Weighted k-Server Bounds via Combinatorial Dichotomies

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The k-Server Problem

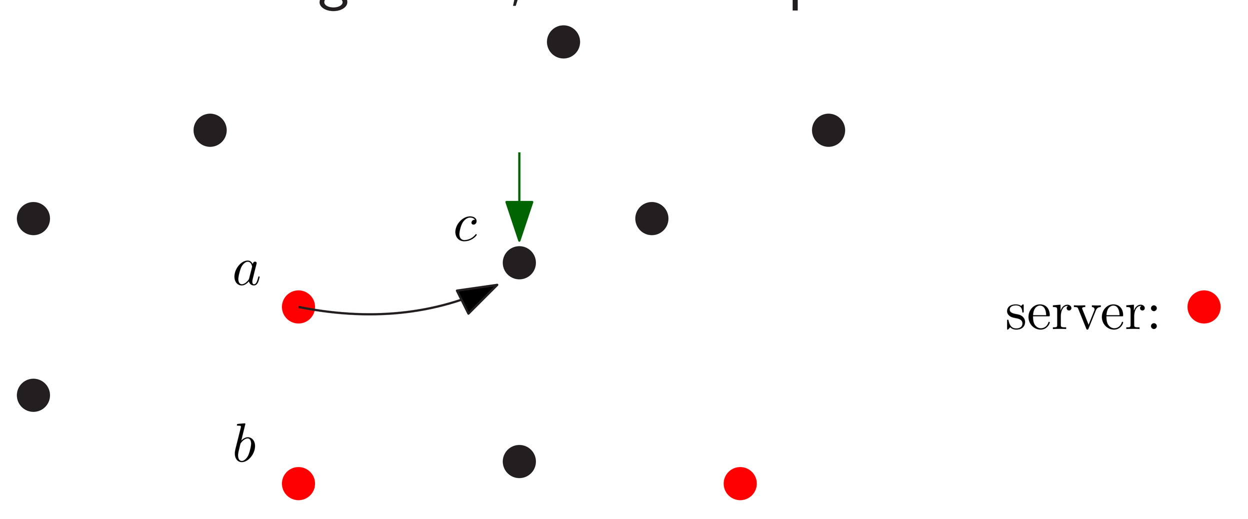
- Generalization of many online problems

Definition of the problem:

- Metric Space of n points, k servers.
- In each time step a point is requested.
- To serve the request, need to move one server there.
- **Goal:** Minimize total distance traveled by the servers.

Example:

- At time 1, request arrives at point c
- we serve it by moving the server from the point a, which might be a wrong move, if a is requested at time 2



$\sigma = (c, ?, ?, ?, ?, \cdots)$

Competitive ratio:

$$R(\sigma) = \frac{ALG(\sigma)}{OPT(\sigma)}; \qquad \text{Competitive ratio} = \max_{\sigma}(R(\sigma))$$

- $ALG(\sigma)$: cost of the online algorithm on σ
- $OPT(\sigma)$: cost of the optimal offline solution on σ

Goal: Achieve the smallest possible competitive ratio

Classic k-Server Results

1. For metrics of $n = k + 1$ points, comp. ratio $\geqslant k$.
   Conjecture that k is the right bound [Manasse, McGeoch, Sleator '88]
2. Conjecture true for $k = 2$, $n = k + 1$ [Manasse, McGeoch, Sleator '88]
3. [Fiat, Rabani, Ravid '90]: $O((k!)^3)$ (smart, ad-hoc)
4. [Koutsoupias, Papadimitriou '94] : $(2k - 1)$ (Work Function Algorithm - WFA)

Morally: Arbitrary n is not much harder than $n = k + 1$.

The Weighted k-Server Problem

- Servers have different weights $w_1 \leqslant w_2 \leqslant \ldots \leqslant w_k$
- Cost of moving server i distance d is $w_i \cdot d$

Focus on uniform metrics (all distances 1)

- Even this case not well understood

Known Results:

1. For $n = k + 1$, comp. ratio $\geqslant (k + 1)! - 1$ [Fiat, Ricklin '94]
   Conjecture that $(k + 1)! - 1$ is the right bound
2. Conjecture true for $k = 2$, $n = k + 1$. [Chrobak, Sgall 2000]
3. [Fiat, Ricklin '94]: $2^{2^{O(k)}}$ (smart, ad-hoc)

