. We assume that X is irreducible and reduced. Let n be the dimension of X and let p be the codimension of Y. We have differential operators d, d^c [GH, p. 109] acting on differential forms and currents. For any $2n-2p$ form η, we define

$$\delta_Y(\eta) := \int_Y \eta .$$

This gives a current of bidegree (p,p). If ω is a L^1-form of type (p,p), then it may be viewed as a current $[\omega]$ of type (p,p) by

$$[\omega](\eta) := \int_X \omega \wedge \eta .$$

A Green current for Y is a current g of bidegree $(p-1, p-1)$ such that we have a smooth (p,p)-form $\omega(g)$ with

$$dd^c g = [\omega(g)] - \delta_Y .$$

Now let L be a holomorphic line bundle with a smooth hermitian metric $\| \ \|$. If s is a non-zero meromorphic section of L, then $[\log \|s\|^{-2}]$ is a Green current for $\mathrm{div}(s)$ and

$$\omega([\log \|s\|^{-2}]) = c_1(L, \| \ \|) .$$

This is the Poincaré-Lelong equation. Suppose that $\mathrm{div}(s)$ intersects Y properly. For any $2n-2p$ form η, let

$$(\log \|s\|^{-2} \wedge \delta_Y)(\eta) := \int_Y \log \|s\|^{-2} \eta$$

and

$$(c_1(L, \| \ \|) \wedge g)(\eta) := g(c_1(L, \| \ \|) \wedge \eta) .$$

By linearity, we extend all definitions to cycles instead of subvarieties. The star product of $[\log \|s\|^{-2}]$ with the Green current g for Y is given by

$$[\log \|s\|^{-2}] * g := \log \|s\|^{-2} \wedge \delta_Y + c_1(L, \| \ \|) \wedge g .$$

It is a Green current for $\mathrm{div}(s).Y$ with

$$\omega([\log \|s\|^{-2}] * g) = c_1(L, \| \ \|) \wedge \omega(g) .$$

This $*$-product should be viewed as an analogue of the action of Cartier divisors on cycles in intersection theory [Fu]. It has the same properties if we pass to equivalence classes modulo the ranges of d and d^c. So up to now, we consider only such equivalence classes of Green currents and if two currents T_1, T_2 are equivalent, we write $T_1 \equiv T_2$.

If T is a current of degree $2n$, then we define

$$\langle T|X\rangle := \frac{1}{2}T(1) \in \mathbb{C}$$

where 1 is the constant function on X. For our purposes, the normalization of d^c is a little bit inconvenient. It leads to the fact that $[\log\|s\|^{-2}]$ is a Green current for $\mathrm{div}(s)$ instead of $[-\log\|s\|]$. This explains the $\frac{1}{2}$ in the definition above. If X is a compact complex space, irreducible and reduced, then the above can be generalized using either Hironaka's resolution of singularities or working with differentiable forms on singular spaces [BH].

Now suppose that X is a compact Kähler manifold with Kähler form ω. The Kähler metric gives rise to harmonic forms on X. For a cycle Y, there is a unique Green current g_Y vanishing on all harmonic forms and with $\omega(g_Y)$ harmonic. Uniqueness means up to the ranges of d and d^c since a Green current is viewed as an equivalence class. We call g_Y the Arakelov current for Y. It was introduced in [Fa].

1.5) Assume now $K = \mathbb{C}$ with the Euclidean absolute value. Let $\mathbb{P} := \mathbb{P}^{n_0} \times \cdots \times \mathbb{P}^{n_t}$ and let X be a t-dimensional closed subvariety of $\mathbb{P}_{\mathbb{C}}$. We denote by $\check{\mathbb{P}}^{n_j}$ the dual projective space, it is the space of hyperplanes in $\mathbb{P}^{n_j}$. For $\check{\mathbb{P}} := \check{\mathbb{P}}^{n_0} \times \cdots \times \check{\mathbb{P}}^{n_t}$, let p_1, p_2 be the projections of $\mathbb{P}_{\mathbb{C}} \times \check{\mathbb{P}}_{\mathbb{C}}$ onto the factors. For $\alpha = 0, \ldots, t$, there is a canonical global section F_α of $p_1^* O_{\mathbb{P}}(e_\alpha) \otimes p_2^* O_{\check{\mathbb{P}}}(e_\alpha)$. If $x_0, \ldots, x_{n_\alpha}$ are coordinates on $\mathbb{P}^{n_\alpha}$ and $\xi_0, \ldots, \xi_{n_\alpha}$ are dual coordinates, then

$$F_\alpha(\boldsymbol{x}, \boldsymbol{\xi}) = x_0\xi_0 + \cdots + x_{n_\alpha}\xi_{n_\alpha} .$$

In fact, the Chern form F_X is defined by

$$\mathrm{div}(F_X) = p_{2*}\big(\mathrm{div}F_0 \ldots \mathrm{div}F_t . p_1^*(X)\big) .$$

On $O_{\mathbb{P}^{n_j}}(1)$, we have the Fubini-Study metric. Using pull back and tensor product, we get a standard metric on any line bundle M of multi-projective space and we denote the metrized line bundle by $\overline{M}$. Recall that g_Y denotes the Arakelov current for a closed subvariety Y. From [So], we get

Lemma 1.6. i) $g_{\mathrm{div}F_\alpha} \equiv \left[\log\|F_\alpha\|^{-2}\right] - \sum_{j=1}^{n_\alpha} \frac{1}{j}$;

ii) $g_{\mathrm{div}F_X} \equiv p_{2*}\big(p_1^* g_X * g_{\mathrm{div}F_0} * \cdots * g_{\mathrm{div}F_t}\big)$.

The next result is a translation of [So, Théorème 3] to the local archimedean case.

Proposition 1.7. For $j = 0, \ldots, t$, let s_j be a non-zero global section of $O_{\mathbb{P}_{\mathbb{C}}}(e_j)$. We suppose that

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap X = \emptyset .$$

Let S be the product of the unit spheres in $\mathbb{C}^{n_i+1}$ $(i = 0, \ldots, t)$. On S, let dP be the product of the Lebesque probability measures. Then we have

$$\begin{aligned} &\langle [\log\|s_0\|^{-2}] * \cdots * [\log\|s_t\|^{-2}] \mid X\rangle \\ &= \int_S \log|F_X(\boldsymbol{\xi})|\, dP(\boldsymbol{\xi}) \\ &\quad - \log|F_X(\boldsymbol{s}_0, \ldots, \boldsymbol{s}_t)| \\ &\quad + \frac{1}{2}\sum_{i=0}^{t} \delta_i(X) \sum_{j=1}^{n_i} \frac{1}{j} \end{aligned}$$

where $\boldsymbol{s}_i$ is the vector of coefficients of s_i and where $\delta_i(X)$ is the degree of X with respect to the line bundles $\big(O_{\mathbb{P}}(e_j)\big)_{j \neq i}$.

Proof: We want to normalize the vector $\boldsymbol{s}_i = (s_{i0}, \ldots, s_{in_i})$ such that it has L_2-norm 1. For example, if we replace s_0 by $s_0 \,/\, \big(\sum_{j=0}^{n_i} |s_{0j}|^2\big)^{1/2}$, then the left hand side of the claim changes by

$$\langle \log\Big(\sum_{j=0}^{n_i} |s_{0j}|^2\Big) * \log\|s_1\|^{-2} * \cdots * \log\|s_t\|^{-2} \,|X\rangle = \frac{1}{2}\log\Big(\sum_{j=0}^{n_i} |s_{0j}|^2\Big)\delta_0(X) .$$

Since the Chow form is multi-homogeneous of degree $\big(\delta_0(X), \ldots, \delta_t(X)\big)$, the right hand side changes by the same quantity. Therefore, we may assume that the vector of coefficients of all s_i have L_2-norm 1.

Let $\boldsymbol{s}$ be the point of $\check{\mathbb{P}}_{\mathbb{C}}$ corresponding to $(\boldsymbol{s}_0, \ldots, \boldsymbol{s}_t)$. The restriction of $\log\|F_\alpha\|^{-2}$ to $\mathbb{P}_{\mathbb{C}} \times \{\boldsymbol{s}\}$ is equal to $\log\|s_\alpha\|^{-2}$. Together with the harmonicity of the first Chern forms, we obtain

$$\begin{aligned} &\langle [\log\|s_0\|^{-2}] * \cdots * [\log\|s_t\|^{-2}] | X\rangle \\ &= \langle [\log\|s_0\|^{-2}] * \cdots * [\log\|s_t\|^{-2}] * g_X | \mathbb{P}_{\mathbb{C}}\rangle \\ &= \langle [\log\|F_0\|^{-2}] * \cdots * [\log\|F_t\|^{-2}] * p_1^* g_X | \mathbb{P}_{\mathbb{C}} \times \{\boldsymbol{s}\}\rangle \\ &= \frac{1}{2}\big([\log\|F_0\|^{-2}] * \cdots * [\log\|F_t\|^{-2}] * p_1^* g_X * p_2^* g_{\boldsymbol{s}}\big)(1) . \end{aligned}$$

Here, we have used that for a Green current g with harmonic $\omega(g)$, we have

$\langle g|Y\rangle = \frac{1}{2}(g * g_Y)(1)$. Note that

$$\Big([\sum_{j=1}^{n_0}\frac{1}{j}] * [\log\|F_1\|^{-2}] * \cdots * [\log\|F_t\|^{-2}] * p_1^* g_X * p_2^* g_S\Big)(1)$$
$$= \int_{\mathbb{P}_{\mathbb{C}}\times\check{\mathbb{P}}_{\mathbb{C}}}\Big(\sum_{j=1}^{n_0}\frac{1}{j}\Big)(c_1+\check{c}_1)\wedge\cdots\wedge(c_t+\check{c}_t)\wedge\omega_X\wedge\omega_S$$
$$= \delta_0(X)\sum_{j=1}^{n_0}\frac{1}{j}\,.$$

where $c_i, \check{c}_i$ are the pull-backs of $c_1(\overline{O_{\mathbb{P}}(e_i)})$, $c_1(\overline{O_{\check{\mathbb{P}}}(e_i)})$ to $\mathbb{P}_{\mathbb{C}}\times\check{\mathbb{P}}_{\mathbb{C}}$ and ω_X, ω_S are the pull backs of $\omega(g_X)$ and $\omega(g_S)$.

By Lemma 6 i), we get

$$\langle[\log\|s_0\|^{-2}] * \cdots * [\log\|s_t\|^{-2}] \mid X\rangle$$
$$= \frac{1}{2}(g_{\mathrm{div} F_0} * \cdots * g_{\mathrm{div} F_t} * p_1^* g_X * p_2^* g_S)(1)$$
$$+ \frac{1}{2}\sum_{i=0}^{t}\delta_i(X)\sum_{j=1}^{n_i}\frac{1}{j}\,.$$

By Lemma 6 ii) and projection formula, the first summand equals

$$\frac{1}{2}(g_{\mathrm{div} F_X} * g_S)(1) = \langle g_{\mathrm{div} F_X} \mid \{s\}\rangle\,.$$

Let now $\check{c}_i = c_1(\overline{O_{\check{\mathbb{P}}}(e_i)})$. Then we have

$$[\log\|F_X\|^{-2}] - g_{\mathrm{div} F_X}$$
$$\equiv \Big[\int_{\check{\mathbb{P}}_{\mathbb{C}}}\log\|F_X\|^{-2}\,\check{c}_0^{n_0}\wedge\cdots\wedge\check{c}_t^{n_t}\Big]$$
$$\equiv \Big[\int_S \log|F_X(\xi)|^{-2}\,dP(\xi)\Big]\,.$$

We obtain

$$\langle g_{\mathrm{div} F_X} \mid \{s\}\rangle = \int_S \log|F_X(\xi)|\,dP(\xi) - \log|F_X(s)|\,.$$

This leads immediately to the claim. □

Up to now, the complete absolute value $|\ |$ on K may be either archimedean or non-archimedean. Fix $t \in \mathbb{N}$.

We use the following notation. Let $Z_1, \dots, Z_r$ be pure dimensional cycles on the algebraic variety X and let p_i be the codimension of Z_i in X. The support of Z_i is denoted by $|Z_i|$. We say that $Z_1, \dots, Z_r$ intersect properly if for any subset $I \subset \{1, \dots, r\}$ every irreducible component of $\cap_{i\in I} Z_i$ has codimension $\sum_{i\in I} p_i$. The support of a Cartier divisor is denoted by $|D|$.

