# BARYON - BARYON INTERACTIONS

by

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## SUMMARY

The Bethe-Salpeter equation, in the ladder and instantaneous interaction approximations, is investigated fully for the interaction of two nucleons through scalar, pseudo-scalar, and vector fields. It is shown that by using the boundary conditions proposed by Green and Biswas, it is possible to solve the Bethe-Salpeter equation for the instantaneous interaction by a very straight forward method.

A continuity equation for the Bethe-Salpeter wave function is derived by defining a conjugate Bethe-Salpeter wave equation. This equation is used to form an expression independent of the interaction function, which, when integrated over the relative time, reduces to a covariant continuity equation. The probability density is evaluated explicitly in terms of the solution functions and this enables the solutions to be interpreted physically.

It is found that for the uncoupled states, the solution functions correspond directly to Schrodinger wave functions outside the interaction region. For the coupled states, the relations are not as simple, but a method using the probability density expression is given, whereby the scattering parameters for these states can be calculated.

A significant feature of the potentials is that the first order terms are the one boson exchange potentials, and this provides a basis for comparison with the one boson exchange theory of nucleon forces.

Nearly all of the work is carried out in momentum space, and no approximations, non-relativistic or otherwise are made, except to go to the discrete representation for the final numerical calculations. The final equations for the phase shifts can be solved to any arbitrary accuracy by increasing the number of points in the numerical integration procedure.

Results are given for the experimentally verified mesons and a scalar meson, and an attempt is made to fit experimentally derived phase shifts using the meson-nucleon coupling constants as adjustable parameters. The results indicate that either the ladder or instantaneous interaction approximation has a limited range of validity when applied to strong interactions.

STATEMENT

This thesis contains no material which has been accepted for award of any other degree or diploma in any University and that, to the best of my knowledge and belief, the thesis contains no material previously published or written by another person, except when due reference is made in the text of the thesis.

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# CHAPTER I

## INTRODUCTION

### 1.1) The Bethe-Salpeter Equation

The relativistic S-matrix theory of Feynman was first applied to the bound state problem for two interacting Fermi-Dirac particles by Bethe and Salpeter[1]. The Bethe-Salpeter (hereafter referred to as B-S) wave function $\Psi$, is given in their notation by the integral equation

$$\Psi(3,4) = \varphi_{a,b}(3,4) - i\iiiint d\tau_5\, d\tau_6\, d\tau_7\, d\tau_8 \; K_{+a}(3,5)\, K_{+b}(4,6)\, \bar{G}(5,6;7,8)\, \Psi(7,8) \qquad (1.1)$$

The 3 and 4 refer to the four-vectors, $x_3$ and $x_4$, specifying positions of the two particles a and b, and $K_{+a}$, $K_{+b}$ are the nucleon propagators as defined by Feynman[2]. $K_{+a}$ satisfies the equation

$$(i\not{\nabla}_3^a - M)\; K_{+a}(3,5) = \delta_+^4(3,4)$$

where $\not{\nabla} = \nabla . \gamma = \sum_{\nu=1,4} \nabla_\nu \gamma^\nu$, and a similar equation is satisfied by $K_{+b}$. $\bar{G}$ is the interaction function and the inhomogeneous term $\varphi_{a,b}$, is the free particle wave function.

The equation (1.1) can be converted into an integro-differential equation by applying the operator $(i\not{\nabla}_3^a - M_a)\,(i\not{\nabla}_4^b - M_b)$ to give

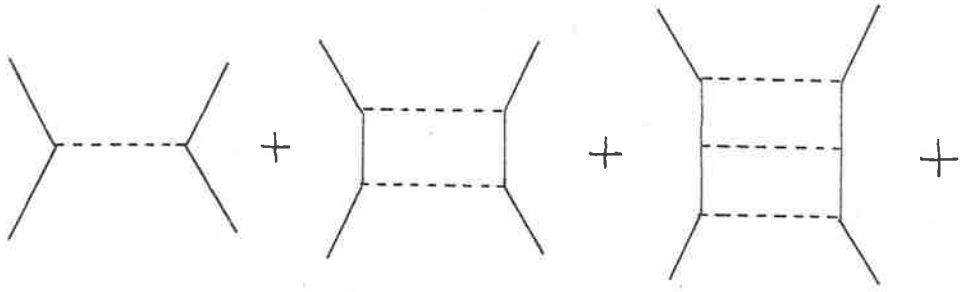
$$(i\not{\partial}_3^a - M_a)\,(i\not{\partial}_4^b - M_b)\;\Psi(3,4) \;=\; i\iint d\tau_5\, d\tau_6$$

$$\bar{G}(3,4;5,6)\;\Psi(5,6) \qquad (1.2)$$

The general interaction function $\bar{G}$, is a sum of the contributions from all possible Feynman graphs that can be constructed for the interaction. We can write

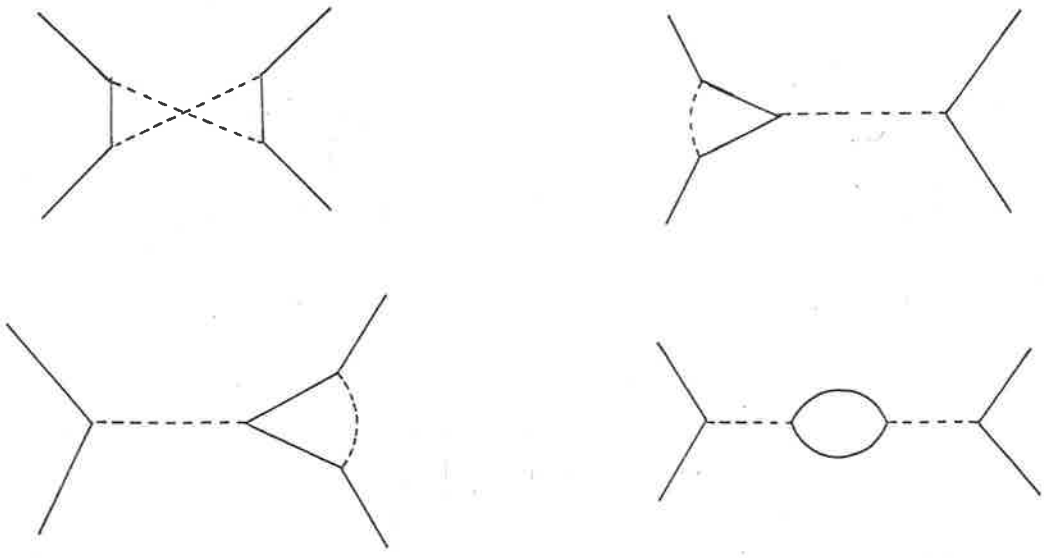
$$\bar{G}(3,4;5,6) \;=\; \sum_m G^{(m)}(3,4;5,6)$$

where the right hand side is the sum over all irreducible graphs, and can be ordered in powers of the coupling constant $g^2$. When the coupling constant is small this may be looked upon as an asymtotic series, and if we only retain the first term $G^{(1)}$, then we take into account all interactions of the type



This is usually known as the ladder approximation. In using this form of interaction, we are assuming that there is only one meson being exchanged at any one time, and graphs involving crossed meson lines are excluded. If we were to include the next higher order term, it would be proportional to $g^4$ and would include all Feynman diagrams which can be reduced to the

following irreducible graphs.



If the coupling constant is small, then including only ladder type graphs should be a good approximation, since the probability of finding two or more virtual mesons in the field will be very small. The currently accepted value of $g_\pi^2 = 14$ for the pion-nucleon coupling constant seems to invalidate this approximation, but recent success of the one boson exchange theory, to be discussed in section 1.4), gives some justification for its use. It is one of the aims of this work to test the validity of the one boson exchange theory by determining the effects of higher order contributions from the ladder diagrams. Actually, by considering vector mesons, which may be looked upon as resonant states of the pseudo-scalar mesons, we are in effect including some forms of crossed meson diagrams. However, to what extent this is effective, is difficult to say.

From now on we shall restrict ourselves to the ladder approximation

where $\overline{G} = G^{(1)} \ldots G^{(1)}$ is of the simple form, $G(3, 4)\ \delta^4(3, 5)$. $\delta^4(4,6)$, and enables the double integral of equation (1.2) to be carried out to give

$$(i \not\partial_3^a - M_a)\ (i \not\partial_4^b - M_b)\ \Psi(3, 4) = i\, G(3, 4)\ \Psi(3, 4) \qquad (1.3)$$

Furthermore, $G(3, 4)$ is given by

$$G(3, 4) = G'(3, 4)\ \Gamma_{a\tau}\ \Gamma_{b\tau}$$

where $\Gamma_a$ and $\Gamma_b$, the vertex parts, are Dirac matrices operating on particles a and b respectively. The index $\tau$ represents summation over different operator components and $G'(3, 4)$ is the quantum propagator, which is a function of the relative coordinates, $x_3 - x_4$ only.

For nucleons interacting through a meson field, $G'(3, 4)$ is the covariant Yukawa potential, which in momentum space can be written as

$$V(\underline{q}, q_o) = \frac{g^2}{2}\ \frac{4\pi}{\underline{q}^2 + \mu^2 - q_o^2} \qquad (1.4)$$

where $\underline{q}$ is the relative momentum, $q_o$ is the relative energy, and $\mu$ is the meson mass. Even with this relatively simple form of interaction, equation (1.3) has not been solved exactly, and is very difficult to work with. Hence we adopt the instantaneous interaction approximation, which corresponds to the neglect of $q_o^2$ with respect to $\underline{q}^2 + \mu^2$ in (1.4). In taking the Fourier transform of (1.4), we find that

$$V(r, t) = \frac{g^2}{2} \delta(t) \frac{e^{-\mu r}}{r}$$

so that $V(r, t)$ is zero except when the relative time $t$ is zero. Physically, this corresponds to neglect of retardation, or the time it takes for the nucleons to exchange the meson, i.e. infinite meson velocity. This should be a valid approximation in the non-relativistic region at least, where the velocity of the nucleons is small relative to the virtual meson velocity. It should be noted that the instantaneous interaction approximation is not in itself a non-covariant approximation and will lead to a covariant continuity equation for the B-S wave function.

In the following work we will use the bi-spinor representation. The operators for particle a act from the right, and of particle b from the left, and the B-S wave equation becomes

$$(i \not{\partial}_1 - M) \Psi (i \not{\partial}_2 - M) = 2 i \delta(t) V(r) \Gamma (\sum_i \tau_i \Psi \tau_i) \Gamma^* \tag{1.5}$$

where the asterisk will always denote hermitian conjugation. In charge independent meson theory, the operator $\sum_i \tau_i --- \tau_i$ has the eigenvalue $-3$ for the isotopic spin singlet state, and $+1$ for the isotopic spin triplet state. $\Gamma$ depends on the type of interaction and is $I$, $i\gamma^5$, or $i\gamma^\mu$, for scalar (scalar), pseudo-scalar (pseudo-scalar) and vector (vector) interactions respectively.

The most general interaction would include the effects of pseudo-vector and tensor mesons, as well as derivative couplings for each of the

fields. There is no experimental evidence for a pseudo-vector meson which could contribute to the nucleon-nucleon potential, but the existence of a number of spin 2 resonances has been experimentally verified. However, such resonances as the B and $f^{\circ}$ would only give very short range forces because of their large masses, and their effects would be negligible except at very high energies.

It is well known that to first order in the coupling constants, the s (v) contribution is zero, and the ps (pv) and ps (ps) contributions are equivalent, so that if these couplings did contribute, their effects would not appear in the non-relativistic region in which we are mainly interested. The only derivative coupling commonly used is that of the vector (tensor) field. However, it is hoped that this will not be required, and we will restrict ourselves to the s(s), ps(ps) and v(v) theories.

The equation (1.5) for a combination of these fields, becomes

$$(i \not\partial_1 - M)\, \Psi\, (i \not\partial_2 - M) = 2 i \delta(t) \Big[ V_1(r)\, \Psi + V_2(r)\, \gamma^5 \Psi \gamma^5 + V_3(r)\, \gamma^\mu \Psi \gamma_\mu \Big] \qquad (1.6)$$

where $V_1(r)$, $V_2(r)$ and $V_3(r)$ are the scalar, pseudo-scalar, and vector meson Yukawa potentials respectively, and the isotopic spin operators have been incorporated in the V(r)'s. Equation (1.6) is written for one meson of each type, but can be extended to include any number of mesons by adding

terms of the appropriate type of interaction and meson mass. This is the equation that will be investigated fully in this work.

## 1.2) Summary of previous work on the solution of the B-S equation

This section can be divided into two parts; (a), when the interacting particles are bosons, and (b), when they are fermions. The scalar particle case has received far more attention because of the relative simplicity of the problem compared with the spin $\frac{1}{2}$ case. However, it has the disadvantage that experimental data from scalar particle interactions, e.g. pion-pion, is scarce, and it cannot be applied realistically to nucleon-nucleon interactions for which there exists a set of reasonably accurate data. This means that we have to turn to the case of interacting fermions to test the B-S theory for strong interactions.

The B-S equation, in the ladder and instantaneous interaction approximations, has been applied successfully to the mass correction of hydrogen-fine-structure by Salpeter[3], and the spectrum of positronium by Klein[4], where the coupling constant is small. However the validity of the ladder approximation for strong interactions has not yet been proved, because in all solutions to date, it has not been possible to estimate the physical effects of other approximations used in solving the B-S equation.

(a) The B-S equation for spinless particles takes on a very simple form in the ladder approximation. The B-S wave function is simply a function of the four vectors of the two particles and does not involve Dirac matrices. The usual way of solving this equation is to use a method based on that of Wick[5] and Cutkosky[6]. Wick considered a bound state and obtained a boundary condition at $t = \pm \infty$ for the four-dimensional wave function. The condition implied that the wave function could be continued analytically to complex values of the relative time variable (or relative energy) and it was allowable to consider the wave function for purely imaginary values of $t$ or $q_o$. A wave function satisfied by this function was obtained by rotation of the integration path in the complex plane of $q_o$. This was applied to two particles bound by a zero mass scalar field and could be reduced to an eigenvalue of the Strum-Liouville type. Cutkosky, in a subsequent paper, found these analytical solutions.

The method of Wick avoids the use of the instantaneous interaction approximation and has been used subsequently with a number of modifications. Early work on the subject may be followed through the works of Scarf[7], Green[8], Levy[9], and Klein[10].

Okubo and Feldman[11] extend the above method to ladder diagrams involving pair annihilation while still using zero mass mesons for spinless nucleons-anti-nucleons in the bound state. They perform a self-energy subtraction and find the energy eigenvalues as a function of the coupling

constant.

Vosko[12] gives a procedure for finding approximate solutions of the B-S equation for scalar particles interacting via a massless field. The method is based on a variational principle and can be applied to an arbitrary interaction function of finite range. The results found suggest that the effects of retardation are large when the field has mass.

Zemach and Schwartz[13] use the Wick rotation to imaginary t and find that the Euclidian wave equation has a solution which behaves as an ordinary Schrodinger scattering wave for large space like distances. They use a modification of the Schwinger variational principle to calculate scattering phase shifts for the interaction of two scalar particles.

(b) For the case of two interacting fermions, the B-S wave function is a 16 component spinor and the B-S equation is much more complicated. This is the more interesting application of the covariant two body equation since it can describe nucleons and electrons, about which we have a considerable amount of experimental data.

Green and Biswas[14] made a fully covariant investigation of the B-S equation for a pair of nucleons interacting through the pseudo-scalar field including pair creation and nucleon recoil. Their method was to reduce the B-S equation to two coupled differential equations and they

obtained matrix solutions of varying degrees of explicitness for instantaneous and delayed interactions, vanishing and non-vanishing meson mass and energy. In a subsequent paper, Biswas[15] used the instantaneous interaction and equated the coefficients of independent $\gamma$-matrices of the coupled equations to obtain a set of 12 equations, which were then integrated over the relative time.

The boundary condition used for the relative time variable was an extension to 4-dimensions of the usual one applied to the 3-dimensional Schrodinger wave function. It said that the B-S wave function and its first spatio-temporal derivative must be finite and continuous everywhere. The use of this boundary condition enabled explicit expressions for the coefficient functions of the independent $\gamma$-matrices to be found.

He identified an equation involving a single function as representing the spin singlet wave equation, and showed that non-relativistically it reduced to a Schrodinger type of equation with a classical Yukawa potential among smaller terms. A feature of the non-relativistic potential was a singularity at $V(r) = M$, where $M$ is the nucleon mass, and $V(r)$ is the Yukawa potential. Here the potential changed from strongly attractive to strongly repulsive via a singularity, and this was interpreted as a "soft" core.

As well as deriving a single equation, (identified with the spin singlet interaction), Biswas[16] also found a pair of coupled equations,

which were identified with the coupled spin triplet $l = j-1$ and $l = j+1$ states. As will be shown later, this identification was not correct. He derived the velocity independent potentials for these states and obtained reasonable fits to Gammel's[17] phenomenological potentials using the pion-nucleon coupling constant as the only adjustable parameter.

Reinfelds[18,19] took the solutions found by Biswas for the pseudo-scalar field, and put them into a simpler and more concise form. His work consists principally of finding numerical solutions of the single equation and the coupled equations given by Biswas. Using the non-relativistic approximation of expanding the operator $T = (M - \nabla^2)^{\frac{1}{2}}$ in powers of $\nabla^2/M^2$, the equations become differential equations, which were solved on a computer using a power series method. First he found a value for the spin triplet "soft" core radius by fitting the deuteron binding energy. The core radius is simply related to the pion-nucleon coupling constant and this was then used to calculate phase shifts from the single and coupled equations.

A probability density for the B-S wave function was also defined. This differs from the one found from more fundamental arguments in this work, and the solution functions could only approximately be related to it. Overall, the significance of the functions in the physical picture were vague and not well understood, so that the results as presented, do not have even qualitative significance, except perhaps for the spin singlet state.

The results of most other authors on the subject of the B-S equation applied to nucleons do not reach a stage where comparison with experiment is possible. We will briefly mention a few of these.

H. Yamamoto[20] starts with the B-S equation in the ladder approximation for bound states of bosons and fermions. For fermions he uses pseudo-scalar coupling with iso-spin. Solution of the exact equation is achieved by replacing the interaction function by an invariant scalar, non-local interaction of the form

$$\int \left[ (p - k)^2 + \mu^2 \right] \Phi(k)\, d^4k = v(p) \int v(k)\, \Phi(k)\, d^4k = C\, v(p)$$

This is unrealistic. $v(p)$ is chosen so that for $\mu = 0$ it approaches Cutkosky's exact solution in the scalar particle case. The results are relations between the coupling constant and the binding energy and for fermions the most favorable bound state is found to be that of total spin 1 (i.e. deuteron). He uses these solutions to test the effect of neglecting retardation and recoil and concludes that the neglect of retardation is qualitatively a good approximation for charge triplet, spin 0, even parity states, and charge singlet, spin 1, even parity states. It is not good for other states. A feature of the solution is that it mixes S and D states of the deuteron with the P state in the relativistic terms, but this is no doubt due to the approximations used.

A. Swift and B. Lee[21] consider a model for fermion-antifermion

scattering mediated by pseudo-scalar bosons in the ladder approximation. They decompose the B-S equation into partial waves and continue the resulting equations into the complex angular momentum plane. They find that Fredholm theory is not applicable because the kernels of their integral equations are not square integrable. However, they obtain solutions in the weak coupling, low energy limit by using an iterative scheme. They also use the optical theorem to calculate the total cross-section for fermion - anti-fermion annihilation.

W. Kummer[22] investigates properties of the B-S equation for two fermions with equal masses, mediated by a massless vector particle. He finds that all continuous solutions have a highly singular behaviour at the origin.

N. Nakanishi[23] finds exact solutions in closed form, of the B-S equation for zero energy bound states of two spinor particles exchanging massless mesons in the ladder approximation. He also calculates the high energy total cross-section.

From these it can be seen that very little of the work is capable of describing the true physical situation for nucleon-nucleon interactions, and most of the solutions are more of mathematical than physical interest. The work of Biswas and Reinfelds attempted to retain only the instantaneous interaction approximation besides the usual ladder approximation, so that if these solutions can be properly interpreted, they may be expected to

give a description of the physical situation. As mentioned before, the effect of these approximations is unknown, so that an exact solution in this case will give an indication of their validity.

## 1.3) Summary of present work

The scope of this work is to solve and interpret fully, the equation (1.6). The method of solution used by Green and Biswas was initially derived to investigate a more general form of interaction than the instantaneous interaction. Their technique was to reduce the B-S equation to two coupled differential equations, and introduce a third. The B-S wave function was then expressed as a sum of the four eigenfunctions of the nucleon-nucleon system, and led to a system of twelve coupled equations. With the introduction of the instantaneous interaction approximation, the integral over the relative time could be carried out and the boundary conditions enabled equations, satisfied by functions depending only on the relative radial coordinate, to be found.

For the pseudo-scalar interaction, their method and ours are equivalent, and would lead to the same results. However, their method cannot be extended to include other types of interactions, and it is in fact unnecessary to go through their lengthy and involved procedure to

obtain solutions in the instantaneous interaction case. In chapter II, the equation (1.6) is solved by a very straightforward method which can be applied to any instantaneous interaction.

To find the physical significance of these solutions, it is necessary to obtain a physical quantity expressed in terms of the solution functions. This is usually done by means of a continuity equation, the derivation of which, for the B-S wave function, has always been an outstanding problem. The method presented in chapter III is basically simple, and leads to a covariant continuity equation involving the physical B-S wave function.

Our method of forming a continuity equation is more fundamental than the one used by Reinfelds, and gives a somewhat different result. The probability density obtained from the continuity equation can be expressed exactly in terms of the solution functions, enabling an accurate identification of our solutions with Schrodinger wave functions.

We find that the solution identified by Biswas as the spin singlet state, does represent this state outside the region of interaction. We also find solutions for the spin triplet $l = j$ state, which were excluded by restrictions imposed on the relative time dependence of the solution functions. Each of these states is described by two functions, $x$ and $w$, and the probability density is given by $\rho = x^*w + w^*x$. Outside the interaction region, $x = w$, and the probability density reduces to the conventional form. When calculating phase shifts, we are only interested

in the wave functions outside the interaction region, so that x and w give the same results, and we are free to identify either with the Schrodinger wave function.

The coupled states are described by four functions which satisfy two pairs of coupled equations. The two coupled equations found by Biswas approximately correspond to one of the pair, and were interpreted as representing the wave equations for the coupled states. This, however, is incorrect as is shown by the probability density, and the fact that unitarity was not satisfied. All four functions contribute to the wave functions of both the coupled states, and in chapter IV a method is given whereby the phase shifts for these states can be calculated using the probability density.

We therefore obtain a complete description of the nucleon-nucleon system as given by the B-S equation in the ladder and instantaneous interaction approximations. By working in momentum space, we avoid making any non-relativistic approximations, so that from our results we should be able to make some conclusive observations regarding the validity of the two approximations used.

It is noted that the "soft" core found by Biswas does not appear in our solutions, and we conclude that it was a result of the non-relativistic approximations used in obtaining the differential equations. The potentials found, can be written as a first order and a second order term in the Yukawa potentials. The first term, proportional to $g^2$, is

readily identified as the one boson exchange (OBE) potential. The second term is proportional to $g^4$, and gives the higher order contributions of the ladder graphs. The appearance of the OBE term is significant, and enables a direct comparison with the OBE theory to be made.

The bosons to be investigated are the ones currently used in the OBE theory of nucleon forces, and except for the scalar boson, are well determined experimentally. Below we list these bosons, with their masses, spins, and parity.

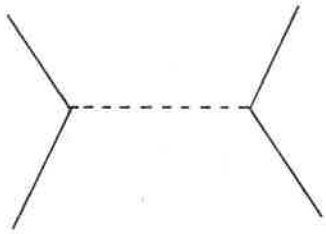
| | Mass (Mev.) | Spin | I-Spin | Parity |
|---|---|---|---|---|
| Scalar boson ($\sigma$) | 500 $\sim$ 700 | 1 | 0 | + |
| $\pi$ | 137 | 0 | 1 | - |
| $\eta$ | 540 | 0 | 0 | - |
| $\rho$ | 750 | 1 | 1 | - |
| $\omega$ | 780 | 1 | 0 | - |

The coupling constants of these mesons to the nucleon will be used as adjustable parameters, since the only coupling constant we have any knowledge of is the pion-nucleon coupling constant $g_{\pi}^{2}$. The scalar boson, which will be denoted by $\sigma$, has no experimental basis, and its mass is usually used as an additional parameter. The OBE theory requires the mass of the scalar meson to be in the range 500 $\sim$ 700 Mev., and it is necessary to cancel the central attractive force of the $\omega$-meson. In the next section, we will briefly discuss the one boson exchange theory of

nucleon forces and its applications.

## 1.4) One boson exchange theory

Because the following theory to first order in $g^2$ gives the one boson exchange potentials, we will briefly discuss to what extent and use the theory is put in nucleon physics. The OBE (one boson exchange) graph is



where the dotted line represents the only meson exchanged in the interaction. Before the discovery of the heavier mesons, the OBE theory was restricted to the one pion exchange (OPE) theory which, because of the large value of the pion-nucleon coupling constant, could only be applied to low energy and high angular momentum interactions.

The OPE theory is used with success in phase shift analysis of nucleon-nucleon scattering cross-sections. It was found by Cziffra et al.[24] that by using the OPE potential phase shifts for the higher partial waves, and OPE contributions to the lower ones, data fits were improved, and considerably reduced the number of parameters required in

the search procedures. This has now become standard practice in phase shift analysis. It also enables a value of the pion coupling constant $g_\pi^2$ to be determined, and is found to have the currently accepted value of about 14.

With the discovery of the $\omega$, $\rho$ and $\eta$ mesons, it was reasonable to calculate potentials based on the OBE interaction. This, to a certain extent, can be considered as including some two and three pion exchanges, since these mesons can be regarded as resonance states of pions. The use of these bosons hence gives some justification to extend the method of OBE to higher energies than allowed for the OPE, and the calculations are usually taken to the non-relativistic limit of 300 Mev.

Probably the most extensive calculations using the OBE theory are by Sawada, Ueda, Watari and Yonezawa[25]. They find rather good fits to phase shifts for $\ell \geqslant 2$ using the above mentioned bosons, plus an undetected scalar boson which is found to be necessary. This scalar boson is also found to be required by others, e.g. Bryan and Scott[26]. The OBE theory is usually restricted to $\ell \geqslant 2$, since below this many more complicated processes are assumed to contribute.

Ino, Matsuda and Sawada[27] have examined the usefulness of more recently found pion resonances such as the $f^\circ$, $\varphi$ and B mesons. They find that the $f^\circ$ meson improves results, but the $\varphi$ and B are not necessary. Because of the success of the OBE and OPE theories, efforts have been made to deduce the effect of the two pion exchange (TPE) inter-

action. e.g. Gupta[28], Furuichi[29], Cottinham[30], Gotsman[31] and Sato[32]. However, the results generally indicate that the TPE effect is not significant except at relativistic energies.

In summarizing, it may be said that there is considerable evidence for the validity of the OPE and OBE model at energies below 300 Mev., and that the pion-nucleon coupling constant is reasonably well established on basis of the OPE theory.

# CHAPTER II

## SOLUTION OF THE BETHE-SALPETER EQUATION

In this chapter we solve equation (1.6) for the B-S wave function $\Psi$. Firstly, the dependence of $\Psi$ on the relative time variable $t$ will be examined. The physical significance of $t$ is not very well understood, but we will use the same method as employed by Biswas and Green [14] to deal with it. This consists of requiring that $\Psi$ is finite and continuous for all $t$, and approaches zero as $t \to \pm \infty$. Finally the equation will be integrated over all $t$.

For the form of $\Psi$ we will take the most general Dirac matrix which is a linear combination of the 16 independent $\gamma$-matrices. We will use the four eigenstates of the nucleon-nucleon system to express the independent Dirac matrices in $\Psi$ so that the coefficient functions will depend on the radial coordinate or the momentum only. By this method it is possible to find equations involving the coefficient functions only and they will be solved numerically at a later stage.

Here, not much will be said about the physical significance of the coefficient functions. The physical interpretation of our solution will be dealt with in the next chapter where a continuity equation is derived and the probability density is expressed in terms of the coefficient functions.

The derivation of a probability density is very important, as it will enable us to relate our coefficient functions to physical wave functions and lead to a method for calculating phase shifts.

Throughout this work, the centre of mass system is used so that the components of the energy-momentum four-vector of the centre of mass are $(0, 0, 0, 2E)$ where $E$ is half the total energy of the system.

The following notation will also be used:

$\tau = t_1 + t_2$ is the physical time of the system

$t = t_1 - t_2$ is the relative time

$r = |\underline{x}| = |\underline{x}_1 - \underline{x}_2|$ is the relative radial coordinate

$q = |\underline{q}| = |\underline{q}_1 - \underline{q}_2|$ is the relative momentum coordinate

$E = i\frac{\partial}{\partial \tau}$. Natural units, $\hbar = c = 1$, will be used throughout.

The operators D and D*, where

$$D = \gamma^4(i\underline{\gamma}.\underline{\nabla} + M)$$

and D* is its hermitian conjugate

$$D^* = (i\underline{\gamma}.\underline{\nabla} + M)\gamma^4$$

will also be used extensively.

In order to avoid the non-relativistic approximation of expanding the operator $T = (M^2 - \Delta)^{\frac{1}{2}}$ in powers of $\Delta/M^2$, we will use the momentum representation for most of the work. This means that the operator D

becomes $\gamma^4(-\underline{\gamma}.\underline{q} + M)$, $T = (q^2 + M^2)^{\frac{1}{2}}$ and the $V(r)$ acting on $\Psi$ become integral operators. The advantages of working in momentum space will become obvious as we proceed.

## 2.1) Dependence on the relative time of the B-S wave function

In section 1.2) it was shown that by use of the co-spinor representation, the B-S equation for two nucleons interacting through the ladder and instantaneous interaction approximations can be written as

$$(i\not{\partial}_1 - M)\Psi(i\not{\partial}_2 - M) = 2i\delta(t)\left[V_1(r)\Psi + V_2(r)\gamma^5\Psi\gamma^5 + V_3(r)\gamma_\mu\Psi\gamma^\mu\right] \qquad (2.1)$$

The $V(r)$ 's are Yukawa potentials of the form

$$V_i(r) = \frac{g_i^2}{4\pi}\frac{e^{-\mu_i r}}{r}\underline{\tau}^{(1)}\ldots\ldots\underline{\tau}^{(2)} \qquad (2.2)$$

where $\mu_i$ is the meson mass, $\underline{\tau}^{(1)}$ and $\underline{\tau}^{(2)}$ are the usual isotopic spin matricies, and $g_i^2$ is the nucleon-meson coupling constant.

