IV.3. Zeros of an Analytic Function

Note. In this section we consider properties of the set of points where an analytic function is zero. We define entire functions, prove Liouville's Theorem (Theorem IV.3.4) for entire functions, and then use this to prove the Fundamental Theorem of Algebra (Theorem IV.3.5). In Corollary IV.3.9 we see that each zero of an analytic function is of finite multiplicity. Finally, we state our first version of the Maximum Modulus Theorem (Theorem IV.3.11).

Definition. If $f : G \to \mathbb{C}$ is analytic and $a \in G$ satisfies $f(a) = 0$, then a is a *zero of multiplicity* $m \geq 1$ if there is analytic $g : G \to \mathbb{C}$ such that $f(z) = (z-a)^m g(z)$ where $g(a) \neq 0$.

Note. Conway states on page 76: "The reader might be pleasantly surprised to know that after many years of studying Mathematics [they are] right now on the threshold of proving the Fundamental Theorem of Algebra." We give a proof based on treating a polynomial as an analytic function with infinite radius of convergence.

Definition. An *entire function* is a function analytic in the entire complex plane. Entire functions are sometimes called *integral functions.*

Note. An area of study in complex analysis is entire function theory. A classical book in this area is Ralph Boas' *Entire Functions* (Academic Press, 1954). Results in this are often concern factorization and rates of growth of such functions. We consider some of their properties in Chapter XI, "Entire Functions." By Theorem

IV.2.8, we have that an entire function has a power series centered at $a = 0$ valid on $B(0; R)$ for arbitrary $R > 0$. This can be summarized as given in the next proposition.

Proposition IV.3.3. If f is an entire function then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with infinite radius of convergence.

Note. The next result, appears in Joseph Liouville's "Leçons sur les fonctions doubelement péiodiques," *Journal für Mathematik Bd.*, **88**(4), 277–310 (1847). It is commonly known as Liouvilles Theorem after the French mathematician Joseph Liouville (March 24, 1809–September 8, 1882). However, the result was published earlier by Augustin Louis Cauchy (August 21, 1789–May 23, 1857) in *Comptes rendus de l'Académie des Sciences*, **19**, 1377–1384 (1844). With it, we can give an easy proof of the Fundamental Theorem of Algebra.

Augustin Louis Cauchy and Joseph Liouville. Images from the MacTutor History of Mathematics websites on Cauchy and Liouville (accessed 11/7/2023).

Theorem IV.3.4. Liouville's Theorem.

If f is a bounded entire function then f is constant.

Note. Notice the quote on page 77:

> "The reader should not be deceived into thinking that this theorem is insignificant because it has such a short proof. We have expended a great deal of effort building up machinery and increasing our knowledge of analytic functions. We have plowed, planted, and fertilized; we shouldn't be surprised if, occasionally, something is available for easy picking."

Of course, Liouville's Theorem does not hold for functions of a real variable: Consider $\sin x$, $\cos x$, $1/(x^2 + 1)$.

Note. Being the Fundamental Theorem of *Algebra*, we might think that there is a purely algebraic proof of the Fundamental Theorem of Algebra. However, as we see in Modern Algebra 2 (MATH 5410), no such proof exists. In that class, a proof which borrows only two results from analysis is given. The two analysis results are: (A) every positive real number has a real positive square root, and (B) every polynomial in $\mathbb{R}[x]$ (that is, every polynomial with real coefficients) of odd degree has a root in $\mathbb{R}$. Result (A) is based on the completeness of the real numbers. An analysis proof of (A) is given in Analysis 1 (MATH 4127/5127); see my online notes for Analysis 1 on Section 1.3. The Completeness Axiom (notice Exercise 1.3.9). Result (B) is the Intermediate Value Theorem. This is also covered in Analysis 1; see my online notes on Section 4.1. Limits and Continuity and notice

Corollary 4-9. For the mostly-algebraic proof, see my online Modern Algebra 2 notes on Section V.3.Appendix. The Fundamental Theorem of Algebra. We now give a proof of the Fundamental Theorem of Algebra which is analytic (i.e., uses analysis) and is based primarily on Liouville's Theorem.

Theorem IV.3.5. Fundamental Theorem of Algebra.
If $p(z)$ is a nonconstant polynomial (with complex coefficients) then there is a complex number a with $p(a) = 0$.