Definition 1.8. A theory of local heights for t-dimensional varieties consists of the following data: For any complete algebraic K-variety X of dimension t, there is a submonoid $\mathfrak{g}_X^+$ of the group of isometry classes of line bundles with bounded metrics. It is required that $\mathfrak{g}^+$ is closed under pull back. For non-zero meromorphic sections s_j of $\hat{L}_j \in \mathfrak{g}_X^+$ such that $|\mathrm{div}(s_0)|, \dots, |\mathrm{div}(s_t)|$ intersect properly, we assume that there is a local height

$$\lambda_{(\hat{L}_0,s_0),\dots,(\hat{L}_t,s_t)}(X) \in \mathbb{R}$$

satisfying the following properties.

i) The local height is multi-linear and symmetric in $(\hat{L}_j, s_j)_{j=0,\dots,t}$.

ii) If $\varphi : X' \to X$ is a morphism of complete t-dimensional K-varieties, then the projection formula

$$\lambda_{(\hat{L}_0,s_0),\dots,(\hat{L}_t,s_t)}(\varphi_*(X')) = \lambda_{\varphi^*(\hat{L}_0,s_0),\dots,\varphi^*(\hat{L}_t,s_t)}(X')$$

holds. Here, we extend the local heights to all cycles by linearity.

iii) We consider a second metric $\|\ \|'$ on L_0 with resulting metrized line bundle $\hat{L}_0' \in \mathfrak{g}_X^+$. Let C_- be the infimum and let C_+ be the supremum of $\log(\|\ \|'/\|\ \|)$ on $X(\bar{K})$ (cf. 1)). Then

$$C_- \deg_{L_1,\dots,L_t}(X) \le \lambda_{(\hat{L}_0,s_0),\dots,(\hat{L}_t,s_t)}(X) - \lambda_{(\hat{L}_0',s_0),\dots,(\hat{L}_t,s_t)}(X)$$
$$\le C_+ \deg_{L_1,\dots,L_t}(X)$$

where $\deg_{L_1,\dots,L_t}(X)$ is the degree of X with respect to $L_1, \dots, L_t$.

iv) Let s_0' be another non-zero meromorphic section of L_0 such that $|\mathrm{div}(s_0')|, |\mathrm{div}(s_1)|, \dots, |\mathrm{div}(s_t)|$ intersect properly. Then

$$\lambda_{(\hat{L}_0,s_0),\dots,(\hat{L}_t,s_t)}(X) - \lambda_{(\hat{L}_0,s_0'),\dots,(\hat{L}_t,s_t)}(X)$$
$$= \log\Big|\Big(\frac{s_0'}{s_0}\Big)(\mathrm{div} s_1 \dots \mathrm{div} s_t)\Big|\,.$$

(Here, for $f \in K(X)$ and $P_j \in X(\bar{K})$, we define $f(\sum_j n_j P_j) := \prod_j f(P_j)^{n_j}$.)

v) Let $\mathbb{P} := \mathbb{P}^{n_0} \times \cdots \times \mathbb{P}^t$ be a multi-projective space. We use the standard metrics on the line bundles $O_{\mathbb{P}}(e_j)$. Then the restriction of the metrized line bundle $\overline{O_{\mathbb{P}}(e_j)}$ to the closed subvariety X is in $\mathfrak{g}_X^+$. For $j = 0, \dots, t$, let s_j be a global section of $O_{\mathbb{P}}(e_j)$. In the notation of Example 3 and Proposition 7, the local height of X with respect to $(\overline{O_{\mathbb{P}}(e_j)}, s_j)_{j=0,\ldots,t}$ equals

$$\log|F_X| - \log|F_X(s_0,\ldots,s_t)|$$

in the non-archimedean case and

$$\int_S \log|F_X(\xi)|\, dP(\xi) - \log|F_X(s_0,\ldots,s_t)| + \frac{\log|e|}{2}\sum_{i=0}^{t}\delta_i(X)\sum_{j=1}^{n_i}\frac{1}{j}$$

in the archimedean case.

Note that $\deg_{L_1,\ldots,L_t}(X) \geq 0$ for any $\hat{L}_1,\ldots,\hat{L}_t \in \mathfrak{g}_X^+$. To prove this, we may assume that X is projective (Chow Lemma) and $t > 0$. We argue by contradiction. If $\deg_{L_1,\ldots,L_t}(X) < 0$, then it follows from iii) that the metric of $\hat{L} \in \mathfrak{g}_X^+$ is unique up to tensor product with O_X and a multiple of the trivial metric. Using an embedding into projective space $\mathbb{P}^n$, this has to be true for the restriction of $\overline{O_{\mathbb{P}^n}(1)}$ to X. By a change of coordinates on $\mathbb{P}^n$, we get easily a contradiction.

Proposition 1.9. Let $(\mathfrak{g}^+,\lambda)$ be a theory of local heights. For a complete K-variety X, let $\mathfrak{P}_X^+$ be the set of isometry classes of metrized line bundles of the form $\varphi^*\overline{O_{\mathbb{P}^n}(1)}$ where $\varphi: X \to \mathbb{P}_K^n$ is a K-morphism and $O_{\mathbb{P}^n}(1)$ has the standard metric. Then $\mathfrak{P}_X^+$ is a submonoid of $\mathfrak{g}_X^+$ and λ induces a theory of local heights $(\mathfrak{P}_X^+,\lambda)$ which is independent of $(\mathfrak{g}_X^+,\lambda)$.

Proof: Let X be a complete K-variety and for $j = 0,\ldots,t$, let s_j be a non-zero meromorphic section of $\hat{L}_j \in \mathfrak{P}_X^+$. It follows from v) that $\mathfrak{P}_X^+$ is a subset of $\mathfrak{g}_X^+$. Using Segre embeddings, it follows that $\mathfrak{P}_X^+$ is a monoid. We have to show that the local height $\lambda(X)$ with respect to $(\hat{L}_j, s_j)_{j=0,\ldots,t}$ is canonically determined by i) - v) of Definition 8. By iv) we may assume that it exists a K-morphism φ from X into a multi-projective space $\mathbb{P} := \mathbb{P}^{n_0}\times\cdots\times\mathbb{P}^{n_t}$ such that $\hat{L}_j = \varphi^*\overline{O_{\mathbb{P}}(e_j)}$ and that s_j is the pull back of a global section of $O_{\mathbb{P}}(e_j)$ for all j. Using ii) and v), we get the claim. □

On a complex variety X, a line bundle $\hat{L}$ with smooth hermitian metric is said to have semi-positive curvature if and only if $\varphi^*(\hat{L})$ has semi-positive curvature for all morphisms φ from smooth complex varieties to X.

Theorem 1.10. In the archimedean case, there is a theory of local heights for t-dimensional varieties over K. For convenience, let $K = \mathbb{C}$ with the usual absolute value. For a proper variety X over K, let $\mathfrak{g}_X$ be the group of isometry classes of line bundles with smooth hermitian metrics. Let $\mathfrak{g}_X^+$ be the subset of elements with semi-positive curvature. For $j = 0,\ldots,t$, let s_j be a non-zero meromorphic section of $\hat{L}_j \in \mathfrak{g}_X^+$. If $|\mathrm{div}(s_0)|,\ldots,|\mathrm{div}(s_t)|$ intersect properly, then we define

$$\lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(X) := \left\langle \left[\log\|s_0\|^{-2}\right] * \cdots * \left[\log\|s_t\|^{-2}\right] \mid X\right\rangle .$$

Then $(\mathfrak{g}^+,\lambda)$ is a theory of local heights.

Proof: Obviously, $\mathfrak{g}_X^+$ is a submonoid of $\mathfrak{g}_X$. We have to check the properties i) - v) in Definition 8. Clearly, i) and ii) are satisfied. To prove iii), note that $\rho := \log(\|\ \|' / \|\ \|)$ is a C^∞-function on the complete K-variety X. Using multi-linearity of $*$-product, we get

$$\begin{aligned}&\left\langle \left[\log\|s_0\|^{-2}\right] * \cdots * \left[\log\|s_t\|^{-2}\right] \mid X\right\rangle \\ &\quad - \left\langle \left[\log\|s_0\|'^{-2}\right] * \cdots * \left[\log\|s_t\|^{-2}\right] \mid X\right\rangle \\ &\quad = \int_X \rho\, c_1(\hat{L}_1)\wedge\cdots\wedge c_1(\hat{L}_t) .\end{aligned}$$

By positivity of the Chern forms, we get iii).

Next, we prove iv). We have $g := \log\|s_0'\| - \log\|s_0\| = \log|s_0'/s_0|$. Using $\omega(g) = 0$, iv) is a consequence of

$$\begin{aligned}&\left\langle \left[\log\|s_0\|^{-2}\right] * \cdots * \left[\log\|s_t\|^{-2}\right] \mid X\right\rangle \\ &\quad - \left\langle \left[\log\|s_0'\|'^{-2}\right] * \cdots * \left[\log\|s_t\|^{-2}\right] \mid X\right\rangle \\ &\quad = \langle g \mid \mathrm{div}(s_1)\ldots\mathrm{div}(s_t)\rangle .\end{aligned}$$

Finally, v) is just Proposition 7. □

Remark 1.11. To prove Theorem 10, we have only used the basic properties of the action $\left[\log\|s\|^{-2}\right] * g$. In [GS], these rules are proved under the hypothesis that X is an algebraic variety. However, the definitions and proofs can be generalized to the case of compact complex spaces. Then Theorem 10 remains true if we replace complete K-varieties by compact complex spaces which are irreducible and reduced. Details will be given in a joint book with E. Bombieri.

We need an analogue of Proposition 7 in the non-archimedean case. As the $*$-product is replaced by intersection product on models over the valuation ring K^o, we have to assume that the valuation is discrete.

Proposition 1.12. Assume that $|\ |$ is a discrete absolute value on K with uniform parameter π_K. Let $\mathbb{P} := \mathbb{P}^{n_0}\times\cdots\times\mathbb{P}^{n_t}$ be a multi-projective space. For $j = 0,\ldots,t$, let s_j be a global section of $O_{\mathbb{P}_K}(e_j)$. Let X be a t-dimensional closed subvariety of $\mathbb{P}_K$ such that

$$|\mathrm{div}(s_0)| \cap\cdots\cap |\mathrm{div}(s_t)| \cap X = \emptyset .$$

Using intersection product on the model $\mathbb{P}_{K^o}$ over K^o, we have in the notation of Example 3 and Proposition 7

$$-\log|\pi_K|\deg\big(\operatorname{div}(s_0)\ldots\operatorname{div}(s_t).\bar{X}\big) = \log|F_X| - \log|F_X(s_0,\ldots,s_t)| \ .$$

Proof: We use the same notation as in the proof of Proposition 7. If we replace s_0 by $s_0/|s_0|$ where $|s_0| := \max_{j=0,\ldots,n_0}|s_{0j}|$, then the left hand side of the claim changes by $\log|s_0|\,\delta_0(X)$ (use projection formula) and so does the right hand side. Therefore we may assume that $|s_i| = 1$ for all i. Using the projection formula two times and a Chow form F_X' of Gauss norm 1, we get

$$\begin{aligned}\deg&\big(\operatorname{div}(s_0)\ldots\operatorname{div}(s_t).\bar{X}\big)\\ &= \deg\big(\operatorname{div}(F_0)\ldots\operatorname{div}(F_t).p_1^*\bar{X}.p_2^*\{s\}\big)\\ &= \deg\big(\operatorname{div}F_X'.\{s\}\big)\\ &= \big(-\log|F_X(s)| + \log|F_X|\big)/(-\log|\pi_K|) \ .\end{aligned}$$

□

Lemma 1.13. Let $\bar{X}$ be a proper flat variety over the complete discrete valuation ring K^o. Suppose that D is a vertical Cartier divisor on $\bar{X}$, i.e. the restriction of D to the generic fibre X of $\bar{X}$ is equal to the zero divisor. For all $x \in X(\bar{K})$, we assume that $|f(x)| \leq 1$ where D is given by a rational function f on a neighbourhood of the closure of x in $\bar{X}$. Then the Weil divisor associated to D is effective.

Proof: Let $\bar{X}'$ be the normalization of $\bar{X}$. Then $\bar{X}'$ is finite over $\bar{X}$. So we may assume that $\bar{X}$ is normal. Let $m(D,W)$ be the order of D in the irreducible component W of the special fibre of $\bar{X}$. Since $\mathcal{O}_{\bar{X},W}$ is a normal local ring of dimension 1, it is a discrete valuation ring. Let π_W be a generator of the maximal ideal. There is $x \in X(\bar{K})$ with specialization $w \in W$ such that π_W is regular in w. Suppose that D is given by the rational function f on a neighbourhood of w in $\bar{X}$. By construction, we have $|\pi_W(x)| < 1$. It follows from

$$|\pi_W(x)|^{m(D,W)} = |f(x)| \leq 1$$

that $m(D,W) \geq 0$. This proves the claim. □

It was noted by Zhang [Zh] that in arithmetic intersection theory, the height of a horizontal cycle depends only on the metrics on the line bundles and not on the particular model. The proof of the next theorem is similar to his considerations.

Theorem 1.14. Let K be a complete field with respect to the discrete absolute value $|\ \ |$. On the algebraic closure $\bar{K}$ of K, we consider the canonical extension of $|\ \ |$. For a complete $\bar{K}$-variety X, let $\mathfrak{P}_X^+$ be the isometry classes of metrized line bundles of the form $\varphi^*\overline{O_{\mathbb{P}^n}(1)}$ where $\varphi : X \to \mathbb{P}^n_{\bar{K}}$ is a morphism. Then there is a unique local height λ such that $(\mathfrak{P}^+, \lambda)$ is a theory of local heights for t-dimensional varieties over $\bar{K}$.