We will write equation (2.1) as

$$\left[\left(E + i\frac{\partial}{\partial t}\right)\gamma^4 + i\underline{\gamma}.\underline{\nabla} - M\right]\Psi\left[\left(E - i\frac{\partial}{\partial t}\right)\gamma^4 - i\underline{\gamma}.\underline{\nabla} - M\right] =$$

$$2i\,\delta(t)\left[V_1(r)\,\Psi + V_2(r)\,\gamma^5\Psi\gamma^5 + V_3(r)\,\gamma_\mu\Psi\gamma^\mu\right] \qquad (2.3)$$

or, on employing the D* operator,

$$\left(E - D^* + i\frac{\partial}{\partial t}\right)\Psi\left(E - D^* - i\frac{\partial}{\partial t}\right) = 2i\,\delta(t)\left[V_1(r)\,\gamma^*\Psi\gamma^* + V_2(r)\,\gamma^*\gamma^5\Psi\gamma^5\gamma^* + V_3(r)\,\gamma^*\gamma_\mu\Psi\gamma^\mu\gamma^*\right] \qquad (2.4)$$

If there is no interaction, then the right hand side of equation (2.4) is zero, and we can write down the t-dependence of $\Psi$ by solving the homogeneous differential equation in t. Keeping in mind that the order of operators is important, the general solution is

$$\Psi = e^{i(E-D^*)t}\,A + B\,e^{-i(E-D^*)t}$$

where A and B are Dirac matrices. However, if we include an instantaneous interaction, a general solution is given by

$$\Psi = e^{i(E-D^*)t}\,(X + Y\ \mathrm{sgn}\ t) + (X' + Y'\ \mathrm{sgn}\ t)\,e^{-i(E-D^*)t} \qquad (2.5)$$

This is the form of solution for any instantaneous interaction and does not depend upon the type of field mediating the interaction. In principle it should be possible to apply this method to other functions than the $\delta$-function which is the inhomogeneous term in our case, but finding the solution may be rather difficult. We will now proceed to

examine the effects of the boundary conditions on $\Psi$.

## 2.2) Boundary Conditions

Firstly, we must have $\Psi \rightarrow 0$ as $|t| \rightarrow +\infty$ . Let the eigenvalue of the operator D* acting on $X + Y \text{ sgn } t$ be $a$ , so that

$$D^* (X + Y \text{ sgn } t) = a (X + Y \text{ sgn } t) \qquad (2.6)$$

Then

$$D^{*2} (X + Y \text{ sgn } t) = a^2 (X + Y \text{ sgn } t)$$

From the definition of D* we know that $D^{*2} = T^2 = q^2 + M^2$ so we must choose $a = \pm T$.

To implement the boundary condition at $t = \pm \infty$ , the nucleon mass is taken to have a small positive imaginary part, so that for the first term of equation (2.5) to go to zero as $|t| \rightarrow \infty$ we must have $a = -T$ for $t > 0$ and $a = +T$ for $t < 0$.

i.e.

$$a = -T \text{ sgn } t$$

Similarly, if

$$(X' + Y' \operatorname{sgn} t)\, D^* = (X' + Y' \operatorname{sgn} t)\, a' \qquad (2.7)$$

we must choose

$$a' = T \operatorname{sgn} t$$

and the solution (2.5) becomes

$$= e^{i(T|t| + Et)} (X + Y \operatorname{sgn} t) + (X' + Y' \operatorname{sgn} t)\, e^{i(T|t| - Et)} \qquad (2.8)$$

Reinfelds put the solution found by Biswas for the pseudoscalar interaction into this concise form, but it is obviously possible for any instantaneous interaction.

The boundary conditions also require the wave function to be continuous for all $t$. Hence for $\Psi$ to be continuous at $t = 0$, we must have

$$Y = -Y' \qquad (2.9)$$

We can use this and equations (2.6) and (2.7) to write $\Psi$ in a more useful form. From equation (2.6) we have

$$D^* (X + Y \operatorname{sgn} t) = -T \operatorname{sgn} t\, (X + Y \operatorname{sgn} t)$$

which gives

$$D^*X = -TY \quad \text{and} \quad D^*Y = -TX$$

since $X$ and $Y$ are independent of $t$. Therefore

$$Y = -\frac{D^*}{T} X \tag{2.10}$$

and similarly from equations (2.7) and (2.9)

$$Y = -X' \frac{D^*}{T} \tag{2.11}$$

Using these, the expression for $\Psi$ now becomes

$$\begin{aligned}\Psi &= e^{i(T|t|+Et)}\left(1 - \frac{D^*}{T} \operatorname{sgn} t\right) X + X' \left(1 + \frac{D^*}{T} \operatorname{sgn} t\right) e^{i(T|t|-Et)} \\ &= e^{i(E-D^*)t}\left(1 - \frac{D^*}{T} \operatorname{sgn} t\right) X + X' \left(1 + \frac{D^*}{T} \operatorname{sgn} t\right) e^{-i(E-D^*)t}\end{aligned} \tag{2.12}$$

Thus, by use of the boundary conditions, we have shown that $\Psi$ depends on two Dirac matrices: $X$ and $X'$ which are to be determined. One relation between $X$ and $X'$ is given by equations (2.10) and (2.11). This is

$$D^* X = X' D^* \tag{2.13}$$

Another relation can be found by substituting the $\Psi$ given by (2.12) into equation (2.4) and integrating over $t$ from $t = -\infty$ to $t = +\infty$. This gives

$$\begin{aligned}T(X + X') - \frac{E}{T}(D^*X + X'D^*) &= V_1 \gamma^4(X + X')\gamma^4 \\ &+ V_2 \gamma^4 \gamma^5(X + X')\gamma^5 \gamma^4 + V_3 \gamma^4 \gamma_\mu(X + X')\gamma^\mu \gamma^4\end{aligned} \tag{2.14}$$

The two equations (2.13) and (2.14) are sufficient to determine the Dirac matrices X and X'.

This in some ways compares with the results found by Biswas and expressed by Reinfelds. We have derived a more general result by a much simpler and straight forward method. In the following section we will express X and X' as a linear combination of independent Dirac matrices multiplied by coefficient functions depending on the momentum q only. The notation used will differ from that used by Reinfelds and will enable explicit expressions to be found for all the relevant functions. Also, no restrictions are imposed on the t-parity of the solution functions as was done by Biswas, since this eliminates certain solutions that are physically meaningful.

We note that it should be possible to expand the interaction potential in powers of the relative energy $q_o^2$ of which the first term is the instantaneous interaction approximation. The four dimensional propagator in momentum space is given by

$$V(\underline{q}, q_o) = \frac{g^2}{2} \frac{4\pi}{\underline{q}^2 + \mu^2 - q_o^2}$$

Assuming that $\underline{q}^2 + \mu^2 \gg q_o^2$, we can expand this in powers of $q_o^2$ to give

$$V(\underline{q}, q_o) = \frac{g^2}{2} \frac{4\pi}{(\underline{q}^2 + \mu^2)} \left[ 1 + \frac{q_o^2}{\underline{q}^2 + \mu^2} + \frac{q_o^4}{(\underline{q}^2 + \mu^2)^2} + - - - \right]$$

and upon taking the transform with respect to the relative energy, arrive at

$$V(\underline{q},\ t) \;=\; \frac{g^2}{2}\,\frac{4\pi}{\underline{q}^2+\mu^2}\left[\,\delta(t) + \frac{1}{\underline{q}^2+\mu^2}\;\delta''(t) \;+\; \cdots\;\right]$$

If the second order term is included, then it results in the addition of terms of the form $V'\left[(E - D^*)^2\, X + X'\,(E - D^*)^2\right]$ to terms of order $V(X + X')$ which form the left hand side of equation (2.14). Here $V'$ is the integral operator $\dfrac{g^2}{2}\displaystyle\int \frac{4\pi\ d\underline{q}'}{\left[(\underline{q} - \underline{q}')^2 + \mu^2\right]^2}$ and $V$ is $\dfrac{g^2}{2}\displaystyle\int \frac{4\pi\ d\underline{q}'}{(\underline{q} - \underline{q}')^2 + \mu^2}$ acting on functions of $\underline{q}'$. Also, we know that $D^{*2} = T^2$ so that the operator $D^*$ acting on $X$ and $X'$ produces a factor of order $T$ (e.g. see section 3.2) and hence the expression $(E - D^*)^2\, X + X'\,(E - D^*)^2$ is of order $\dfrac{\underline{q}^4}{M^2}\,(X + X')$.

From this it can be seen that non-relativistically we are justified in neglecting the second and higher order terms in the expansion with respect to the relative energy. This approximation involves the neglect of terms of order $\dfrac{\underline{q}^4}{M^2}\,V^2$ with respect to $V$, where $V$ is the Yukawa potential, and will not affect the non-relativistic interaction.

## 2.3) Form of X and X'

The most general form of X and X' is a linear combination of the sixteen independent $\gamma$-matrices. These will be expressed in terms of the eigenfunctions of $\underline{J}^2$, where $\underline{J}$ is the total angular momentum of the system. Biswas[16] gives the eigenfunctions in the coordinate representation corresponding to the four different states the nucleon-nucleon system. Transformed to momentum space, for a given angular momentum $j$, these are

$$
\begin{aligned}
& l = j \qquad S = 0 \qquad \text{spin singlet state} \\
& \psi_0 = P_j \gamma^4 \\
& \qquad\qquad S = 1 \qquad \text{spin triplet state} \\
& l = j \qquad \psi_1 = (q_1\gamma' + q_2\gamma^2)\, q^{-1} P'_j \gamma^3 \gamma^4 \\
& l = j-1 \qquad \psi_2 = j\gamma^3 P_{j-1} - (q_1\gamma' + q_2\gamma^2)\, q^{-1} P'_{j-1} \\
& l = j+1 \qquad \psi_3 = (j+1)\gamma^3 P_{j+1} + (q_1\gamma' + q_2\gamma^2)\, q^{-1} P_{j+1}
\end{aligned}
\qquad (2.15)
$$

where $l$ and $j$ are the orbital and total angular momentum respectively, S is the total spin of the system, and

$$
\begin{aligned}
P_{j+1} &= P_{j+1}(q_3/q) \\
P'_j(x) &= \frac{\partial}{\partial x} P_j(x)
\end{aligned}
$$

In order to obtain an expression involving 16 independent $\gamma$-matrices using the functions $\psi_0$, $\psi_1$, $\psi_2$ and $\psi_3$, we have to take a linear combination of these, each multiplied by $I$, $\gamma^4$, $\gamma^5$ and $\gamma^4\gamma^5$. This

gives us 16 independent Dirac matrices expressed in terms of the eigenfunctions of the nucleon-nucleon system. They will be multiplied by sixteen unknown functions depending on the momentum $q$ only since the angular dependence is contained in the $\Psi$'s.

Therefore we will write

$$X = \sum_{i=0}^{3} \left[ w_i \Psi_i + x_i \Psi_i \gamma^4 + y_i \Psi_i \gamma^5 + z_i \Psi_i \gamma^4 \gamma^5 \right] \qquad (2.16)$$

$$X' = \sum_{i=0}^{3} \left[ w_i' \Psi_i + x_i' \Psi_i \gamma^4 + y_i' \Psi_i \gamma^5 + z_i' \Psi_i \gamma^4 \gamma^5 \right] \qquad (2.17)$$

where the $w_i$ , $w_i'$ , $x_i$ , $x_i'$ , etc. are functions of $q$ only.

We will now proceed to substitute expressions (2.16) and (2.17) into equations (2.13) and (2.14), and by equating coefficients of the independent Dirac matrices, find equations involving the coefficient functions. The expressions involving the first two terms, $\Psi_i$ and $\Psi_i \gamma^4$, are independent of the expressions involving the terms in $\Psi_i \gamma^5$ and $\Psi_i \gamma^4 \gamma^5$ That is, the interaction mixes terms in $\Psi_i$ and $\Psi_i \gamma^4$ but not in $\Psi_i$ and $\Psi_i \gamma^5$ or $\Psi_i \gamma^4 \gamma^5$. Physically this corresponds to parity conservation of the nucleon-nucleon interaction.

For ease of handling, the two cases will be considered separately so that firstly, we will consider $X$ and $X'$ as a function of the $\Psi_i$ and $\Psi_i \gamma^4$ only.

## 2.4) The $\Psi$ and $\Psi\gamma^4$ solution

In this case we use

$$X = \sum_{i=0}^{3} \left[ w_i \Psi_i + x_i \Psi_i \gamma^4 \right]$$
$$X' = \sum_{i=0}^{3} \left[ w_i' \Psi_i + x_i' \Psi_i \gamma^4 \right] \qquad (2.18)$$

Substituting these into (2.13) and (2.14) and equating coefficients of the $\Psi$'s and $\Psi\gamma^4$'s, leads to 16 equations in 16 unknown functions of $q$.

We will employ the notation that

$$x_0 + x_0' = x_0^+ , \quad w_1 - w_1' = w_1^- , \quad \text{etc.}$$

The effect of $\underline{\gamma}.\underline{q}$ in $D^* = (-\underline{\gamma}.\underline{q} + M)\gamma^4$ on the $\Psi$'s is also required. This is given by the relations

$$\underline{\gamma}.\underline{q}\,\Psi_0 = -\Psi_0\,\underline{\gamma}.\underline{q} = \frac{q}{2j+1}(\Psi_2 + \Psi_3)\gamma^4$$

$$\underline{\gamma}.\underline{q}\,\Psi_1 = \Psi_1\,\underline{\gamma}.\underline{q} = \frac{q}{2j+1}(j\Psi_3 - (j+1)\Psi_2)\gamma^4$$

$$\underline{\gamma}.\underline{q}\,\Psi_2 = -q(j\Psi_0 - \Psi_1) \qquad \Psi_2\,\underline{\gamma}.\underline{q} = -q(j\Psi_0 + \Psi_1)$$
$$\underline{\gamma}.\underline{q}\,\Psi_3 = -q((j+1)\Psi_0 + \Psi_1) \qquad \Psi_3\,\underline{\gamma}.\underline{q} = -q((j+1)\Psi_0 - \Psi_1) \qquad (2.19)$$

On putting (2.18) into (2.13) we obtain the following eight equations.

$$M x_0^- = q(j w_2^+ - (j+1) w_3^+) \qquad (2.20)$$

$$M x_1^- = -q(w_2^- - w_3^-) \qquad (2.21)$$

$$M x_2^+ = -\frac{q}{2j+1}(w_0^+ - (j+1) w_1^-) \qquad (2.22)$$

$$M x_3^+ = -\frac{q}{2j+1}(w_0^+ + j w_1^-) \qquad (2.23)$$

$$M w_0^- = q\,(j\,x_2^- + (j+1)\,x_3^-) \tag{2.24}$$

$$M w_1^- = -q\,(x_2^+ - x_3^+) \tag{2.25}$$

$$M w_2^+ = -\frac{q}{2j+1}\,(x_0^- - (j+1)\,x_1^+) \tag{2.26}$$

$$M w_3^+ = -\frac{q}{2j+1}\,(x_0^- + j\,x_1^+) \tag{2.27}$$

Before we can write down the equations from equation (2.14), the Yukawa potential terms have to be transformed to momentum space. Now the right hand side of equation (2.4) involves a sum of terms of the form

$$V(r)\,a(r)\,\Psi_i = \frac{g^2}{2}\frac{e^{-\mu r}}{r}\,a(r)\,\Psi_i \tag{2.28}$$

where $\Psi_i$ may be any of the functions in coordinate space defined in (2.15), and is a function of the angular coordinates only. The three dimensional Fourier transform of $V(r)$ is

$$V(\underline{q}) = \frac{g^2}{2}\frac{4\pi}{(\underline{q}^2 + \mu^2)}$$

and we expand $V(\underline{q} - \underline{q}')$ as follows:

$$V(\underline{q} - \underline{q}') = \frac{g^2}{2}\frac{4\pi}{(\underline{q} - \underline{q}')^2 + \mu^2} = \frac{g^2}{2}\frac{4\pi}{2qq'(z - \cos\theta)}$$

where $\cos\theta = \underline{q}.\underline{q}' / |\underline{q}.\underline{q}'|$

and $z = \dfrac{q^2 + q'^2 + \mu^2}{2qq'}$

Upon using the relation

$$\frac{1}{z - t} = \sum_n (2n + 1)\, Q_n(z)\, P_n(t)$$

where $P_n$ is the Legendre polynomial and $Q_n$ is the Legendre function of the second kind, we can write $V(\underline{q} - \underline{q}')$ as

$$V(\underline{q} - \underline{q}') = \frac{g^2}{2} \frac{4\pi}{2qq'} \sum_n (2n + 1)\, Q_n(z)\, P_n(\cos\theta)$$

The transform of a term such as $V(r)\, a(r)\, P_j(\cos\theta')$, where $\cos\theta' = x_3/r$, then becomes

$$\frac{1}{(2\pi)^3} \int V(\underline{q} - \underline{q}')\, a(\underline{q}')\, P_j(\cos\theta')\, d\underline{q}'$$

$$= \frac{g^2}{2} \frac{4\pi}{(2\pi)^3} \int \frac{1}{2qq'} \sum_n (2n + 1)\, Q_n(z)\, P_n(\cos\theta)\, a(q')\, P_j(\cos\theta')\, d\underline{q}'$$

When the integration over the angular variables is carried out with help of the addition theorem, the expression reduces to

$$\frac{1}{\pi q} \int_0^\infty q'\, dq'\, Q_j(z)\, a(q')\, P_j(q_3/q)$$

We have used $P_j$ instead of the $\Psi_i$ to show this, but the results hold for the four different $\Psi$'s given in (2.15). The subscript $j$ of $Q_j(z)$ is determined by the spin angle function multiplying the $V(r)$. It will be $j$ for $\Psi_0$ and $\Psi_1$, and $j-1$ and $j+1$ for the terms involving $\Psi_2$ and $\Psi_3$ respectively, so that the transform of $V(r)\, w_2(r)\, \Psi_2$ is given

by $\frac{1}{\pi q}\int_0^\infty q'\,dq'\,Q_{j-1}(z)\,w_2(q')\,\psi_2$ etc.

We will write the transform of $V_1(r)\,x_0(r)\,\gamma^4\psi_1\gamma^4$ as $\frac{1}{q}W_{1,j}\,q\,\gamma^4\psi_0\gamma^4$, so that $W_{1,j}$ is the integral operator $\frac{g_1^2}{2}\frac{1}{\pi}\int_0^\infty dq'\,Q_j(z_1)$ and the argument of the Legendre function is

$$z_1 = \frac{q^2 + q'^2 + \mu_1^2}{2qq'}$$

The notation employed will be to write the function $x_0 = x_0(q)$ on the right of the operator if it is to be integrated over, and to the left if it is not included in the integral. Hence

$$\frac{1}{q}W_{1,j}\,q\,x_0\gamma^4\psi_0\gamma^4 = \frac{g_1^2}{2}\frac{1}{\pi q}\int_0^\infty q'\,dq'\,Q_j(z_1)\,x_0(q')\,\gamma^4\psi_0\gamma^4$$

By using the commutation relations of the $\gamma$'s, we can eliminate the $\gamma^4$'s and $\gamma^5$'s from the right hand side of equation (2.14) and it becomes convenient to define new operators as follows:

$$\begin{aligned}
A_j &= W_{1,j} - W_{2,j} - 4W_{3,j}\\
B_j &= W_{1,j} + W_{2,j} + 2W_{3,j}\\
C_j &= W_{1,j} - W_{2,j}\\
D_j &= W_{1,j} + W_{2,j} - 2W_{3,j}
\end{aligned} \qquad (2.29)$$

We are now in a position to equate the coefficients of the $\psi$'s and

$\psi\gamma$'s in (2.14) to obtain eight more equations

$$(T - \frac{1}{q} B_j\, q)\, w_o^+ = \frac{E}{T}\left[ M\, x_o^+ - q(j\, w_2^- + (j+1)\, w_3^-)\right] \tag{2.30}$$

$$(T - \frac{1}{q} D_j\, q)\, w_1^+ = \frac{E}{T}\left[ M\, x_1^+ + q\,(w_2^+ - w_3^+)\right] \tag{2.31}$$

$$(T + \frac{1}{q} B_{j-1}\, q)\, w_2^+ = \frac{E}{T}\left[ - M\, x_2^- - \frac{q}{2j+1}\,(w_o^- - (j+1)\, w_1^+)\right] \tag{2.32}$$

$$(T + \frac{1}{q} B_{j+1}\, q)\, w_3^+ = \frac{E}{T}\left[ - M\, x_3^- - \frac{q}{2j+1}\,(w_o^- + j\, w_1^+)\right] \tag{2.33}$$

$$(T - \frac{1}{q} A_j\, q)\, x_o^+ = \frac{E}{T}\left[ M\, w_o^+ - q\,(j\, x_2^+ + (j+1)\, x_3^+)\right] \tag{2.34}$$

$$(T - \frac{1}{q} C_j\, q)\, x_1^+ = \frac{E}{T}\left[ M\, w_1^+ + q\,(x_2^- - x_3^-)\right] \tag{2.35}$$

$$(T + \frac{1}{q} C_{j-1}\, q)\, x_2^+ = \frac{E}{T}\left[ - M\, w_2^- - \frac{q}{2j+1}\,(x_o^+ - (j+1)\, x_1^-)\right] \tag{2.36}$$

$$(T + \frac{1}{q} C_{j+1}\, q)\, x_3^+ = \frac{E}{T}\left[ - M\, w_3^- - \frac{q}{2j+1}\,(x_o^+ + j\, x_1^-)\right] \tag{2.37}$$

The equations (2.20) - (2.27) and (2.30) - (2.37) can now be solved to obtain individual equations in $w_o^+$, $w_1^+$, $x_o^+$ and $x_1^+$, which will be shown to form the spin singlet and $1 = j$ spin triplet probability densities. Care must be taken to preserve the order of the functions and operators. In our notation two operators together represent a double integral e.g. if

$$B_j\, w_o^+ = \int_o^\infty B_j(q, q')\, w_o^+(q')\, dq'$$

then

$$C_j \frac{1}{T} B_j\, w_o^+ = \iint_o^\infty C_j(q,q') \frac{1}{(q'^2 + M^2)^{\frac{1}{2}}} B_j(q', q'')\, w_o^+(q'')\, dq''\, dq'$$

$$= \int_0^\infty \left[ \int_0^\infty C_j(q,q'') \frac{1}{(q''^2 + M^2)^{\frac{1}{2}}} B_j(q'',q')\, dq'' \right] w_o^+(q')\, dq'$$

Using equations (2.36), (2.37), (2.22), (2.23), 2.21), (2.25), (2.34) and (2.30) in that order, we find that $x_o^+$ satisfies an equation which will be written as

$$(E^2 - T^2)\, x_o^+ = \frac{1}{q} K\, q\, x_o^+ \tag{2.38}$$

where

$$K = -\frac{1}{T}\left\{ T^2 A_j + \left[ M^2 B_j - q\left( \frac{j}{2j+1} C_{j-1} + \frac{j+1}{2j+1} C_{j+1} \right) q \right] \frac{1}{T}(T - A_j) \right\} \tag{2.39}$$

Making use of equation (2.34) we also find an equation for $w_o^+$ which we will write as

$$(E^2 - T^2)\, (T\, w_o^+) = \frac{1}{q} K^T q (T\, w_o^+) \tag{2.40}$$

In equation (2.39) K is an integral operator that may be written as $\int_0^\infty dq'\, K(q,q')$. K then corresponds to the same integral operator with the q and q' interchanged in the kernel, i.e. $K^T$ is the operator $\int_0^\infty dq'\, K(q',q)$. In our notation $K^T$ would be written as

$$K^T = -\left\{ A_j T^2 + (T - A_j) \frac{1}{T} \left[ M^2 B_j - q\left( \frac{j}{2j+1} C_{j-1} + \frac{j+1}{2j+1} C_{j+1} \right) q \right] \right\} \frac{1}{T}$$

For the numerical work the integral operators are taken as square matrices, in which case $K^T$ is the transpose of the matrix K. Written in full, equation (2.38) is

$$(E^2 - T^2)\, x_0^+(q) = -\frac{1}{\pi q(q^2+M^2)^{\frac{1}{2}}}\int_0^\infty q' \Big\{ (q^2+M^2)\, A_j(q,q') + M^2 B_j(q,q') - q\,q' \Big[ \frac{j}{2j+1} C_{j-1}(q,q') + \frac{j+1}{2j+1} C_{j+1}(q,q') \Big] - \frac{1}{\pi}\int_0^\infty q'' \Big[ M^2 B_j(q,q'') - q\,q'' \Big( \frac{j}{2j+1} C_{j-1}(q,q'') + \frac{j+1}{2j+1} C_{j+1}(q,q'') \Big) \Big] \cdot \frac{1}{(q''^2+M^2)^{\frac{1}{2}}} A_j(q'',q')\, dq'' \Big\} x_0^+(q')\, dq'$$

In a similar way, using equations (2.26), (2.27), (2.36), (2.20) and (2.32) in (2.33) gives an equation for $w_1^+$.

$$(E^2 - T^2)\, w_1^+ = \frac{1}{q} H\, q\, w_1^+ \tag{2.42}$$

where

$$H = -\frac{1}{T}\Big\{ T^2 D_j + \Big[ M^2 C_j - q \Big( \frac{j+1}{2j+1} B_{j-1} + \frac{j}{2j+1} B_{j+1} \Big) q \Big] \frac{1}{T} (T - D_j) \Big\} \tag{2.42}$$

and using equation (2.31) gives the following equation involving $x_1^+$.

$$(E^2 - T^2)\, (T\, x_1^+) = \frac{1}{q} H^T q\, (T\, x_1^+) \tag{2.43}$$

where the superscript on $H^T$ has the same meaning as before.

In the next chapter, it will be shown that $x_0^+$ or $T\, w_0^+$ can be directly identified with the spin singlet wave function outside the region of interaction, whereas $w_1^+$ or $T\, x_1^+$ corresponds to the spin triplet $l = j$ wave function. The operators $K$ and $H$ thus correspond to the momentum space potentials of the spin singlet and $l = j$ spin triplet

states respectively.

It can be seen that the potentials consist of first order and second order terms in the Yukawa potential operators. It will be shown that the first order terms may readily be identified with the one boson exchange potentials, and the Born approximation phase shifts are listed in section 4.3). The second order terms must correspond to higher order processes than the OBE, but are not readily identifiable with the two boson exchange potential which may be expected to be next most important process in the interaction.

To obtain solutions for the opposite parity states, we have to take the X and X' as a sum of the $\psi$'s multiplied by $\gamma^5$ and $\gamma^4\gamma^5$. This we proceed to do now.

## 2.5) The $\psi\gamma^5$ and $\psi\gamma^4\gamma^5$ solution

This time we take X and X' to be

$$X = \sum_{i=0}^{3} \left[ y_i \psi_i + z_i \psi_i \gamma^4 \right] \gamma^5$$

$$X' = \sum_{i=0}^{3} \left[ y'_i \psi_i + z'_i \psi_i \gamma^4 \right] \gamma^5 \qquad (2.44)$$

Putting these into equation (2.13) gives

$$M z_0^+ = q (j y_2^+ - (j + 1) y_3^+) \tag{2.45}$$

$$M z_1^+ = -q (y_2^- - y_3^-) \tag{2.46}$$

$$M z_2^- = -\frac{q}{2j+1} (y_0^+ - (j + 1) y_1^-) \tag{2.47}$$

$$M z_3^- = -\frac{q}{2j+1} (y_0^+ + j y_1^-) \tag{2.48}$$

$$M y_0^+ = q (j z_2^- + (j + 1) z_3^-) \tag{2.49}$$

$$M y_1^+ = -q (z_2^+ - z_3^+) \tag{2.50}$$

$$M y_2^- = -\frac{q}{2j+1} (z_0^- + (j + 1) z_1^+) \tag{2.51}$$

$$M y_3^- = -\frac{q}{2j+1} (z_0^- + j z_1^+) \tag{2.52}$$

For the integral operators, we will now use the notation

$$\begin{aligned} A_j &= -W_{1,j} + W_{2,j} - 4W_{3,j} \\ B_j &= -W_{1,j} - W_{2,j} + 2W_{3,j} \\ C_j &= W_{1,j} - W_{2,j} \\ D_j &= W_{1,j} + W_{2,j} - 2W_{3,j} \end{aligned} \tag{2.53}$$

where the W's have the same meaning as for the first solution, but the A, B, C and D are defined differently. The equation (2.14) then gives the following equations.

$$(T + \frac{1}{q} D_j q) y_0^+ = \frac{E}{T}\left[M z_0^- - q(j y_2^- + (j + 1) y_3^-)\right] \tag{2.54}$$

$$(T - \frac{1}{q} B_j q) y_1^+ = \frac{E}{T}\left[M z_1^- + q(y_2^+ - y_3^+)\right] \tag{2.55}$$

$$(T - \frac{1}{q} D_{j-1}\, q)\, y_2^+ = \frac{E}{T}\left[- M z_2^+ - \frac{q}{2j+1}(y_0^- - (j+1)\, y_1^+)\right] \tag{2.56}$$

$$(T - \frac{1}{q} D_{j+1}\, q)\, y_3^+ = \frac{E}{T}\left[- M z_3^+ - \frac{q}{2j+1}(y_0^- + j\, y_1^+)\right] \tag{2.57}$$

$$(T - \frac{1}{q} A_j\, q)\, z_0^+ = \frac{E}{T}\left[M y_0^- - q(j\, z_2^+ + (j+1)\, z_3^+)\right] \tag{2.58}$$

$$(T + \frac{1}{q} C_j\, q)\, z_1^+ = \frac{E}{T}\left[M y_1^- + q(z_2^- - z_3^-)\right] \tag{2.59}$$

$$(T - \frac{1}{q} C_{j-1}\, q)\, z_2^+ = \frac{E}{T}\left[- M y_2^+ - \frac{q}{2j+1}(z_0^+ - (j+1)\, z_1^-)\right] \tag{2.60}$$

$$(T - \frac{1}{q} C_{j+1}\, q)\, z_3^+ = \frac{E}{T}\left[- M y_3^+ - \frac{q}{2j+1}(z_0^+ + j\, z_1^-)\right] \tag{2.61}$$

The equations (2.45) - (2.52) and (2.54) - (2.61) yield two sets of coupled equations for $y_2^+$, $y_3^+$ and $z_2^+$, $z_3^+$. The elimination of the other variables is straight forward but quite lengthy. It is found that $y_2^+$ and $y_3^+$ satisfy the two coupled equations

$$(E^2 - T^2)\, y_2^+ = \frac{1}{q} G\, q\, y_2^+ + \frac{j+1}{2j+1}\frac{1}{q} H\, q\, y_3^+ \tag{2.62}$$

$$(E^2 - T^2)\, y_3^+ = \frac{j}{2j+1}\frac{1}{q} E\, q\, y_2^+ + \frac{1}{q} F\, q\, y_3^+ \tag{2.63}$$

and that $z_2^+$ and $z_3^+$ satisfy

$$(E^2 - T^2)\,(T\, z_2^+) = \frac{1}{q} G^T\, q\,(T\, z_2^+) + \frac{j+1}{2j+1}\frac{1}{q} E^T\, q\,(T\, z_3^+) \tag{2.64}$$

$$(E^2 - T^2)\,(T\, z_3^+) = \frac{j}{2j+1}\frac{1}{q} H^T\, q\,(T\, z_2^+) + \frac{1}{q} F^T\, q\,(T\, z_3^+) \tag{2.65}$$

The operators G, H, E and F are given by

$$G = -\frac{1}{T}\left\{ M^2(C_{j-1} + D_{j-1}) + \frac{j}{2j+1}\,(q\,A_j\,q + C_{j-1}\,q^2) \right.$$
$$+ \frac{j+1}{2j+1}\,(q\,B_j\,q + q^2\,D_{j-1}) - M^2\,C_{j-1}\,\frac{1}{T}\,D_{j-1}$$
$$\left. - \frac{j+1}{2j+1}\,q\,B_j\,\frac{q}{T}\,D_{j-1} - \frac{j}{2j+1}\,C_{j-1}\,\frac{q}{T}\,A_j\,q \right\} \qquad (2.66)$$

$$H = -\frac{1}{T}\left\{ q(A_j - B_j)\,q - q^2\,D_{j+1} + C_{j-1}\,q^2 \right.$$
$$\left. + q\,B_j\,\frac{q}{T}\,D_{j+1} - C_{j-1}\,\frac{q}{T}\,A_j\,q \right\} \qquad (2.67)$$

$$E = -\frac{1}{T}\left\{ q(A_j - B_j)\,q - q^2\,D_{j-1} + C_{j+1}\,q^2 \right.$$
$$\left. + q\,B_j\,\frac{q}{T}\,D_{j-1} - C_{j+1}\,\frac{q}{T}\,A_j\,q \right\} \qquad (2.68)$$

$$F = -\frac{1}{T}\left\{ M^2(C_{j+1} + D_{j+1}) + \frac{j+1}{2j+1}\,(q\,A_j\,q + C_{j+1}\,q^2) \right.$$
$$+ \frac{j}{2j+1}\,(q\,B_j\,q - q^2\,D_{j+1}) - M^2\,C_{j+1}\,\frac{1}{T}\,D_{j+1}$$
$$\left. - \frac{j+1}{2j+1}\,C_{j+1}\,\frac{q}{T}\,A_j\,q - \frac{j}{2j+1}\,B_j\,\frac{q}{T}\,D_{j+1} \right\} \qquad (2.69)$$

These are the only useful relations that can be obtained from the 16 equations. All the other variables are zero, or can be given in terms of $y_2^+$, $y_3^+$, $z_2^+$ and $z_3^+$, but these relations will not be required.