Note. With this version of the Fundamental Theorem of Algebra, other versions follow. For example, it follows by the Factor Theorem (see my online notes for Introduction to Modern Algebra [MATH 4217/5217] on Section IV.23. Factorizations of Polynomial; notice Corollary 23.3) that an nth degree complex polynomial can be factored into a product of n linear terms. In other words, an n degree complex polynomial has n zeros (counting multiplicity). In the terminology of modern algebra, the field $\mathbb{C}$ is *algebraically closed* (see Definition 31.14 in my online notes for Introduction to Modern Algebra 2 [MATH 4237/5137] on Section VI.31. Algebraic Extensions).

Note. The following result puts some restrictions on analytic functions in terms of the zeros of the function.

Theorem IV.3.7. Let G be a connected open set and let $f : G \to \mathbb{C}$ be analytic. The following are equivalent.

(a) $f \equiv 0$ on G,

(b) there is a point $a \in G$ such that $f^{(n)}(a) = 0$ for all $n \in \mathbb{Z}$, $n \geq 0$, and

(c) the set $\{z \in G \mid f(z) = 0\}$ has a limit point in G.

Note. Theorem IV.3.7 does not hold for functions of a real variable (where we take "analytic" to mean continuously differentiable). Recall that

$$f(x) = \begin{cases} e^{-1/x^2}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

is infinitely differentiable for all $x \in \mathbb{R}$ and $f^{(n)}(0) = 0$ for all $n \in \mathbb{Z}$, $n \geq 0$, but $f \not\equiv 0$ on $\mathbb{R}$ (so (b) does not imply (a)).

Recall

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

has zeros $\{x \in \mathbb{R} \mid x = 1/(n\pi), n \in \mathbb{Z}\} \cup \{0\}$. So g is continuously differentiable on $\mathbb{R}$ and the set of zeros has a limit point, but $g \not\equiv 0$ on $\mathbb{R}$. That is, (c) does not imply (a).

Note. We extract several corollaries from Theorem IV.3.7.

Corollary IV.3.8. If f and g are analytic on a region G (where G is an open connected set), then $f \equiv g$ if and only if $\{z \in G \mid f(z) = g(z)\}$ has a limit point in G.

Note. Corollary IV.3.8 does not hold in $\mathbb{R}$. Consider $f(x) \equiv 0$ and

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Note. With Theorem IV.3.7, we can factor an analytic function in much the same way that we factor a polynomial. Recall that if p is a polynomial with a zero a of multiplicity m, then $p(z) = (z-a)^m t(z)$ for a polynomial $t(z)$ such that $t(a) \neq 0$. This result for polynomials follows from the Factor Theorem; see, for example, my online notes for Introduction to Modern Algebra (MATH 4127/5127) on Section IV.23. Factorizations of Polynomials and notice Corollary 23.3.

Corollary IV.3.9. If f is analytic on an open connected set G and f is not identically zero then for each $a \in G$ with $f(a) = 0$, there is $n \in \mathbb{N}$ and an analytic function $g : G \to \mathbb{C}$ such that $g(a) \neq 0$ and $f(z) = (z-a)^n g(z)$ for all $z \in G$. That is, each zero of f has finite multiplicity.

Corollary IV.3.10. If $f : G \to \mathbb{C}$ is analytic and not constant, then for any $a \in G$ where $f(a) = 0$, there is an $R > 0$ such that $B(a; R) \subseteq G$ and $f(z) \neq 0$ for $0 < |z - a| < R$.

Note. The following is extremely important! At least, I often use it in my complex research. In the proof, we will use Exercise III.3.17 which says: "Let G be a region and suppose that $f : G \to \mathbb{C}$ is analytic such that $f(G)$ is a subset of a circle or a line. Then f is constant." Now for the Maximum Modulus Theorem.

Theorem IV.3.11. Maximum Modulus Theorem.
If G is a region and $f : G \to \mathbb{C}$ is an analytic function such that there is a point $a \in G$ with $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant.

Note. There are several other results and generalizations related to the Maximum Modulus Theorem in Chapter VI, "The Maximum Modulus Theorem," especially in Section VI.1. The Maximum Principle. One such theorem is the following.

Theorem VI.1.2. Maximum Modulus Theorem—Second Version.
Let G be a bounded open set in $\mathbb{C}$ and suppose f is a continuous function on $\overline{G}$ which is analytic in G. Then

$$\max\{|f(z)| \mid z \in \overline{G}\} = \max\{|f(z)| \mid z \in \partial G\}.$$

($\overline{G}$ is G closure and ∂G is the boundary of G.)

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