Proof: Uniqueness is clear from Proposition 9. To define the local heights, we use intersection theory on models over the valuation ring. Since the valuation ring of $\bar{K}$ is not noetherian, we have to pass to finite subextensions of $\bar{K}/K$.

Let X be a complete $\bar{K}$-variety. For $j = 0,\ldots,t$, let s_j be a non-zero meromorphic section of $\hat{L}_j \in \mathfrak{P}_X^+$. Using Segre embeddings, we construct a morphism φ from X into a multi-projective space P such that $\hat{L}_j = \varphi^*O_{\mathbb{P}}(e_j)$ We choose a finite subextension L/K of $\bar{K}/K$ such that all the objects introduced above are defined over L.

By Chow's Lemma, there is a birational L-morphism π from a projective L-variety X' onto X. Using blow up, we find a projective variety $\bar{X}'$ over the valuation ring L^o with generic fibre X' and a morphism $\bar{\varphi} : \bar{X}' \to \mathbb{P}_{L^o}$ over L^o with restriction $\varphi \circ \pi$ to the generic fibres.

To define the local height $\lambda(X)$ of X with respect to $(\hat{L}_j, s_j)_{j=0,\ldots,t}$, we assume that $|\operatorname{div}(s_0)|,\ldots,|\operatorname{div}(s_t)|$ intersect properly. If Z is a cycle on $\bar{X}'$ of (topological) dimension 0, then we define

$$\langle Z|\bar{X}'\rangle := -\log|\pi_L|\deg Z$$

where π_L is a uniform parameter on L. Note that the pairing is invariant under base change. Then the local height is defined by

$$\lambda(X) := \langle\operatorname{div}(\pi^*s_0)\ldots\operatorname{div}(\pi^*s_t)|\bar{X}'\rangle \ .$$

Here π^*s_j is considered as a non-zero meromorphic section of $\bar{\varphi}^*O_{\mathbb{P}}(e_j)$. The intersection product on $\bar{X}'$ is well defined modulo rational equivalence in the special fibre. The local height is independent of L. Using blow up and projection formula, we see that $\lambda(X)$ does not depend on the choice of X' and the model $\bar{X}'$.

We have to show that $\lambda(X)$ depends only on the metrics and not on the choice of φ. Simultaneously, we prove property iii) of Definition 8. Let $\varphi' : X \to \mathbb{P}'$ be a second morphism into a multi-projective space with $L_j \cong \varphi'^*O_{\mathbb{P}'}(e_j)$. We may assume that these objects are also defined over L and that $\varphi' \circ \pi$ extends to a morphism $\bar{\varphi}' : \bar{X}' \to \mathbb{P}'_{L^o}$ over L^o. On the common model $\bar{X}'$, we have two line bundles $\bar{\varphi}^*O_{\mathbb{P}}(e_j)$ and $\bar{\varphi}'^*O_{\mathbb{P}'}(e_j)$. Then $\pi^*(s_j)$ extends to meromorphic sections r_j and r_j', respectively. Let $\lambda'(X)$ be the local height of X with respect to $\bar{\varphi}'$. Then we have

$$\lambda'(X) - \lambda(X) = \sum_{j=0}^{t}\langle\operatorname{div}(r_0')\ldots\operatorname{div}(r_{j-1}').\operatorname{div}(r_j'/r_j).\operatorname{div}(r_{j+1})\ldots\operatorname{div}(r_t)|\bar{X}'\rangle \ .$$

Let $\alpha_j \in \bar{K}^*$ with

$$|\alpha_j| \geq \sup_{x \in X'(\bar{K})} \|s_j(x)\|' / \|s_j(x)\| .$$

Here $\| \ \|'$ denotes the metric of $\varphi'^*\overline{O_{\mathbb{P}'}(e_j)}$. We may assume that all $\alpha_j \in L$.

Let $\bar{U}$ be an open affine subset of $\bar{X}'$ whose images in $\mathbb{P}_{L^o}$ and $\mathbb{P}'_{L^o}$ are contained in standard open subsets. Then the bundles $\bar{\varphi}^* O_{\mathbb{P}}(e_j)$ and $\bar{\varphi}'^* O_{\mathbb{P}'}(e_j)$ are trivial on $\bar{U}$. Let γ_j, γ_j' be local equations on $\bar{U}$ of r_j, r_j' with respect to trivializations. If $x \in \bar{U}(\bar{K}^o)$, then it follows from the definition of standard metric that

$$|\gamma_j(x)| = \|\pi^* s_j(x)\| \ , \quad |\gamma_j'(x)| = \|\pi^* s_j(x)\|' \ .$$

Applying Lemma 13, we see that

$$\mathrm{div}(r_j'/r_j) \geq \mathrm{div}(\alpha_j)$$

on $\bar{X}'$.

Now assume that the metrics $\| \ \|$ and $\| \ \|'$ agree on L_j. Together with a similar upper bound, we get $\mathrm{div}(r_j'/r_j) = 0$. We conclude $\lambda(X) = \lambda'(X)$, i.e. the local height depends only on the metrics.

To prove iii) of Definition 8, let us choose two different metric $\| \ \|, \| \ \|'$ on L_0 as above. Using the notation above, we get

$$\begin{aligned} \lambda'(X) - \lambda(X) &= \langle c_1 \ldots c_t . \mathrm{div}(r_0'/r_0) \mid \bar{X}' \rangle \\ &\geq \langle c_1 \ldots c_t . \mathrm{div}(\alpha_0) \mid \bar{X}' \rangle \end{aligned}$$

where c_i is the first Chern class of $\bar{\varphi}^* O_{\mathbb{P}}(e_j)$. Using projection formula, we get

$$\lambda(X) - \lambda'(X) \leq \log|\alpha_0| \deg_{L_1, \ldots, L_t}(X) \ .$$

Since the value group of $\bar{K}$ is dense in $\mathbb{R}_+^*$ and using a similar lower bound, we get iii). The properties i) and ii) are immediate from the corresponding properties of intersection product.

To prove iv), let s_0' be a second non-zero meromorphic section of L_0. Let $\lambda(X)$ be as above and let $\lambda'(X)$ be the local height with s_0' replacing s_0. Using the model $\bar{X}'$, we have

$$\begin{aligned} \lambda(X) - \lambda'(X) &= \langle \mathrm{div}\Big(\pi^* \frac{s_0}{s_0'}\Big) . \mathrm{div}(\pi^* s_1) \ldots \mathrm{div}(\pi^* s_t) \mid \bar{X}' \rangle \\ &= \langle \mathrm{div}(\pi^* f_0) \mid Z_0 \rangle \end{aligned}$$

where $f_0 := s_0/s_0'$ and

$$Z_0 := \mathrm{div}(\pi^* s_1) \ldots \mathrm{div}(\pi^* s_t) . \bar{X}' \ .$$

Since f_0 is a rational function, the vertical components of Z_0 give no contribution to $\langle \mathrm{div}(\pi^* f_0) | Z_0 \rangle$. Let $\sum_j n_j P_j$ be the horizontal part of Z_0. We may assume that $P_j \in X'(\bar{K})$. Then

$$\langle \mathrm{div}(\pi^* f) \mid \bar{P}_j \rangle = -\log|f_0(P_j)| \ .$$

This proves iv). Clearly, v) follows from Proposition 12. Therefore $(\mathfrak{P}^+, \lambda)$ is a theory of local heights for t-dimensional $\bar{K}$-varieties. □

Remark 1.15. In a subsequent work, Theorem 14 will be proved for all non-archimedean fields $(K, | \ |)$.

Remark 1.16. Let K be any field with a complete absolute value $| \ |$ and let $(\mathfrak{g}^+, \lambda)$ be a theory of local heights for t-dimensional varieties over K. For a complete K-variety X, let $\sqrt{\mathfrak{g}_X^+}$ be the set of isometry classes $\hat{L}$ of line bundles on X such that some power $\hat{L}^{\otimes n} \in \mathfrak{g}_X^+$. Clearly, $\sqrt{\mathfrak{g}_X^+}$ is a monoid closed under pull back.

Proposition 1.17. Under the hypothesis above, there is a unique λ' such that $(\sqrt{\mathfrak{g}^+}, \lambda')$ is a theory of local heights extending $(\mathfrak{g}^+, \lambda)$.

Proof: Let $\hat{L}_j^{\otimes n_j} \in \mathfrak{g}_X^+$ and let s_j be non-zero meromorphic sections of L_j such that $|\mathrm{div}(s_0)|, \ldots, |\mathrm{div}(s_t)|$ intersect properly. Then we define the local height $\lambda(X)$ with respect to $(\hat{L}_j, s_j)$ by

$$\lambda(X) := \frac{1}{n_0 \cdots n_t} \lambda_{(\hat{L}_0^{\otimes n_0}, s_0^{\otimes n_0}), \ldots, (\hat{L}_t^{\otimes n_t}, s_t^{\otimes n_t})}(X) \ .$$

It is easily verified that this is well-defined and satisfies i) - v). □

Let $(\mathfrak{g}^+, \lambda)$ be a theory of local heights for t-dimensional varieties over K. For a complete K-variety X, we consider the set $\overline{\mathfrak{g}_X^+}$ of isometry classes of metrized line bundles $(L, \| \ \|)$ on X satisfying the following property: For $n \in \mathbb{N}$, there is a morphism φ_n from a complete K-variety X_n of dimension t onto X and a metric $\| \ \|_n$ on $\varphi_n^*(L)$ such that $(\varphi_n^*(L), \| \ \|_n) \in \mathfrak{g}_{X_n}^+$ and the sequence

$$\sup_{x \in X_n(\bar{K})} \left| \log(\| \ \|_n / \varphi_n^* \| \ \|) \right|$$

converges to 0 for $n \to \infty$. Note that $\| \ \|$ is a bounded metric since φ_n is proper and since $\varphi_n^* \| \ \|$ is bounded for large n (cf. Proposition 2.18).

Proposition 1.18. There is a unique $\bar{\lambda}$ such that $(\overline{\mathfrak{g}^+}, \bar{\lambda})$ is a theory of local heights extending $(\mathfrak{g}^+, \lambda)$. An isometry class $\hat{L}$ on the complete K-variety X is in $\overline{\mathfrak{g}_X^+}$ if and only if there is a morphism φ' from a complete

K-variety X' onto X such that $(\varphi')^*\hat{L} \in \overline{\mathfrak{g}^+_{X'}}$. Moreover, if $(\| \ \|_n)_{n\in\mathbb{N}}$ is a sequence of metrics on L inducing elements of $\overline{\mathfrak{g}^+_X}$ and converging uniformly to the metric $\| \ \|$, then the isometry class of $(L, \| \ \|)$ is in $\overline{\mathfrak{g}^+_X}$.

Proof: Let φ and ψ be morphisms of complete t-dimensional K-varieties X' and X'' onto X. Then there is a commutative diagram

$$\begin{array}{ccccc} & & X''' & & \\ & \swarrow & & \searrow & \\ X' & & & & X'' \\ & \varphi\searrow & & \swarrow\varphi & \\ & & X & & \end{array}$$

of surjective morphisms of t-dimensional K-varieties. Since φ and ψ are finite over an open dense part of X, we can use an appropriate irreducible component of $X' \times_X X''$. Hence $\overline{\mathfrak{g}^+_X}$ is a monoid and $\overline{\mathfrak{g}^+}$ is closed under pull back. For $j = 0, \ldots, t$, let s_j be a non-zero meromorphic section of $\hat{L}_j = (L_j, \| \ \|_j) \in \overline{\mathfrak{g}^+_X}$ such that $|\mathrm{div}(s_0)|, \ldots, |\mathrm{div}(s_t)|$ intersect properly. Using the remark above, there is a morphism φ_n from a complete t-dimensional K-variety X_n onto X and, for all j, metrics $\| \ \|_{j,n}$ on $\varphi_n^*(L_j)$ with $\hat{L}_{j,n} := (\varphi_n^*(L_j), \| \ \|_{j,n}) \in \mathfrak{g}^+_{X_n}$ such that the sequence

$$\sup_{x \in X_n(\bar{K})} \left| \log(\| \ \|_{j,n} / \varphi_n^* \| \ \|_j) \right|$$

converges to 0. Let $\lambda_n(X_n)$ be the local height of X_n with respect to $\left(\hat{L}_{j,n}, \varphi_n^*(s_j)\right)_{j=0,\ldots,t}$. By the projection formula and Definition 8iii), $\lambda(X_n)/[X_n : X]$ is a Cauchy sequence where $[X_n : X]$ is the degree of $K(X_n)$ over $K(X)$. The limit $\bar{\lambda}(X)$ is called the local height of X with respect to $(\hat{L}_j, s_j)_{j=0,\ldots,t}$. By 8ii) and 8iii), this is the only possibility to define $\bar{\lambda}(X)$. This proves uniqueness. Clearly, the definition of $\bar{\lambda}(X)$ is independent of the choice of the sequence. It is immediately verified that $(\overline{\mathfrak{g}^+}, \bar{\lambda})$ is a theory of local heights extending $(\mathfrak{g}^+, \lambda)$. The other statements in Proposition 18 are now trivial. □

Remark 1.19. Let $(\mathfrak{g}^+, \lambda)$ be a theory of local heights for t-dimensional varieties over K. Then $\left(\sqrt{\overline{\mathfrak{g}^+}}, \overline{\lambda}'\right)$ is a canonical theory of local heights extending $(\mathfrak{g}^+, \lambda)$. It is called the completion of $(\mathfrak{g}^+, \lambda)$ and is denoted by $(\hat{\mathfrak{g}}^+, \hat{\lambda})$. If we have $\mathfrak{g}^+ = \hat{\mathfrak{g}}^+$, then $(\mathfrak{g}^+, \lambda)$ is called a complete theory of local heights.