In the next chapter we will show how $y_2^+$, $y_3^+$, $z_2^+$ and $z_3^+$ form the probability density for the $l = j-1$ and $l = j+1$ spin triplet states in a rather complicated way. One of the equations, (2.38) or (2.40) is

approximately analogous to the spin singlet equation derived by Biswas, and solved by Reinfelds for the pseudo-scalar interaction case. In their notation the other one could not be written down explicitly. An equation corresponding to (2.41) or (2.43), which will give the $l = j$ spin triplet state, was not given at all because restrictions imposed on the relative time dependence of the solution functions eliminated it.

A pair of coupled equations approximately analogous to (2.62) and (2.63) with $y_2^+$ and $y_3^+$ identified with the physical $l = j-1$ and $l = j+1$ wave functions, were also found by Biswas, and solved numerically by Reinfelds. As we will see, $y_2^+$ and $y_3^+$ cannot even approximately be identified with the physical wave functions, since the four functions $y_2^+$, $y_3^+$, $z_2^+$ and $z_3^+$ all contribute to both the spin triplet coupled states.

# CHAPTER III

## THE BETHE-SALPETER PROBABILITY DENSITY

The finding of a probability density for the B-S wave function has always been an outstanding problem. A knowledge of the probability density it vital to the physical interpretation of the theory, since the meaning of the functions found in the last chapter is not at all obvious.

Allcock[33] and Allcock and Hooton[34] derived a general covariant expression for the scalar product of two bound states from general considerations of translational and Lorentz invariance. This scalar product was eventually expressed in terms of the B-S wave function and its conjugate. Green[35] derived a general theory of normalization and interpretation of Feynman amplitudes from which the Allcock-Hooton method may be derived as a special case. The difficulty of using either of these two methods arises from the fact that they do not indicate what the conjugate B-S wave function should be. It cannot be formed in the conventional way because the interaction function is purely imaginary and the hermitian conjugate of the B-S equation does not lead to an acceptable continuity equation.

Reinfelds[18] attempted to overcome this problem by using equation (2.14) and its hermitian conjugate to construct a continuity equation.

Using $\Psi = X + X'$ and $\chi = \frac{1}{T}(D^*X + X'D^*)$, he took the trace of

$\psi^* \times (2.14) - (2.14)^* \times \Psi$ which is

$$\mathrm{Tr}\left(\Psi^* \frac{\partial \chi}{\partial \tau} + \frac{\partial \chi^*}{\partial \tau} \Psi\right) + i\,\mathrm{Tr}\left(\Psi^* (T\Psi) - (T\Psi^*)\Psi\right) = 0 \qquad (3.0)$$

He then identified equation (3.0) with

$$\frac{\partial \rho}{\partial \tau} + \underline{\nabla}.\underline{j} = 0$$

where $\rho$ is taken as $\frac{1}{2}\,\mathrm{Tr}(\Psi^*\chi + \chi^*\Psi)$. To do this, he had to add a perfect divergence to the first term of (3.0) and subtract it from the second. This is rather unsatisfactory as the term added is by no means unique and the method does not lead to a unique probability density. The method given in this chapter uses a more fundamental approach and gives a somewhat different probability density. This probability density will enable us to identify the functions found in Chapter II with physical wave functions of the nucleon-nucleon system and will also lead to a method for calculating phase-shifts for the scattering.

Firstly, we define a "conjugate" B-S equation which is conjugate with respect to the physical variables $\underline{x}$ and $\tau$ only, but not with respect to the relative time variable $t$. This equation will be satisfied by the function $\Phi$ and will enable a continuity equation to be found. The continuity equation, depending on the relative time, will then be integrated over $t$ from $-\infty$ to $+\infty$. This is possible by virtue of

our boundary condition that the wave functions go to zero as $t \rightarrow \pm \infty$ .

It will be found that the wave function $\phi$ obtained from the integrated "conjugate" B-S wave equation, corresponds to $\Psi^*$, where $\Psi$ is the wave function obtained from the integrated B-S equation. This proves that our definition of the conjugate B-S equation is valid from the physical viewpoint.

## 3.1) Derivation of the Probability Density

Here we work in the more familiar coordinate representation so that the B-S equation will be written in the form

$$(i\frac{\partial}{\partial\tau} + i\frac{\partial}{\partial t} - D^*)\,\Psi\,(i\frac{\partial}{\partial\tau} - i\frac{\partial}{\partial t} - D^*) = 2i\,V_1(r)\,\delta(t)\gamma^\mu\Psi\gamma^\mu \qquad (3.1)$$

The right hand side will be written for the scalar interaction only, but the following applies to the general form of interaction as given by equation (2.4). We define a "conjugate" B-S Wave function $\Phi$ which satisfies a similar equation to that satisfied by $\Psi$, but with $D^* \rightarrow D$ and $i\frac{\partial}{\partial\tau} \rightarrow -i\frac{\partial}{\partial\tau}$, so that

$$-i\frac{\partial\Phi}{\partial\tau} = E\,\Phi \qquad i\underline{\nabla}.\underline{\gamma}\,\Phi = \underline{q}.\underline{\gamma}\,\Phi$$

i.e. $\bar{\Phi}$ satisfies

$$\left( - i \frac{\partial}{\partial \tau} + i \frac{\partial}{\partial t} - D \right) \bar{\Phi} \left( - i \frac{\partial}{\partial \tau} - i \frac{\partial}{\partial t} - D \right) = 2i \, \delta (t) \, V_1 \gamma^4 \bar{\Phi} \gamma^4 \quad (3.2)$$

This corresponds to $\bar{\Phi}$ being the conjugate of $\Psi$ in the physical variables $\underline{x}$ and $\tau$, but not in the relative time variable $t$.

As $\Psi$ and $\Phi$ have to be equally capable of describing the physical situation, we impose the same boundary conditions on the function $\bar{\Phi}$ as were used for $\Psi$. These being that $\bar{\Phi} \rightarrow 0$ as $t \rightarrow \pm \infty$ and that it is continuous for all $t$. Hence the equation (3.2) for $\bar{\Phi}$ can be solved in exactly the same way as equation (3.1) was solved for $\Psi$ in sections 2.1) and 2.2). The solutions of (3.1) and (3.2) are

$$\begin{aligned} \Psi &= e^{i(E-D^*)t} \left(1 - \frac{D^*}{T} \operatorname{sgn} t\right) X + X' \left( 1 + \frac{D^*}{T} \operatorname{sgn} t\right) e^{-i(E-D^*)t} \\ \bar{\Phi} &= e^{i(E-D)t} \left(1 - \frac{D}{T} \operatorname{sgn} t\right) R + R' \left(1 + \frac{D}{T} \operatorname{sgn} t\right) e^{-i(E-D)t} \end{aligned} \quad (3.3)$$

where $R$ and $R'$ satisfy the equations

$$D R = R' D \quad (3.4)$$

and

$$T(R + R') - \frac{E}{T} (D R + R' D) = V_1 \gamma^4 (R + R') \gamma^4 \quad (3.5)$$

and $X$ and $X'$ satisfy (2.13) and (2.14). If we now take the hermitian conjugate of equations (2.13) and (2.14) we find

$$D\,X'^{*} = X^{*}D$$

and

$$T(X^{*} + X'^{*}) - \frac{E}{T}(D\,X'^{*} + X^{*}D) = V_{4}\,\gamma^{4}\,(X^{*} + X'^{*})\,\gamma^{4}$$

so that on identifying these with (3.3) and (3.4) we find that $R = X'^{*}$ and $R' = X^{*}$.

We will call $\Psi = X + X'$ the physical or integrated B-S wave function because it depends only on the physical variables $\underline{x}$ and $\tau$. Similarly, the equation for $\Phi$ leads to the physical function $\phi = R + R'$, where $R$ and $R'$ satisfy the same equations as $X'^{*}$ and $X^{*}$. Therefore we have identified $\phi$ with the hermitian conjugate of the physical B-S wave function $\Psi^{*}$, and have shown that our definition of a conjugate B-S equation is physically acceptable.

We will now use equations (3.1) and (3.2) to construct a continuity equation. This is essential for the interpretation of the various coefficient functions found in the last chapter and further demonstrates the usefulness of our treatment of the relative time variable. Using

$$A = -M\gamma^{4} + i\frac{\partial}{\partial t}$$

$$B = -M\gamma^{4} - i\frac{\partial}{\partial t}$$

we can rewrite (3.1) and (3.2) as

$$\left(i\frac{\partial}{\partial x}.\gamma\gamma^4 + A\right)\Psi\left(i\frac{\partial}{\partial x}.\gamma\gamma^4 + B\right) = 2\,i\,V_1(r)\,\delta(t)\gamma^4\Psi\gamma^4 \qquad (3.6)$$

and

$$\left(-i\frac{\partial}{\partial x}.\gamma\gamma^4 + A\right)\Phi\left(-i\frac{\partial}{\partial x}.\gamma\gamma^4 + B\right) = 2\,i\,V_1(r)\,\delta(t)\gamma^4\Phi\gamma^4 \qquad (3.7)$$

where

$$i\frac{\partial}{\partial x}.\gamma = i\frac{\partial}{\partial t}\gamma^4 - i\underline{\nabla}.\underline{\gamma}$$

To obtain the continuity equation, we take $\mathrm{Tr}\left[\Phi \times (3.6) - (3.7) \times \Psi\right]$ which gives

$$\mathrm{Tr}\left[\Phi\left(i\frac{\vec{\partial}}{\partial x}.\gamma\gamma^4 + A\right)\Psi\left(i\frac{\overleftarrow{\partial}}{\partial x}.\gamma\gamma^4 + B\right) - \left(-i\frac{\vec{\partial}}{\partial x}.\gamma\gamma^4 + A\right)\Phi\left(-i\frac{\overleftarrow{\partial}}{\partial x}.\gamma\gamma^4 + B\right)\right] = 0 \qquad (3.8)$$

On expanding, this becomes

$$\mathrm{Tr}\left[\Phi\, i\frac{\vec{\partial}}{\partial x}.\gamma\gamma^4\Psi\, i\frac{\overleftarrow{\partial}}{\partial x}.\gamma\gamma^4 - i\frac{\vec{\partial}}{\partial x}.\gamma\gamma^4\Phi\, i\frac{\overleftarrow{\partial}}{\partial x}.\gamma\gamma^4\Psi + \Phi(A\Psi)\, i\frac{\overleftarrow{\partial}}{\partial x}.\gamma\gamma^4 + \Phi\left(i\frac{\vec{\partial}}{\partial x}.\gamma\gamma^4\Psi\, B\right) + (A\Phi)\, i\frac{\overleftarrow{\partial}}{\partial x}.\gamma\gamma^4\Psi + i\frac{\vec{\partial}}{\partial x}.\gamma\gamma^4(\Phi B)\Psi + \Phi(A\Psi B) - (A\Phi B)\Psi\right] = 0 \qquad (3.9)$$

The physical continuity equation is to be obtained by integrating this equation over all $t$. Together with the boundary condition that $\Psi$ and $\Phi$ vanish at $t = \pm\infty$, we can write

$$\Phi\, i\frac{\partial\Psi}{\partial t} = -i\frac{\partial\Phi}{\partial t}\,\Psi \qquad (3.10)$$

and hence, under the trace

$$(A\,\Phi)\,\Psi = \Phi\,(\Psi\,B) \qquad \text{etc.}$$

Therefore, the last two terms of equation (3.9) cancel, and it can be rewritten as

$$\mathrm{Tr}\left\{\frac{i}{2}\frac{\partial}{\partial x}\cdot\left[\Phi\left\{(i\frac{\vec{\partial}}{\partial x}\cdot\gamma\gamma^4 + A)\,\Psi\gamma\gamma^4 + \gamma\gamma^4\Psi\,(i\frac{\overleftarrow{\partial}}{\partial x}\cdot\gamma\gamma^4 + B)\right\}\right.\right.$$
$$\left.\left. + \left\{\gamma\gamma^4\Phi\,(-i\frac{\overleftarrow{\partial}}{\partial x}\cdot\gamma\gamma^4 + B) + (-i\frac{\vec{\partial}}{\partial x}\cdot\gamma\gamma^4 + A)\,\Phi\,\gamma\gamma^4\right\}\Psi\right]\right\} = 0 \qquad (3.11)$$

On substituting back for A and B, we finally obtain

$$\mathrm{Tr}\left\{\frac{i}{2}\frac{\partial}{\partial x}\cdot\left[\Phi\left\{(E - \vec{D}^* + i\frac{\partial}{\partial t})\,\Psi\gamma\gamma^4 + \gamma\gamma^4\Psi\,(E - \overleftarrow{D}^* - i\frac{\partial}{\partial t})\right\}\right.\right.$$
$$\left.\left. + \left\{\gamma\gamma^4\Phi\,(E - \overleftarrow{D} - i\frac{\partial}{\partial t}) + (E - \vec{D} + i\frac{\partial}{\partial t})\,\Phi\,\gamma\gamma^4\right\}\Psi\right]\right\} = 0 \qquad (3.12)$$

Thus we have shown that our continuity equation (3.8), can be written in the covariant form

$$\frac{\partial}{\partial x}\cdot j = 0$$

where $j$ is the four-vector current, and integration over all $t$ is implied. This result, which could have been expected from our fully

covariant theory, further justifies our methods. The probability density is given by the fourth component of $j$, i.e. $\rho = j_4$, and the probability current by $\underline{j} = (j_1, j_2, j_3)$. From (3.12), the explicit expressions for $\rho$ and $\underline{j}$ are

$$\rho = \frac{i}{2} \mathrm{Tr} \left\{ \bar{\Phi} \left[ (E - \vec{D}^*) \Psi + \Psi (E - \overleftarrow{D}^*) \right] + \left[ \bar{\Phi} (E - \overleftarrow{D}) + (E - \vec{D}) \bar{\Phi} \right] \Psi \right\} \tag{3.13}$$

and

$$\underline{j} = \frac{i}{2} \mathrm{Tr} \left\{ \bar{\Phi} \left[ (E - \vec{D}^* + i \frac{\partial}{\partial t}) \Psi \underline{\gamma} \gamma^4 + \underline{\gamma} \gamma^4 \Psi (E - \overleftarrow{D}^* - i \frac{\partial}{\partial t}) \right] \right.$$

$$\left. + \left[ \underline{\gamma} \gamma^4 \bar{\Phi} (E - \overleftarrow{D} - i \frac{\partial}{\partial t}) + (E - \vec{D} + i \frac{\partial}{\partial t}) \bar{\Phi} \underline{\gamma} \gamma^4 \right] \Psi \right\} \tag{3.14}$$

This derivation applies equally well to a general form of interaction, and is not restricted to the ladder or instantaneous interaction approximations. It is interesting to note that although we did not define the equation for $\bar{\Phi}$ to be conjugate in the variable $t$, conjugation is in a sense implied by the integration over $t$, as is shown by equation (3.10).

We will now integrate (3.13) to obtain the physical probability density. Substituting the expressions (3.3) for $\Psi$ and $\bar{\Phi}$ into (3.13), it is a straight forward matter to evaluate the integral over $t$. It is found that parts of $X$ and $X'$ that anticommute with $D^*$ cancel, because the operators $(1 + \frac{\underline{D}^*}{T} \mathrm{sgn}\ t)$ and $(1 - \frac{\underline{D}^*}{T} \mathrm{sgn}\ t)$ are orthogonal. From the relation $D^*X = X'D^*$, it can be seen that $\Psi = X + X'$ commutes with $D^*$,

and hence the probability density is expressible in terms of $\Psi$ and $\Psi^*$ only. This may be expected as $\Psi$ corresponds to our physical B-S wave function.

When the integral over (3.13) is carried out, the expression for the probability density reduces to

$$\rho = \mathrm{Tr}\left[\frac{\vec{D}}{T}\Psi^*.\Psi + \Psi^*.\frac{\vec{D}^*}{T}\Psi\right] \tag{3.15}$$

and since $D^*$ and $\Psi$ commute, this is a hermitian quantity. The probability density derived by Reinfelds was

$$\mathrm{Tr}\left[2\,X^*\frac{\overleftarrow{D}}{T}.\Psi^* + \Psi^*.\frac{\vec{D}^*}{T}(2\,X)\right]$$

and almost agrees with ours. However, this expression cannot be obtained from a covariant continuity equation, and is not necessarily unique as mentioned before.

In the next section it will be shown how $\rho$ reduces to the conventional form $2\,\mathrm{Tr}\,(\Psi^*\,\Psi)$ outside the interaction region. The occurrence of the factor D/T in the probability density is rather interesting and cannot be neglected, as was done by Reinfelds who used the approximation $\rho = 2\,\mathrm{Tr}\,(\Psi^*\,\Psi)$. This made it difficult to explain certain terms in $\Psi$ and they were eventually neglected. Our probability density (3.21), will eventually lead to some very concise expressions in terms of the coefficient functions found in Chapter II, although the

algebra is rather lengthy.

## 3.2) Probability density in limit of zero interaction

In the limit $V \to 0$, $T \to E$ and equation (2.14) becomes

$$X + X' = \frac{1}{E}(D^*X + X'D^*) \tag{3.22}$$

Operating from the left with $D^*/E$ gives

$$\frac{D^*}{E}(X + X') = \frac{1}{E}(E^2X + D^*X'D^*) = 2X$$

and operating from the right with $D^*/E$ gives

$$(X + X')\frac{D^*}{E} = \frac{1}{E}(D^*X\,D^* + X'E^2)\frac{1}{E} = 2X'$$

Therefore

$$X = X' \quad \text{and} \quad D^*X = E\,X \tag{3.23}$$

This means that as $V \to 0$, $\frac{D^*}{T}\psi \to \psi$, and the probability density reduces to the conventional expression $2\,\mathrm{Tr}(\psi^*\psi)$. We will use this fact in the next chapter, where we find expressions for phase shifts, to determine the relations between the incoming amplitudes $\psi$ and $\eta = \frac{D^*}{T}\psi$.

## 3.3) Probability density for the first solution

As already indicated by the definition of the spin-angle Dirac matrices $\psi_0$, $\psi_1$, $\psi_2$ and $\psi_3$, it would be expected that the coefficients of $\psi_0$ should be in some way related to the spin singlet interaction, whereas those of $\psi_1$ should be related to the spin triplet $l = j$ part of the interaction. In section 2.4) the equations satisfied by these coefficients were derived, and $\psi$ was given by

$$\psi = (w_0^+ + x_0^+ \gamma^4)\psi_0 + (w_1^+ + x_1^+ \gamma^4)\psi_1 + (w_2^+ + x_2^+ \gamma^4)\psi_2 + (w_3^+ + x_3^+ \gamma^4)\psi_3 \qquad (3.24)$$

The functions $w_2^+$, $x_2^+$, $w_3^+$ and $x_3^+$, although small relative to $w_0^+$, $x_0^+$, $w_1^+$ and $x_1^+$, are not zero, and their significance is not obvious. It would seem that the spin triplet $l = j \pm 1$ states contribute to this $\psi$. This, however, is excluded by the fact that the terms involving $\psi_2$ and $\psi_3$ are of the same parity as those involving $\psi_0$ and $\psi_1$, and hence must in some way contribute to the probability density of the $l = j$ states. How this happens will become apparent as we proceed.

Writing the probability density as $\rho = \mathrm{Tr}(\eta^* \psi + \psi^* \eta)$ where $\eta = \frac{D^*}{T}$, and again working in momentum space, we evaluate $\eta$ in terms of the coefficient functions to obtain

$$T\eta = M x_0^+ \psi_0 + \left[M w_0^+ - q\,(j\, x_2^+ + (j+1)\, x_3^+)\right]\psi_0 \gamma^4$$

$$+\left[M x_1^+ + q\,(w_2^+ - w_3^+)\right]\psi_1 \quad + M w_1^+ \psi_1 \gamma^4$$

$$+ q \frac{j+1}{2j+1} w_1^+ \psi_2 - \frac{q}{2j+1} x_0^+ \psi_2 \gamma^4$$

$$- q \frac{j}{2j+1} w_1^+ \psi_3 - \frac{q}{2j+1} x_0^+ \psi_3 \gamma^4 \qquad (3.25)$$

This is found by evaluating $D^* \psi$ where $\psi$ is given by (3.24) and we have used the relations (2.19).

Using equations (2.22) and (2.23) we find

$$j x_2^+ + (j+1) x_3^+ = -\frac{q}{M} w_0^+$$

so that

$$M w_0^+ - q(j x_2^+ + (j+1) x_3^+) = \frac{T^2}{M} w_0^+ \qquad (3.26)$$

Also, from (2.26) and (2.27)

$$w_2^+ - w_3^+ = \frac{q}{M} x_1^+$$

and hence

$$M x_1^+ + q\,(w_2^+ - w_3^+) = \frac{T^2}{M} x_1^+ \qquad (3.27)$$

Putting these into (2.25) gives

$$T \eta = \psi_0 \left[ M x_0^+ + \frac{T^2}{M} w_0^+ \gamma^4 \right]$$

$$+ \quad \psi_1 \left[ \frac{T^2}{M} x_1^+ + M w_1^+ \gamma^4 \right]$$

$$- \frac{q}{2j+1} \left( j \psi_3 - (j+1) \psi_2 \right) w_1^+ - \frac{q}{2j+1} \left( \psi_2 + \psi_3 \right) \gamma^4 x_0^+$$

$$= \quad \psi_0 \left[ (M + \underline{\gamma} . \underline{q}) x_0^+ + \frac{T^2}{M} w_0^+ \gamma^4 \right]$$

$$+ \quad \psi_1 \left[ \frac{T^2}{M} x_1^+ + (M - \underline{\gamma} . \underline{q}) w_1^+ \gamma^4 \right] \qquad (3.28)$$

where the relations (2.19) have again been used to express $\psi_2$ and $\psi_3$ in terms of $\psi_0$ and $\psi_1$. The $\psi$ in (3.24), with use of

$$w_2^+ = \frac{q}{M} \frac{j+1}{2j+1} x_1^+ \quad , \quad w_3^+ = - \frac{q}{M} \frac{j}{2j+1} x_1^+$$

and

$$x_2^+ = x_3^+ = - \frac{1}{M} \frac{q}{2j+1} w_0^+$$

becomes

$$\psi = \frac{1}{M} \psi_0 \left[ (M + \underline{\gamma} . \underline{q}) w_0^+ + M x_0^+ \gamma^4 \right]$$

$$+ \frac{1}{M} \psi_1 \left[ M w_1^+ + (M - \underline{\gamma} . \underline{q}) x_1^+ \gamma^4 \right] \qquad (3.29)$$

and the probability density is

$$\rho = \frac{1}{M} \left[ \frac{M}{T} x_0^{+*} M w_0^+ + \frac{1}{T} q^2 x_0^{+*} w_0^+ + \frac{T}{M} w_0^{+*} M x_0^+ \right] \psi_0^2$$

$$+\frac{1}{M}\left[\frac{T}{M}\,x_1^{+*}\,.\,M\,w_1^{+}+\frac{M}{T}\,w_1^{+*}\,.\,M\,x_1^{+}+\frac{1}{T}\,q^2\,w_1^{+*}\,.\,x_1^{+}\right]\psi_1^{\,2}$$

$$=\frac{1}{M}\left[x_0^{+*}\,.\,(T\,w_0^{+})+(T\,w_0^{+})^{*}\,.\,x_0^{+}\right]\psi_0^{\,2}+\frac{1}{M}\left[(T\,x_1^{+})^{*}\,.\,w_1^{+}+w_1^{+*}\,.\,(T\,x_1^{+})\right]\psi_1^{\,2}$$

Here we have written $\mathrm{Tr}\,(\psi_0^{*}\,\psi_0)=\psi_0^{\,2}$ and $\mathrm{Tr}\,(\psi_1^{*}\,\psi_1)=\psi_1^{\,2}$. In order to obtain a closer correspondence with Schrodinger wave functions we will simplify the notation by writing $x_0$ for $x_0^{+}$, $w_0$ for $\frac{T}{M}\,w_0^{+}$, $x_1$ for $\frac{T}{M}\,x_1^{+}$ and $w_1$ for $w_1^{+}$ so that

$$\rho=(x_0^{*}\,w_0+w_0^{*}\,x_0)\,\psi_0^{\,2}+(x_1^{*}\,w_1+w_1^{*}\,x_1)\,\psi_1^{\,2} \tag{3.30}$$

and the equations satisfied by $x_0$, $w_0$, $x_1$ and $w_1$ from (2.38), (2.40), (2.41) and (2.43) are

$$(E^2-T^2)\,x_0=\frac{1}{q}\,Kq\,x_0 \tag{3.31}$$

$$(E^2-T^2)\,w_0=\frac{1}{q}\,K^{T}q\,w_0 \tag{3.32}$$

$$(E^2-T^2)\,w_1=\frac{1}{q}\,Hq\,w_1 \tag{3.33}$$

$$(E^2-T^2)\,x_1=\frac{1}{q}\,H^{T}q\,x_1 \tag{3.34}$$

In view of the fact that $\psi_0$ is an eigenfunction of the nucleon-

nucleon spin singlet state, we will identify $x_0^* w_0 + w_0^* x_0$ with the probability density of this state. Similarly $x_1^* w_1 + w_1^* x_1$ is identified with the probability density of the spin triplet $l = j$ state. From the results it can be seen that the coefficients of $\Psi_2$ and $\Psi_3$ only occur as an intermediate step in the calculation, and physically have nothing to do with the spin singlet or $l = j$ spin triplet interactions.

It has already been shown in the last section that outside the interaction region, the probability density reduces to the conventional form, so that $x_0 \rightarrow w_0$ and $x_1 \rightarrow w_1$ as $V \rightarrow 0$. This means that in calculating phase shifts, the functions $x_0$ and $w_0$ should give the same results, as should $x_1$ and $w_1$. This is in fact borne out by numerical calculations, so that for scattering states we are free to identify either (3.31) or (3.32) with the Schrodinger equation for the spin singlet state, and (3.33) or (3.34) with the Schrodinger equation for the spin triplet $l = j$ state.

We next examine the probability density for the second solution which reduces just as neatly, but to not as simple a result.

3.4) Probability density for the second solution

In this case $\Psi$ is given by

$$\psi = \Psi_0 (y_0^+ + z_0^+ \gamma^4) \gamma^5 + \Psi_1 (y_1^+ + z_1^+ \gamma^4) \gamma^5 + \Psi_2 (y_2^+ + z_2^+ \gamma^4) \gamma^5 + \Psi_3 (y_3^+ + z_3^+ \gamma^4) \gamma^5 \qquad (3.35)$$

and the functions $y_0^+$, $z_0^+$, $y_1^+$ and $z_1^+$ are small compared with $y_2^+$, $z_2^+$, $y_3^+$ and $z_3^+$, but are not zero. As before, on inspection it can be seen that all the terms are of the same parity and the terms in $\Psi_0$ and $\Psi_1$ should contribute to the probability density of the coupled spin triplet states.

Evaluating $D^* \psi$ and $\Psi$, we find

$$\begin{aligned} T\eta = &- \Psi_0 q (j z_2^+ + (j+1) z_3^+) \gamma^4 \gamma^5 + \Psi_1 q (y_2^+ - y_3^+) \gamma^5 \\ &+ \Psi_2 \left[ - M y_2^+ - \frac{1}{2j+1} \frac{q^2}{M} (j y_2^+ + (j+1) y_3^+) \right] \gamma^4 \gamma^5 \\ &+ \Psi_2 \left[ - M z_2^+ - \frac{j+1}{2j+1} \frac{q^2}{M} (z_2^+ - z_3^+) \right] \gamma^5 \\ &+ \Psi_3 \left[ - M z_3^+ - \frac{1}{2j+1} \frac{q^2}{M} (j y_2^+ + (j+1) y_3^+) \right] \gamma^4 \gamma^5 \\ &+ \Psi_3 \left[ - M z_3^+ + \frac{j}{2j+1} \frac{q^2}{M} (z_2^+ - z_3^+) \right] \gamma^5 \end{aligned} \qquad (3.36)$$

and

$$\psi = \Psi_0 \frac{q}{M} (j y_2^+ + (j+1) y_3^+) \gamma^4 \gamma^5 - \Psi_1 \frac{q}{M} (z_2^+ - z_3^+) \gamma^5$$

$$+ \Psi_2 (y_2^+ + z_2^+ \gamma^4) \gamma^5 + \Psi_3 (y_3^+ + z_3^+ \gamma^4) \gamma^5 \qquad (3.37)$$

To eliminate the functions $\Psi_0$ and $\Psi_1$, we express them in terms of $\Psi_2$ and $\Psi_3$ using the relations (2.19) to obtain

$$\begin{aligned} \mathrm{Tr}\,(\Psi_0^* \Psi_0) &= \mathrm{Tr}\frac{1}{(2j+1)}\,(\Psi_2^* \Psi_2 + \Psi_3^* \Psi_3) \\ \mathrm{Tr}\,(\Psi_1^* \Psi_1) &= \mathrm{Tr}\frac{1}{(2j+1)}\,((j+1)^2 \Psi_2^* \Psi_2 + j^2 \Psi_3^* \Psi_3) \end{aligned} \qquad (3.38)$$

Cross terms of the type $\Psi_2^* \Psi_3$ are neglected because $\Psi_2$ and $\Psi_3$ are orthogonal. They do not contribute to the total probability since their integral over angular coordinates vanishes.