Goal: Show that competitive ratio of WFA is $O((k + 1)!)$

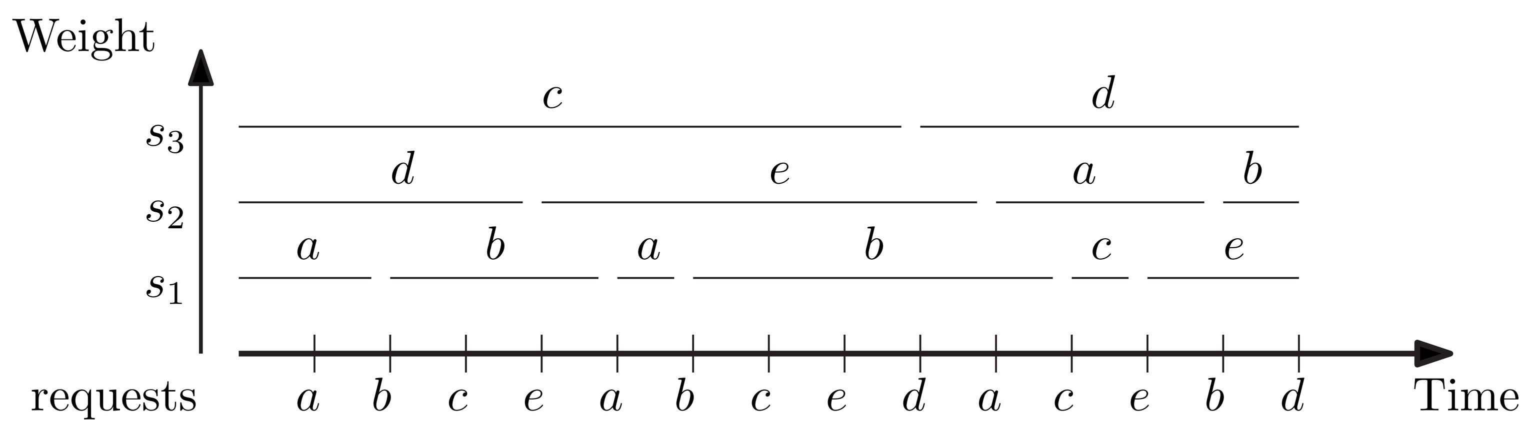
Our Results

1. **Lower Bound $2^{2^{k-4}}$ for deterministic algorithms**
   - Almost matches the $2^{2^{k+1}}$ upper bound
   - For metric spaces of $\geqslant 2^{2^{k-4}}$ points.
2. **Upper Bound $2^{2^{k+O(\log k)}}$ for generalized (WFA)**
3. **For d distinct weights, generalized WFA is $2^{O(d \cdot k^{d+3})}$-competitive**

Our Approach

Combinatorial View:

Solution to weighted k-server = collection of labeled intervals



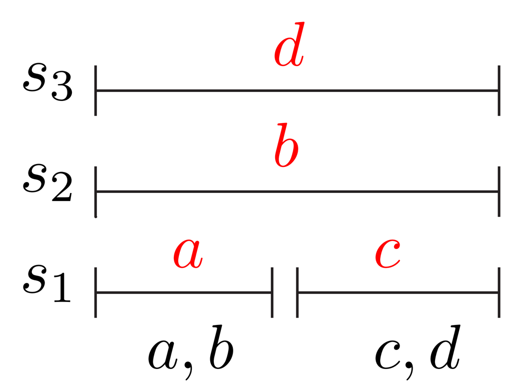
- Intervals at level i = Moves of ith server.
- Label of an interval = Location of the server

A solution has two aspects:

1. Pattern of the intervals (service pattern)
2. Label of each interval

Main Ideas:

- Most crucial decision: Location of server with weight w_{max}
- Focus on service patterns with tree structure
- Assume we know the optimal service pattern
- Problem: It might have many feasible labelings for the root
  $\Rightarrow$ Many "good" locations for server w_{max}



Combinatorial Problem: Given the optimal service pattern, how many feasible choices exist for the root?

Dichotomy Property: Root has either $\leqslant f(k)$ feasible choices or n (i.e location of heavy server does not matter)

Goal: Estimate $f(k)$

- Lower Bound on $f(k) \Rightarrow$ Lower Bound for weighted k-server
- Upper Bound on $f(k) \Rightarrow$ Upper Bound for generalized WFA.

Core Results: $f(k) \geqslant 2^{2^{k-4}}$, $f(k) \leqslant 2^{2^{k+O(\log k)}}$

Open Problems

Randomized Algorithms?

- Best known $2^{2^{k+O(1)}}$, lower bound $\Omega(\log k)$.
- Natural to guess $2^{O(k)}$ bound

Weighted k-server on other metrics? (line, trees, arbitrary)

- No $f(k)$-competitive known for $k > 2$