For a complete K-variety X of dimension t, let $\hat{\mathfrak{g}}_X$ be the set of isometry classes $\hat{L}$ of metrized line bundles on X with the following property: There is a morphism φ from a complete t-dimensional K-variety X' onto X such that $\varphi^*\hat{L}$ is the difference of two elements of $\hat{\mathfrak{g}}^+_{X'}$. Clearly, $\hat{\mathfrak{g}}_X$ is a group containing $\hat{\mathfrak{g}}^+_X$.

For $j = 0, \ldots, t$, let s_j be a non-zero meromorphic section of $\hat{L}_j \in \hat{\mathfrak{g}}_X$ such that $|\mathrm{div}(s_0)|, \ldots, |\mathrm{div}(s_t)|$ intersect properly. The local height $\hat{\lambda}_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(X) \in \mathbb{R}$ is uniquely determined by the properties:

i) $\hat{\lambda}$ is multilinear and symmetric in the variables $(\hat{L}_j, s_j)_{j=0,\ldots,t}$;
ii) The projection formula is satisfied;
iii) On $\hat{\mathfrak{g}}^+_X$, $\hat{\lambda}$ agrees with the completion of λ.

Moreover, the change of section formula 8iv) remains true on $\hat{\mathfrak{g}}_X$. Using the remark at the beginning of the proof of Proposition 18, this is easily deduced from the properties of $(\hat{\mathfrak{g}}^+, \hat{\lambda})$.

Example 1.20. If the absolute value on K is archimedean, then we consider the completion $(\hat{\mathfrak{g}}^+, \hat{\lambda})$ of the theory of local heights defined in Theorem 10. On a complete K-variety X, let $\hat{L}$ be a line bundle with a smooth hermitian metric. By Chow's Lemma, there is a birational morphism φ of a projective K-variety X' onto X. The tensor product of $\varphi^*(\hat{L})$ with a large power of a very ample line bundle on X' with a standard metric is in $\mathfrak{g}^+_{X'}$. We conclude that the isometry class of $\hat{L}$ is in $\hat{\mathfrak{g}}_X$.

Now assume that the given absolute value is discrete. Then we consider the completion of the theory of local heights $(\mathfrak{P}^+, \lambda)$ for t-dimensional varieties over $\bar{K}$ (Theorem 14). Let $\bar{X}$ be a proper flat variety over the discrete valuation ring K^o with generic fibre X. If $\mathcal{L}$ is a line bundle on $\bar{X}$ whose restriction to X equals L, then $\mathcal{L}$ induces in the following way a metric $\| \ \|$ on L. Note that $P \in X(\bar{K})$ has a canonical extension to a $\bar{K}^o$-valued point $\bar{P}$ of $\bar{X}$. This follows from the valuative criterion of properness. For a local section s of $\mathcal{L}$ corresponding to a regular function γ under a trivialization defined in a neighbourhood of $\bar{P}$, let $\|s(P)\| := |\gamma(P)|$. It is easily checked that this defines a metric on L. This metric was considered in [Zh], it is called algebraic. We claim that the induced metrized line bundle $\hat{L}$ is in $\hat{\mathfrak{P}}_X$. By Chow's Lemma we may assume that $\bar{X}$ is projective over K^o. Then $\mathcal{L}$ may be written as the difference of two very ample line bundles on $\bar{X}$. Since the claim is clear on projective space, we get immediately $\hat{L} \in \hat{\mathfrak{P}}_X$.

For line bundles $\mathcal{L}_0, \ldots, \mathcal{L}_t$ on $\bar{X}$ with restrictions $L_0, \ldots, L_t$ and non-zero meromorphic sections $s_0, \ldots, s_t$ such that $|\mathrm{div}(s_0)|, \ldots, |\mathrm{div}(s_t)|$ intersect properly on X, it follows from projection formula that

$$\lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(X) = \langle \mathrm{div}(s_0) \ldots \mathrm{div}(s_t) | \bar{X} \rangle$$

where on L_j we use the algebraic metric induced by $\mathcal{L}_j$ (cf. Theorem 14). Note that the intersection product on the right hand side is computed on $\bar{X}$.

2. M-Fields

Definition 2.1. An M-field K consists of a field K and a measure space (M,μ) together with μ almost everywhere(μ-ae) defined maps

$$M \to \mathbb{R}_+ \ , v \mapsto |x|_v$$

for $x \in K$. They are supposed to satisfy the conditions

a) $|x+y|_v \leq |x|_v + |y|_v \quad \mu\text{-ae}$
b) $|xy|_v = |x|_v\,|y|_v \quad \mu\text{-ae}$
c) $\log|z|_v \in L^1(M,|\mu|)$ and $|0|_v = 0 \quad \mu\text{-ae}$

for all $x, y \in K$ and $z \in K^*$.

Example 2.2. Suppose K is a field with a set M of absolute values such that

$$\{v \in M; |\alpha|_v \neq 1\}$$

is finite for all $\alpha \in K^*$. On M we consider a point measure μ, i.e. finite sets are measurable and have finite positive measure. Obviously, K is an M-field.

Definition 2.3. Let K be an M-field. For $\alpha \in K^*$, let

$$d_\alpha := \int_M \log|\alpha|_v \, d\mu(v) \ .$$

If $d_\alpha = 0$ for all $\alpha \in K^*$, then we say that the product formula is satisfied. So d_α measures the defect of the product formula.

Example 2.4. Let $R > 0$ and let M_R be the closed disc in $\mathbb{C}$ with radius R. The field of meromorphic functions on M_R is denoted by K_R. On the boundary ∂M_R of M_R, we consider the probability Lebesque-measure. In the interior, the following point measure is considered:

$$\mu(\{v\}) := \begin{cases} \log(R/|v|) & \text{if } 0 < |v| < R \\ \log R & \text{if } v = 0 \ . \end{cases}$$

This gives a real measure μ on M_R which is positive if and only if $R \geq 1$. For $f \in K_R^*$ and $v \in M_R$, let

$$|f|_v := \begin{cases} |f(v)| & \text{if } v \in \partial M_R \\ e^{-\mathrm{ord}(f,v)} & \text{if } |v| < R \end{cases}$$

where $\mathrm{ord}(f,v)$ is the order of the meromorphic function f in v. It is easy to check that K_R is an M_R-field. Let c_f be the leading coefficient of the Laurent series of f in 0, then the defect d_f of product formula equals $\log|c_f|$.

This is a consequence of Jensen's formula. For details, the reader is refered to [La2, p. 163].

Remark 2.5. Let K, M and μ be as in Example 2 and suppose that the product formula is satisfied. Let F/K be an algebraic extension. For any subextension L/K, we denote by M_L the set of absolute values L which extend an element of M. The purpose is to define a measure on M_F such that the product formula is satisfied.

For a finite subextension L/K and for $w \in M_L$, let

$$A_w := \{u \in M_F : u|w\} \ .$$

Here $u|w$ means that the restriction of the absolute value $|\ |_u$ to L equals $|\ |_w$. Using that, we identify any subset of M_L with a subset of M_F.

Let $v \in M$ be the restriction of the element $w \in M_L$ and let K_v be the completion of K with respect to v. Up to isomorphism, there is a unique decomposition of $L \otimes_K K_v$ into a product of artinian noetherian local rings R_j. There is a bijective correspondence between factors R_j and the extensions w' of $|\ |_v$ to L given by the fact that the completion of L with respect to w' is isomorphic (as an extension of K_v) to the residue field of a unique R_j. Let R_w be the factor corresponding to w, then we define

$$\mu(A_w) := \big([R_w : K_v] \ / \ [L:K]\big)\mu(\{v\}) \ .$$

Let L'/K be a subextension of L/K and let $w' \in M_{L'}$. It is easy to check that

$$\mu(A_{w'}) = \sum_{\substack{w|w' \\ w \in M_L}} \mu(A_w) \ .$$

Let $\mathcal{A}_L$ be the set of finite subsets of M_L, then $\mathcal{A}_L$ is closed under intersection and forming $A \backslash B$. The above consideration proves that μ induces a positive additive function on $\mathcal{B} = \cup_L \mathcal{A}_L$ where L ranges over all finite subextensions of F/K. Let $(B_n)_{n \in \mathbb{N}}$ be a decreasing sequence in $\mathcal{B}$ with empty intersection. Note that $M_F = \varprojlim M_L$ where L ranges over all finite subextensions of F/K. Obviously, we may suppose that B_1 is contained in the fibre $M_{F,v}$ over some $v \in M$. Then we have $M_{F,v} = \varprojlim M_{L,v}$. We endow every $M_{L,v}$ with the discrete topology, then $M_{F,v}$ is compact and the A_w's form a basis of topology. The A_w's are closed as well, hence any B_n is closed. By compactness, only finitely many B_n's are non-empty.

Therefore, we have a unique extension of μ to a positive measure on $(M, \mathcal{A})$ where $\mathcal{A}$ is the σ-algebra generated by $\mathcal{B}$. Then F is an M_F-field and

$$\left|N_{L/K} f\right|_v = \prod_{w \in M_{L,v}} |f|_v^{[R_w:K_v]} \qquad (f \in L)$$

implies that the product formula is satisfied.

Example 2.6. The standard example is $K = \mathbb{Q}$ with the set M of non-trivial absolute values. They are normalized in the following way. In the archimedean case, we take the Euclidean absolute value. For a prime number p, the p-adic absolute value is assumed to satisfy $|p|_p = \frac{1}{p}$. Together with the counting measure, this gives an M-field satisfying the product formula. Using Remark 5, any algebraic extension $F/\mathbb{Q}$ will be in a canonical way an M_F-field satisfying the product formula.

Example 2.7. Let $c > 1$ and let X be a projective variety over the field k. Denote by deg the degree with respect to an ample line bundle. We assume that X is regular in codimension 1. Let K be the function field of X and let M be the set of prime divisors together with the counting measure μ. For $v \in M$, the order in v is a discrete valuation on K. For $f \in K$, let

$$|f|_v := c^{-\operatorname{ord}(f,v)\deg(v)}$$

be the corresponding absolute value. Then K is an M-field satisfying the product formula.

Example 2.8. Let K_R be the M_R-field of Example 4. We are going to define an M-field structure on the algebraic closure $\bar{K}_R$.

Let L/K_R be a finite field extension of degree n. There is a primitive element f_0 of L/K_R. The coefficients of its minimum polynomial are defined on an open neighbourhood U of M_R. By the theory of Riemann surfaces, there is, up to isomorphism, a unique finite holomorphic covering $\pi : X_U \to U$ of U by a smooth complex curve X_U with field of meromorphic functions isomorphic to the subfield of L generated by f_0 and the meromorphic functions on U. The degree of the covering equals n. If U' is an open subset of U containing M_R, then the covering $\pi' : X_{U'} \to U'$ is isomorphic to the restriction of π to $\pi^{-1}(U')$ since they have isomorphic fields of meromorphic functions. It follows that $M_L := \pi^{-1}(M_R)$ is a compact complex Riemann surface with boundary not depending on the choice of U. If we define a meromorphic function on M_L as the restriction to M_L of a meromorphic function on a neighbourhood of M_L in X_U, then the field of meromorphic functions on M_L is isomorphic to L. Up to isomorphisms, M_L is determined by L and we identify L with the field of meromorphic functions on M_L.

Next we look for an M_L-field structure on L. First, we define the measure μ_L on the boundary. Outside a set E of measure zero, the covering is a local isomorphism. Let $w \notin E$ be a local paramter on ∂M_L and let v be the corresponding local parameter on ∂M_R. Then we define $\mu_L(dw) := \frac{1}{n}\mu(dv)$. In the interior of M_L, μ_L will be a point measure. For $w \in M_L \backslash \partial M_L$ with image v in $M_R \backslash \partial M_R$, let $\mu_L(\{w\}) := \frac{r(w)}{n}\mu(\{v\})$ where $r(w)$ is the order of

ramification in w. For $f \in L$, we define

$$|f|_w := \begin{cases} |f(w)| & \text{if } w \in \partial M_L \\ e^{-\operatorname{ord}(f,w)/r(w)} & \text{if } w \in M_L \backslash \partial M_L \,. \end{cases}$$

If $f \in K_R$, then we have $|f|_v = |f|_w$. It is easily shown that L is an M_L-field.