Therefore

$$\begin{aligned} \mathrm{Tr}\,(\eta^* \psi) = \Bigg\{ & y_2^{+*} \left[ \frac{M}{T} z_2^+ + \frac{j+1}{2j+1} \frac{a^2}{MT} (z_2^+ - z_3^+) \right] \\ & + z_2^{+*} \left[ \frac{M}{T} y_2^+ + \frac{1}{2j+1} \frac{a^2}{MT} (j\, y_2^+ + (j+1)\, y_3^+) \right] \\ & + \frac{1}{(2j+1)^2} \frac{a^2}{MT} (j\, y_2^{+*} + (j+1)\, y_3^{+*})(j\, z_2^+ + (j+1)\, z_3^+) \\ & + \frac{(j+1)^2}{(2j+1)^2} \frac{a^2}{MT} (z_2^{+*} - z_3^{+*})(y_2^+ - y_3^+) \Bigg\} \Psi_2^2 \\ & + \Bigg\{ y_3^{+*} \left[ \frac{M}{T} z_3^+ - \frac{j}{2j+1} \frac{a^2}{MT} (z_2^+ - z_3^+) \right] \\ & + z_3^{+*} \left[ \frac{M}{T} y_3^+ + \frac{1}{2j+1} \frac{a^2}{MT} (j\, y_2^+ + (j+1)\, y_3^+) \right] \end{aligned}$$

$$+ \frac{1}{(2j+1)^2}\frac{a^2}{MT}\,(j\,y_2^{+*} + (j+1)\,y_3^{+*})\,(j\,z_2^{+} + (j+1)\,z_3^{+})$$

$$+ \frac{j^2}{(2j+1)^2}\frac{a^2}{MT}\,(z_2^{+*} - z_3^{+*})\,(y_2^{+} - y_3^{+})\Big\}\,\Psi_3^2 \qquad (3.39)$$

By rearranging terms, this expression can be greatly simplified. We will also change the notation in a similar way as was done in the last section by writing $y_2^+$ as $y_2$, $y_3^+$ as $y_3$, $\frac{T}{M}z_2^+$ as $z_2$ and $\frac{T}{M}z_3^+$ as $z_3$. The probability density then reduces to

$$\rho = \Big\{y_2^*\,z_2 + z_2^*\,y_2 - \frac{j+1}{(2j+1)^2}\frac{a^2}{T^2}\Big[(y_2^* - y_3^*)\,(j\,z_2 + (j+1)\,z_3)$$

$$+ (j\,z_2^* + (j+1)\,z_3^*)\,(y_2 - y_3)\Big]\Big\}\,\Psi_2^2$$

$$+ \Big\{y_3^*\,z_3 + z_3^*\,y_3 + \frac{j}{(2j+1)^2}\frac{a^2}{T^2}\Big[(y_2^* - y_3^*)\,(j\,z_2 + (j+1)\,z_3)$$

$$+ (j\,z_2^* + (j+1)\,z_3^*)\,(y_2 - y_3)\Big]\Big\}\,\Psi_3^2 \qquad (3.40)$$

where $y_2$, $y_3$, $z_2$ and $z_3$ satisfy the equations

$$(E^2 - T^2)\,y_2 = \frac{1}{q}\,G\,q\,y_2 + \frac{j+1}{2j+1}\,\frac{1}{q}\,H\,q\,y_3 \qquad (3.41)$$

$$(E^2 - T^2)\,y_3 = \frac{j}{2j+1}\,\frac{1}{q}\,E\,q\,y_2 + \frac{1}{q}\,F\,q\,y_3 \qquad (3.42)$$

$$(E^2 - T^2)\,z_2 = \frac{1}{q}\,G^T\,q\,z_2 + \frac{j+1}{2j+1}\,\frac{1}{q}\,E^T\,q\,z_3 \qquad (3.43)$$

$$(E^2 - T^2)\,z_3 = \frac{j}{2j+1}\,\frac{1}{q}\,H^T\,q\,z_2 + \frac{1}{q}\,F^T\,q\,z_3 \qquad (3.44)$$

and G, H, E and F are given by (2.66) - (2.69).

In this case we identify the quantity multiplying the $\psi_2^2$ with the probability density of the spin triplet $l = j-1$ state, and that multiplying the $\psi_3$ with the spin triplet $l = j+1$ state. From the expression (3.40) it can be seen that the $l = j-1$ state depends on $y_3$ and $z_3$ as well as $y_2$ and $z_2$. Similarly the $l = j+1$ state depends mainly on $z_3$ and $y_3$ but has a $y_2$ and $z_2$ contribution. Here it is not possible to identify our functions with Schrodinger wave functions in any simple way. However, in the next chapter we develop a method that enables us to numerically calculate phase shifts by using the probability density to obtain a connection between our functions and Schrodinger wave functions.

The approximation of taking $y_2$ and $y_3$ or $z_2$ and $z_3$ as the physical wave functions is invalid. As $V \longrightarrow 0$, $y_2 \rightarrow z_2$ plus a term involving $z_3$ which is small compared to $z_2$, but nevertheless important in the mixing terms, and the theory, using say, equations (3.41) and (3.42), does not satisfy unitarity.

Before proceeding to the calculation of phase shifts, we will consider the normalization of $\psi_2$ and $\psi_3$, which has so far been neglected.

## 3.5) Normalization of the $\Psi_2$ and $\Psi_3$

The spin angle functions are as yet unnormalized. For the $\Psi_0$ and $\Psi_1$ states it is unimportant since we will not be using the normalized wave functions, and $\Psi_0$ and $\Psi_1$ occur independently. However, for the coupled states, it is necessary to have correct relative magnitudes of $\Psi_2$ and $\Psi_3$.

From (2.19) it is readily shown that

$$\mathrm{Tr} \int_{-\pi}^{\pi} \Psi_2^* \, \Psi_2 \, \sin\theta \, d\theta = 2j$$

$$\mathrm{Tr} \int_{-\pi}^{\pi} \Psi_3^* \, \Psi_3 \, \sin\theta \, d\theta = 2(j+1)$$

Therefore we will put

$$\mathrm{Tr} \, (\Psi_2^* \, \Psi_2) = \Psi_2^2 = 2j \, \Psi_2'^2$$

and

$$\mathrm{Tr} \, (\Psi_3^* \, \Psi_3) = \Psi_3^2 = 2(j+1) \, \Psi_3'^2$$

where $\Psi_2'$ and $\Psi_3'$ are now orthonormal. Since we are not interested in the absolute value of the probability density, the factor 2 may be dropped and the expression (3.40) for $\rho$ for the coupled states becomes

$$\rho = \left\{ j \, (y_2^* \, z_2 + z_2^* \, y_2) - \frac{j(j+1)}{(2j+1)^2} \frac{a^2}{T^2} \left[ (y_2^* - y_3^*) \, (j \, z_2 + (j+1) \, z_3) \right. \right.$$

$$\left. \left. + \, (j \, z_2^* + (j+1) \, z_3^*) \, (y_2 - y_3) \right] \right\} \Psi_2'^2$$

$$+\left\{(j+1)\,(y_3^*\,z_3+z_3^*\,y_3)+\frac{j(j+1)}{(2j+1)^2}\,\frac{\sigma^2}{T^2}\left[(y_2^*-y_3^*)\,(j\,z_2+(j+1)\,z_3)\right.\right.$$

$$\left.\left.+\,(j\,z_2^*+(j+1)\,z_3^*)\,(y_2-y_3)\right]\right\}\psi_3'^{\,2} \qquad (3.45)$$

# CHAPTER IV

## DETERMINATION OF PHASE SHIFTS

In this chapter, the methods by which the phase shifts will be calculated, are given. For the uncoupled states the method is straight forward and we simply use the conventional procedure for finding phase shifts from a Schrodinger equation. For the coupled states, things are more complicated, and we will use the probability density found in the last chapter to establish relations between the functions $y_2$, $y_3$, $z_2$ and $z_3$, and the physical wave functions of Schrodinger theory.

The calculation of phase shifts in momentum space for the uncoupled states is quite often used. However, the case of the coupled states is not well known. Signell's[36] derivation of the scattering parameters is not general enough for our purpose, so here we will treat the theory fully for conventional wave functions as well as for the functions of the B-S wave function. The first section, 4.1), is included to establish the notation and for sake of completeness, since analysis in momentum space is not widely used.

Although the complete solutions for the phase shifts have to be carried out numerically because of the complexity of the interaction potential, for the uncoupled states we will be able to write down some

results in the first Born approximation. This gives two terms, the first of which is the one boson exchange result, and the second corresponds to higher order processes. The identification of the OBE terms is significant in a number of ways. Firstly, it provides a check on the theory presented so far. Secondly, it will help us in understanding the potentials we have derived, and thirdly, it gives a basis of comparison between this theory and others, particularly the OBE theory, which was briefly discussed in Chapter I.

## 4.1) Phase shifts from a momentum space wave equation

In configuration space, the scattered wave as $r \to \infty$ for a finite range potential, is given by

$$\Psi(r) \sim e^{ikz} + \frac{e^{ikr}}{r} f(\theta) \tag{4.1}$$

If the two terms are expanded in partial waves, we obtain

$$e^{ikz} = \sum_{\ell} (2\ell+1)\, i^{\ell}\, \frac{1}{kr} \sin(kr - \tfrac{1}{2}\ell\pi)$$

and

$$\frac{e^{ikr}}{r} f(\theta) = \sum_{\ell} (2\ell+1)\, i^{\ell}\, \frac{i\, a_{\ell}}{2kr} \left[\cos(kr - \tfrac{1}{2}\ell\pi)\right.$$

$$+ i \sin (kr - \tfrac{1}{2}\ell\pi)\Big]$$

so that

$$\psi_\ell(r) \sim (2\ell+1)\, i\, \frac{1}{kr}\Big[(1 - \tfrac{1}{2}a_\ell)\sin(kr - \tfrac{1}{2}\ell\pi) + \tfrac{1}{2} i\, a_\ell \cos(kr - \tfrac{1}{2}\ell\pi)\Big] \qquad (4.2)$$

where $a_\ell = 1 - e^{2i\delta_\ell}$ and $\delta_\ell$ is the phase shift for the $\ell$ th partial wave.

The Fourier transform of $r\,\psi_\ell(r)$ in the neighbourhood of $q = k$, corresponding to the region of observation, is

$$\psi_\ell(q) \sim (1 - \tfrac{1}{2}a_\ell)\,\delta(q-k) - \tfrac{1}{2} i\, a_\ell \frac{1}{\pi} P\left(\frac{1}{k-q}\right) + R(q) \qquad (4.3)$$

where $R(q)$ is the continuous part of $\psi_\ell(q)$, and $P$ denotes the Cauchy principal value.

Now, the wave equation in momentum space for the $\ell$ th partial wave is given by

$$(k^2 - q^2)\,\psi_\ell(q) = f(q) = \int_0^\infty V_\ell(q,q')\,\psi_\ell(q')\,dq' \qquad (4.4)$$

from which we obtain

$$\psi_\ell(q) = (m - \frac{i\pi}{2k} f(k))\,\delta(q-k) + P\frac{f(q)}{k^2 - q^2} \qquad (4.5)$$

where $m\ \delta(q-k)$ is the incident wave. In the region $q = k$, (4.3) and (4.5) are equivalent except for the constant factor $m$ which is equal to 1 in equation (4.3). From (4.3) we can see that $\tan \delta_\ell$ is given by the ratio of imaginary part/real part of the coefficient of the $\delta(q-k)$, which in (4.5) is $m - \frac{i\pi}{2k} f(k)$. This is the quantity that will be used to calculate $\delta_\ell$, since it explicitly includes the magnitude of the incident wave.

As an example of a spin singlet equation we will consider equation (3.31), where the wave function $\Psi_\ell(q)$ is $x_\circ$, and the integral operator

$$\int_\circ^\infty V_\ell(q, q')\, d\, q' \quad \text{is} \quad \frac{1}{q} K\, q.$$

$$(k^2 - q^2)\, x_\circ(q) = \frac{1}{q} K\, q\, x_\circ(q) = f(q) \tag{4.6}$$

This equation is solved for $f(q)$ by firstly dividing through by the complex factor $(m - \frac{i\pi}{2k} f(k))$ so that

$$x_\circ(q) = (m - \frac{i\pi}{2k} f(k))\ \delta(q-k) + P\, \frac{f(k)}{k^2 - q^2} \tag{4.7}$$

becomes

$$x'_\circ(q) = \delta(q-k) + P\, \frac{f'(q)}{k^2 - q^2} \tag{4.8}$$

The equation (4.8) is now real, and the prime denotes division by the factor $m - \frac{i\pi}{2k} f(k)$. i.e.

$$x_o(q) = x'_o(q)\left(m - \frac{i\pi}{2k} f(k)\right)$$

$$f(q) = f'(q)\left(m - \frac{i\pi}{2k} f(k)\right) \qquad (4.9)$$

If we apply the integral operator $\frac{1}{q} K q = \int_o^\infty dq' K'(q, q')$ to equation (4.8), we obtain

$$f'(q) = K'(q, k) + P \int_o^\infty \frac{K'(q, q') f'(q')}{k^2 - q'^2} dq' \qquad (4.10)$$

This is the integral equation that will be solved numerically for $f'(q)$. The numerical techniques will be discussed in the next Chapter.

From equation (4.9) it is found that

$$m - \frac{i\pi}{2k} f(k) = \frac{m}{1 + \frac{i\pi}{2k} f'(k)}$$

where $f'(k)$ is real, so that the phase shift $\delta$ is given by

$$\tan \delta = -\frac{\pi}{2k} f'(k) \qquad (4.11)$$

This method will give phase shifts for the spin singlet and the $l = j$ spin triplet scattering states. We now turn to the more complicated problem of the coupled states.

4.2) Phase shift analysis for the coupled states

The equations (3.41) - (3.44) will be written as

$$(E^2 - T^2)\, y_2 = \underline{G}\, y_2 + \frac{j+1}{2j+1}\, \underline{H}\, y_3 = f(q) \tag{4.12}$$

$$(E^2 - T^2)\, y_3 = \frac{j}{2j+1}\, \underline{E}\, y_2 + \underline{F}\, y_3 = g(q) \tag{4.13}$$

$$(E^2 - T^2)\, z_2 = \underline{G}^T z_2 + \frac{j+1}{2j+1}\, \underline{E}^T z_3 = f'(q) \tag{4.14}$$

$$(E^2 - T^2)\, z_3 = \frac{j}{2j+1}\, \underline{H}^T z_2 + \underline{F}^T z_3 = g'(q) \tag{4.15}$$

where $\underline{G} = \frac{1}{q} G\, q$, $\underline{H}^T = \frac{1}{q} H^T q$ etc., and G, H, E and F are the integral operators (2.66) - (2.69).

We write the solutions of these equations as

$$y_2 = (m - i\, a\, f)\, \delta(q - k) + P \frac{f(q)}{k^2 - q^2} \tag{4.16}$$

$$y_3 = (n - i\, a\, g)\, \delta(q - k) + P \frac{g(q)}{k^2 - q^2} \tag{4.17}$$

$$z_2 = (m' - i\, a\, f')\, \delta(q - k) + P \frac{f'(q)}{k^2 - q^2} \tag{4.18}$$

$$z_3 = (n' - i\, a\, g')\, \delta(q - k) + P \frac{g'(q)}{k^2 - q^2} \tag{4.19}$$

where the notation employed is as follows: $a = \frac{\pi}{2k}$, f and g are the values of f (q) and g (q) at q = k, the unprimed variables f and g

arise from the $y_1$ and $y_3$ solutions(4.16) and (4.17), the single primed variables $f'$ and $g'$ arise from the $z_1$ and $z_3$ solutions (4.18) and (4.19), and the double primed variables will be used for the Schrodinger wave function solutions to be introduced. We will also use the notation that $\underline{G}\ \delta(q-k) = \underline{G}_o$. i.e. if $\underline{G} = \int dq'\ G'(q, q')$, then $\underline{G}_o = G'(q, k)$ and similarly for the operators $\underline{E}$, $\underline{H}$ and $\underline{F}$.

On operating on (4.16) with $\underline{G}$, and on (4.17) with $\frac{j+1}{2j+1}\ \underline{H}$, and adding, we obtain

$$f(q) = (m - i\,a\,f)\,\underline{G}_o + (n - i\,a\,g)\,\frac{j+1}{2j+1}\,\underline{H}_o$$

$$+ P\left[\underline{G}\,\frac{f(q)}{k^2 - q^2} + \frac{j+1}{2j+1}\,\underline{H}\,\frac{g(q)}{k^2 - q^2}\right] \qquad (4.20)$$

Similarly, by operation on (4.16) with $\frac{j}{2j+1}\ \underline{E}$ and on (4.17) with $\underline{F}$, we find

$$g(q) = (m - i\,a\,f)\,\frac{j}{2j+1}\,\underline{E}_o + (n - i\,a\,g)\,\underline{F}_o$$

$$+ P\left[\frac{j}{2j+1}\,\underline{E}\,\frac{f(q)}{k^2 - q^2} + \underline{F}\,\frac{g(q)}{k^2 - q^2}\right] \qquad (4.21)$$

These two coupled integral equations have to be solved numerically. In the next chapter we show how to calculate four real functions, $f_1(q)$, $f_2(q)$, $g_1(q)$ and $g_2(q)$ which are related to $f(q)$ and $g(q)$ as follows:

$$f(q) = (m - i a f) f_1(q) + (n - i a g) f_2(q) \tag{4.22}$$

$$g(q) = (m - i a f) g_1(q) + (n - i a g) g_2(q) \tag{4.23}$$

The equations (4.18) and (4.19) may be treated analogously to give similar expressions with all the variables primed. If we put $q = k$ in equations (4.22) and (4.23), then they can be solved for $f$ and $g$ in terms of the $m$, $n$, $f_1$, $f_2$, $g_1$ and $g_2$, where again $f_1 = f_1(k)$ etc. This enables us to write the quantities $m - i a f$ and $n - i a g$, which are the wave amplitudes outside the interaction region, in terms of $m$, $n$, $f_1$, $f_2$, $g_1$, and $g_2$. These are

$$m - i a f = \frac{m + i a (m g_2 - n f_2)}{A + i B} \tag{4.24}$$

$$n - i a g = \frac{n + i a (n f_1 - m g_1)}{A + i B} \tag{4.25}$$

where

$$A = 1 + a^2 (g_1 f_2 - g_2 f_1) \tag{4.26}$$

$$B = a(g_2 + f_1) \tag{4.27}$$

Similarly, for $z_2$ and $z_3$ we have

$$m' - i a f' = \frac{m' + i a (m' g_2' - n' f_2')}{A' + i B'} \tag{4.28}$$

$$n' - i a g' = \frac{n' + i a (n' f_1' - m' g_1')}{A' + i B'} \tag{4.29}$$

and

$$A' = 1 + a^2 (g_1' f_2' - g_2' f_1') \tag{4.30}$$

$$B' = a (g_2' + f_1') \tag{4.31}$$

These are the relevant parts of the functions in the region of observation.

We now introduce the Schrodinger wave functions $b_2$ and $b_3$ such that

$$\rho = 2 b_2^* b_2 \psi_2'^2 + 2 b_3^* b_3 \psi_3'^2 \tag{4.32}$$

where $b_2$ and $b_3$ satisfy the coupled equations

$$(E^2 - T^2) b_2 = G'' b_2 + H'' b_3 \tag{4.33}$$

$$(E^2 - T^2) b_3 = E'' b_2 + F'' b_3 \tag{4.34}$$

Following the same procedure as before, we may write

$$b_2 = (m'' - i a f'') \delta (q - k) + P \frac{f'' (q)}{k^2 - q^2} \tag{4.35}$$

$$b_3 = (n'' - i a g'') \delta (q - k) + P \frac{g'' (q)}{k^2 - q^2} \tag{4.36}$$

and

$$m'' - i a f'' = \frac{m'' + i a (m'' g_2'' - n'' f_2'')}{A'' + i B''} \tag{4.37}$$

$$n'' - i a g'' = \frac{n'' + i a (n'' f_1'' - m'' g_1'')}{A'' + i B''} \tag{4.38}$$

where

$$A'' = 1 + a^2 (g_1'' f_2'' - g_2'' f_1'')$$

$$B'' = a (g_2'' + f_1'')$$

Of course we do not know $f_1''$, $f_2''$, $g_1''$ and $g_2''$, since the integral operators G", H", E" and F" are not known. However, it is possible to calculate $f_1''$, $f_2''$,$g_1''$ and $g_2''$ by comparing the probability density expressions (3.45) and (4.32). Firstly, we will derive the eigenstate phase shifts and mixing parameter for the Schrodinger wave functions $b_2$ and $b_3$.

As in the case of the uncoupled states, the phase shifts for the $l = j-1$ and $l = j+1$ states are given by

$$\tan \delta_2 = \text{Im./Real of } (m'' - i a f'')$$

and

$$\tan \delta_3 = \text{Im./Real of } (n'' - i a g'')$$

respectively. In the eigenstate as defined by Blatt and Beidenharn[37], these phase shifts are equal. i.e.

$$\frac{a (m'' g_2'' - n'' f_2'') A'' - m'' B''}{m'' A'' + a (m'' g_2'' - n'' f_2'') B''} = \frac{a (n'' f_1'' - m'' g_1'') A'' - n'' B''}{n'' A'' + a (n'' f_1'' - m'' g_1'') B''}$$

$$= \tan \delta \qquad (4.39)$$

If we substitute for A" and B", this can be simplified to

$$\frac{a(m'' g_2'' - n'' f_2'')}{m''} = \frac{a(n'' f_1'' - m'' g_1'')}{n''} \qquad (4.40)$$

This is also equal to $\tan \delta$, but this cannot be derived directly from (4.39). The mixing parameter, which is the ratio of $b_2$ to $b_3$ incoming amplitudes in the eigenstate, is easily calculated from (4.40) to be

$$\frac{m''}{n''} = \frac{f_1'' - g_2'' \pm \sqrt{(f_1'' - g_2'')^2 + 4 g_1'' f_2''}}{2 g_1''} \qquad (4.41)$$

Unitarity of the S-matrix, or equivalently, conservation of probability, requires that $g_1'' = f_2''$, since $(m''/n'')_1 = -(n''/m'')_2$ where 1 and 2 refer to the two eigenstates, and are given by the + or - sign in (4.41). From this it can be seen that a knowledge of $f_1''$, $g_1'' = f_2''$ and $g_2''$ is sufficient to describe the scattering.

Next, we look for relations between m, n, m' and n' since the incident wave functions are determined by only two quantities, m" and n", and not by four. For zero interaction $y_2 = m\,\delta(q-k)$, $y_3 = n\,\delta(q-k)$, $z_2 = m'\,\delta(q-k)$, $z_3 = n'\,\delta(q-k)$, and also $\eta = \psi$ from section 3.2). Using (3.36) and (3.37) to evaluate $\eta$ and $\psi$ in terms of the m's and n's, we obtain, on equating coefficients of the independent $\gamma$-matrices of $\eta = \psi$, six relations between m, n, m' and n':

$$-\frac{k}{E}(j\,m' + (j+1)\,n') = \frac{k}{M}(j\,m + (j+1)\,n) \qquad (4.42)$$

$$\frac{k}{E}(m - n) = -\frac{k}{M}(m' - n') \qquad (4.43)$$

$$-\frac{M}{E}\left[m + \frac{1}{2j+1}\frac{k^2}{M^2}\left(j\,m + (j+1)\,n\right)\right] = m' \tag{4.44}$$

$$-\frac{M}{E}\left[m' + \frac{j+1}{2j+1}\frac{k^2}{M^2}(m' - n')\right] = m \tag{4.45}$$

$$-\frac{M}{E}\left[n + \frac{1}{2j+1}\frac{k^2}{M^2}(j\,m + (j+1)\,n)\right] = n' \tag{4.46}$$

$$-\frac{M}{E}\left[n' - \frac{j}{2j+1}\frac{k^2}{M^2}(m' - n')\right] = n \tag{4.47}$$

Equations (4.42) and (4.43) follow from the rest and the equations (4.44) - (4.47) yield two non-trivial relations. They are

$$m = \left(1 + \frac{j+1}{2j+1}\frac{k^2}{M^2}\right) m' \tag{4.48}$$

$$n = \left(1 + \frac{j}{2j+1}\frac{k^2}{M^2}\right) n'$$

We are now in a position to substitute into the probability density expressions (3.45) and (4.32), the relevant parts of the various functions. These are the coefficients of $\delta(q - k)$ and are the wave amplitudes outside the interaction region. Equating expressions (3.45) and (4.32), we find that for the $l = j-1$ state

$$\Big[j\Big\{\left[m\,A + a\,(m\,g_2 - n\,f_2)\,B\right]\left[m'\,A' + a\,(m'\,g_2' - n'\,f_2')\,B'\right]$$

$$+ \left[a\,(m\,g_2 - n\,f_2)\,A - m\,B\right]\left[a\,(m'\,g_2' - n'\,f_2')\,A - m'\,B'\right]\Big\}$$

$$-\frac{j\,(j+1)}{(2j+1)^2}\,\frac{k^2}{E^2}\left\{\left[(m-n)\,A+a\,(m\,(g_1+g_2)-n\,(f_1+f_2))B\,\right].\right.$$

$$\left[(j\,m'+(j+1)\,n')\,A'+a\,(j\,(m'\,g_2'-n'\,f_2')+(j+1)\,(n'\,f_1'-m'g_1'))B'\right]$$

$$+\left[a\,(m\,(g_1+g_2)-n\,(f_1+f_2))\,A-(m-n)\,B\,\right].$$

$$\left.\left[a\,(\,j\,(m'g_2'-n'f_2')+(j+1)\,(n'f_1'-m'g_1'))\,A'-(jm'+(j+1)n')B'\right]\right\}\Bigg]$$

$$/\,(A^2+B^2)\,(A'^2+B'^2)$$

$$=\frac{m''^2+a^2(m''\,g_2''-n''\,f_2'')^2}{A''^2+B''^2} \qquad (4.49)$$

and for the $l = j+1$ state

$$\Bigg[(j+1)\left\{\left[n\,A+a\,(n\,f_1-m\,g_1)\,B\right]\left[n'A'+a\,(n'f_1'-m'g_1')\,B'\right]\right.$$

$$+\left[a\,(n\,f_1-m\,g_1)\,A-n\,B\,\right]\left[\,a\,(\,n'f_1'-m'g_1')\,A'-n'B'\,\right]\Big\}$$

$$+\frac{j(j+1)}{(2j+1)^2}\,\frac{k^2}{E^2}\left\{\left[(m-n)\,A+a\,(m\,(g_1+g_2)-n\,(f_1+f_2))B\,\right].\right.$$

$$\left[(j\,m'+(j+1)\,n')\,A'+a\,(j\,(m'g_2'-n'f_2')+(j+1)\,(n'f_1'-m'g_1'))B'\right.$$

$$+\left[a\,(m\,(g_1+g_2)-n(f_1+f_2))\,A-(m-n)\,B\,\right].$$

$$\left.\left.\left[a\,(j\,(m'g_2'-n'f_2')+(j+1)\,(n'f_1'-m'g_1'))\,A'-(j\,m'+(j+1)n')B'\right]\right\}\Bigg]$$

$$/\,(A^2+B^2)\,(A'^2+B'^2)$$

$$=\frac{n''^2+a^2\,(n''\,f_1''-m''\,g_1'')^2}{A''^2+B''^2} \qquad (4.50)$$

For zero interaction, all the f's and g's are zero, so that (4.49) and (4.50) reduce to

$$m''^2 = j\, m\, m' - \frac{j(j+1)}{(2j+1)^2}\,\frac{k^2}{E^2}\,(m-n)\,(j\,m' + (j+1)\,n') \tag{4.51}$$

and

$$n''^2 = (j+1)\, n\, n' + \frac{j(j+1)}{(2j+1)}\,\frac{k^2}{E^2}\,(m-n)\,(j\,m' + (j+1)\,n') \tag{4.52}$$

Relations (4.51) and (4.52) also hold for non-zero interactions, since the incoming amplitudes are not changed by the interaction.

The overall normalization is arbitrary, and for convenience we choose $n' = 1$, so that

$$n = 1 + \frac{j}{2j+1}\,\frac{k^2}{M^2} \tag{4.53}$$

Using these and the relation (4.48) to eliminate n, n' and m from (4.49) and (4.50), the left hand sides reduce to functions of m' of known form. The numerator of the right hand side of equation (4.49) consists of $m''^2$, which is independent of the interaction, plus $a^2(m''\, g''_2 - n''\, f''_2)^2$, the magnitude of which is determined by the interaction potential. This second term can be very small compared to $m''^2$, but it contains the quantities $g''_1$ and $f''_2$, which we want to calculate. Therefore, to forestall inaccuracies in the numerical work, the parts of the functions depending only on the incident wave have to be eliminated from the equations (4.49) and (4.50). The denominator can also be

expressed as an interaction dependent plus an interaction independent part, and from (4.26) and (4.27) it can be seen that $A = A' = A'' = 1$ and $B = B' = B'' = 0$ if the potential is zero. Hence, we will write

$$1 + \zeta = A''^2 + B''^2$$

Using (4.51) we can separate out the factor $m''^2$ in the numerator of the left hand side of equation (4.49), and writing the denominator as

$$1 + \varepsilon = (A^2 + B^2)(A'^2 + B'^2)$$

equation (4.49) becomes

$$\frac{m''^2 + s(m')}{1 + \varepsilon} = \frac{m''^2 + a^2(m'' g_2'' - n'' f_2'')^2}{1 + \zeta} \qquad (4.54)$$

Similarly, equation (4.50) is written as

$$\frac{n''^2 + u(m')}{1 + \varepsilon} = \frac{n''^2 + a^2(n'' f_1'' - m'' g_1'')^2}{1 + \zeta} \qquad (4.55)$$

where $s$ and $u$ are known functions of $m'$, and $m''^2$ and $n''^2$ can be expressed in terms of $m'$ by using the relation (4.48) in (4.51) and (4.52).

In the scattering eigenstate, equations (4.54) and (4.55) give

$$\frac{m''^2 + s(m')}{n''^2 + u(m')} = \frac{m''^2 + a^2(m'' g_2'' - n'' f_2'')^2}{n''^2 + a^2(n'' f_1'' - m'' g_1'')^2}$$

$$= \frac{m''^2}{n''^2} \frac{1 + a^2(m'' g_2'' - n'' f_2'')^2/m''^2}{1 + a^2(n'' f_1'' - m'' g_1'')^2/n''^2} = \frac{m''^2}{n''^2} \qquad (4.56)$$

by virtue of (4.40). Hence $m'$ must be such that

$$\frac{s(m')}{u(m')} = \frac{m''^2(m')}{n''^2(m')} \qquad (4.57)$$

The functions $m''^2$, $n''^2$, $s$ and $u$ are quadratic in $m'$, and hence (4.57) is a quartic equation for $m'$ which has to be solved numerically.

The equation (4.57) is found to have four roots in all cases. Two extra roots may be expected since we are working with quadratic terms. We should also expect the two pairs of roots to lead to the same result because the scattering parameters are unique for any given potential. This is actually found to be the case. If the four roots are $m_1'$, $m_2'$, $m_3'$ and $m_4'$, then by substituting these into the expressions for $m''^2/n''^2$, we find

$$\frac{m''^2(m_1')}{n''^2(m_1')} = \frac{m''^2(m_2')}{n''^2(m_2')} = \theta_1^2$$

and

$$\frac{m''^2(m_3')}{n''^2(m_3')} = \frac{m''^2(m_4')}{n''^2(m_4')} = \theta_2^2$$

where $\theta_1$ and $\theta_2$ are the mixing parameters for the two eigenstates of the scattering.

The conservation of probability requires that $\theta_1 = -1/\theta_2$, and provides a sensitive check of the theory and numerical work. We do not

know the signs of $\theta_1$ and $\theta_2$ since we only calculate the values $\theta_1^2$ and $\theta_2^2$, but it is found that $\theta_1^2 = 1/\theta_2^2$ to within a few percent in all cases, demonstrating the accuracy of our numerical procedures.

From (4.57), we introduce a function $\lambda_i = (m_i')$ such that

$$s_i = s_i(m_i') = m_i''^2 \lambda_i \tag{4.58}$$

Then

$$u_i = n_i''^2 \lambda_i \tag{4.59}$$

where $i = 1, 2$ and refers to the two eigenstates. Using these in (4.54) and (4.55) gives

$$m_i''^2 (1+\lambda_i)(1+\zeta) = \left[ m_i''^2 + a^2(m_i'' g_2'' - n_i'' f_2'')^2 \right] (1+\xi) \tag{4.60}$$

$$n_i''^2 (1+\lambda_i)(1+\zeta) = \left[ n_i''^2 + a^2(n_i'' f_1'' - m_i'' g_1'')^2 \right] (1+\xi) \tag{4.61}$$

If we also introduce $\alpha_i$ such that

$$(1+\lambda_i)(1+\zeta) = (1+\alpha_i^2)(1+\xi) \tag{4.62}$$

then

$$m_i''^2 (1+\alpha_i^2) = m_i''^2 + a^2 (m_i'' g_2'' - n_i'' f_2'')^2$$

$$n_i''^2 (1+\alpha_i^2) = n_i''^2 + a^2 (n_i'' f_1'' - m_i'' g_1'')^2$$

from which we obtain

$$m_i'' \alpha_i = a (m_i'' g_2'' - n_i'' f_2'')$$

$$n_i'' \alpha_i = a (n_i'' f_1'' - m_i'' g_1'')$$

and putting $m_i''/n_i'' = \theta_i$, gives

$$\theta_i \alpha_i = a ( \theta_i g_2'' - f_2'') \tag{4.63}$$

$$\alpha_i = a (f_1'' - \theta_i g_1'') \tag{4.64}$$

The equations (4.63) and (4.64), for $i = 1$ and 2, give $f_1''$, $g_1''$, $f_2''$ and $g_2''$ in terms of the $\theta$'s and $\alpha$'s. The expressions are

$$f_1'' = \frac{\theta_1 \alpha_2 - \theta_2 \alpha_1}{a(\theta_1 - \theta_2)} \qquad f_2'' = \frac{\theta_1 \theta_2 (\alpha_1 - \alpha_2)}{a(\theta_1 - \theta_2)}$$

$$g_1'' = -\frac{(\alpha_1 - \alpha_2)}{a(\theta_1 - \theta_2)} \qquad g_2'' = \frac{\theta_1 \alpha_1 - \theta_2 \alpha_2}{a(\theta_1 - \theta_2)} \tag{4.65}$$

and $f_2'' = g_1''$ since $\theta_1 \theta_2 = -1$.

To calculate the $\alpha$'s, we will use equation (4.62). The $\lambda$'s can be obtained numerically from (4.58) and (4.59), but $\xi$ is still a function of $f_1''$, $g_1''$, $f_2''$ and $g_2''$ and we solve for it by using equations (4.65).

$$A'' = + a^2 (g_1'' f_2'' - g_2'' f_1'')$$

$$= 1 + \frac{(\alpha_1 - \alpha_2)^2 - (\theta_1 \alpha_1 - \theta_2 \alpha_2)(\theta_1 \alpha_2 - \theta_2 \alpha_1)}{(\theta_1 - \theta_2)^2}$$

$$= 1 - \alpha_1 \alpha_2 \tag{4.66}$$

and

$$B'' = a (g_2'' + f_1'') = \alpha_1 + \alpha_2 \tag{4.67}$$

Therefore

$$1 + \xi = A''^2 + B''^2 = (1 + \alpha_1^2)(1 + \alpha_2^2) \qquad (4.68)$$

and substituting from (4.62) for $1 + \alpha_1^2$ and $1 + \alpha_2^2$ gives

$$1 + \xi = \frac{(1 + \xi)^2}{(1 + \lambda_1)(1 + \lambda_2)}$$

On putting this back into (4.62) we obtain

$$\alpha_1^2 = \frac{\xi - \lambda_2}{1 + \lambda_2} \quad , \qquad \alpha_2^2 = \frac{\xi - \lambda_1}{1 + \lambda_1} \qquad (4.69)$$

It is found convenient to evaluate $\tan \delta$ of (4.39) in terms of the $\alpha$'s. When equations (4.65) - (4.67) are used in (4.39), we obtain the simple relations

$$\tan \delta_1 = - \alpha_2$$

and

$$\tan \delta_2 = - \alpha_1$$

Since we can calculate the numerical values of $\xi$ and the $\lambda$'s, we can find $\alpha_1^2$ and $\alpha_2^2$, which together with $\theta_1^2$ or $\theta_2^2$ determine the scattering. The difficulty now is that we have an ambiguity in sign in each of the scattering parameters.

The numerical results presented in chapter VI show that in some cases our numerical solutions lie close to the one boson exchange results, so that we will use the OBE results as a guide to choose the signs of the parameters. In the cases where our results differ markedly from the OBE

values, it is possible to establish the signs when the coupling constant is small enough to give agreement with the OBE theory. This is the only way that the signs can be found, since we have obtained all our results through quadratic expressions.

## 4.3) One boson exchange results

If only first order terms in the Yukawa potential are retained, then for the uncoupled states we have the one boson exchange (OBE) potential, which is usually found by other methods. This result is not surprising, since the OBE diagram is based on the ladder approximation, and other approximations not used in the preceding analysis.

The proof of this is straight forward if we use the first order Born approximation. The equation (4.10) for $f'(q)$ becomes

$$f'(q) = K'(q, k) \tag{4.70}$$

and hence the phase shift for the spin singlet interaction is given by

$$\tan \delta = -\frac{\pi}{2k} f'(k) = -\frac{\pi}{2k} K'(k, k)$$

Neglecting term of second order in the integral operators, we find from

equation (2.39) that

$$K'(k, k) = -\frac{1}{E}\left[E^2 A_j + M^2 B_j - k^2\left(\frac{j}{2j+1} C_{j-1} + \frac{j+1}{2j+1} C_{j+1}\right)\right]$$

and

$$\tan\delta = \frac{\pi}{2kE}\left[E^2 A_j + M^2 B_j - k^2\left(\frac{j}{2j+1} C_{j-1} + \frac{j+1}{2j+1} C_{j+1}\right)\right]$$

Using (2.29) to find expressions for the individual types of fields, we have

Scalar field

$$\tan\delta = \frac{g^2}{4kE}\left[\left((E^2 + M^2) - \frac{2k^2 + \mu^2}{2}\right) Q_j + k^2 \delta_{jo}\right]$$

Pseudo-scalar field

$$\tan\delta = \frac{g^2}{4kE}\left[\frac{\mu^2}{2} Q_j - k^2 \delta_{jo}\right]$$

Vector field

$$\tan\delta = -\frac{g^2}{4kE}\left[2M^2 + 4k^2\right] Q_j$$

where the argument of the Legendre functions is $(1 + \mu^2/2k^2)$, and $\mu$ is the meson mass. Similarly the spin triplet $l = j$ equation gives

$$\tan\delta = \frac{\pi}{2kE}\left[E^2 D_j + M^2 C_j - k^2\left(\frac{j+1}{2j+1} B_{j-1} + \frac{j}{2j+1} B_{j+1}\right)\right]$$

and for individual types of interactions this becomes

Scalar field

$$\tan\delta = -\frac{g^2}{4kE}\left[-(E^2+M^2)\,Q_j + \frac{k^2}{2j+1}\left((j+1)\,Q_{j-1} + j\,Q_{j+1}\right)\right]$$

Pseudo-scalar field

$$\tan\delta = -\frac{g^2}{4kE}\,\frac{1}{2j+1}\left[j\,Q_{j+1} - (2j+1)\,Q_j + (j+1)\,Q_{j-1}\right]$$

Vector field

$$\tan\delta = -\frac{g^2}{4kE}\left[E^2 Q_j + \frac{k^2}{2j+1}\left((j+1)\,Q_{j-1} + j\,Q_{j+1}\right)\right]$$

More will be said about the significance of the OBE terms when the results are given and discussed in Chapter VI.

Because of the complicated technique necessary to obtain phase shifts for the coupled states, it is not possible to derive analytical expressions even in the first Born approximation. However, it may be of interest to note that the OBE phase shifts would be obtained from the following coupled equations.

$$(E^2 - T^2)\,b_2 = \underline{G}\,b_2 + \frac{\sqrt{j(j+1)}}{2j+1}\,\tfrac{1}{2}\,(\underline{E}+\underline{H})\,b_3 \qquad (4.71)$$

$$(E^2 - T^2)\,b_2 = \frac{\sqrt{j(j+1)}}{2j+1}\,\tfrac{1}{2}\,(\underline{E}+\underline{H})\,b_2 + \underline{F}\,b_3 \qquad (4.72)$$

These may be compared with equations (4.12) - (4.15) where a factor $\sqrt{j(j+1)}\,/\,(2j+1)$ arises from normalizing $\Psi_2$ and $\Psi_3$. In the Born approximation, $\underline{G}^T = \underline{G}$ etc., and the resemblance is obvious; however, the

results of this chapter do not lead to the results of the OBE theory as (4.71) and (4.72) would. Terms arising from the operator $\frac{1}{2}(\underline{E} + \underline{H})$ can be found, but they are not the most significant. Despite the strong resemblance between equations (4.12) - (4.15) and (4.71) - (4.72), there is no direct way of obtaining the second set from the first, and it is not possible to find analytical results in the first Born approximation using the method of the previous section.

We will next briefly summarize the allowable states of the nucleon-nucleon system.

## 4.4) Allowable states of the nucleon-nucleon system

The total wave function for two nucleons is a product of the radial (or momentum), spin, and isotopic spin wave functions, and must be anti-symmetric. The parity of the radial wave function depends on the orbital angular momentum $\ell$, and is $(-1)^{\ell}$, and the spin and iso-spin singlet states are anti-symmetric, whereas the triplet states are symmetric.

To obtain an overall anti-symmetric wave function, we have to use the following combinations:

| Spin singlet | $\underline{\tau}_1 . \underline{\tau}_2$ | |
|---|---|---|
| even $\ell$ | +1 | (iso-triplet) |
| odd $\ell$ | -3 | (iso-singlet) |
| Spin triplet | | |
| even $\ell$ | -3 | (iso-singlet) |
| odd $\ell$ | +1 | (iso-triplet) |

Since the p-p and n-n systems are in the iso-triplet state, they can only exist for even $\ell$ in the spin singlet states, i.e. $^1S_0$, $^1D_2$, $^1G_4$, etc., and for odd $\ell$ in the spin triplet states, i.e. $^3P_0$, $^3P_1$, $^3P_2$, $^3F_2$, etc. The n-p system can be in the iso-spin singlet or triplet state, and therefore can exist in all states. The effect of the $\underline{\tau}_1 . \underline{\tau}_2$ eigenvalue is included in the computer programmes.

# CHAPTER V

## NUMERICAL METHODS

The purpose of this chapter is to explain the techniques used in solving the singular integral equations from which the phase shifts are determined. A listing of the final Fortran programmes used on the University of Adelaide's CDC 6400 computer is included in the appendix.

### 5.1) Uncoupled equation solutions

The type of equation to be solved in the single equation case is

$$f(q) = K(q, k) + P \int_0^\infty \frac{K(q, q')}{k^2 - q'^2} f(q')\, dq' \qquad (5.1)$$

The existence of a solution depends on $K(q,q')$, which is the interaction function in momentum space and consists of first and second order terms in the Yukawa potentials.

The kernel of equation (5.1) is singular and Fredholm theory cannot be applied to it as it stands. An iterative method using $f_1(q) = K(q,k)$ as the first approximation and estimating the value of the integral across

the singularity analytically, as was done by Signell[38] for the OPE potential, does not converge for our more general interaction. The method therefore adopted is to separate the singularity by introducing the function $\ell(q)$ such that

$$f(q) = \frac{\ell(q)}{1 - P\int_0^\infty \frac{\ell(q')}{k^2 - q'^2}\,dq'} = \frac{\ell(q)}{1 - C} \qquad (5.2)$$

With this substitution, equation (5.1) becomes

$$\ell(q) = K(q,k) + \int_0^\infty \frac{K(q,q') - K(q,k)}{k^2 - q'^2}\,\ell(q')\,dq' \qquad (5.3)$$

and the kernel is no longer singular. This is now approximated by a system of simultaneous equations and solved for $\ell(q')$. This method has been used successfully for the non-static OPE potential by Goto[39]. A different subtraction, giving the kernel $\left[K(k,q)\,K(q',k) - K(q,q')\,K(k,k)\right]/(k^2 - q'^2)$, is used by Noyes[40], but this method leads to the same results.

In the discrete representation, equation (5.3) becomes

$$K(q_i,k) = \sum_{j=1}^{N}\left[\delta_{ij} + \frac{K(q_i,k) - K(q_i,q_j)}{k^2 - q_j^2}\,\delta_{q_j}\right]\ell(q_j) \qquad (5.4)$$

where $i = 1, \text{----}, N$ and $q_i$ is the value of $q$ at the $i$ th point. Initially, the $q_i$ were chosen so as to increase exponentially with $i$ to give good sensitivity at low momenta around the incident energy, and

progressively increase the step size as $i$ becomes large. However, this was found to be inaccurate once the step size became greater than the smallest meson mass used, i.e., 1.0 for the pion. This is because the Legendre function of the second kind, with the argument $z = (q^2 + q'^2 + \mu^2)/2qq'$, has a peak of width of order $\mu$ about $q = q'$, and if the step size becomes greater than $\mu$, we get a poor representation of the function.

This factor, together with available computer facilities, determines to what value we can take the upper limit of the integral (or summation). The upper limit to the step size is therefore 1.0, which is used for $q > 3.0$ for most of the calculations. For $q < 3.0$, or energy less than about 300 Mev., we use a smaller step size to obtain more sensitivity in the incident momentum region where the kernel is likely to vary more rapidly. It also enables the constant $C$ in equation (5.2) to be evaluated more accurately. In this region the step size is usually taken to be 0.3.

The programmes have been written to handle a maximum of 40 points, giving an upper limit that we may use for the integral of $q = 32.0 \sim 4.10^4$ Mev., which should be more than ample to describe any physical situation. The procedure used is to solve (5.4) for say, $N = 20$, and $N = 30$. If the solutions give the same function for $\ell(q)$ in both cases, then the integral converges and we have a solution. If the coupling constants are small enough, we can get consistent solutions in all states, but the potentials may become singular if the coupling

constants are made too large. The results for all states are given and discussed in the next chapter.

Once the function $\ell(q)$ is found, the constant $C = P\int_0^\infty \frac{\ell(q')}{k^2 - q'^2}\, dq'$ is calculated by the subroutine SENT. This calculates the integral excluding the singular region by Simpson's rule, and evaluates the singular region analytically by approximating $\ell(q)$ to a quadratic function. The accuracy of this is found to be better than 1% for the step sizes given.

In the discrete representation, all the integral operators become matrices and an integral of the form

$$\int C(q, q')\, D(q', q'')\, dq' = E(q, q'') \qquad (5.6)$$

is evaluated by taking the matrix product of $C(q, q')\, \delta q'$ and $D(q, q')$. i.e. (5.6) becomes

$$C\, \delta q\, D = E$$

where $\delta q$ is a diagonal matrix with the elements $\delta q_1$, $\delta q_2$, etc. Therefore to evaluate the operators K, H, G, etc. we take matrix products of the Yukawa potential operators $A_j$, $B_j$, etc. as required by the equations (2.39), (2.42) and (2.66) - (2.69).

The integrals, that give rise to second order terms in the potentials, can be shown to converge in all cases. However, care must be taken that the range is taken large enough so that the rest of the integral to $\infty$ can

be neglected. If (5.6) represents such an integral, then to obtain an accurate value of the integral at $q = q'' = q_N$, the limit of integration must be taken to at least $2q_N$ because both functions C and D have a maximum when the two arguments, q and q", are equal. Therefore, to obtain accurate values of q and q" up to the value $q_N$ in the function E (q, q") of equation (5.6), we have to take products of 2N × 2N matrices. This is included in the programmes, and results in an accuracy of better than 1%.

## 5.2) Coupled equation solutions

For the coupled equations we employ a similar technique to that used for the single equation. To avoid iteration, we solve a system of 2N simultaneous equations in order to obtain N points for each function.

We will write a pair of coupled equations representative of either (4.17) and (4.18) or (4.19) and (4.20), in the matrix form

$$\begin{bmatrix} E^2 - T^2 & 0 \\ 0 & E^2 - T^2 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} G & H \\ E & F \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \qquad (5.7)$$

where G, H, E and F are integral operators, and $a_2$, $a_3$, f and g are functions of q. Now (5.7) gives

$$\begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} x\ \delta(q-k) \\ y\ \delta(q-k) \end{bmatrix} + \begin{bmatrix} (E^2 - T^2)^{-1} & 0 \\ 0 & (E^2 - T^2)^{-1} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \qquad (5.8)$$

where $x$ and $y$ are numbers and would correspond to $(m - i\,a\,f)$ and $(n - i\,a\,g)$ in equations (4.21) and (4.22). If we call

$$\int G(q, q')\ \delta(q' - k)\, dq' = G_o \quad \text{etc., then}$$

$$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} x\,G_o + y\,H_o \\ x\,E_o + y\,F_o \end{bmatrix} + \begin{bmatrix} G & H \\ E & F \end{bmatrix} \begin{bmatrix} (E^2 - T^2)^{-1} & 0 \\ 0 & (E^2 - T^2)^{-1} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \qquad (5.9)$$

and

$$\begin{bmatrix} x\,G_o + y\,H_o \\ x\,E_o + y\,F_o \end{bmatrix} = \left[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} G & H \\ E & F \end{bmatrix} \begin{bmatrix} (k^2 - q^2)^{-1} & 0 \\ 0 & (k^2 - q^2)^{-1} \end{bmatrix} \right] \begin{bmatrix} f \\ g \end{bmatrix}$$

$$= \left[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} G - G_o & H - H_o \\ E - E_o & F - F_o \end{bmatrix} \begin{bmatrix} (k^2 - q^2)^{-1} & 0 \\ 0 & (k^2 - q^2)^{-1} \end{bmatrix} \right] \begin{bmatrix} f \\ q \end{bmatrix}$$

$$- \begin{bmatrix} G_o & H_o \\ E_o & F_o \end{bmatrix} \begin{bmatrix} (k^2 - q^2)^{-1} & 0 \\ 0 & (k^2 - q^2)^{-1} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \qquad (5.10)$$

If we put

$$P \int_o^\infty \frac{f(q)}{k^2 - q^2}\, dq = a \qquad P \int_o^\infty \frac{g(q)}{k^2 - q^2}\, dq = b \qquad (5.11)$$

the last term becomes $- \begin{bmatrix} G_o\,a + H_o\,b \\ E_o\,a + F_o\,b \end{bmatrix}$

so that equation (5.10) is now

$$\begin{bmatrix} (x+a)\,G_o + (y+b)\,H_o \\ (x+a)\,E_o + (y+b)\,F_o \end{bmatrix} = \left[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} G-G_o & H-H_o \\ E-E_o & F-F_o \end{bmatrix} \begin{bmatrix} (k^2-q^2)^{-1} & 0 \\ 0 & (k^2-q^2)^{-1} \end{bmatrix} \right] \begin{bmatrix} f \\ g \end{bmatrix} \quad (5.12)$$

This is now approximated by a system of $2N$ simultaneous equations, so that $G$, $H$, $E$ and $F$ become square matrices, $1$ is replaced by the unit matrix, and $G$ , $H$ , $E$ , $F$ , $f$ and $g$ are column matrices. These are solved firstly, for $x + a = 1$, $y + b = 0$ to give a solution we call $\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}$, and secondly for $x + a = 0$, $y + b = 1$ to give $\begin{bmatrix} f_2 \\ g_2 \end{bmatrix}$.

The general solution is then given by

$$\begin{bmatrix} f \\ g \end{bmatrix} = (x+a) \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} + (y+b) \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \quad (5.13)$$

and if we integrate these according to (5.11), we find

$$a = (x+a)\, f_1' + (y+b)\, f_2' \quad (5.14)$$

$$b = (x+a)\, g_1' + (y+b)\, g_2' \quad (5.15)$$

where $f_1' = P\int_0^\infty \frac{f_1(q)}{k^2-q^2}\,dq$ etc.

The equations (5.14) and (5.15) can be solved for $a$ and $b$,

$$a = \frac{y\, f_2' + x\left[\, g_1' f_2' + f_1' (1 - g_2')\right]}{(1 - g_2')(1 - f_1') - f_2' g_1'} \tag{5.16}$$

$$b = \frac{x\, g_1' + y\left[\, f_2' g_1' + g_2' (1 - f_1')\right]}{(1 - g_2')(1 - f_1') - f_2' g_1'} \tag{5.17}$$

from which $x + a$ and $y + b$ can be obtained. These are

$$x + a = \frac{x (1 - g_2') + y f_2'}{(1 - g_2')(1 - f_1') - f_2' g_1'} \tag{5.18}$$

$$y + b = \frac{x g_1' + y (1 - f_1')}{(1 - g_2')(1 - f_1') - f_2' g_1'} \tag{5.19}$$

Putting these into (5.13), it is found that if

$$\begin{bmatrix} f \\ g \end{bmatrix} = x \begin{bmatrix} f_1'' \\ g_1'' \end{bmatrix} + y \begin{bmatrix} f_2'' \\ g_2'' \end{bmatrix} \tag{5.20}$$

then

$$f_1'' = \frac{(1 - g_2') f_1 + g_1' f_2}{(1 - g_2')(1 - f_1') - f_2' g_1}$$

$$f_2'' = \frac{f_2' f_1 + (1 - f_1') f_2}{(1 - g_2')(1 - f_1') - f_2' g_1'}$$

$$g_1'' = \frac{(1 - g_1') g_1 + g_1' g_2}{(1 - g_2')(1 - f_1') - f_2' g_1'}$$

$$g_2'' = \frac{f_2' g_1 + (1 - f_1') g_2'}{(1 - g_2')(1 - f_1') - f_2' g_1'} \tag{5.21}$$

where the functions $f_1''$, $f_2''$, $g_1''$ and $g_2''$ would correspond to $f_1$, $f_2$, $g_1$ and

$g_2$ of equations (4.27) and (4.28).

This method gives solutions of the two sets of coupled equations, which are then used to calculate the phase shifts and mixing parameters by the technique explained in the last chapter.

## 5.3) Units

Initially, the units $\hbar = c = 1$ were adopted. We also work in meson mass units. i.e. $\mu_\pi = 1.00$. In these units

$$1 \text{ Mev.} = .00740466$$

and the nucleon mass is

$$M = 6.73035$$

E is half the total energy in the centre of mass system, so that for two nucleons of mass M

$$E = M + \tfrac{1}{2} (\text{K.E. of C. of M. system})$$

and

$$E^2 - M^2 = k^2 = M (\text{K.E.}) + \tfrac{1}{4} (\text{K.E.})^2$$

Therefore

$$k = \left[ M.\ (K.E.) + \tfrac{1}{4}\ (K.E.)^2 \right]^{\frac{1}{2}}$$

This is the relation between the momentum and the kinetic energy in the centre of mass system. A simpler, approximate relation, is given by

$$k = \sqrt{.025 \times (K.E.\ (Mev.))}$$

# CHAPTER VI

## RESULTS AND DISCUSSION

### 6.1) General observations

In this work we have succeeded in exactly reducing the B-S equation in the ladder and instantaneous interaction approximations for nucleons interacting through scalar, pseudo-scalar and vector fields, to Schrodinger type of wave equations. The physical significance of the solutions became obvious once the B-S probability density was derived in chapter III, which also enabled a method to be developed for finding phase shifts from the two pairs of coupled equations representing the $l = j-1$ and $l = j+1$ states. The probability density also showed that outside the interaction region, we can identify the functions satisfying the single equations, with the Schrodinger wave functions for the uncoupled states.

Using the above theory we propose to try to answer a number of questions and to fit our results to experimentally derived phase shifts. The questions to which we hope to provide some answers are:

(i) How valid is the ladder approximation?

(ii) How valid is the instantaneous interaction approximation?

(iii) How valid is the OBE potential as compared with taking into account all graphs of the ladder type?

(iv) What is the effect of mixing mesons in the ladder approximation?

In the next two sections we give numerical results and graphs of phase shifts, and discuss (i), (iii) and (iv).

The effect of the instantaneous approximation is difficult to assess accurately. It is generally assumed to be of a semi-relativistic nature. i.e., it is valid in the non-relativistic region, but somewhat doubtful relativistically. That it is valid in the non-relativistic region is indicated by the argument in section 2.2), and also by our results which correctly reproduce the OBE potential as a first order term. The fact that these OBE terms are the ones usually obtained by taking the non-relativistic approximation of the most general OBE potentials, e.g. Hoshizaki and Machida[41], Goto and Machida[42], indicates that our instantaneous interaction approximation is actually non-relativistic. This of course assumes that an exact ladder diagram solution would give the relativistic OBE potential for the first term.

The term of second order in the Yukawa potential is the one of greatest interest, since the OBE potential is well understood in the way it gives rise to central, tensor, and spin-orbit forces. Even though this term is not a very complicated expression, the integral cannot be carried out analytically, which makes it difficult to interpret physically. This term must somehow represent the rest of the ladder graphs excluding the OBE term and possibly relativistic effects. The next higher order terms after the OBE term are expected to be due to the exchange of two mesons. In the

ladder approximation, these would be given by the graphs of the type

but graphs involving crossed meson lines are excluded. The fact that the above diagrams give matrix elements proportional to $g^4$, tends to indicate that our second order terms are largely due to two meson exchange effects, although it is not possible to directly identify our potential with these two diagrams. Before discussing the numerical results, we can make two observations about the interaction potentials.

The second term in all cases is proportional to $(\underline{\tau}_1.\underline{\tau}_2)^2$. i.e., in the iso-spin triplet case it gives 1, and in the iso-singlet case it gives the factor 9. This is because the iso-spin factor appears with each Yukawa factor, so that if the theory led to higher orders in the propagators, we would have a factor of $(\underline{\tau}_1.\underline{\tau}_2)^n$ for the n th order, which would be very large indeed in the iso-spin singlet states.

The other thing to note is that the magnitude of the OBE term in the pseudo-scalar case may be expected to be much smaller relative to the second order term than in the scalar or vector interaction case. This is because the large terms, which have a factor $M^2$, cancel in the OBE term to give the OPE potential, but they do not cancel in the second order term.

This is the result of the commutation relations of $\gamma^5$ with the spin-angle Dirac matrices $\Psi_0$ , $\Psi_1$ , $\Psi_2$ and $\Psi_3$ .

We will now give and discuss numerical results in terms of phase shifts for the individual meson fields.

## 6.2) Results for individual meson fields

### (a) Scalar field

The $I = 0$ scalar boson is found to be necessary in the OBE theory of nucleon forces. In our case, the scalar meson, which we will denote by $\sigma$ , does not give greatly different results from the OBE theory. At energies below 300 Mev., the second order term contribution is almost negligible.

In all the computations in this section, we calculate phase shifts in the energy range 0 - 300 Mev. and for coupling constants of $g^2 = 2$, 6, 10 and 14. This is the range of coupling constants we would expect to have to deal with. The existence of a solution, in the sense that increasing the cut-off of the integral produces no change in result, depends on the magnitude of the coupling constant. If the coupling constant is small enough we can always find a solution, but if it is made too large in any

state, the potential becomes singular and there is no solution.

The results for only scalar meson exchange are given by Fig. 1 where we plot the phase shift in degrees against the incident energy in the centre of mass system. The meson mass used is 500 Mev. In most cases we give three curves. The one labelled OBE is the one boson exchange phase shift in the Born approximation. In brackets we include the coupling constant used. The curve labelled BORN is the Born approximation result when the full potential is used and the difference between this and the OBE curve indicates the relative effect of the second order term. The third one, labelled with just the coupling constant, is the full solution and when compared with the BORN curve, indicates the validity of using the Born approximation.

The relevant features of each of the graphs are as follows:

$^1S_0$: No solution exists for $g_\sigma^2 \geqslant 2$

$^1P_1$: Here we obtain solutions for $g_\sigma^2 = 2$ but not for 6. Fig. 1 shows that the second order term is negligible compared with the OBE term and that it is in fact the OBE term that makes the potential singular. The potential is attractive in all states.

$^1D_2$: Solutions exist for $g_\sigma^2 = 2, 6$ but not 10. The OBE and BORN results again coincide showing the negligible effect of the second term, and at low energies or small coupling constant the Born approximation gives a good representation of the complete solution.

$^1F_3$: For the $^1F_3$ state solutions exist for all coupling constants up to

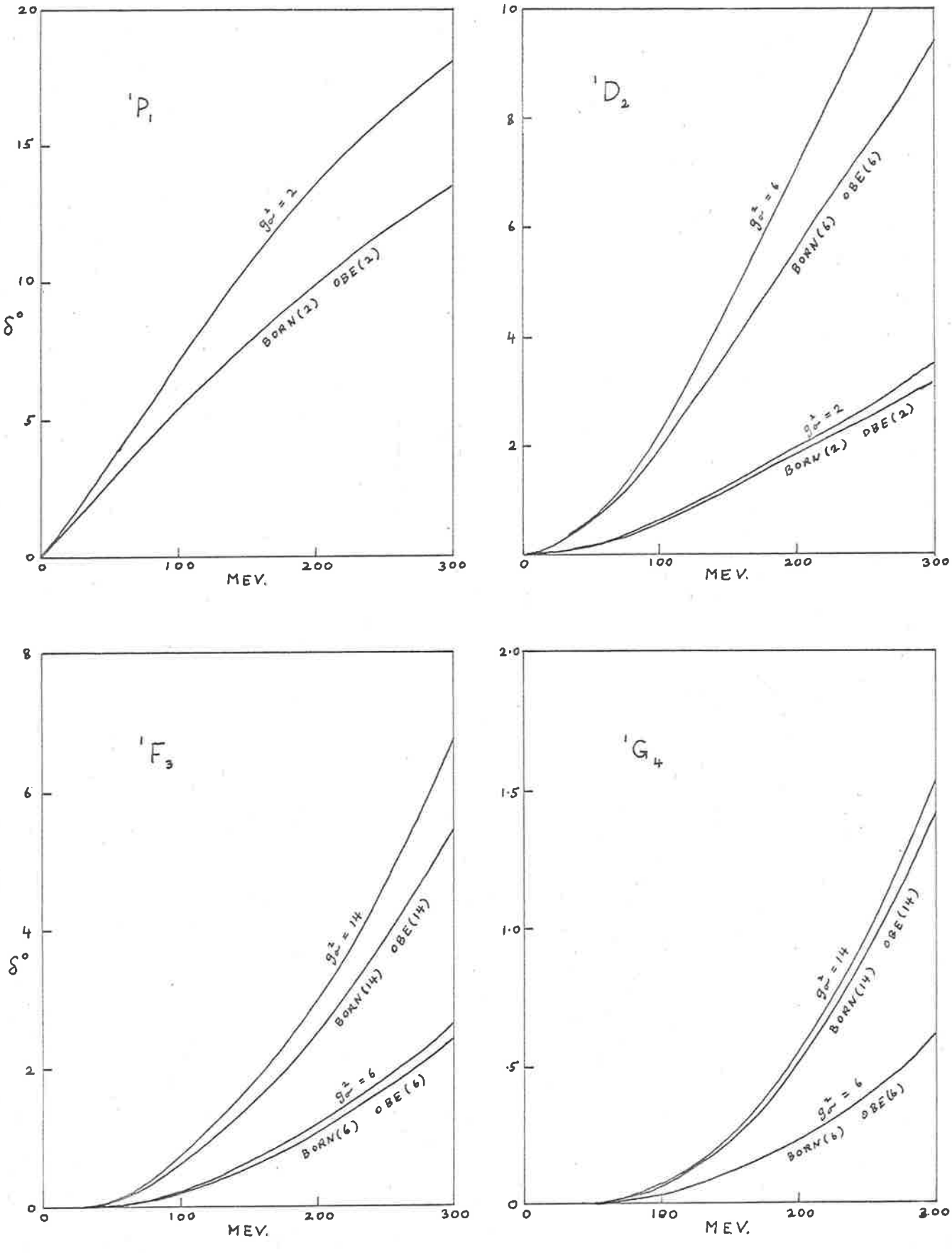

$^1P_1$
$g^2_{\sigma\sigma'} = 2$
BORN(2) OBE(2)
$\delta^\circ$
MEV.
$^1D_2$
$g^2_{\sigma\sigma'} = 6$
BORN(6) OBE(6)
$g^2_{\sigma\sigma'} = 2$
BORN(2) OBE(2)
MEV.
$^1F_3$
$g^2_{\sigma\sigma'} = 14$
BORN(14) OBE(14)
$g^2_{\sigma\sigma'} = 6$
BORN(6) OBE(6)
$\delta^\circ$
MEV.
$^1G_4$
$g^2_{\sigma\sigma'} = 14$
BORN(14) OBE(14)
$g^2_{\sigma\sigma'} = 6$
BORN(6) OBE(6)
MEV.