Now let L'/L be a finite extension of degree m. Then we get a finite covering $M_{L'} \to M_L$ of degree m. If $w' \in M_{L'}$ maps to $w \in M_L$, then $|\ |_{w'}$ is an extension of $|\ |_w$ to L'. For $f \in L$, we have

$$\int_{M_{L'}} \log|f|_{w'}\, d\mu_{L'}(w') = \int_{M_L} \log|f|_w\, d\mu_L(w)\,.$$

Let d_f be the defect of product formula on X_L, then we claim that $d_f = \frac{1}{n}d_{Nf}$ where Nf is the norm of f with respect to L/K_R. The above consideration shows that we can assume that L/K_R is a Galois extension. Then the Galois group of this extension is isomorphic to the transformation group G of the covering π. We have

$$\begin{aligned} \int_{M_L\backslash \partial M_L} \log|f|_w\, d\mu_L(w) &= \sum_{v\in M_R\backslash \partial M_R} \sum_{\pi(w)=v} \log|f|_w\, \mu_L(\{w\}) \\ &= -\sum_{v\in M_R\backslash \partial M_R} \sum_{\pi(w)=v} \operatorname{ord}(f,w)\mu(\{v\})/n \\ &= \frac{1}{n}\int_{M_R\backslash \partial M_R} \log|Nf|_v\, d\mu(v) \end{aligned}$$

where in the last step we have used $\pi_*\operatorname{div}(f) = \operatorname{div}(Nf)$. Using the action of G over non-ramified points, we get easily

$$\int_{\partial M_L} \log|f|_w\, d\mu_L(w) = \frac{1}{n}\int_{\partial M_R} \log|Nf|_v\, d\mu(v)\,.$$

This proves $d_f = \frac{1}{n}d_{Nf}$.

We define $M_{\bar{K}_R}$ as the inverse limit of M_L where L ranges over all finite subextensions of $\bar{K}_R/K_R$. The Riemann surfaces induce a compact topology on $\bar{M}_R := M_{\bar{K}_R}$. But this is not the right topology. On the interior of M_L, we have to replace the usual topology by the discrete topology to ensure that μ_L is a regular Borel measure on M_L. Then $\bar{M}_R$ is locally compact and its boundary $\partial \bar{M}_R := \varprojlim \partial M_L$ is compact. Similarly as in Remark 5, the measures μ_L on $M_L\backslash \partial M_L$ induce a measure μ on $\bar{M}_R \backslash \partial \bar{M}_R$. Let U_L be a measurable subset of ∂M_L and let π_L be the projection of $\bar{M}_R$ onto M_L. Then we define $\mu(\pi_L^{-1}U_L) := \mu_L(U_L)$. It is easily shown that this is independent of the choice of L and gives an additive function μ on some class of subsets of $\partial\bar{M}_R$ containing a basis of open sets. This class is closed under forming intersection, union and complement. If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence in this class with empty intersection, then $\lim_{n\to\infty}\mu(A_n)=0$. To see this, one approximates $\mu(A_n)$ by a sufficiently large compact subset K_n of A_n such that $\mu(A_n\backslash K_n)<\varepsilon 2^{-n}$. There is a sufficiently large m such that $\bigcap_{n\le m}K_m=\emptyset$. We conclude that $\mu(A_m)<\varepsilon$. Now μ has a continuation to a positive measure μ on $\partial\bar{M}_R$. Together with the measure on $\bar{M}_R\backslash\partial\bar{M}_R$, we obtain a measure μ a M. Finally, $\bar{K}_R$ is an $\bar{M}_R$-field.

Remark 2.9. Let S be a countable subset of the M-field K. There is a countable subfield K' of K containing S. We can choose a subset M' of M whose complement is a null set such that the restriction of $|\ |_v$ to K' is a well-defined absolute value for any $v\in M'$.

Remark 2.10. The main problem in the context of M-fields is that the maps $|\ |_v$ have not to be absolute values. Since we are concerned with problems arising from algebraic geometry where finitely many varieties, line bundles or meromorphic sections are given over the M-field K, we can find a finitely generated subfield K' of K such that all these objects are defined over K'. Skipping a null-set $M\backslash M'$, we may assume that the restriction of $|\ |_v$ to K' is an absolute value for all $v\in M'$ (Remark 9). Clearly, K' is an M'-field.

2.11) At the end of this section, we want to introduce boundedness in the context of M-fields. Let us assume for a moment that all $|\ |_v$ are absolute values on K. For $v\in M$, let $\bar{K}^v$ be the completion of the algebraic closure of the completion of K with respect to the place v. Note that $\bar{K}^v$ is a complete algebraically closed field [BGR, Proposition 3.4.1/3]. For a K-variety X, let

$$X(M):=\coprod_{v\in M}X(\bar{K}^v)\times\{v\}\,.$$

Assume that X is affine. Then $E\subset X(M)$ is called M-bounded in X if and only if for any regular function a on X, there is an integrable function c on M with

$$|a(P)|_v\le c(v)$$

for all $(P,v)\in E$. Then there is a well-known procedure to define M-bounded subsets of $X(M)$ for any variety X over K, and one deduces similarly as in the number field case that $X(M)$ is M-bounded if X is proper over K. We shall prove that below, but first we have to deal with the problem that $|\ |_v$ has not to be an absolute value on K. As indicated in Remark 10, we have to pass to finitely generated subfields K' of K. If the reader is not interested in these technical details, then he should skip 12) and Definition 13 and he should read the rest of this section under the assumptions of this paragraph.

2.12) Let K be an M-field. We introduce the following category $\mathfrak{C}$. An object is a family $A:=(A_v)_{v\in M'}$ of sets where M' is a subset of M such that $M\backslash M'$ is a null-set. Two such families A and B are defined to be equal if $A_v=B_v$ for almost all v. A morphism φ from an object A to an object B is a family of maps $\varphi_v:A_v\to B_v$ for almost all v. An object A is called a subset of the object B if $A_v\subset B_v$ for almost all v. For two subsets A_1,A_2 of B, we can form well-defined objects $A_1\cup A_2$, $A_1\cap A_2$, $A_1\backslash A_2$ by using the corresponding operations in almost all components of the family B. Similarly, we define the image and the inverse image of an object under a morphism in the category $\mathfrak{C}$. Let X be a K-variety. Then there is a finitely generated subfield K_0' such that X is defined over K_0', i.e. it exists a K_0'-variety X_0 such that $X\cong X_0\otimes_{K_0'}K$. For a finitely generated subfield K' of K containing K_0', let $M_{K'}$ be the set of $v\in M$ such that the restriction of $|\ |_v$ to K' is an absolute value. By Remark 10, the family

$$X_0(M_{K'}):=\left(X_0(\overline{K'}^v)\right)_{v\in M_{K'}}$$

is an object of $\mathfrak{C}$. Note that it depends on the choice of K_0' and of X_0.

We form a new category $\mathfrak{C}'$. An object in $\mathfrak{C}'$ is a family $(A_{K'})$ of objects in $\mathfrak{C}$ (with K' running over all sufficiently large, finitely generated subfields of K) such that $A_{K'}$ is a subset of $A_{K''}$ for $K'\subset K''$. Two such families $(A_{K'})$ and $(B_{K'})$ are viewed to be equal if and only if $A_{K'}=B_{K'}$ for sufficiently large K'. A morphism φ of the objects $(A_{K'})$ and $(B_{K'})$ in $\mathfrak{C}'$ is a family of morphisms $\varphi_{K'}:A_{K'}\to B_{K'}$ in $\mathfrak{C}$ for sufficiently large K' compatible with $A_{K'}\subset A_{K''}$ and $B_{K'}\subset B_{K''}$ for $K'\subset K''$. An object $(A_{K'})$ is called a subset of an object $(B_{K'})$ if $A_{K'}$ is a subset of $B_{K'}$ for K' sufficiently large. For two subsets $(A_{K'})$ and $(\tilde{A}_{K'})$, we get well-defined objects $(A_{K'})\cup(\tilde{A}_{K'})$, $(A_{K'})\cap(\tilde{A}_{K'})$, $(A_{K'})\backslash(\tilde{A}_{K'})$ in $\mathfrak{C}'$ by using these operations componentwise for sufficiently large K'. Similarly, we define the image and the inverse image of an object in $\mathfrak{C}'$. A real function on an object $(A_{K'})$ is a morphism from $(A_{K'})$ to the object $(\mathbb{R})$ of $\mathfrak{C}'$. With real functions on an object, we can form the usual algebraic operations, the maximum and the minimum. Moreover, if f is a real function on an object of $\mathfrak{C}'$, then it is easy to define a subset $\{f\ge 0\}$ by proceeding componentwise.

For example, the family

$$X(M):=\left(X_0(M_{K'})\right)\,,$$

where K' runs over all finitely generated subfields of K containing K_0', is an object of $\mathfrak{C}'$. It does not depend on the choice of K_0' and X_0 up to isomorphisms. A morphism $\varphi:X\to X'$ of K-varieties induces a natural morphism $X(M)\to X'(M)$ in $\mathfrak{C}'$ also denoted by φ, i.e. we get a functor from the category of K-varieties to $\mathfrak{C}'$. Let a be a regular function on X. If K' is a finitely generated subfield of K containing K_0 such that a is defined over K', then we have a real function on $X_0(M_{K'})$ given by $(P,v) \mapsto |a(P)|_v$. This induces a real function $|a|$ on $X(M)$. If E is a subset of $\{|a| \geq 0\}$, then we say $|a(P)|_v \geq 0$ on E.

Definition 2.13. Let X be an affine K-variety. Then $E \subset X(M)$ is said to be M-bounded if for any regular function a on X, there is an integrable function c on M such that

$$|a(P)|_v \leq c(v)$$

on E.

Note that this definition coincides with the one in 11) under the assumption that all $v \in M$ give absolute values on K.

Definition 2.14. For any variety X over K, $E \subset X(M)$ is said to be M-bounded if there is a covering of X by affine open subsets $(U_j)_{j=1,\dots,r}$ and a decomposition $E = \bigcup_{j=1}^r E_j$ with $E_j \subset U_j(M)$ such that E_j is M-bounded in U_j for all j (in the sense of Definition 13).

Obviously, both definitions agree on affine varieties.

Lemma 2.15. Let E be an M-bounded subset of $X(M)$ and let $(V_k)_{k=1,\dots,s}$ be an open affine covering of X. Then there is a decomposition $E = \bigcup_{k=1}^s E'_k$ with E'_k M-bounded subsets of $V_k(M)$.

Proof: Let $(U_j, E_j)_{j=1,\dots,r}$ be a decomposition as in Definition 14. We may assume that the covering (V_k) is a refinement of (U_j). So we may assume that X is affine and that $V_k = \{x_k \neq 0\}$ for some regular function x_k on X. Since (V_k) is a covering of X, we have regular functions $a_1, \dots, a_s$ on X with

$$a_1 x_1 + \cdots + a_s x_s = 1 .$$

We define a subset of E by

$$E_k := \{|x_k(P)|_v = \max_{j=1,\dots,s} |x_j(P)|_v\} .$$

Obviously, we have $E = \bigcup_{k=1}^s E_k$ and $E_k \subset V_k(M)$. We have to prove that E_k is M-bounded in V_k. It is enough to check the inequality in Definition 13 for a set of generators of $\mathcal{O}_X(V_k)$ and hence for $1/x_k$. Using the relation above, we obtain

$$|(1/x_k)(P)|_v \leq |s|_v^{\delta_v} \max_{j=1,\dots,s} |a_j(P)|_v$$

on E_k, where δ_v is one if v is archimedean and zero otherwise. Since E is bounded in X, there is $c \in L^1(M,\mu)$ with

$$\max_{j=1,\dots,s} |a_j(P)|_v \leq c(v)$$

on E. This proves the claim. □

Proposition 2.16. Let $\varphi : X \to X'$ be a morphism of varieties over K and let $E \subset X(M)$. If E is M-bounded, then $\varphi(E)$ is M-bounded in X'. If φ is a closed immersion and $\varphi(E)$ is M-bounded in X', then E is bounded in X.

Proof: The first claim follows immediately from Lemma 15. The second claim is trivial. □

Proposition 2.17. If X is a projective variety over K, then $X(M)$ is M-bounded.

Proof: By Proposition 16, we may assume $X = \mathbb{P}^n_K$. Let $x_0, \dots, x_n$ be a set of coordinates on $\mathbb{P}^n$. We consider the decomposition $\mathbb{P}^n_K(M) = \bigcup_{j=0}^n E_j$ and $U_j := \{x_j \neq 0\}$, where

$$E_j := \{|x_j(P)|_v = \max_{k=0,\dots,n} |x_k(P)|_v\} .$$

Obviously, this decomposition satisfies the assumption of Definition 15. □

Proposition 2.18. Let $\varphi : X \to X'$ be a proper morphism of varieties over K and let E' be an M-bounded subset of $X'(M)$. Then $\varphi^{-1}(E')$ is M-bounded in X. In particular, $X(M)$ is M-bounded for a complete K-variety X.