Fig. 1

and including 14. The curves for $g_\sigma^2 = 6$ and 14 only are given as they give a good indication of behaviour at other values of the coupling constant.

'$G_4$: Here again we have solutions for all coupling constants, and we plot the curves for $g_\sigma^2 = 6$ and 14. The same comments apply to this and the '$F_3$ state, as were made about the '$D_2$ state. As we increase the angular momentum, so the Born approximation approaches the complete solution.

We do not give graphs of the spin triplet $l = j$ states. The only difference between these and the spin singlet results in each state are that the spin triplet phase shifts are about 10% smaller than the corresponding spin singlet ones. All other features are the same.

Thus we can see that the scalar interaction is almost entirely due to the OBE potential, which is attractive in all states, and that the Born approximation gives a reasonably good result provided the coupling constant is not too large.

(b) Pseudo-scalar field

It is well known that because of the low mass of the $\pi$-meson, it is responsible for nearly all the nucleon-nucleon force in the outer interaction region. In analogy with the results for scalar and vector mesons, it would be expected that the OPE effect dominates in the low energy and

high angular momentum region, and the more complicated effects of the ladder diagram only come into force at high energies and low angular momentum. However, this does not seem to be the case. The reason for this can be attributed to two factors. The most important is the relative smallness of the pion mass. This has the effect of making the integral in the second order term of the potential very large compared with the OPE term. This factor, together with the cancellation of terms of order $M^2$ in the OBE term, makes the overall potential attractive for nearly all energies in all states.

Just to what extent the second order term dominates the OPE potential, can be seen from the phase shifts which are given in Figs. 2 and 3. A curve given by a broken line indicates the results of phase shift analysis given by Arndt and MacGregor[43] and is included for comparison.

$^1S_0$: Fig. 2(a) gives the $g_\pi^2 = 2$ solution which is the only one we obtain. Here the OPE phase shift is negative (repulsive potential), but the second term is very strongly attractive and the final phase shifts are positive.

$^1P_1$: No solutions found for $g_\pi^2 \gg 2$.

$^1D_2$: Here we find solutions for $g_\pi^2 \leq 6$. The phase shifts increase very rapidly with $g_\pi^2$ since the second order term proportional to $g_\pi^4$ is the dominant one. The phase shifts for $g_\pi^2 = 2$ and 6 are given in Fig. 2(b).

$^1F_3$: In this state we have the effect of the $(\underline{\tau}_1.\underline{\tau}_2)^2$ factor coming in,

which now multiplies the second order term by 9. The result of this is a very strongly attractive force so that we obtain solutions for $g_\pi^2 = 2$ in the range 0 - 300 Mev., but only in the range 0 - 100 Mev. for $g_\pi^2 = 6$.

$^1G_4$: Fig. 2(d) shows the phase shifts obtained for the $^1G_4$ state. Here again the Born approximation, which lies quite close to the complete solution, is much greater than the OPE result.

$^3P_1$: Only a solution for $g_\pi^2 = 2$ is obtained, (Fig. 3(a) ). Even at this small value of the coupling constant, the second term makes the potential attractive for energies above 100 Mev.

$^3D_2$: No solutions for $g_\pi^2 \gg 2$. This is because the OPE term is attractive as well as the second order term, which is now multiplied by a factor of 9.

$^3F_3$: We can find solutions for $g_\pi^2 \lesssim 10$ and some of these are given in Fig. 3(b). The Born approximation corresponds to the full solution very closely.

$^3G_4$: Solutions exist for $g_\pi^2 = 2$ and 6. (Fig. 3(c) )

From this it can be seen that our pion exchange potentials differ radically from the OPE result which is generally accepted as the dominating interaction in low energy nucleon-nucleon interactions. It appears that it may be difficult to fit our results to experimentally derived phase shifts because of this large attractive potential. This will be discussed further in the next section.

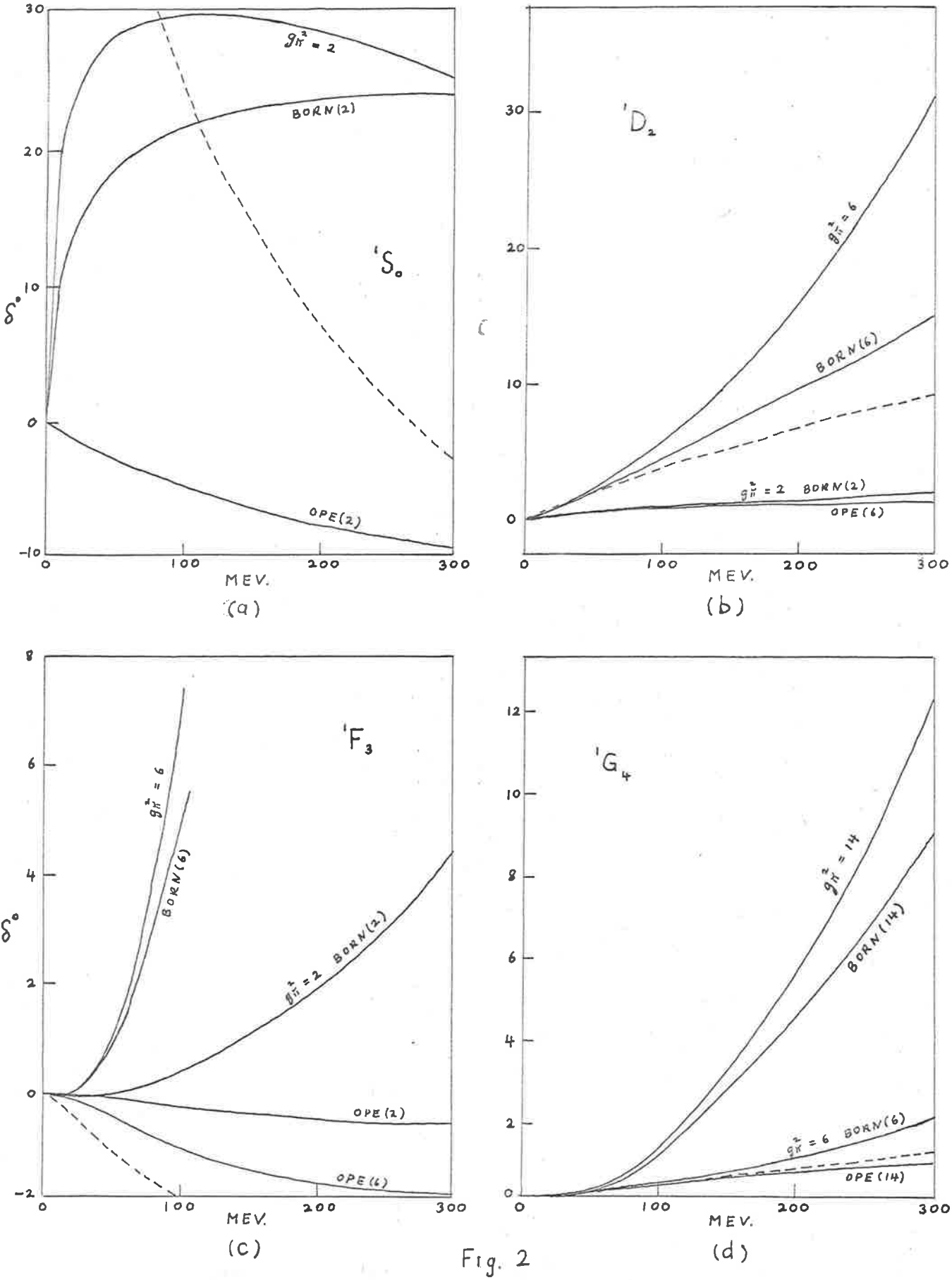
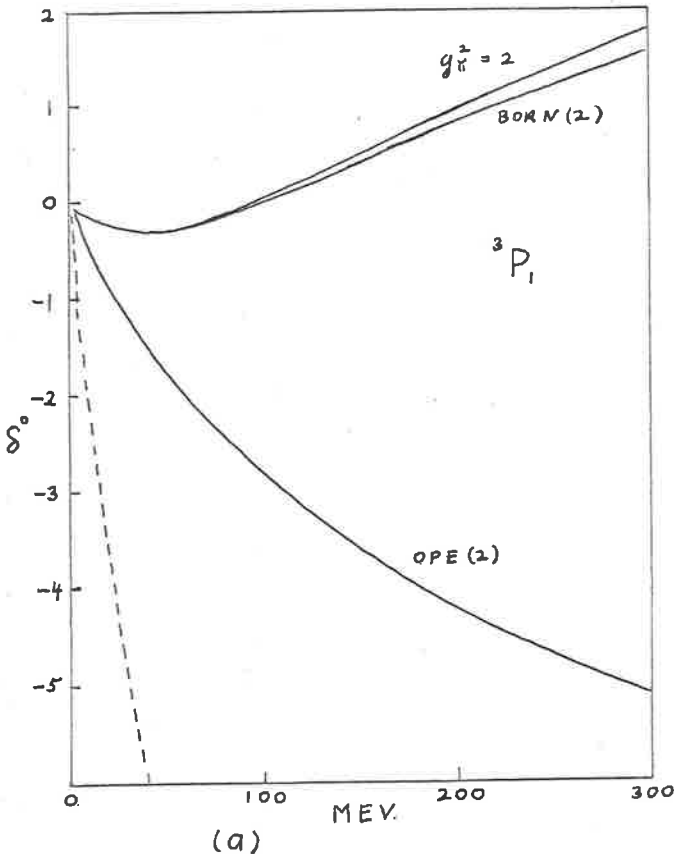
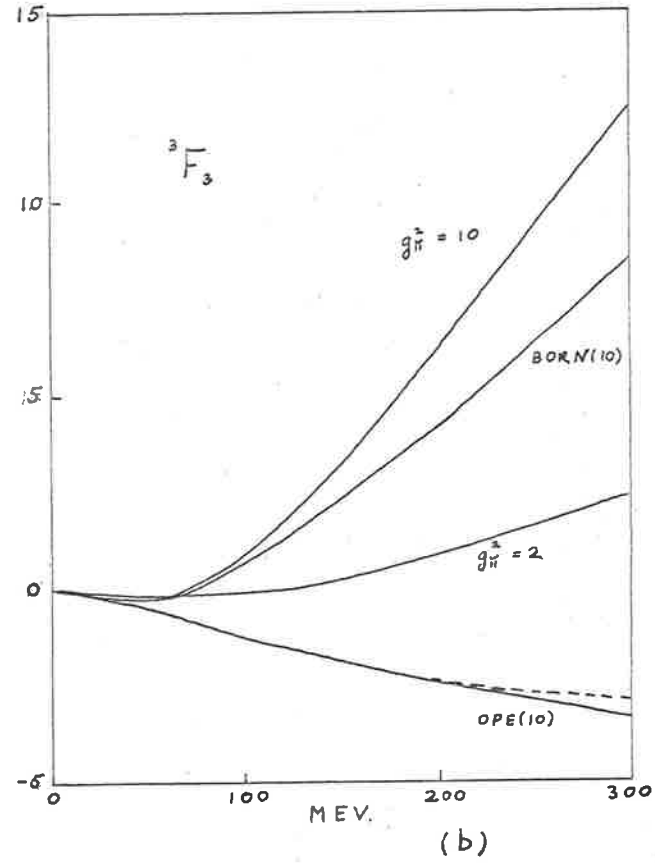

30
20
10
0
-10
δ°
$g_\pi^2 = 2$
BORN(2)
$^1S_0$
OPE(2)
0
100
200
300
MEV.
(a)
30
20
10
0
$^1D_2$
$g_\pi^2 = 6$
BORN(6)
$g_\pi^2 = 2$ BORN(2)
OPE(6)
0
100
200
300
MEV.
(b)
8
6
4
2
0
-2
δ°
$^1F_3$
$g_\pi^2 = 6$
BORN(6)
$g_\pi^2 = 2$ BORN(2)
OPE(2)
OPE(6)
0
100
200
300
MEV.
(c)
12
10
8
6
4
2
0
$^1G_4$
$g_\pi^2 = 14$
BORN(14)
$g_\pi^2 = 6$ BORN(6)
OPE(14)
0
100
200
300
MEV.
(d)

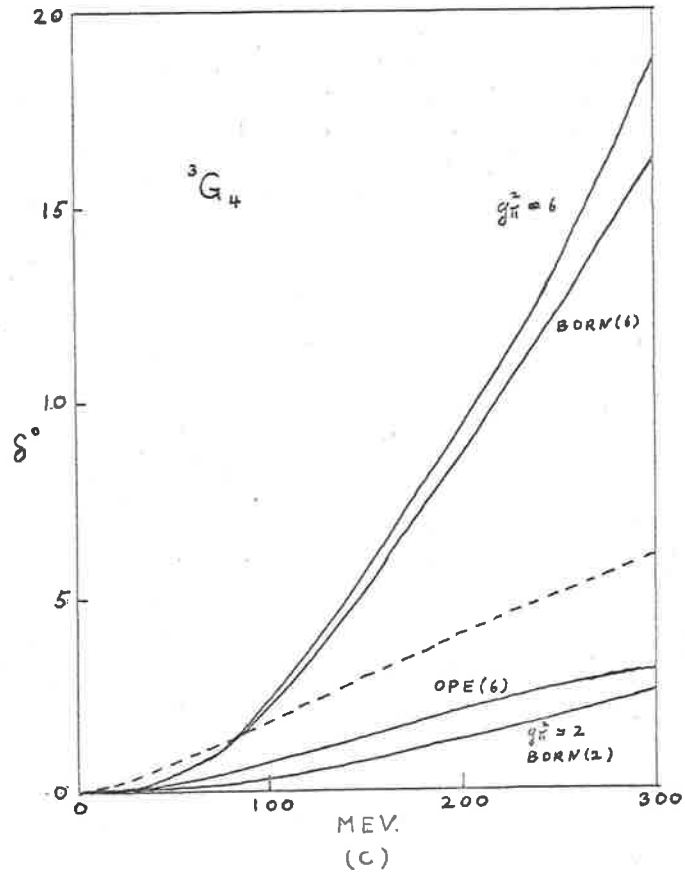
Fig. 2


$g_\pi^2 = 2$
BORN (2)
$^3P_1$
OPE (2)
$\delta°$
MEV.

(a)


$^3F_3$
$g_\pi^2 = 10$
BORN(10)
$g_\pi^2 = 2$
OPE(10)
MEV.

(b)


$^3G_4$
$g_\pi^2 = 6$
BORN(6)
OPE(6)
$g_\pi^2 = 2$
BORN(2)
$\delta°$
MEV.

(c)

Fig. 3

We also give results for the iso-spin triplet $\eta$-meson in Figs. 4 and 5. Comparing these with the pion phase shifts should yield some information regarding the origin of the very strong attractive force in the pion exchange potential. This is because in the spin singlet even states and the spin triplet odd states, the only difference between the $\pi$-meson and $\eta$-meson potentials is due to their difference in mass.

$^1S_0$: A solution for $g_\eta^2 = 2$ only is obtained. Here the OBE phase shift is negative, while the BORN result is positive, and lies fairly close to the full solution.

$^1P_1$: Again we only find a solution for $g_\eta^2 = 2$. The Born approximation is a good one, and the second order term contributes significantly, but is not dominant.

$^1D_2$: In this state, solutions are found for $g_\eta^2 = 2$ and 6 but not 10. The full solution is not very different from the OBE result.

$^1F_3$: The short range of the force is apparent, and the exact solution is approaching the OBE curve as the angular momentum increases.

Fig. 5 gives the phase shifts for the $l = j$ triplet interaction. The potential is attractive in all these states, and does not differ very greatly from the OBE potential.

From these, it can be seen that with a larger meson mass, the second term does not dominate the OBE term in the pseudo-scalar interaction, so that the difficulty with the $\pi$-meson arises mainly from its smaller mass. This indicates that in our potential the higher order contributions from

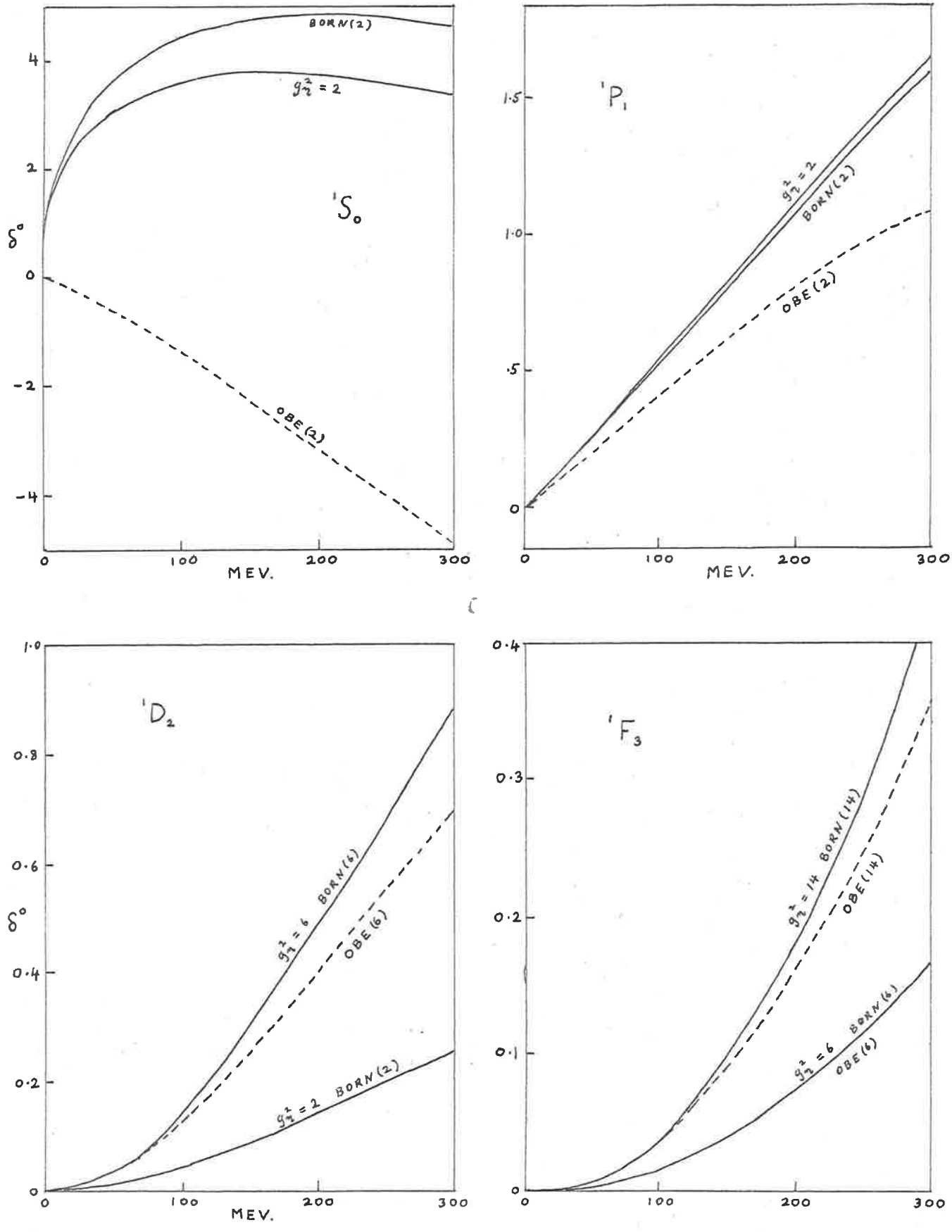

BORN(2)
$g_2^2 = 2$
$^1S_0$
OBE(2)
δ°
MEV.
$^1P_1$
$g_2^2 = 2$
BORN(2)
OBE(2)
MEV.
$^1D_2$
$g_2^2 = 6$ BORN(6)
OBE(6)
$g_2^2 = 2$ BORN(2)
δ°
MEV.
$^1F_3$
$g_2^2 = 14$ BORN(14)
OBE(14)
$g_2^2 = 6$ BORN(6)
OBE(6)

Fig. 4

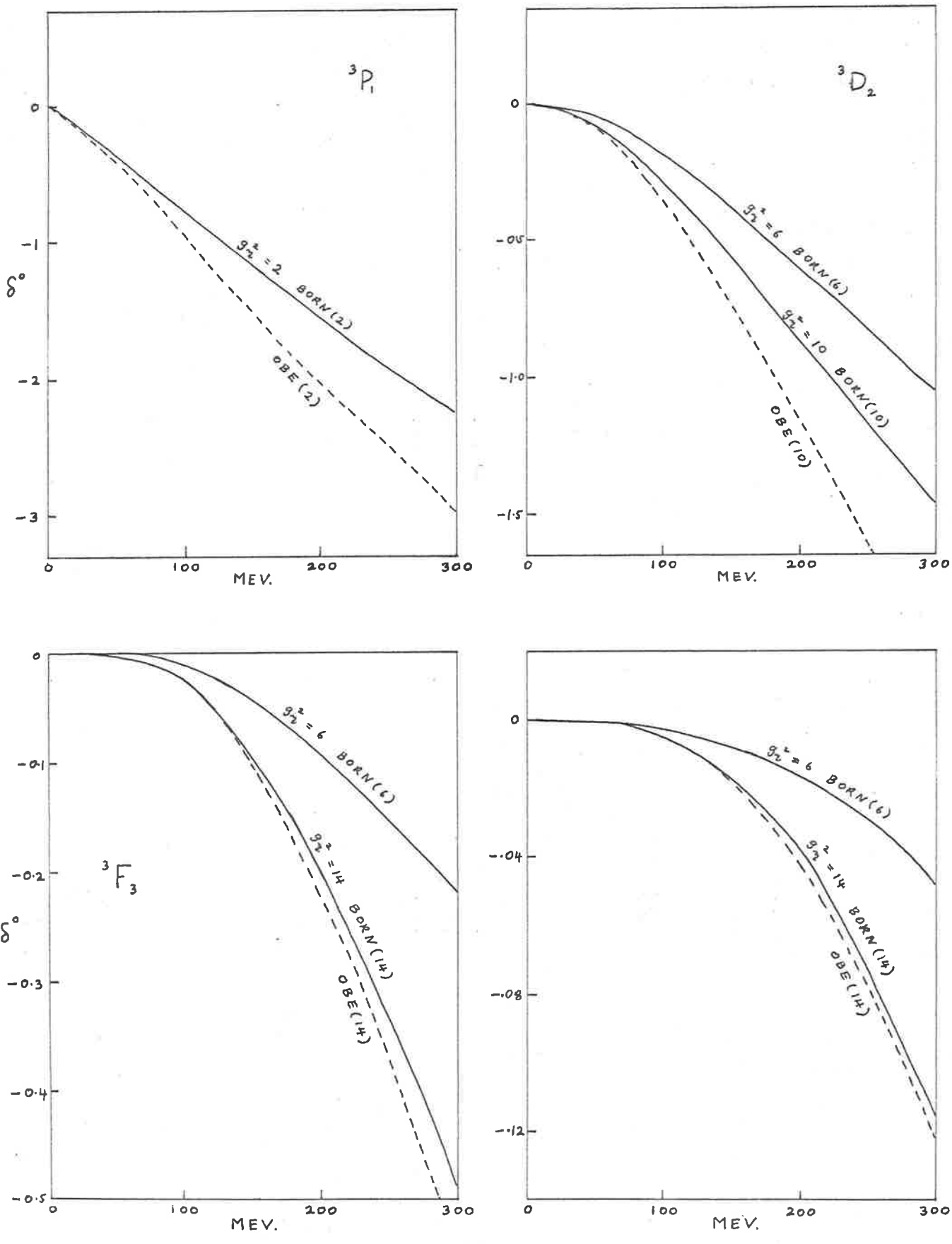

$^3P_1$
$g_2^2 = 2$ BORN(2)
OBE(2)
δ°
MEV.
$^3D_2$
$g_2^2 = 6$ BORN(6)
$g_2^2 = 10$ BORN(10)
OBE(10)
MEV.
$^3F_3$
$g_2^2 = 6$ BORN(6)
$g_2^2 = 14$ BORN(14)
OBE(14)
δ°
MEV.
$g_2^2 = 6$ BORN(6)
$g_2^2 = 14$ BORN(14)
OBE(14)
MEV.

Fig. 5

the $\pi$-mesons provide a strong, long range force, and is in disagreement with the generally accepted view that the range of force due to the exchange of n mesons is proportional to $1/n\mu$.

(c) Vector field

Figs. 6 and 7 show the results if we only employ vector mesons. The dominant features of these are the small effect the vector potential has at low energies due to the short range, and that except for the S and P states the second term is attractive, but has no appreciable effect except for higher energies and large coupling constants.

We give graphs for the iso-vector $\rho$-meson only. In the odd spin triplet states, and the even spin singlet states, the phase shifts are the same for the iso-scalar $\omega$- meson as for the $\rho$-meson. Since the $\omega$-meson interaction is independent of the iso-spin, the trend of phase shifts is similar in all states. i.e. negative and somewhat larger than the OBE result.

$^1S_0$: Solutions can be found for all coupling constants up to $g_\rho^2 = 14$ and although the BORN result is large and positive, the exact phase shifts are negative.

$^1P_1$: No solution for $g_\rho^2 \geqslant 2$. This is an iso-singlet state, and the $\underline{\tau}_1.\underline{\tau}_2$ factor makes the OBE potential singular. By comparison with the other states, it can be seen that the second order term probably has little effect.

$^1D_2$: The small phase shift at low energies indicates the short range of the potential, and OBE, BORN and full solutions do not differ greatly.

$^1F_3$: This is again an iso-singlet state and it is not possible to obtain a solution for $g_\rho^2 \geqslant 6$. Below 100 Mev. the potential has almost no effect, but above this, the phase shift increases very rapidly. The three curves are very close, showing that the OBE term is the most significant.

$^1G_4$: The results for $g_\rho^2 = 6$ and 14 are shown. The three curves coincide in both cases.

$^3P_1$: (Fig. 7). We give results for $g_\rho^2 = 6$ and 14. For this state, the second order term differs from the OBE term by about 20%, and the Born approximation is not a good one.

$^3D_2$: No solutions can be found for this iso-spin singlet state. This is due to the low angular momentum and not so much to the second order term.

$^3F_3$: The potential has no appreciable effect for incident energies below 100 Mev., and second order term is negligible compared with the OBE term.

$^3G_4$: Here we can only obtain solutions for $g_\rho^2 = 2$ and 6. The OBE potential is again the dominant one, and the Born approximation is valid. For larger coupling constants, the OBE potential becomes singular.

In summing up the results for the vector meson interaction, it can be

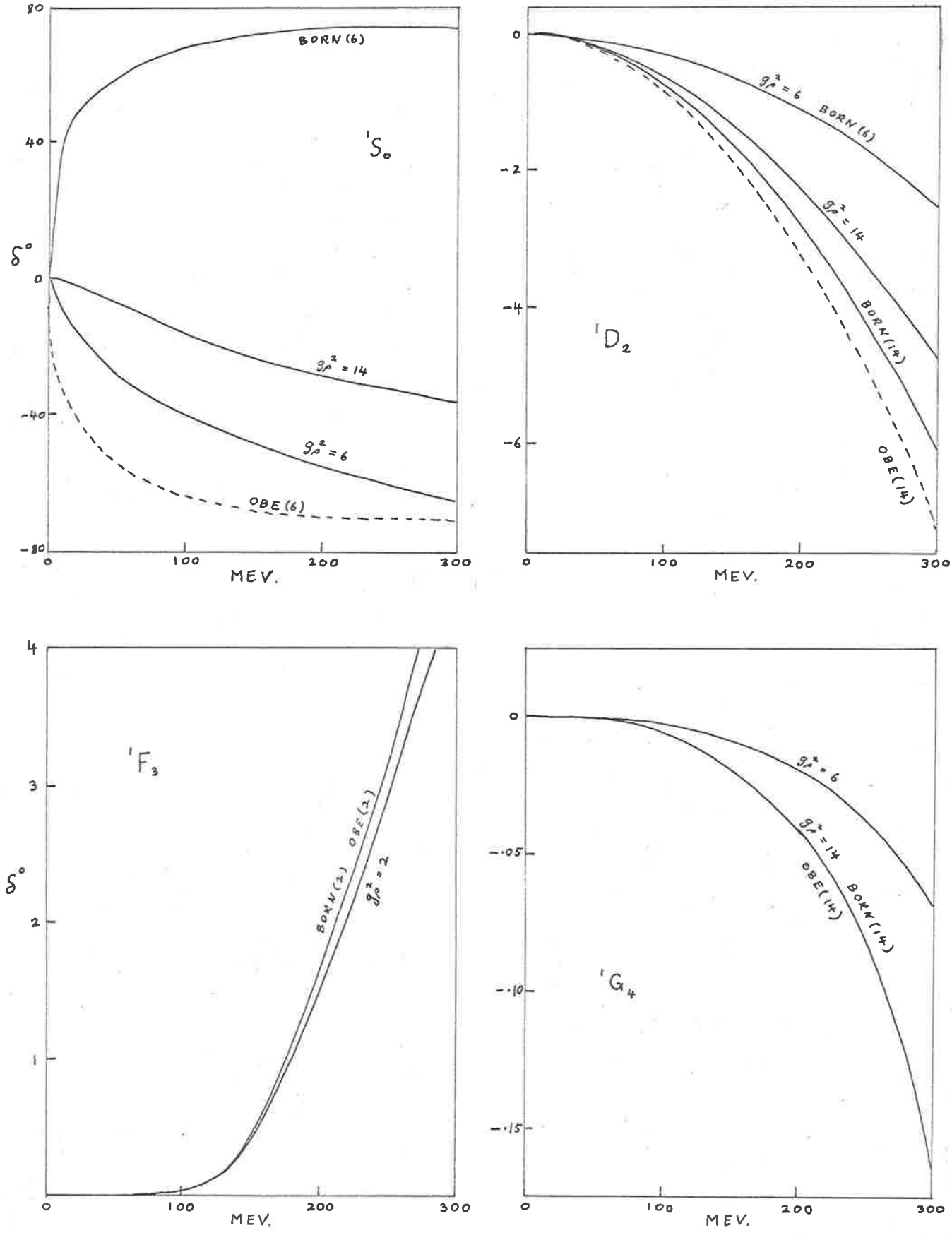
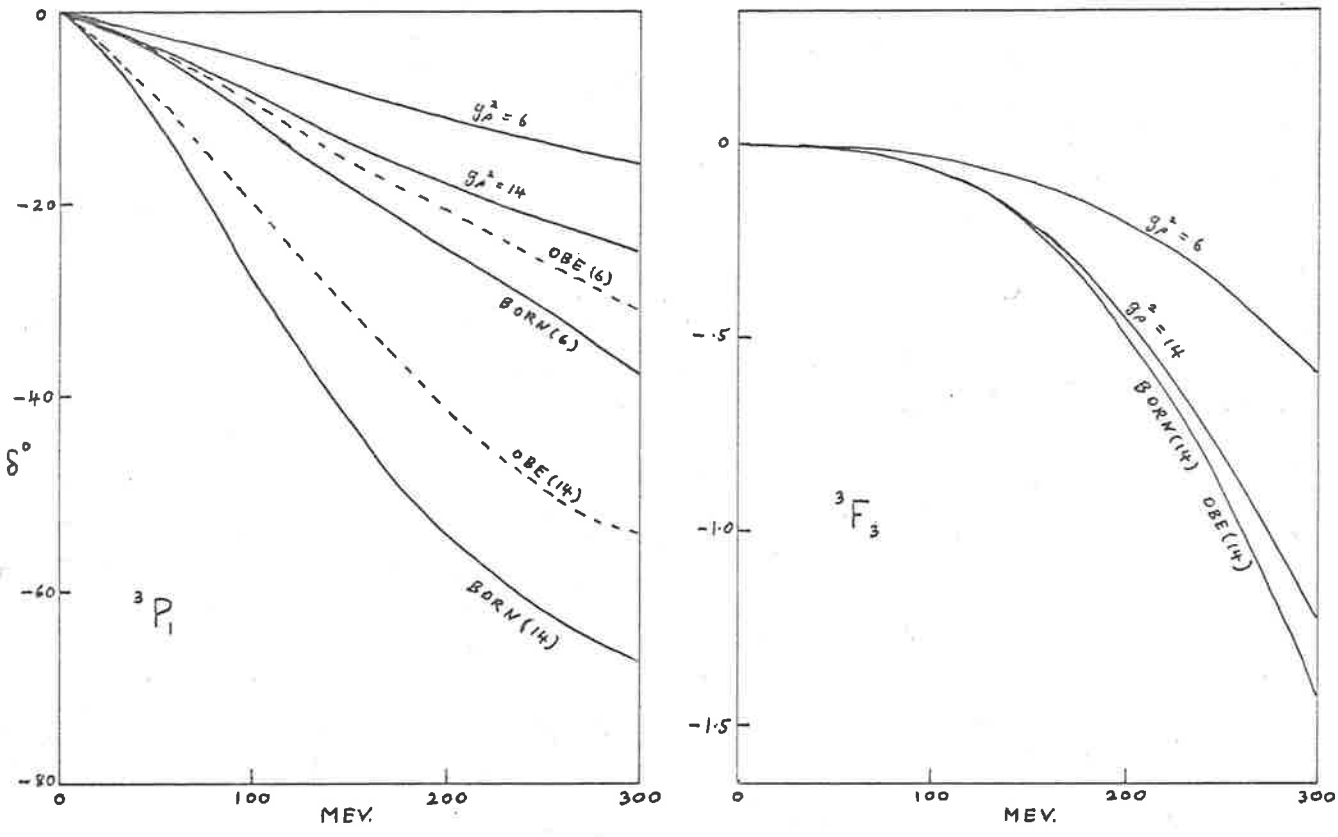

80
40
0
-40
-80
δ°
BORN (6)
$^1S_0$
$g_\rho^2 = 14$
$g_\rho^2 = 6$
OBE (6)
0
100
200
300
MEV.
0
-2
-4
-6
$g_\rho^2 = 6$
BORN (6)
$g_\rho^2 = 14$
BORN (14)
OBE (14)
$^1D_2$
0
100
200
300
MEV.
4
3
2
1
δ°
$^1F_3$
BORN (2)
OBE (2)
$g_\rho^2 = 2$
0
100
200
300
MEV.
0
-.05
-.10
-.15
$g_\rho^2 = 6$
$g_\rho^2 = 14$
OBE (14)
BORN (14)
$^1G_4$
0
100
200
300
MEV.