Proof: By Chow's Lemma and Proposition 16, we may assume that φ is projective, i.e. it exists a closed immersion i with a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathbb{P}^n_{X'} \\ & \varphi \searrow \quad \swarrow & \\ & X' & \end{array}$$

By Proposition 16, we may assume that $X = \mathbb{P}^n_{X'}$. Moreover, Lemma 15 shows that we may assume X' affine. So we have

$$\mathbb{P}^n_{X'} = \operatorname{Proj} \mathcal{O}_{X'}(X')[x_0, \dots, x_n] .$$

Let $U_j := \{x_j \neq 0\}$ and let $E_j \subset U_j(M)$ be given by

$$E_j := \varphi^{-1}(E') \cap \{|x_j(P)|_v = \max_{k=0,\dots,n} |x_k(P)|_v\} .$$

It is immediately checked that E_j is M-bounded in U_j. □

Definition 2.19. Let X be a variety over the M-field K and let α be a real function on $X(M)$. Then α is said to be locally M-bounded if and only if for any M-bounded subset E of $X(M)$, there is an integrable function c on M such that

$$|\alpha(v,P)| \leq c(v)$$

on E. The function α is called M-bounded if there is $c \in L^1(M,\mu)$ such that the above inequality is true on $X(M)$.

Corollary 2.20. For a complete K-variety X, a real function on $X(M)$ is M-bounded if and only if it is locally M-bounded.

Proof: This follows from Proposition 18. □

2.21) Let L be a line bundle on the variety X over the M-field K. Suppose that K_0' and X_0 are as in 11). Moreover, we may assume that it exists a line bundle L_0 on X_0 with $L \cong L_0 \otimes_{K_0'} K$ compatible with $X \cong X_0 \otimes_{K_0'} K$. Recall that $L_0(M_{K'})$ is given by the family $\left(L_0(\overline{K'}^v)\right)_{v \in M_{K'}}$ for any finitely generated subfield K' of K containing K_0'. We define a metric on $L_0(M_{K'})$ as a family $\| \ \|_v$ of metrics on $L_0(\overline{K'}^v)$ for almost all v. Recall that $L(M)$ is given by the family $(L_0(M_{K'}))$.

Definition 2.22. An M-metric $\| \ \|$ on L is a real function on $L(M)$ inducing a metric on $L_0(M_{K'})$ for all sufficiently large, finitely generated subfields K'. Moreover, we assume that for any open subset U and any non-vanishing section $s \in L(U)$, the real function $\log\|s\|$ is locally M-bounded on U.

Remark 2.23. Clearly, the trivial metrics on $O_X(M_{K'})$ induce an M-metric on the trivial bundle O_X called the trival M-metric. It is easy to define the tensor product, the dual and the pull back of M-metrized line bundles. The details are left to the reader.

Now let $\| \ \|$, $\| \ \|'$ be two M-metrics on L. Using the above, we get a metric $\| \ \| / \| \ \|'$ on O_X. Evaluating at the section 1, we get a real function on $X(M)$ also denoted by $\| \ \| / \| \ \|'$. It is easy to see from the definitions that this real function is locally M-bounded. By Corollary 20, we get:

Corollary 2.24. If X is a complete K-variety, then $\log(\| \ \|' / \| \ \|)$ is M-bounded.

Example 2.25. Consider the line bundle $L := O_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n_K$. The family of standard metrics (Examples 1.2, 1.5) induces a metric on $L(M_{K'})$ for any finitely generated subfield K' of K. This gives rise to an M-metric on L called the standard M-metric. The resulting M-metrized line bundle is denoted by $\overline{O_{\mathbb{P}^n}(1)}$.

3. Global Heights

Let K be an M-field. For any $v \in M$, we want to fix a complete theory of local heights $(\mathfrak{g}^+, \lambda)$ for varieties over K. Since the maps $| \ |_v$ are not necessarily absolute values, we have to pass to sufficiently large, finitely generated subfields as indicated in Remarks 2.9 and 2.10. This will be made more precise in the next paragraph.

3.1) Let $v \in M$ and let K' be a finitely generated subfield of K such that the restriction of $| \ |_v$ to K' is an absolute value. We denote the corresponding place of K' by v'. Then we fix a complete theory of local heights $(\mathfrak{g}^+_{v'}, \lambda(\cdot, v'))$ for t-dimensional varieties over the completion $K'_{v'}$ (Definition 1.8, Example 1.20).

Now we consider M-metrized line bundles $\hat{L}$ on a t-dimensional complete variety X over K. Then X and L are defined over a finitely generated subfield K_0' by a complete variety X_0 and a line bundle L_0 on X_0. By definition, $\hat{L}$ induces a metrized line bundle $\hat{L}(K', v)$ on $X_0 \otimes_{K_0'} K'_{v'}$ for sufficiently large, finitely generated subfields K' of K and almost all v. We require that the restriction of $\hat{L}(K', v)$ to any irreducible component Y of $X_0 \otimes_{K_0'} K'_{v'}$ is in $(\mathfrak{g}_{v'})_Y$ at least for sufficiently large K' and almost all $v \in M$. The set of isometry classes of such M-metrized line bundles is denoted by $\mathfrak{g}_X$. Similarly, we define $\mathfrak{g}_X^+$. Then $\mathfrak{g}_X^+$ is a submonoid and $\mathfrak{g}_X$ is a subgroup of the group of isometry classes of M-metrized line bundles. Moreover, they are closed under pull back with respect to K-morphisms.

Definition 3.2. Let $\hat{L}_0, \ldots, \hat{L}_t \in \mathfrak{g}_X$ with non-zero meromorphic sections $s_0, \ldots, s_t$. We say that the local height of a t-dimensional subvariety Y of X is well-defined if and only if $|\mathrm{div}(s_0)|, \ldots, |\mathrm{div}(s_t)|$, Y intersect properly and if the local height of Y in the theory $(\mathfrak{g}_{v'}, \lambda(\cdot, v'))$ does not depend on the choice of the sufficiently large, finitely generated subfield K' of K for almost all $v \in M$. For such a subvariety Y, the local height is a $\mu - ae$ defined function

$$v \mapsto \lambda_{(\hat{L}_0, s_0), \ldots, (\hat{L}_t, s_t)}(Y, v)$$

on M. By additivity, we extend this definition to all t-dimensional cycles.

Proposition 3.3. For a complete K-variety X,let $\mathfrak{P}_X^+$ be the set of isometry classes $\varphi^*\overline{O_{\mathbb{P}^n}(1)}$ (Example 2.25) where φ is ranging over all K-morphisms of X into some projective space. Then $\mathfrak{P}_X^+$ is a submonoid of $\mathfrak{g}_X^+$. Let $\mathfrak{P}_X$ be the subgroup of $\mathfrak{g}_X$ generated by $\mathfrak{P}_X^+$ and let $Y, (\hat{L}_j, s_j)_{j=0,\ldots,t}$ as above. If $\hat{L}_j \in \mathfrak{P}_X$ for all j, then the local height of Y with respect to $(\hat{L}_j, s_j)_{j=0,\ldots,t}$ is well-defined if and only if $|\mathrm{div}(s_0)|, \ldots, |\mathrm{div}(s_t)|$, $|Y|$ intersect properly.

Proof: This follows from Proposition 1.9. For compatibility of local heights, it is enough to note that this is true for multi-projective spaces. □

Lemma 3.4. Let $\mathbb{P} := \mathbb{P}^{n_0} \times \cdots \times \mathbb{P}^{n_t}$ and let Z be a t-dimensional subvariety of $\mathbb{P}_K$. Suppose that the local height $\lambda(Z,v)$ of Z with respect to the standard M-metrized line bundles $\overline{O_{\mathbb{P}}(e_j)}$ with global sections s_j ($j = 0,\ldots,t$) is well-defined. Then the function $v \mapsto \lambda(Z,v)$ is integrable on M.

Proof: Skipping a null-set in M and passing to a sufficiently large, finitely generated subfield of K, we may assume that all elements of M give absolute values on K. The non-archimedean part M_{fin} and the archimedean part M_∞ of M are measurable as they are given by $\{\log|2|_v \leq 1\}$ and $\{\log|2|_v > 1\}$. Using 1.8v), it is clear that the restriction of $\lambda(Z,v)$ to M_{fin} is integrable. So we may assume that all absolute values in M are archimedean. Replacing the measure μ on M by an equivalent one, we may assume that all absolute values in M are the usual ones. Then M has finite measure with respect to $|\mu|$.

Let F_Z be the Chow form of Z and let S be the product of unit spheres in $\mathbb{C}^{N_k+1}$ ($k = 0,\ldots,t$). For $v \in M$, we have an isometry $\overline{K_v} \xrightarrow{\sim} \mathbb{C}$ and we denote the image of F_Z by F_Z^v. We have to show that

$$v \mapsto \int_S \log|F_Z^v(x)|\, dP(x)$$

is integrable on M (Definition 1.8v).

First, we prove that the function above is measurable on M. Using approximation by step functions, it's enough to see that $\log|F_Z^v(x)|$ is measurable for any $x \in S$. We can approximate x by points with coordinates in $\mathbb{Q}(\sqrt{-1})$ (lying not necessarily on S). So we may assume that $x \in P\big(\mathbb{Q}(\sqrt{-1})\big)$. Then we have

$$\log|F_Z^v(x)| = C\log\Big|N_{K(\sqrt{-1})/K}F_Z(x)\Big|_v$$

where

$$C = \begin{cases} 1 & \text{if } \sqrt{-1} \in K \\ \frac{1}{2} & \text{if } \sqrt{-1} \notin K \,. \end{cases}$$

Anyway, this proves measurability.

Let $|F_Z|_v$ be the maximum of the absolute values of all coefficients. Then

$$\int_S \log|F_Z^v(x)|\, dP(x) - \log|F_Z|_v$$

is bounded in absolute value by a constant independent of v (for a proof, see for example [BGS, 4.3]). Since M has finite measure and since $\log|F_X|_v$ is integrable on M, we get the claim. □

Let X be a complete variety over the M-field K.

Lemma 3.5. Let Z be a t-dimensional cycle on X. Suppose that $(s_0,\ldots,s_t)$ and $(s_0',\ldots,s_t')$ are non-zero meromorphic sections of $(\hat{L}_0,\ldots,\hat{L}_t) \in (\mathfrak{g}_X)^{t+1}$ such that $(|\mathrm{div}(s_0)| + |\mathrm{div}(s_0')|,\ldots,|\mathrm{div}(s_t)| + |\mathrm{div}(s_t')|, |Z|$ intersect properly. If the local height $\lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z,v)$ is well-defined and integrable on M, then $\lambda_{(\hat{L}_0,s_0'),\ldots,(\hat{L}_t,s_t')}(Z,v)$ is also well-defined and integrable on M.

Proof: From the theory of local heights, we have for almost all $v \in M$

$$\lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z,v) - \lambda_{(\hat{L}_0,s_0'),\ldots,(\hat{L}_t,s_t')}(Z,v) = \sum_{j=0}^{t} \log|f_j(Z_j)|_v \;,$$

where $f_j := s_j'/s_j$ and

$$Z_j := \mathrm{div}(s_0')\ldots\mathrm{div}(s_{j-1}').\mathrm{div}(s_{j+1})\ldots\mathrm{div}(s_t).Z \,.$$

This implies the claim. □

Lemma 3.6. Let F be an infinite field and let Y be a proper variety over F. Let $y_1,\ldots,y_n$ be not necessarily closed points of Y and let $\mathcal{M}$ be a line bundle on Y. Then there is a non-zero meromorphic section s of $\mathcal{M}$ such that

$$\{y_1,\ldots,y_n\} \cap |\mathrm{div}(s)| = \emptyset \,.$$

Proof: By Chow's Lemma, we may assume that Y is projective. There is a very ample line bundle $\mathcal{L}$ on Y such that $\mathcal{L}\otimes\mathcal{M}$ is also very ample. Since F is infinite, for any embedding of Y into projective space, there is a hyperplane section not containing $y_1,\ldots,y_n$. Therefore we have non-zero global sections s_1, s_2 of $\mathcal{L}\otimes\mathcal{M}$ and $\mathcal{M}$, respectively, with

$$y_j \notin |\mathrm{div}(s_1)| \cup |\mathrm{div}(s_2)| \quad (j = 1,\ldots,n) \,.$$

The non-zero meromorphic section $s = s_1/s_2$ satisfies our claim. □

Definition 3.7. A t-dimensional prime cycle Z on X is called integrable with respect to $\hat{L}_0,\ldots,\hat{L}_t \in \mathfrak{g}_X$ if the local height $\lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z,v)$ is well-defined and integrable for some set $(s_0,\ldots,s_t)$ of non-zero meromorphic sections of $(L_0,\ldots,L_t)$. Then the global height of Z with respect to $(\hat{L}_j,s_j)_{j=0,\ldots,t}$ is defined by

$$h(Z) := h_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z) := \int_M \lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z)d\mu(v) \,.$$

By additivity, we extend this definition to cycles.