Fig. 6


0
-20
-40
-60
-80
δ°
$g_\rho^2 = 6$
$g_\rho^2 = 14$
OBE(6)
BORN(6)
OBE(14)
BORN(14)
$^3P_1$
0
100
200
300
MEV.
0
-.5
-1.0
-1.5
$g_\rho^2 = 6$
$g_\rho^2 = 14$
BORN(14)
OBE(14)
$^3F_3$
0
100
200
300
MEV.

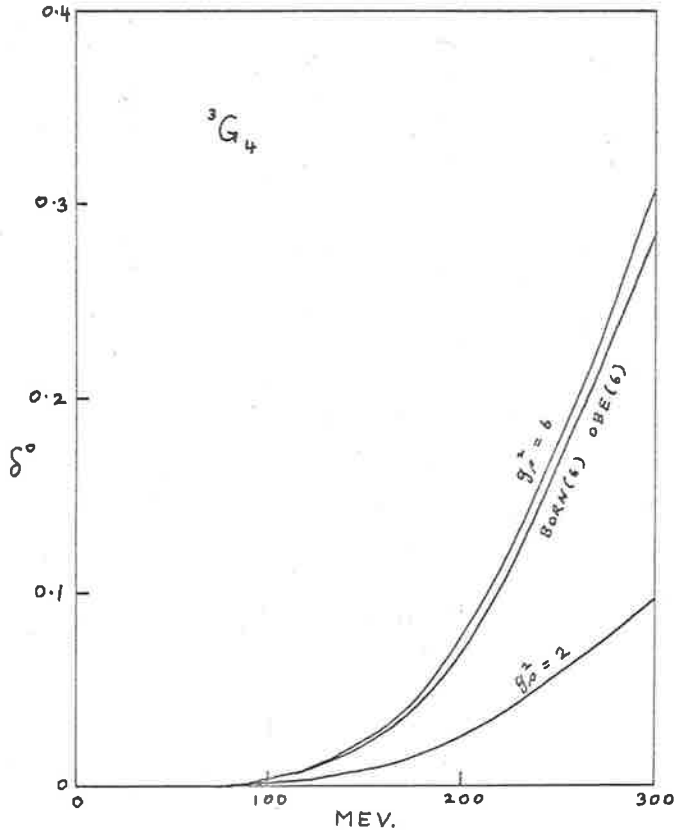

0.4
0.3
0.2
0.1
0
δ°
$^3G_4$
$g_\rho^2 = 6$
BORN(6) OBE(6)
$g_\rho^2 = 2$
0
100
200
300
MEV.

Fig. 7

said that the second order term has very little effect for $\ell > 2$, and the full solution shows that the potential for the iso-spin singlet states is singular if the coupling constant is too large. This is sometimes not taken into account when the Born approximation is used with the OBE theory to fit experimental data.

We will now examine the effect of using more than one type of meson to describe the nucleon-nucleon potential.

## 6.3) Combination of mesons

Here we investigate the possibility that a combination of mesons may allow us to obtain some correspondence between our results and experimentally derived ones. In OBE theory with the Born approximation, the effects of the various mesons simply add up to produce the total potential, as do the phase shifts. This makes it a very simple model to work with.

In our case, the second term is quadratic in the coupling constants, and if we use two mesons, say the $\pi$ and $\rho$, then the total effect is not equal to the effect of the $\pi$ plus the effect of the $\rho$. We may get considerable effects from the cross terms which are proportional to $g_\pi^2 \, g_\rho^2$. That this is the case, can be seen by looking at the Born approximation, which gives an indication of the complete solution in the

higher angular momentum states. We must somehow try to reduce the large attractive force of the $\pi$-meson, especially in the iso-singlet states, which should be repulsive. The only particle that could help do this is the iso-scalar $\omega$-meson since this gives a repulsive potential in all states. The $\eta$-meson is repulsive in the spin singlet states, but attractive in the spin triplet states and hence would make matters worse in the triplet states.

The spin singlet equation has in the second term, the factor

$$\left[ M^2 B_j - q \left( \frac{j}{2j+1} C_{j-1} + \frac{j+1}{2j+1} C_{j+1} \right) q \right] \frac{1}{T} A_j \qquad (6.1)$$

which is obtained from (2.39), and integration is implied. The second term in the square brackets can be ignored relative to $M^2 B_j$, non-relativistically at any rate, and we will write

$$\frac{M^2}{4\pi} \int_0^\infty Q_j \left( \frac{k^2 + q'^2 + \mu_1^2}{2kq'} \right) \frac{1}{(q'^2 + M^2)^{\frac{1}{2}}} Q_j \left( \frac{k^2 + q'^2 + \mu_2^2}{2kq'} \right) dq'$$

as $Q_{12} = Q_{21}$, where we have now taken the Born approximation. Also, the masses of all the mesons except the $\pi$ are approximately equal, $\sim 5 \mu_\pi$, so for this qualitative argument we will use $Q_j(z_\sigma) = Q_j(z_\eta) = Q_j(z_\rho) = Q_j(z_\omega)$ where $z_\omega = \frac{q^2 + q'^2 + \mu_\omega^2}{2qq'}$ etc. Using relations (2.29), the expression (6.1) then gives

$$- g_\pi^4 Q_{22} + g_\pi^2 (-6 g_\rho^2 - 6 g_\omega^2 - 2 g_\eta^2) Q_{23}$$

$$+ (g_\sigma^2 + g_\eta^2 + 2\, g_\rho^2 + 2\, g_\omega^2)\,(g_\sigma^2 - g_\eta^2 - 4\, g_\rho^2 - 4\, g_\omega^2)\, Q_{33} \qquad (6.2)$$

for the iso-spin triplet case. For the iso-singlet case we have to multiply each $g_\pi^2$ and $g_\rho^2$ by $-3$. The values of $Q_{22}$, $Q_{23}$ and $Q_{33}$ were calculated in the $^1F_3$ state at 100 and 300 Mev. and were found to be

| | $Q_{22}$ | $Q_{23}$ | $Q_{33}$ |
|---|---|---|---|
| 100 Mev. | $1.72.10^{-3}$ | $2.7.10^{-6}$ | $5.2.10^{-7}$ |
| 300 Mev. | $2.36.10^{-2}$ | $1.7.10^{-4}$ | $4.9.10^{-6}$ |

In the $^1F_3$ state, which is an iso-spin singlet state, (6.2) becomes

$$- 9\, g_\pi^4\, Q_{22} - 3\, g_\pi^2\,(18\, g_\rho^2 - 6\, g_\omega^2 - 2\, g_\eta^2)\, Q_{23}$$

$$+ (g_\sigma^2 + g_\eta^2 - 6\, g_\rho^2 + 2\, g_\omega^2)\,(g_\sigma^2 - g_\eta^2 + 12\, g_\rho^2 - 4\, g_\omega^2)\, Q_{33} \qquad (6.3)$$

From this it can be seen that the most helpful term for cancelling the large pion term is $g_\pi^2\, g_\omega^2\, Q_{23}$. The overall effect of the $g_\rho^2$ is not helpful, and the $g_\sigma^2$ and $g_\eta^2$ contribute little compared with the vector meson terms. Even if we choose the pion-nucleon coupling constant as $g_\pi^2 = 4$, we cannot cancel the $g_\pi^4\, Q_{22}$ term, since the expression (6.3) cannot be made positive by any choice of $g_\omega^2$.

Hence it is not possible to bring our results into agreement with experimental ones because of the very strong second order $\pi$-meson term. As an example of the effect of the $\omega$-meson used with the pion, in Fig. 8(a) we show the phase shifts of the $^1F_3$ state for $g_\pi^2 = 6$ and $g_\omega^2 = 12$, 16 and 20. From this it can be seen that the $\omega$-meson has an appreciable

effect in the higher energy region, but it is nowhere near big enough to compensate the strong attraction. Thus, it seems that if we restrict ourselves to the derived potentials, it is not possible to obtain agreement with experiment.

A possible way of obtaining potentials which are not singular, would be to introduce some sort of high momentum cut-off. This would enable us to give solutions in all cases, but would not eliminate the effect of the second order pion term. This is because the Born approximation result is independent of the cut-off, and hence in the higher angular momentum states where the Born approximation is valid, we would find the same results.

## 6.4) Coupled spin triplet equation solutions

In view of the inability of the theory to fit experimental phase shifts and the additional difficulty of interpreting the coupled equation potentials, we only give a few brief results for the coupled spin triplet equations. The computer programme has been written for a general combination of mesons, but because of the poorly defined experimental phase shifts above $\ell = 2$ there is little point in trying to compare and fit our results. For $\ell \leqslant 2$ the potentials are again singular unless the coupling constant is small, and the results follow the same trend as

for the $l = j$ states.

For the scalar boson interaction, the potential is given almost entirely by the first order terms. In the OBE Born approximation, there is no mixing of the $l = j+1$ and $l = j-1$ states. Our solutions give some mixing (Fig. 8(d) ), because the operators E and H of equations (4.12) - (4.15) are non-zero. Figs. 8(b) and 8(c) shows the $^3F_2$ and $^3F_4$ phase shifts for a scalar meson coupling constant of 10. In the low energy region the full solution is quite close to the OBE result, which shows that our method of finding phase shifts through the probability density does give the OBE results in analogy with the uncoupled equations. The unitarity condition, which provides a very sensitive check on the theory and the programme, is found to be satisfied to better than 1%.

In the pseudo-scalar interaction case we have much the same situation as for the uncoupled states. The second order term again dominates the interaction, so that we get little correspondence between the OPE and the full solution. In both states the potentials are strongly attractive as can be seen from Fig. 9.

For the $\rho$-meson, we give the results for the $^3F_2$ and $^3F_4$ states in Fig. 10. The short range force is again apparent, since below 100 Mev. the force has practically no effect on the phase shifts. Above 150 Mev. the phase shifts decrease rapidly, but do not differ greatly from the OBE result. The mixing parameter $m''/n''$ differs considerably from the OBE

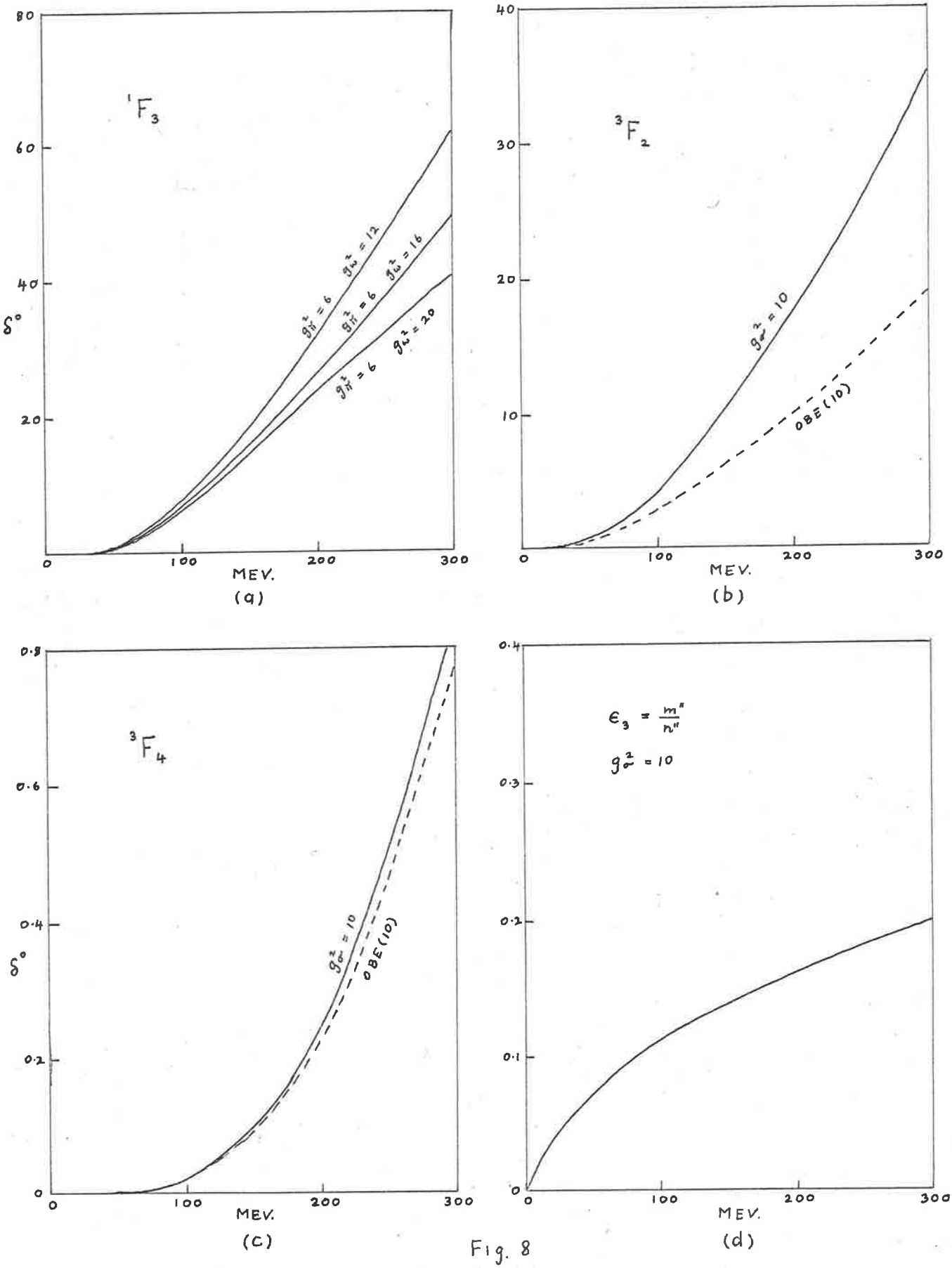
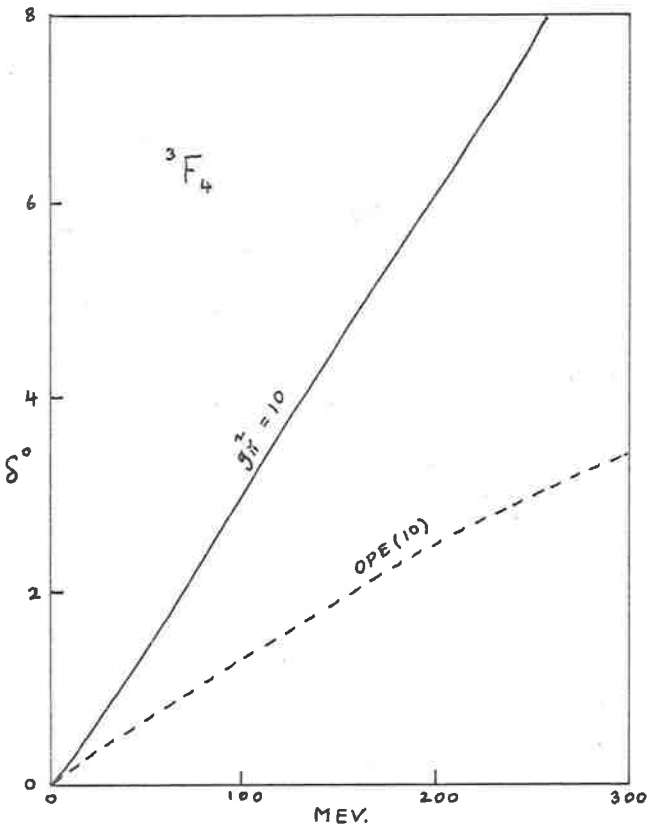
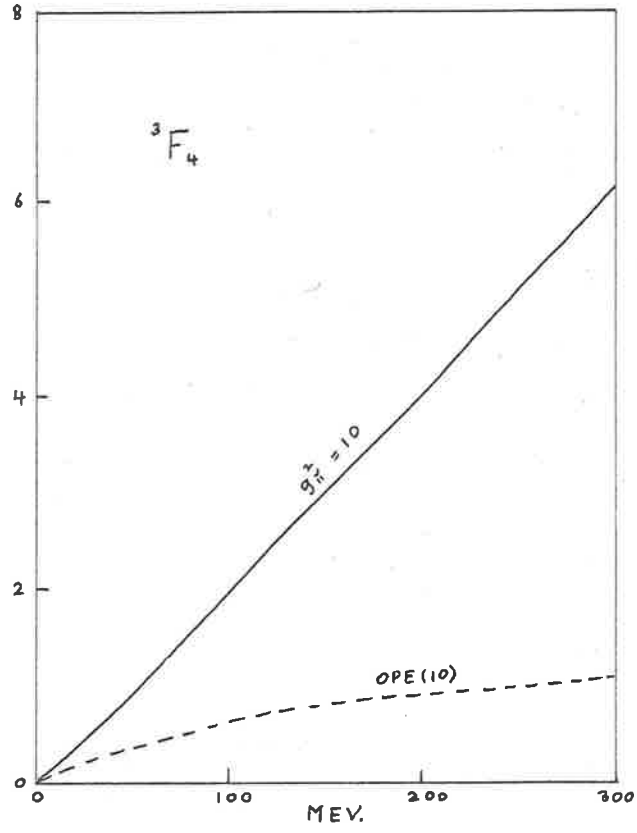

80
60
40
20
δ°
$^1F_3$
$g_\pi^2 = 6$ $g_\omega^2 = 12$
$g_\pi^2 = 6$ $g_\omega^2 = 16$
$g_\pi^2 = 6$ $g_\omega^2 = 20$
0
100
200
300
MEV.
(a)
40
30
20
10
$^3F_2$
$g_\sigma^2 = 10$
OBE (10)
0
100
200
300
MEV.
(b)
0.8
0.6
0.4
0.2
0
δ°
$^3F_4$
$g_\sigma^2 = 10$
OBE (10)
0
100
200
300
MEV.
(c)
0.4
0.3
0.2
0.1
0
$\epsilon_3 = \frac{m^*}{n^*}$
$g_\omega^2 = 10$
0
100
200
300
MEV.
(d)

Fig. 8


8
6
4
2
0
$\delta^0$
$^3F_4$
$g_\pi^2 = 10$
OPE (10)
0
100
200
300
MEV.


8
6
4
2
0
$^3F_4$
$g_\pi^2 = 10$
OPE (10)
0
100
200
300
MEV.

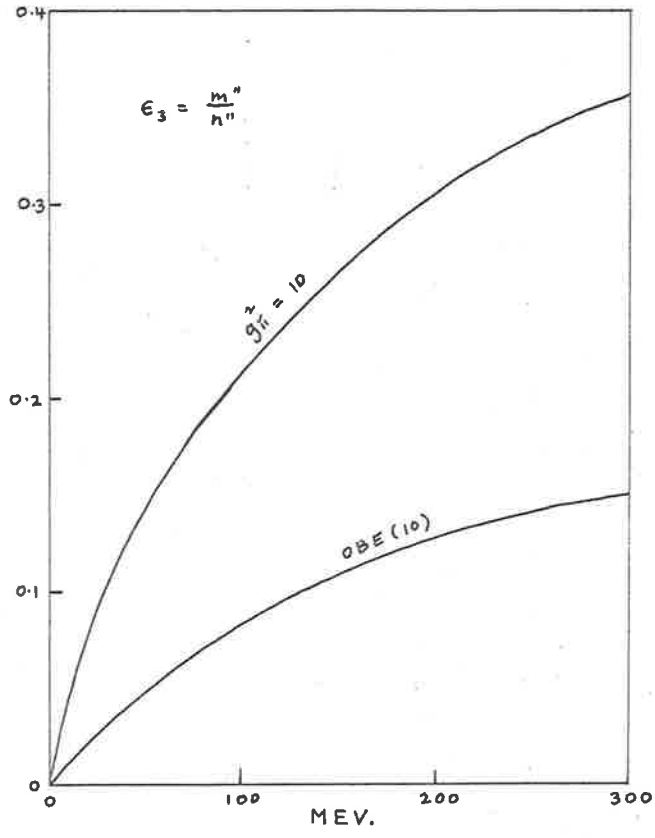
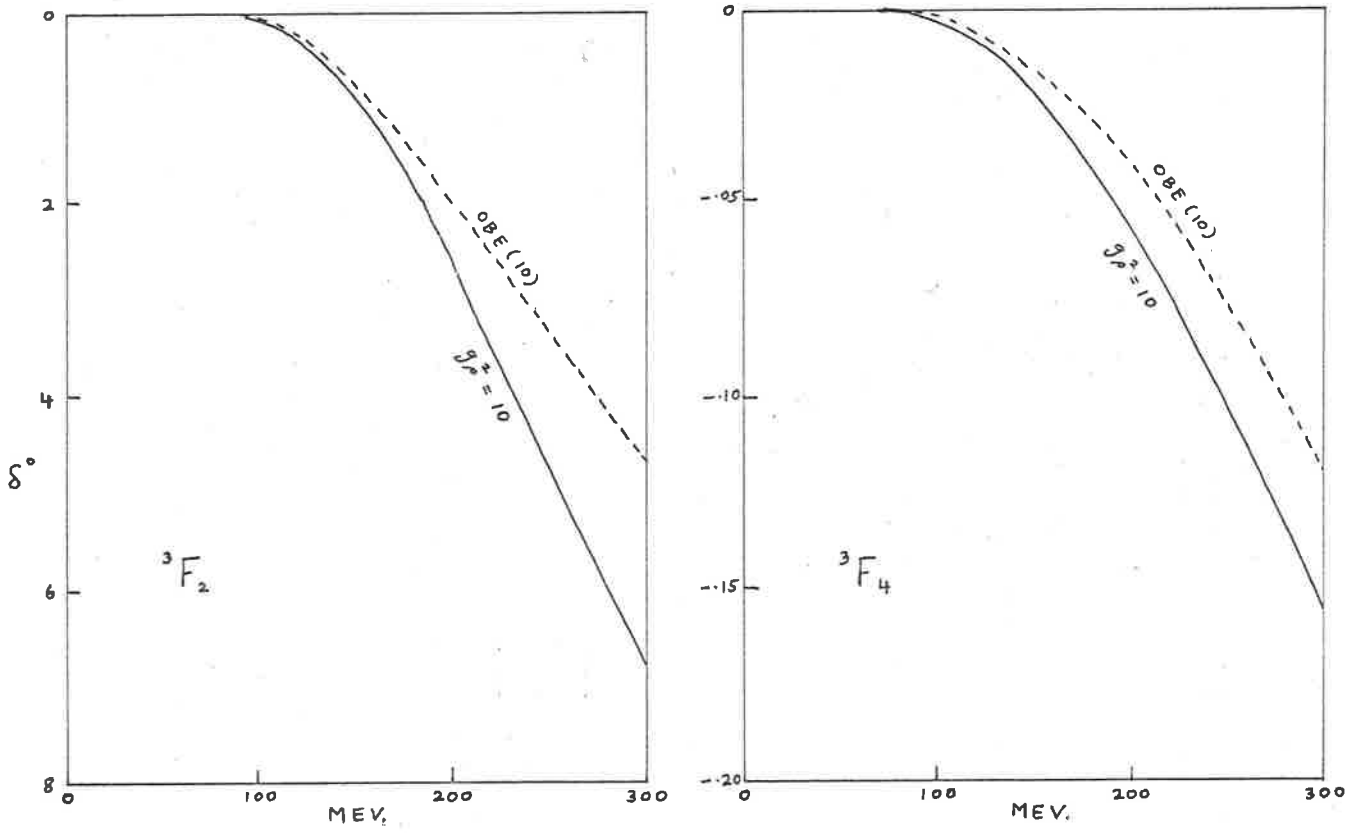

0.4
0.3
0.2
0.1
0
$\epsilon_3 = \frac{m''}{n''}$
$g_\pi^2 = 10$
OBE (10)
0
100
200
300
MEV.

Fig. 9


δ°
0
2
4
6
8
$^3F_2$
OBE (10)
$g_\rho^2 = 10$
0
100
200
300
MEV.
$^3F_4$
-.05
-.10
-.15
-.20

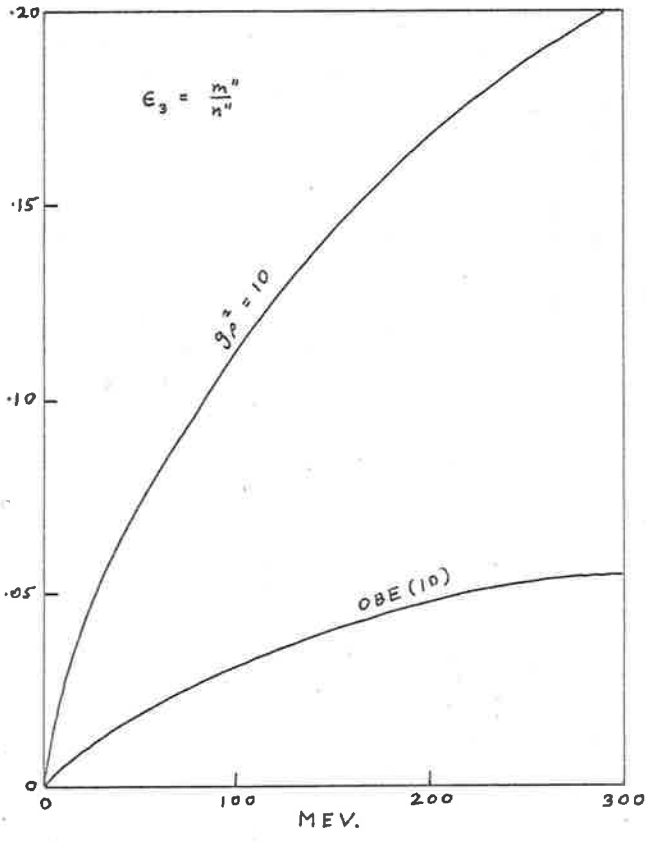

$\epsilon_3 = \frac{m}{\pi^2}$
.20
.15
.10
.05
0
$g_\rho^2 = 10$
OBE (10)
100
200
300
MEV.

Fig. 10

result for the scalar and vector mesons, even though the phase shifts do not. The reason for this can possibly be attributed to the sensitivity of the mixing parameter to the operators E and H which appear in a different way in the OBE theory.

The few results we have presented in this section demonstrates the validity of our method for deriving phase shifts from the two pairs of coupled equations (4.12) - (4.15). In all cases, the unitarity condition discussed in 4.3) is satisfied to better than 2% which speaks for the accuracy of the numerical work. We have also shown that in the cases we would expect the OBE phase shifts to be near to the full solution, i.e. for scalar and vector meson interaction, we get a close correspondence between the two solutions. This indicates that although we cannot obtain the OBE results analytically as we did for the uncoupled states, they do come out of the numerical calculations, and shows that the OBE is the basis of these potentials as it was for the uncoupled states.

## 6.5) Conclusions

In the course of this work we have succeeded in solving the B-S equation in the ladder approximation using an instantaneous interaction without recall to any other approximations. The method given for solving the B-S equation is very direct, and applies quite generally to any instantaneous interaction. The difficulty of physical interpretation of the various functions found was overcome in Chapter III, where we defined a conjugate B-S equation which enabled us to derive a continuity equation. Integrating the continuity equation over the relative time gave expressions for the probability density in the various states.

That our treatment of the relative time variable is valid, is borne out by the results, which show that the probability density reduces to some remarkably simple expressions involving the functions obtained from the B-S wave function. The further fact that the potentials of the equivalent Schrodinger equations yield the well known one boson exchange potential in the first order, strengthens the argument.

The probability density is our link with the physical situation through which we identify the B-S functions with physical wave functions of the nucleon-nucleon system. In the case of uncoupled states, the correspondence is direct, but for the coupled states a method had to be developed to find phase shifts from two pairs of coupled equations involving four functions, each of which contributed to both states. That

this method is sound, is demonstrated by the numerical results. These reproduce features of the OBE theory even though they cannot be derived analytically, and the solution of the quartic equation (4.57) gives the ratio of the mixing parameters to within 2% of that required by unitarity.

Calculations with the scalar meson showed that in all states the ladder diagrams gave no appreciable difference from the OBE result, thus leading us to conclude that in this case the OBE approximation is a good one.

The OBE approximation for the vector meson is quite good for $\ell > 2$, but for the S, P and D states, the contribution from the second order term is quite significant. We also noted that the Born approximation is not a good one in the S and P states.

To explain the results for the pseudo-scalar interaction is difficult. In all states, the variance of our results with the OPE theory is so large, that it is not possible to even qualitatively fit experimental phase shifts. By considering the $\eta$-meson, we saw that this is principally due to the long range and magnitude of the second order term in the pseudo-scalar potential. We know that the first order terms, which give the OBE potentials, are theoretically correct, and if used alone are capable of explaining experimental data. This would lead us to conclude that our second order term does not correctly describe the higher order interactions between nucleons. This is possibly limited to the pseudo-scalar inter-

action, since only in this case terms of order $M^2$ cancel in the OBE term, thus increasing the relative contribution of the second order term. To explain these results we have two alternatives corresponding to the two approximations used.