Corollary 3.8. Let Z be a t-dimensional cycle on X, integrable with respect to $\hat{L}_0, \ldots, \hat{L}_t \in \mathfrak{g}_X$. Then $\lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z,v)$ is well-defined and integrable on M for all non-zero meromorphic sections $(s_0, \ldots, s_t)$ of $(L_0, \ldots, L_t)$ with $|\mathrm{div}(s_0)|, \ldots, |\mathrm{div}(s_t)|$, $|Z|$ intersecting properly. Moreover, for non-zero meromorphic sections $(s_0', \ldots, s_t')$ of $(L_0, \ldots, L_t)$ such that $|\mathrm{div}(s_0)| + |\mathrm{div}(s_0')|, \ldots, |\mathrm{div}(s_t)| + |\mathrm{div}(s_t')|$, $|Z|$ intersect properly, we have

$$h_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z) - h_{(\hat{L}_0,s_0'),\ldots,(\hat{L}_t,s_t')}(Z) = \sum_{j=0}^{t} d_{f_j(Z_j)}$$

where d is the defect of product formula, $f_j := s_j'/s_j$ and

$$Z_j := \mathrm{div}(s_0') \ldots \mathrm{div}(s_{j-1}').\mathrm{div}(s_{j+1}) \ldots \mathrm{div}(s_t).Z\ .$$

Proof: Let $(s_0'', \ldots, s_t'')$ be non-zero meromorphic sections of $(L_0, \ldots, L_t)$ satisfying the hypothesis of Definition 7. We have to prove that the local height $\lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z,v)$ is integrable on M. On a finite field, any absolute value is trivial and we easily deduce that any local height is zero. So we may assume that K is infinite. Using Lemma 6, it is easy to construct non-zero meromorphic sections $(s_0''', \ldots, s_t''')$ of $(L_0, \ldots, L_t)$ such that $|\mathrm{div}(s_0)| + |\mathrm{div}(s_0''')|, \ldots, |\mathrm{div}(s_t)| + |\mathrm{div}(s_t''')|$, $|Z|$ and $|\mathrm{div}(s_0'')| + |\mathrm{div}(s_0''')|, \ldots, |\mathrm{div}(s_t'')| + |\mathrm{div}(s_t''')|$, $|Z|$ intersect properly

Applying Lemma 5 two times, we get integrability. Finally, the last claim is obvious from the proof of Lemma 5. □

Lemma 3.9. Let X be a proper variety over the M-field K and let $\hat{L}_0, \widehat{L_0'}, \ldots, \hat{L}_t, \widehat{L_t'} \in \mathfrak{g}_X$.

a) If a t-dimensional cycle Z of X is integrable with respect to $(\hat{L}_0, \ldots, \hat{L}_t)$ and $(\widehat{L_0'}, \ldots, \widehat{L_t'})$, then Z is integrable with respect to $(\hat{L}_0 \otimes \widehat{L_0'}, \ldots, \hat{L}_t \otimes \widehat{L_t'})$. Moreover, if $(s_0, \ldots, s_t)$ and $(s_0', \ldots, s_t')$ are non-zero meromorphic sections of $(L_0, \ldots, L_t)$ and $(L_0', \ldots, L_t')$, respectively, such that $|\mathrm{div}(s_0)| + |\mathrm{div}(s_0')|, \ldots, |\mathrm{div}(s_t)| + |\mathrm{div}(s_t')|$, $|Z|$ intersect properly, then

$$\begin{aligned} h_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z) &+ h_{(\widehat{L_0'},s_0'),\ldots,(\widehat{L_t'},s_t')}(Z) \\ &= h_{(\hat{L}_0\otimes\widehat{L_0'},s_0\otimes s_0'),\ldots,(\hat{L}_t\otimes\widehat{L_t'},s_t\otimes s_t')}(Z)\ . \end{aligned}$$

b) If $\varphi : X' \to X$ is a morphism of proper varieties over K and if Z' is a t-dimensional cycle on X', then Z' is integrable with respect to $(\varphi^*\hat{L}_0, \ldots, \varphi^*\hat{L}_t)$ if and only if $\varphi_*(Z')$ is integrable with respect to $(\hat{L}_0, \ldots, \hat{L}_t)$. Moreover, if $(s_0, \ldots, s_t)$ are non-zero meromorphic sections of $(L_0, \ldots, L_t)$ such that $|\mathrm{div}(s_0)|, \ldots, |\mathrm{div}(s_t)|$, $|\varphi_*(Z')|$ intersect properly, then

$$h_{(\varphi^*\hat{L}_0,\varphi^*s_0),\ldots,(\varphi^*\hat{L}_t,\varphi^*s_t)}(Z') = h_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(\varphi_*(Z'))\ .$$

Proof: Let us first prove a). As in the proof of Corollary 8, we may assume that K is infinite. Similarly as in the proof of Corollary 8, we see that non-zero meromorphic sections (s_j, s_j') of (L_j, L_j') $(j = 0, \ldots, t)$ satisfying the support condition in a) always exist. Then a) is obvious from Corollary 8 and the multi-linearity of local heights. Claim b) is a consequence of Corollary 8 and the projection formula for local heights. □

Example 3.10. Let $t = 0$ and let X be a proper variety over the algebraically closed M-field K. Then a point $P \in X(K)$ is integrable with respect to the M-metrized line bundle $\bar{L}$ of X if and only if, for any non-zero λ in the fibre over P, $\|\lambda\|_v$ is measurable on M. (Note that $\log \|\lambda\|_v$ is bounded by an integrable function $c(v)$, but no continuity assumption was made in the definition of M-metrics.)

Proposition 3.11. Let X be a complete K-variety and let $\mathfrak{P}_X$ be as in Proposition 3. Then any t-dimensional cycle on X is integrable with respect to $\hat{L}_0, \ldots, \hat{L}_t \in \mathfrak{P}_X$.

Proof: Let Z be a prime cycle of dimension t. Using multi-linearity of local heights, we can replace $\hat{L}_j$ by $\hat{L}_j^{-1}$ without changing integrability. So we may assume $\hat{L}_0, \ldots, \hat{L}_t \in \mathfrak{P}_X^+$. By Lemma 9 and the definition of $\mathfrak{P}_X^+$, we may assume $X = \mathbb{P}_K^{n_0} \times \cdots \times \mathbb{P}_K^{n_t}$ and $\hat{L}_j = \overline{O_{\mathbb{P}}(e_j)}$. Then integrability of Z follows from Lemma 4 and Corollary 8. □

3.12) Let X be a proper variety over the M-field K with $\hat{L}_0, \ldots, \hat{L}_t \in \mathfrak{g}_X^+$. Next we consider the effect of metric change. So let us consider a second set of M-metrics on $L_0, \ldots, L_t$ with resulting M-metrized line bundles $\hat{L}_0', \ldots, \hat{L}_t' \in \mathfrak{g}_X^+$. Let $(s_0, \ldots, s_t)$ be non-zero meromorphic sections of $(L_0, \ldots, L_t)$ and let Z be a t-dimensional cycle on X such that $|\mathrm{div}(s_0)|, \ldots, |\mathrm{div}(s_t)|$, $|Z|$ intersect properly. Suppose that Z is integrable with respect to $(\bar{L}_0, \ldots, \bar{L}_t)$ and $(\hat{L}_0', \ldots, \hat{L}_t')$. Let $\| \ \|_j$ and $\| \ \|_j'$ be the M-metrics of $\hat{L}_j$ and $\hat{L}_j'$, respectively. Then we have real functions $\log(\| \ \|_j' / \| \ \|_j)$ on $X(M)$ bounded by an integrable function c_j on M (Corollary 2.24). From 1.8iii), we get

$$\begin{aligned} &\left| h_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z) - h_{(\hat{L}_0',s_0),\ldots,(\hat{L}_t',s_t)}(Z) \right| \\ &\leq \sum_{j=0}^{t} C_j \deg_{L_0,\ldots,L_{j-1},L_{j+1},\ldots,L_t}(Z) \end{aligned}$$

where the constants $C_j := \int c_j(v)d\mu(v)$ depend only on the metrics on L_j. In other words, the difference of global heights is in $O(\delta_{L_0,\dots,L_t})(Z)$ where

$$\delta_{L_0,\dots,L_t}(Z) := \sum_{j=0}^{t} \deg_{L_0,\dots,L_{j-1},L_{j+1},\dots,L_t}(Z) .$$

Theorem 3.13. For any proper variety X over an M-field K and any base-point free isomorphism classes of line bundles $(L_0,\dots,L_t)$ with non-zero meromorphic sections $(s_0,\dots,s_t)$, there is a real function $h_{(L_0,s_0),\dots,(L_t,s_t)}$, defined for all t-dimensional cycles Z on X such that $|\mathrm{div}(s_0)|,\dots,|\mathrm{div}(s_t)|$, $|Z|$ intersect properly, which is uniquely determined up to $O(\delta_{L_0,\dots,L_t})$ by the following condition:

i) If $(\hat{L}_0,\dots,\hat{L}_t)$ is a set of base-point free M-metrized line bundles in $\mathfrak{g}_X^+$, then

$$h_{(\hat{L}_0,s_0),\dots,(\hat{L}_t,s_t)}(Z) - h_{(L_0,s_0),\dots,(L_t,s_t)}(Z) = O(\delta_{L_0,\dots,L_t})(Z) .$$

where Z ranges over all t-dimensional cycles on X which are integrable with respect to $\hat{L}_0,\dots,\hat{L}_t$ and which satisfy the support condition above.

These global heights have the following properties:

ii) If $\varphi : X' \to X$ is a morphism of proper varieties over K such that $s_0,\dots,s_t$ are non-zero on $\varphi(X')$, then

$$h_{(L_0,s_0),\dots,(L_t,s_t)} \circ \varphi_* - h_{(\varphi^*L_0,\varphi^*s_0),\dots,(\varphi^*L_t,\varphi^*s_t)} = O(\delta_{L_0,\dots,L_t} \circ \varphi_*) .$$

iii) Let $j \in \{0,\dots,t\}$. If L_j' is also a base-point free line bundle on X with non-zero meromorphic section s_j', then

$$\begin{aligned} &h_{(L_0,s_0),\dots,(L_{j-1},s_{j-1}),(L_j\otimes L_j',s_j\otimes s_j'),(L_{j+1},s_{j+1}),\dots,(L_t,s_t)}(Z) \\ &\quad = h_{(L_0,s_0),\dots,(L_j,s_j),\dots,(L_t,s_t)}(Z) \\ &\qquad + h_{(L_0,s_0),\dots,(L_j',s_j'),\dots,(L_t,s_t)}(Z) \\ &\qquad + O(\delta_{L_0,\dots,L_j\otimes L_j',\dots,L_t})(Z) . \end{aligned}$$

where Z ranges over all t-dimensional cycles such that $|\mathrm{div}(s_0)|,\dots,|\mathrm{div}(s_j)|+|\mathrm{div}(s_j')|,\dots,|\mathrm{div}(s_t)|$, $|Z|$ intersect properly.

iv) The global height $h_{(L_0,s_0),\dots,(L_t,s_t)}$ is symmetric in the variables (L_j,s_j) up to $O(\delta_{L_0,\dots,L_t})$.

v) If $(s_0',\dots,s_t')$ is another set of non-zero meromorphic sections of $(L_0,\dots,L_t)$, then in the notation of Corollary 8, we have

$$h_{(L_0,s_0),\dots,(L_t,s_t)}(Z) - h_{(L_0,s_0'),\dots,(L_t,s_t')}(Z) = \sum_{j=0}^{t} d_{f_j(Z_j)} + O(\delta_{L_0,\dots,L_t})(Z)$$

where Z ranges over all t-dimensional cycles on X such that $|\mathrm{div}(s_0)|+|\mathrm{div}(s_0')|),\dots,|\mathrm{div}(s_t)|+|\mathrm{div}(s_t')|$, $|Z|$ intersect properly.

vi) If the product formula is satisfied for (K,M), then the global heights are independent of the choice of $s_0,\dots,s_t$ up to $O(\delta_{L_0,\dots,L_t})$, and they are defined for all t-dimensional cycles.

Proof: From Proposition 11, we know that $L_0,\dots,L_t$ have M-metrics in $\mathfrak{P}_X^+$ such that all t-dimensional cycles are integrable. Let $h_{(L_0,s_0),\dots,(L_t,s_t)}$ be the global height with respect to these M-metrized line bundles. By 12), this is well defined up to $O(\delta_{L_0,\dots,L_t})$ and satisfies i). Properties ii) and iii) are a consequence of Lemma 9. Claim iv) follows from the corresponding statement for local heights. Corollary 8 implies v), and vi) follows from v). In vi), the global height is defined for all t-dimensional cycles Z, since we may assume K infinite as before and then we find non-zero meromorphic sections $(s_0,\dots,s_t)$ of $(L_0,\dots,L_t)$ such that $|\mathrm{div}(s_0)|,\dots,|\mathrm{div}(s_t)|$, $|Z|$ intersect properly by Lemma 6. □

Note that the global heights $h_{(L_0,s_0),\dots,(L_t,s_t)}(X)$ in Theorem 13 do not depend on the choice of the theory of local heights since we have used only $\mathfrak{P}_X$ in the definition.

Example 3.14. Let K be a number field with ring of integers O_K. We want to show that the absolute heights considered in arithmetic intersection theory are the same as the global heights considered above for the M_K-field structure defined in Example 2.6. Let X be a smooth projective variety over K. Suppose that $\mathfrak{X}$ is an O_K-model of X which is projective and flat over O_K. If $\mathcal{L}$ is a line bundle on $\mathfrak{X}$ with restriction L on X, then it induces a line bundle $\mathcal{L}_v$ on $\mathfrak{X} \otimes_{O_K} K_v^\circ$ over the valuation ring K_v° of the completion K_v for any finite place v. The corresponding algebraic metric (Remark 1.23) on $L \otimes_K K_v$ is denoted by $\| \ \|_v$. We denote by X_∞ the disjoint union of the complex manifolds corresponding to $X \otimes_\sigma \mathbb{C}$ where σ runs over all complex embeddings of K in $\mathbb{C}$. On X_∞, F_∞ operates by complex conjugation of points. We consider an F_∞-invariant smooth hermitian metric $\| \ \|$ on the pull back of L to X_∞. For any archimedean place v of K, it induces a smooth hermitian metric $\| \ \|_v$ on $L \otimes_K \bar{K}_v$. Clearly, the family $(\| \ \|_v)_{v\in M_K}$ gives an M_K-metric on L. If we consider the same complete theories of local heights as in Example 1.20, then the M_K-metrized line bundle induced by the hermitian line bundle $(\mathcal{L}, \| \ \|)$ above is in $\mathfrak{g}_X$.

Let t be the dimension of X and let $(\mathcal{L}_j, \| \|)_{j=0,\dots,t}$ be hermitian line bundles on X. Using arithmetic intersection theory, the absolute height of X with respect to the hermitian line bundles is defined by

$$\frac{1}{[K:Q]}\langle \widehat{c}_1(\mathcal{L}_0, \| \|)\dots\widehat{c}_1(\mathcal{L}_t, \| \|)|X\rangle$$

[BGS]. By definition, this expression equals the global height of X with respect to the corresponding M_K-metrized line bundles. Under some restrictions on the metrics, Theorem 13 was proved in the number field case in [Gu].

Example 3.15. Using Theorem 13, we can generalize the first main theorem of Nevanlinna theory. Usually, it says that the height of a holomorphic map from $\mathbb{C}$ into a complex projective variety X with respect to the closed disc with center 0 and radius R is canonical up to $O(1)$ with respect to R (cf. [La2]). The generalization (Theorem 18) is in two ways. First, we consider holomorphic maps from any finite covering of $\mathbb{C}$ to X and then we replace the maps, which are consider as points of X, by subvarieties of X defined over the field of meromorphic functions on the covering.

We use the notation of Examples 2.4 and 2.8. For any $R > 0$, we have an $\bar{M}_R$-field $\bar{K}_R$. If $v \in \bar{M}_R$ induces an absolute value on a finitely generated subfield K' of $\bar{K}_R$, then this restriction is either archimedean or discrete. For the corresponding place v' on K', we use the complete theory of local heights for t-dimensional varieties over $K'_{v'}$ considered in Example 1.20.

Let L be a line bundle on the t-dimensional proper variety X over $\mathbb{C}$. Next, it is shown that any smooth hermitian metric $\| \|$ on L induces a canonical $\bar{M}_R$-metric on $L \otimes_{\mathbb{C}} \bar{K}_R$. Let $v \in \partial\bar{M}_R$ and K', v' as above. Since $| |_{v'}$ is an absolute value on K', we have a homomorphism $K' \to \mathbb{C}$ given by $f \mapsto f(v)$. So we may view K' as a subfield of $\mathbb{C}$ and $| |_{v'}$ is the restriction of the usual absolute value. For almost all $v \in \bar{M}_R$ and K' sufficiently large, we may assume that $X \otimes_{\mathbb{C}} \bar{K}_R$ and $L \otimes_{\mathbb{C}} \bar{K}_R$ are defined over K'. Since $\bar{K'}^{v} = \mathbb{C}$, the hermitian metric on L induces a $\partial\bar{M}_R$-metric on $L \otimes_{\mathbb{C}} \bar{K}_R$ (Definition 2.22).

Now assume $v \notin \partial\bar{M}_R$. Then $X \otimes_{\mathbb{C}} \bar{K}_R^o$ is a canonical model of $X \otimes_{\mathbb{C}} \bar{K}_R$ over the valuation ring $\bar{K}_R^o$ and $L \otimes_{\mathbb{C}} \bar{K}_R^o$ is a canonical line bundle extending $L \otimes_{\mathbb{C}} \bar{K}_R$. Replacing $\bar{K}_R$ by sufficiently large, finitely generated subfields K', we get algebraic metrics $\| \|_{v'}$ on $L(\overline{K'}^{v'})$ (Example 1.20). Hence we get a real function $\| \|$ on $L(\bar{M}_R \backslash \partial\bar{M}_R)$ called the canonical metric. In the special case $\mathbb{P}^n_{\mathbb{C}}$ and $O_{\mathbb{P}^n}(1)$, it is easily checked that the canonical metric agrees with the standard metric induced by coordinates $x_0, \dots, x_n$ of $\mathbb{P}^n_{\mathbb{C}}$.

We claim that the canonical metric on $L \otimes_{\mathbb{C}} \bar{K}_R$ is an $\bar{M}_R \backslash \partial\bar{M}_R$-metric. We have to prove that the real function $\| \|$ is $\bar{M}_R \backslash \partial\bar{M}_R$-bounded. This is clear for $X = \mathbb{P}^n_{\mathbb{C}}$ and $L = O_{\mathbb{P}^n}(1)$ (Example 2.25). In general, the canonical metric is stable under pull back induced by complex morphisms. By Chow's Lemma, we may assume that X is projective. Since the canonical metric is preserved under tensor product, we can reduce the question to base-point free line bundles. But then the claim follows from the special case above.

We conclude that a smooth hermitian metric on L induces an $\bar{M}_R$-metric on $L \otimes_{\mathbb{C}} \bar{K}_R$.

Lemma 3.16. For $j = 0, \dots, t$, let $\hat{L}_j$ be a line bundle with a smooth hermitian metric on the proper variety X over $\mathbb{C}$. Then any t-dimensional cycle on $X \otimes_{\mathbb{C}} \bar{K}_R$ is integrable with respect to the $\bar{M}_R$-metrized line bundles induced by $(\hat{L}_j)_{j=0,\dots,t}$.

Proof: We may assume that Z is a subvariety of $X \otimes_{\mathbb{C}} \bar{K}_R$. By projection formula, we may assume that X is projective. In $\mathrm{Pic}(X)$, any line bundle is difference of two very ample ones. If we use standard metrics on very ample line bundles, then the induced local height is integrable on $\bar{M}_R$ by Proposition 11. So we may assume that all L_j are very ample and have metrics $\| \|_j$ with positive Chern forms. There are non-zero global sections s_j of L_j such that $|\mathrm{div}(s'_0)|, \dots, |\mathrm{div}(s'_t)|$, $|Z|$ intersect properly where s'_j is the pullback of s_j to $L_j \otimes_{\mathbb{C}} \bar{K}_R$. The local height $\lambda(Z, v)$ of Z induced by $(\widehat{L}_j, s_j)_{j=0,\dots,t}$ is well-defined. The compatibility of local heights required in Definition 2 is clear in the archimedean case and follows from 1.8v) and the projection formula in the non-archimedean case. By Proposition 11, $\lambda(Z, v)$ is integrable on $\bar{M}_R \backslash \partial\bar{M}_R$.

Finally, we have to prove that $\lambda(Z, v)$ is integrable on $\partial\bar{M}_R$. Let $\| \|'_j$ be a standard metric on L_j and let $\lambda'(Z, v)$ be the local height of Z with respect to $(L_j, \| \|'_j)_{j=0,\dots,t}$. By arithmetic intersection theory, we have for almost all $v \in \partial\bar{M}_R$

$$\lambda(Z,v) - \lambda'(Z,v)$$

$$= \sum_{j=0}^{t} \int_{Z_v} \log(\| \|'_j / \| \|_j) c_1(\hat{L}'_0) \wedge \dots \wedge c_1(\hat{L}'_{j-1}) \wedge c_1(\hat{L}_{j+1}) \wedge \dots \wedge c_1(\hat{L}_t)$$

where Z_v is the specialization of Z in v. It is the closed subvariety of X obtained in the following way: For almost all $v \in \partial\bar{M}_R$, Z is defined over a finitely generated field K' where the restriction of v induces an absolute value. Using base change with respect to the canonical embedding $K' \subset \mathbb{C}$, we get Z_v. It is enough to show that all integrals $I_j(v)$ on the right hand side are integrable on $\partial\bar{M}_R$. Using positivity of the Chern forms and the well-known

$$\int_{Z_v} c_1(\hat{L}'_0) \wedge \dots \wedge c_1(\hat{L}'_{j-1}) \wedge c_1(\hat{L}_{j+1}) \wedge \dots \wedge c_1(\hat{L}_t)$$

$$= \deg_{L_0,\dots,L_{j-1},L_{j+1},\dots,L_t}(Z) ,$$

the function $I_j(v)$ is bounded on $\partial\bar{M}_R$. So it is enough to prove measurability of I_j on $\partial\bar{M}_R$. Let V be an open subset of $\partial\bar{M}_R$ such that $\lambda(Z,v)$ and $\lambda'(Z,v)$ are defined for all $v \in V$. Then the family $(Z_v)_{v\in V}$ defines a holomorphic subvariety of $X \times V$. It follows from a classical result of Federer, Stoll and King (cf.[Ki, 3.3]) that $I_j(v)$ is a continuous function on V. This proves measurability. □

3.17) Let K be the field of meromorphic functions on $\mathbb{C}$. For any finite subextension L/K of the algebraic closure $\bar{K}$ over K, we have a canonical finite holomorphic covering of $\mathbb{C}$ by a smooth holomorphic Riemann surface M_L^∞. For $R>0$, let $M_L^R \subset M_L^\infty$ be the inverse image of the closed ball with radius R. Similarly as in Example 2.8, we get an $\bar{M}^R := \varprojlim M_L^R$-structure on $\bar{K}$ where L/K is ranging over all finite subextensions of $\bar{K}/K$.

Under the assumptions of Lemma 16, we fix a t-dimensional cycle Z on $X \otimes_{\mathbb{C}} \bar{K}$. We choose non-zero meromorphic sections s_j of L_j such that the base change of $\mathrm{div}(s_0),\ldots,\mathrm{div}(s_t)$ to $\bar{K}$ and Z form a proper intersection, i.e. the intersection of supports is empty. Similarly as in 15), the holomorphic line bundles $\hat{L}_j$ induce $\bar{M}^R$-metrized line bundles on $X \otimes_{\mathbb{C}} \bar{K}$ and Lemma 16 is true for $\bar{K}$, $\bar{M}^R$ replacing $\bar{K}_R$, $\bar{M}_R$. Let $h_R(Z)$ be the global height of Z induced by $(\hat{L}_j, s_j)_{j=0,\ldots,t}$ using the $\bar{M}^R$-structure on K. An equivalent way to define $h_R(Z)$ is to use an embedding $\bar{K} \subset \bar{K}_R$ so that we can apply Lemma 16 directly for the $\bar{M}_R$-field structure on $\bar{K}_R$.

Theorem 3.18. With Z fixed as above, the global height $h_R(Z)$ neither depends on the choice of the hermitian metrics on L_j nor on the choice of the meromorphic sections s_j up to $O(1)$ with respect to varying R.

Proof: To estimate the difference of global heights of Z for a metric change, we use the considerations in 12). Hence, we have to lock for an integrable function $c_j(v)$ on $\bar{M}^R$ bounding the real function $\log(\|\ \|_j' / \|\ \|_j)$ on $X(\bar{M}^R)$. On the boundary $\partial\bar{M}^R$, this quantity is bounded by the same constant c_j as the corresponding term with the original metrics on L_j, i.e. c_j neither depends on v nor on R. For $v \notin \partial\bar{M}^R$, the metric is canonical. Hence, we can use $c_j = 0$ on the complement of the boundary. By 12), this proves the claim for the change of hermitian metrics.

To get the last claim, it is enough to replace s_0 by a non-zero meromorphic section s_0' of L_0 such that the base change of $\mathrm{div}(s_0'),\ldots,\mathrm{div}(s_t)$ to $\bar{K}$ and Z form a proper intersection. By Corollary 8, the difference of global heights $h_R(Z) - h_R'(Z)$ equals $d^R_{f_0(Z_0)}$ where the defect d^R is computed with respect to the $\bar{M}^R$-field structure of $\bar{K}$. We may view $f_0(Z_0)$ as a meromorphic function on some finite covering M_L^∞ as in 17). By Example 2.8, we have

$$d^R_{f_0(Z_0)} = \frac{1}{n} d^R_{N f_0(Z_0)}$$

where n is the degree and N is the norm of L/K. Using the formula for the defect in Example 2.4, we conclude that $d^R_{f_0(Z_0)}$ is independent of R. This proves the claim. □

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