The first possibility is that the ladder approximation is insufficient for describing the nucleon-nucleon interaction. The ladder diagrams represent only a small class of all possible Feynman graphs which could contribute to the nucleon potential. Crossed meson lines and self energy processes which would be included in the higher order terms arising from the expansion of the general interaction function discussed in 1.1), could substantially modify the potential. If the discrepancy was due to crossed meson lines, and it is assumed that the vector mesons describe multiple $\pi$-meson processes including crossed meson lines, then we should be able to use the vector mesons to cancel the large attractive force. However, we saw that the $\rho$ and $\omega$-meson potentials were of too short range to have any effect in the low energy region, so that it is unlikely that the difficulty arises from the exclusion of crossed meson lines.

The second possibility is that despite our observations regarding the validity of the instantaneous interaction approximation, the higher order terms neglected are somehow significant. The inclusion of the second term from the expansion with respect to the relative energy, raises certain problems, and is beyond the scope of this work. However, we note that terms of higher order than $g^4$ would appear, and these although

relativistic in nature, could be important because of the factor $(g^2 \underline{\tau}_1 . \underline{\tau}_2)^n$ associated with the n th order term.

That the instantaneous interaction approximation is not very good for strong interactions, is perhaps the most logical conclusion that can be drawn from our results. From the $\pi$-meson interaction phase shifts, it can be seen that the second order term is significant even at a few Mev., and tends to rule out the argument that it could be solely due to the exchange of two or more mesons in the ladder diagram. This is because it is reasonably well established that the range of force due to the exchange of two pions is much smaller than the OPE range of force, and is only significant in the relativistic region.

On the other hand it is noted by some workers, Vosko[12], and Yamamoto[20], who have investigated the effects of neglect of retardation, that the approximation would not be as good for heavy bosons as it would be for smaller mass bosons. This is quite feasible since for a given energy, a lighter meson would have greater velocity and hence result in less time for the exchange. Our results seem to indicate the opposite of this.

The instantaneous interaction approximation is a result of certain assumptions about an unphysical quantity, the relative energy or time. Because of this, it is not possible to estimate its effects directly and the significance of the approximation can only be found by obtaining a physical theory and comparing it with experiment. The method used here is to integrate over the relative time variable, and we have shown how to

obtain a consistent, physical theory, which to first order is in agreement with that obtained by other field theoretical methods. This proves that our treatment of the relative time variable is correct, even though one of the approximations used has a limited range of validity. In particular, it is concluded that there is a necessity for examining more closely the effects of neglecting retardation in strong interactions, and that the B-S equation is capable of describing, in a direct way, the interaction between nucleons.

```
C     SPIN SINGLET EQUATION SOLUTIONS.
C
      PROGRAM SNGL (INPUT,OUTPUT,TAPE10)
      DIMENSIONP(80),R(80),E(80),Q(40,40),QA(40,40),QB(40,40),QC(40,40),
     1 HP(20),UM(5),CA(5),GP(40),ENERGY(20),Y(40),PZ(20),U(2),F(2),
     2 CXA(4),CXB(4),QK(40),FU(2)
      COMMONP,UM,R
      READ207,KA,(ENERGY(I),I=1,KA)
      READ203,(UM(I),I=1,5)
      READ    204,KG
      DO45KK=1,KG
      READ204,NS,N,XS,XL,JL
  200 FORMAT(1X,10E13.5)
  203 FORMAT(5F6.2)
  204 FORMAT(2I3,2F6.2, I3,5F6.2)
  206 FORMAT( )
  207 FORMAT(I3,9F6.1)
  209 FORMAT(18H       ENERGY(MEV)=F6.1,6E13.5)
  211 FORMAT(24X,6E13.5,2I7)
C
      PMASS=45.2976
      PI=1/(3.1415927*2.)
C        NS MUST BE ODD
      NN=2*N    $ NSM1=NS-1 $ NSM2=NS-2 $ NSP1=NS+1 $ NSP2=NS+2
      OJ=JL/(2.0*JL+1) $ PJ=(JL+1)/(2.0*JL+1)
      NX=1600
      DO20I=1,NS
   20 P(I)=(I-.999)*XS
      DO96I=NSP1,NN
   96 P(I)=P(NS)+(I-NS)*XL
      DO91I=2,NSM1,2
   91 R(I)=XS*1.333333333
      DO92I=3,NSM2,2
   92 R(I)=XS*0.666666666
      DO93I=NSP1,NN,2
   93 R(I)=XL*1.3333333333
      DO94I=NSP2,NN,2
   94 R(I)=XL*0.66666666665
      R(1)=XS/3.0 $ R(NS)=R(1)+XL/3.0
      DO13I=1,NN
   13 E(I)=(PMASS+P(I)*P(I))**0.5
      PRINT200,     (P(I),I=1,NN)
      PRINT206
      PRINT200,     (R(I),I=1,NN)
      PRINT206
      PRINT200,     (E(I),I=1,NN)
      PRINT206
C
      REWIND10
      DO97K=1,4
      CALLQFN(JL,K,N,Q)
```

```
 97 WRITE(10)(Q(I),I=1,NX)
    DO506I=1,NX
506 Q(I)=0.
    DO181K=1,3
    IF(JL.EQ.0)550,551
551 CALLQFN(JL-1,K,N,0)
550 CALLQFN(JL+1,K,N,QC)
    DO182I=1,N $ DO182J=1,N
    M=I+J-1
182 Q(I,J)=P(I)*P(M)*(QJ*Q(I,J)+PJ*QC(I,J))
181 WRITE(10)(Q(I),I=1,NX)
    ENDFILE10
C
    DO40MIST=2,14,4
    REWIND10
    CA(1)=0. $ CA(2)=0. $   CA(3)=0. $ CA(4)=0.0$ CA(5)=0.
    CA(3)=MIST
    PRINT204,NS,N,XS,XL,JL,    (CA(I),I=1,5)
    PRINT203,(UM(I),I=1,5)
    ZET=JL
    J=JL/2
    IF(ZET/2.0-J   )70,70,71
 71 CA(2)=-3.0*CA(2)
    CA(4)=-3.0*CA(4)
 70 CA(4)=CA(4)+CA(5)
    CXA(1)=CA(1)*PI $ CXB(1)=CA(1)*PI
    CXA(2)=-CA(2)*PI $ CXB(2)=CA(2)*PI
    CXA(3)=-CA(3)*PI $ CXB(3)=CA(3)*PI
    CXA(4)=4*CA(4)*PI $ CXB(4)=-2*CA(4)*PI
C
    DO87I=1,NX
    QC(I)=0.
 87 Q(I)=0.
    DO183K=1,4
    READ(10)(QA(I),I=1,NX)
    DO183I=1,N $ DO183J=1,N
    QC(I,J)=QC(I,J)+CXB(K)*QA(I,J)
183 Q(I,J)=Q(I,J)+CXA(K)*QA(I,J)
    DO60I=1,N $ DO60J=1,N
 60 QC(I,J)=QC(I,J)*PMASS
    DO184K=1,3
    READ(10)(QA(I),I=1,NX)
    DO184I=1,N $ DO184J=1,N
184 QC(I,J)=QC(I,J)-CXA(K)*QA(I,J)
    DO61I=2,N $ IM1=I-1
    DO61J=1,IM1
 61 QB(I,J)=QC(I,N+J-I+1)
    DO62I=1,N
    DO62J=I,N
 62 QA(I,J)=QC(I,J-I+1)
```

```
      DO63I=1,N $ DO63J=I,N
   63 QA(J,I)=QA(I,J)
      DO64I=2,N $ IM1=I-1
      DO64J=1,IM1
   64 QB(J,I)=Q(I,N+J-I+1)*R(J+M)/E(J+N)
      DO65I=1,N $ DO65J=I,N
   65 QC(I,J)=Q(I,J-I+1)
      DO66I=1,N $ DO66J=I,N
   66 QC(J,I)=QC(I,J)
      DO67I=1,N $ DO67J=1,N
      Q(I,J)=E(I)*E(I)*QC(I,J)+QA(I,J)
   67 QC(I,J)=QC(I,J)*R(I)/E(I)
      DO77I=1,20
   77 HP(I)=-Q(I,I)/E(I)
      DO68I=1,N $ DO68J=1,N $ DO68K=1,N
   68 Q(I,J)=Q(I,J)-QA(I,K)*QC(K,J)
      DO69I=2,N $ DO69J=2,N
      IF(I-J)72,72,74
   72 L=I-1 $ GOTO75
   74 L=J-1
   75 DO69K=1,L
   69 Q(I,J)=Q(I,J)-QB(I,K)*QB(K,J)
      DO76I=1,N $ DO76J=1,N
   76 QC(I,J)=-Q(I,J)/E(I)
C
      DO40IM=1,KA
      PZ(IM)=SQRT(.624918*ENERGY(IM)+.00000343*ENERGY(IM)**2)
      J=0
   21 J=J+1
      IF(P(J)-PZ(IM))21,21,22
   22 JEST=J
      IF(IM.EQ.2.OR.IM.EQ.5)700,701
  701 IJ=1 $ GOTO703
  700 IJ=2
  703 DO40IJK=1,IJ
      NT=N+10*(IJK-2)
      DO86IN=1,1
      IF(IN-1)80,80,81
   80 DO82I=1,N
      DO82JM=1,N
   82 Q(I,JM)=QC(I,JM)*P(JM)/P(I)
      GOTO83
   81 DO84I=1,N
      DO84JM=1,N
   84 Q(I,JM)=QC(JM,I)*P(JM)/P(I)
   83 J=JEST
      IF((P(J-1)+P(J))/2-PZ(IM))24,24,25
   24 M=J
      GOTO26
   25 M=J-1
```

```
   26 DO102I=1,N
  102 Y(I)=1.0/(PZ(IM)*PZ(IM)-P(I)*P(I))
      DO23L=1,N
   23 CALL INTERP(Q(L,M-1),Q(L,M),Q(L,M+1),P(M-1),P(M),P(M+1),PZ(IM),
     1QK(L),A,B,C)
      CALL INTERP(QK(M-1),QK(M),QK(M+1),P(M-1),P(M),P(M+1),PZ(IM),
     1Z,A,B,C)
C
      DO30I=1,N
      DO30J=1,N
   30 QA(I,J)=QK(I)-Q(I,J)
      DO33I=1,N
      DO32J=1,N
   32 QA(I,J)=QA(I,J)*Y(J)*R(J)
   33 QA(I,I)=QA(I,I)+1.0
      DO42I=1,NT
      QB(I,NT+1)=QK(I)
      DO42J=1,NT
   42 QB(I,J)=QA(I,J)
      CALL SIMLEQ(QB,NT,NT+1,IRR)
      DO103I=1,NT
  103 GP(I)=QB(I,NT+1)
      PRINT200,(GP(I),I=1,NT)
      PRINT206
      CALL INTERP(GP(M-1),GP(M),GP(M+1),P(M-1),P(M),P(M+1),
     1PZ(IM),U(IN),A,B,C)
   86 F(IN)=1.0-SENT(GP,PZ(IM),M,NT,NS)
      CALL INTERP(HP(M-1),HP(M),HP(M+1),P(M-1),P(M),P(M+1),PZ(IM),C,
     1 AY,BY,CY)
C
      SH=-1.0/(4.0*PZ(IM)*PI)
      C= C*SH
      O= Z*SH
      U(2)=U(1)   % F(2)=F(1)
      H1=U(1)/F(1)*SH
      H2=U(2)/F(2)*SH
      W=ATANF(V)
      Z=180*W*PI*2.
      PRINT 209,   ENERGY(IM),C,O,V,H1,H2
      O=ATAN(O)*180*PI*2.
      C=ATAN(C)*180*PI*2.
      PRINT211,C,O,Z,F(1),F(2),PZ(IM),M,NT
      FU(IJK)=Z
      IF(IJK-1)40,40,501
  501 IF(FU(1)+FU(2))502,40,502
  502 IF(ABS((FU(1)-FU(2))/(.5*(FU(1)+FU(2))))-0.2)40,40,45
   40 PRINT206
   45 PRINT 206
      STOP
      END
```

```
C      SPIN TRIPLET L=J EQUATION SOLUTIONS.
C
       PROGRAM SNGL (INPUT,OUTPUT,TAPE10)
       DIMENSIONP(80),R(80),E(80),Q(40,40),QA(40,40),QB(40,40),QC(40,40),
      1 HP(20),UM(5),CA(5),GP(40),ENERGY(20),Y(40),PZ(20),U(2),F(2),
      2 CXD(4),CXB(4),CXC(4),FU(2),QK(40)
       COMMONP,UM,R
C
       READ207,KA,(ENERGY(I),I=1,KA)
       READ203,(UM(I),I=1,5)
       READ    204,KG
       DO45KK=1,KG
       READ204,NS,N,XS,XL,JL
  200 FORMAT(1X,10E13.5)
  203 FORMAT(5F6.2)
  204 FORMAT(2I3,2F6.2, I3,5F6.2)
  206 FORMAT( )
  207 FORMAT(I3,9F6.1)
  209 FORMAT(18H       ENERGY(MEV)=F6.1,6F13.5)
  211 FORMAT(24X,6E13.5,2I7)
C
       PMASS=45.2976
       PI=1/(3.1415927*2.)
C        NS MUST BE ODD
       NN=2*N    $ NSM1=NS-1 $ NSM2=NS-2 $ NSP1=NS+1 $ NSP2=NS+2
       OJ=JL/(2.0*JL+1) $ PJ=(JL+1)/(2.0*JL+1)
       NX=1600
C
       DO20I=1,NS
    20 P(I)=(I-.999)*XS
       DO90I=NSP1,NN
    90 P(I)=P(NS)+(I-NS)*XL
       DO91I=2,NSM1,2
    91 R(I)=XS*1.333333333
       DO92I=3,NSM2,2
    92 R(I)=XS*0.666666666
       DO93I=NSP1,NN,2
    93 R(I)=XL*1.3333333333
       DO94I=NSP2,NN,2
    94 R(I)=XL*0.66666666666
       R(1)=XS/3.0 $ R(NS)=R(1)+XL/3.0
       DO13I=1,NN
    13 E(I)=(PMASS+P(I)*P(I))**0.5
       PRINT200,     (P(I),I=1,NN)
       PRINT206
       PRINT200,     (R(I),I=1,NN)
       PRINT206
       PRINT200,     (E(I),I=1,NN)
       PRINT206
C
```

```
      REWIND10
      DO97K=1,4
      CALLQFN(JL,K,N,Q)
   97 WRITE(10)(Q(I),I=1,NX)
      DO506I=1,NX
  506 Q(I)=0.
      DO181K=1,4
      IF(JL.EQ.0)550,551
  551 CALLQFN(JL-1,K,N,Q)
  550 CALLQFN(JL+1,K,N,QC)
      DO182I=1,N $ DO182J=1,N
      M=I+J-1
  182 Q(I,J)=P(I)*P(M)*(PJ*Q(I,J)+QJ*QC(I,J))
  181 WRITE(10)(Q(I),I=1,NX)
      ENDFILE10
C
      DO40MIST=2,14,4
      REWIND10
      CA(1)=0. $ CA(2)=0. $   CA(3)=0. $ CA(4)=0.0$ CA(5)=0.
      CA(3)=MIST
      PRINT204,NS,N,XS,XL,JL,    (CA(I),I=1,5)
      PRINT203,(UM(I),I=1,5)
      ZET=JL
      J=JL/2
      IF(ZET/2.0-J    )71,71,70
   71 CA(2)=-3.0*CA(2)
      CA(4)=-3.0*CA(4)
   70 CA(4)=CA(4)+CA(5)
      CXD(1)=CA(1)*PI $ CXB(1)=CA(1)*PI  $ CXC(1)=CA(1)*PI
      CXD(2)=+CA(2)*PI $ CXB(2)=CA(2)*PI $ CXC(2)=-CA(2)*PI
      CXD(3)=+CA(3)*PI $ CXB(3)=CA(3)*PI $ CXC(3)=-CA(3)*PI
      CXD(4)=2*CA(4)*PI $ CXB(4)=-2*CA(4)*PI $ CXC(4)=0.
C
      DO87I=1,NX
      QC(I)=0.
   87 Q(I)=0.
      DO183K=1,4
      READ(10)(QA(I),I=1,NX)
      DO183I=1,N $ DO183J=1,N
      QC(I,J)=QC(I,J)+CXC(K)*QA(I,J)
  183 Q(I,J)=Q(I,J)+CXD(K)*QA(I,J)
      DO60I=1,N $ DO60J=1,N
   60 QC(I,J)=QC(I,J)*PMASS
      DO184K=1,4
      READ(10)(QA(I),I=1,NX)
      DO184I=1,N $ DO184J=1,N

  184 QC(I,J)=QC(I,J)-CXB(K)*QA(I,J)
      DO61I=2,N $ IM1=I-1
      DO61J=1,IM1
```

```
   61 QB(I,J)=QC(I,N+J-I+1)
      DO62I=1,N
      DO62J=I,N
   62 QA(I,J)=QC(I,J-I+1)
      DO63I=1,N $ DO63J=I,N
   63 QA(J,I)=QA(I,J)
      DO64I=2,N $ IM1=I-1
      DO64J=1,IM1
   64 QB(J,I)=Q(I,N+J-I+1)*R(J+N)/E(J+N)
      DO65I=1,N $ DO65J=I,N
   65 QC(I,J)=Q(I,J-I+1)
      DO66I=1,N $ DO66J=I,N
   66 QC(J,I)=QC(I,J)
      DO67I=1,N $ DO67J=1,N
      Q(I,J)=F(I)*E(I)*QC(I,J)+QA(I,J)
   67 QC(I,J)=QC(I,J)*R(I)/E(I)
      DO77I=1,20
   77 HP(I)=-Q(I,I)/E(1)
      DO68I=1,N $ DO68J=1,N $ DO68K=1,N
   68 Q(I,J)=Q(I,J)-QA(I,K)*QC(K,J)
      DO69I=2,N $ DO69J=2,N
      IF(I-J)72,72,74
   72 L=I-1 $ GOTO75
   74 L=J-1
   75 DO69K=1,L
   69 Q(I,J)=Q(I,J)-QB(I,K)*QB(K,J)
      DO76I=1,N $ DO76J=1,N
   76 QC(I,J)=-Q(I,J)/E(I)
C
      DO40IM=1,KA
      PZ(IM)=SQRT(.02491B*ENERGY(IM)+.00000343*ENERGY(IM)**2)
      J=0
   21 J=J+1
      IF(P(J)-PZ(IM))21,21,22
   22 JEST=J
      IF(IM.EQ.2.OR.IM.EQ.5)700,701
  701 IJ=1 $ GOTO703
  700 IJ=2
  703 DO40IJK=1,IJ
      NT=N+10*(IJK-2)
      DO86IN=1,1
      IF(IN-1)80,80,81
   80 DO82I=1,N
      DO82JM=1,N
   82 Q(I,JM)=QC(I,JM)*P(JM)/P(I)
      GOTO83

   81 DO84I=1,N
      DO84JM=1,N
   84 Q(I,JM)=QC(JM,I)*P(JM)/P(I)
```

```
 83 J=JFST
    IF((P(J-1)+P(J))/2-PZ(IM))24,24,25
 24 M=J
    GOTO26
 25 M=J-1
 26 DO102I=1,N
102 Y(I)=1.0/(PZ(IM)*PZ(IM)-P(I)*P(I))
    DO23L=1,N
 23 CALLINTERP(Q(L,M-1),Q(L,M),Q(L,M+1),P(M-1),P(M),P(M+1),PZ(IM),
   1QK(L),A,B,C)
    CALLINTERP(QK(M-1),QK(M),QK(M+1),P(M-1),P(M),P(M+1),PZ(IM),
   1Z,A,B,C)
    DO31I=1,N
    DO30J=1,N
 30 QA(I,J)=QK(I)-Q(I,J)
    DO33I=1,N
    DO32J=1,N
 32 QA(I,J)=QA(I,J)*Y(J)*R(J)
 33 QA(I,I)=QA(I,I)+1.0
    DO42I=1,NT
    QB(I,NT+1)=QK(I)
    DO42J=1,NT
 42 QB(I,J)=QA(I,J)
    CALLSIMLEQ(QB,NT,NT+1,IRR)
    DO103I=1,NT
103 GP(I)=QB(I,NT+1)
    CALLINTERP(GP(M-1),GP(M),GP(M+1),P(M-1),P(M),P(M+1),
   1PZ(IM),U(IN),A,B,C)
 86 F(IN)=1.0-SENT(GP,PZ(IM),M,NT,NS)
    CALLINTERP(HP(M-1),HP(M),HP(M+1),P(M-1),P(M),P(M+1),PZ(IM),C,
   1 AY,BY,CY)
    SH=-1.0/(4.0*PZ(IM)*PI)
    C= C*SH
    Q= Z*SH
    U(2)=U(1)   % F(2)=F(1)
    H1=U(1)/F(1)*SH
    H2=U(2)/F(2)*SH
    IF(H1*H2)45,45,99
 99 V=SQRT ((1.+H1*H1)*(1.+H2*H2)/(1.+H1*H2)-1.0)*H1/ABS(H1)
    W=ATANF(V)
    Z=180*W*PI*2.
    PRINT 209,   ENERGY(IM),C,Q,V,H1,H2
    Q=ATAN(Q)*180*PI*2.
    C=ATAN(C)*180*PI*2.
    PRINT211,C,Q,Z,F(1),F(2),PZ(IM),M,NT
    PRINT207,IRR
 40 PRINT206
 45 PRINT 206
    STOP
    END
```

```
      SUBROUTINE SIMLEQ(A,N,M,IRR)
      DIMENSION A(40,41)
      EPS=1.0E-90                                                       000
      NN=N-1                                                            000
      IF(NN)  80,130,10                                                 000
   10 DO 120  L=1,NN                                                    001
      KK=L                                                              001
      DO  40 K=L,N                                                      001
      IF(ABS (A(K,L))-ABS (A(KK,L)))  40,40,30                          001
   30 KK=K                                                              001
   40 CONTINUE                                                          001
      IF(KK-L)  70,70,50                                                001
   50 DO  60  J=L,M                                                     001
      B=A(L,J)                                                          001
      A(L,J)=A(KK,J)                                                    002
   60 A(KK,J)=B                                                         002
   70 IF(ABS (A(L,L))-EPS     ) 80,80,90                                002
   80 IRR=1                                                             002
      RETURN                                                            002
   90 KK=L+1                                                            002
      DO 120 K=KK,N                                                     002
      IF(A(K,L))  100,120,100                                           002
  100 DO 110 J=KK,M                                                     003
  110 A(K,J)=A(K,J)-A(K,L)/A(L,L)*A(L,J)                                003
  120 CONTINUE                                                          003
  130 IF (ABS (A(N,N))- EPS      ) 80,80,140                            003
  140 NN=N+1                                                            003
      DO 170  II=1,M                                                    003
      I=N-II+1                                                          003
      KK=I+1                                                            003
      DO 170 J=NN,M                                                     003
      B=0.0                                                             004
      IF(I-N) 150,170,150                                               004
  150 DO   160  K=KK,N                                                  004
  160 B=B+A(I,K)*A(K,J)                                                 004
  170 A(I,J)=(A(I,J)-B)/A(I,I)                                          004
      IRR=0
      RETURN                                                            004
      END                                                               004
```

```
      FUNCTIONSENT(GP,PZ,M,N,NS)
      DIMENSIONP(40),UMASS(5),R(80),GP(40)
      COMMONP,UMASS,R
      DO11I=1,N
    1 GP(I)=GP(I)/(PZ*PZ-P(I)*P(I))
      NX=M/2
      MM1=M-1 $ MM2=M-2 $ MM3=M-3 $ MP1=M+1 $ MP2=M+2 $ MP3=M+3
      NSP1=NS+1 $ NSM1=NS-1 $ F=0.
      IF(M/2.0-NX)100,100,101
  100 J=1
      IF(M-2)13,13,11
   11 DO2I=1,MM2
    2 F=F+R(I)*GP(I)
      F=F+0.5*R(MM1)*GP(MM1)
      IF(M.EQ.NS-1)20 ,13
   13 DO3I=MP2,NSM1
    3 F=F+R(I)*GP(I)
      F=F+0.5*R(MP1)*GP(MP1) $ F=F+R(1)*GP(NS)   $ GOTO20
  101 J=2
      IF(M-3)15,15,16
   16 DO4I=1,MM3
    4 F=F+R(I)*GP(I)
      F=F+0.5*R(MM2)*GP(MM2)
   15 IF(M.EQ.NS-2)20,18
   18 DO5I=MP3,NSM1
    5 F=F+R(I)*GP(I)
      F=F+0.5*R(MP2)*GP(MP2)$F=F+R(1)*GP(NS)
   20 F=F+(P(NS)-R(1))*GP(NS)
      DO6I=NSP1,N
    6 F=F+R(I)*GP(I)
      DO60I=1,3
      K=M+J*(I-2)
   60 GP(K)=GP(K)*(PZ*PZ-P(K)*P(K))
      CALLINTERP(GP(M-J),GP(M),GP(M+J),P(M-J),P(M),P(M+J),PZ,Z,A,B,C)
      AX=ALOG((P(M+J)+PZ      )/(P(M+J)-PZ      )*(PZ      -P(M-J))/(PZ      +
     1P(M-J)))
      A1=P(M-J)-P(M+J)+PZ      /2.0*AX
      B1=-0.5*ALOG((P(M+J)**2-PZ      **2)/(PZ      **2-P(M-J)**2))
      C1=AX*0.5/PZ
      F=F+A*A1+B*B1+C*C1
      SENT=F
      RETURN
      END
```

```
      SUBROUTINEQFN(J,L,M,Q)
      DIMENSIONQ(40,40),UMASS(5),P(80)
      COMMONP,UMASS
      JT=J+1
      DO52L=1,N $ DO52M=1,N $ K=M+L-1
      Z=(P(L)*P(L)+P(K)*P(K)+UMASS(I)**2)/(2.0*P(L)*P(K))
      IF(Z-10.0)70,70,80
   70 Q(L,M)=0.5*ALOG((Z+1)/(Z-1))
      GOTO(52,53,54,55,56,57)JT
   53 Q(L,M)=Z*Q(L,M)-1.0
      GOTO52
   54 Q(L,M)=0.5*(3*Z*Z-1)*Q(L,M)-1.5*Z
      GOTO52
   55 Q(L,M)=0.5*(5.0*Z**3-3*Z)*Q(L,M)-2.5*Z*Z+2.0/3.0
      GOTO52
   56 Q(L,M)=(35.0*Z**4-30.0*Z*Z+3.0)/8.0*Q(L,M)-35.0/8.0*Z**3+55.0/24.0
     1*Z
      GOTO52
   57 Q(L,M)=(63.*Z**5-70.*Z**3+15.*Z)/8.*Q(L,M)-63.*Z**4/8.+49.*Z*Z/8.
     1-8./15.0
      GOTO52
   80 GOTO(91,92,93,94,95,96)JT
   91 Q(L,M)=1/Z+1/(3*Z**3)
      GOTO52
   92 Q(L,M)=1/(3*Z**2)+1/(5*Z**4)
      GOTO52
   93 Q(L,M)=2/(15*Z**3)+4/(35*Z**5)
      GOTO52
   94 Q(L,M)=2/(35*Z**4)+4/(63*Z**6)
      GOTO52
   95 Q(L,M)=8./(315.*Z**5)+24./(693.*Z**7)
      GOTO52
   96 Q(L,M)=8./(693.*Z**6)+24./(1287.*Z**8)
   52 CONTINUE
      RETURN
      END
```

```
      SUBROUTINEINTERP(Y1,Y2,Y3,X1,X2,X3,X,Y,A,B,C)
      DET=(X1-X3)*(X1-X2)*(X2-X3)
      A=(Y1*(X2-X3)+Y3*(X1-X2)-Y2*(X1-X3))/DET
      B=(Y1*(X3*X3-X2*X2)+Y3*(X2*X2-X1*X1)-Y2*(X3*X3-X1*X1))/DET
      C=(Y1*X2*X3*(X2-X3)+Y3*X1*X2*(X1-X2)-Y2*X1*X3*(X1-X3))/DET
      Y=A*X*X+B*X+C
      RETURN
      END
```

```
C
C       COUPLED SPIN TRIPLET EQUATION SOLUTIONS.
C
        PROGRAMTRIP (INPUT,OUTPUT)
C       A DAINIS MATHS PHYSICS SPIN TRIPLET SOLNS
        DIMENSIONR(20,20),CM(20,20),DM(20,20),CP(20,20),DP(20,20),
       1Q1(20,20),Q2(20,20),Q3(20,20),F(20,20),E(20,20),H(20,20),
       2R23(20,20),Q5(20,20),Q4(20,20),  G(40,40),R(40,42),
       3P(40),       PM(40),UMASS(5),D(5),F1(20),F2(20),G1(20),G2(20)
       4,EN(10),C(5),Z(40),A(20,20),ZF1(2),ZF2(2),ZG1(2),ZG2(2)
        EQUIVALENCE(Q1,E),(Q2,F),(Q3,H),(Q4,R23)
        COMMONZ,UMASS,P,Q1,Q2,Q3,Q4,Q5
        READ    206,KK,    KA,(EN(I),I=1,KA)
C
        DO1RIF=1,KK
        READ 204,N,JL,(D(I),I=1,5),(UMASS(I),I=1,5), XS,XL,NS
        PRINT204,N,JL,(D(I),I=1,5),(UMASS(I),I=1,5),XS,XL,NS
        PRINT 205
C
  200 FORMAT(10E13.5)
  201 FORMAT(I10)
  204 FORMAT(2I3,12F6.2,I2)
  205 FORMAT(/ )
  206 FORMAT(2I3,10F6.1)
  209 FORMAT(18H       ENERGY(MEV)=F6.1)
  203 FORMAT(8E14.5)
  211 FORMAT(17H      UNREAL  ROOT)
  210 FORMAT(17H      ROOT  FAILED,5E13.5)
C
        PI=1/3.1415927
        NN=2*N    $ NSM1=NS-1 $ NSM2=NS-2 $ NSP1=NS+1 $ NSP2=NS+2
        DO20I=1,NS
   20 P(I)=(I-.999)*XS
        DO90I=NSP1,N
   90 P(I)=P(NS)+(I-NS)*XL
        DO91I=2,NSM1,2
   91 Z(I)=XS*1.333333333
        DO92I=3,NSM2,2
   92 Z(I)=XS*0.666666666
        DO93I=NSP1,N ,2
   93 Z(I)=XL*1.3333333333
        DO94I=NSP2,N ,2
   94 Z(I)=XL*0.666666666666
        Z(1)=XS/3.0 $ Z(NS)=Z(1)+XL/3.0
        DO40I=1,N
   40 PM(I)=(P(I)*P(I)+45.2976)**0.5
        ZT=JL
        RJL=SQRT(ZT*ZT+ZT)/(2.0*ZT+1.0)
        PJL=(JL+1.0)/(2*JL+1.0)
        QJL=JL/(2*JL+1.0)
```

```
      NP=N+1
      DO75I=1,5
   75 C(I)=D(I)/2.0
      J=JL/2
      IF(ZT/2.0-J    )70,70,71
   70 C(2)=-3.0*C(2)
      C(4)=-3.0*C(4)
   71 DO95I=1,N
      NIT=N+I
      P(NIT)=P(I) $ Z(NIT)=Z(I)
   95 PM(NIT)=PM(I)
      PRINT 200,    (P (LS),LS=1,N )
      PRINT 200,    (PM(LS),LS=1,N )
      PRINT 200,    (Z (LS),LS=1,N )
C
      CALLQN(JL,N,0.)
      DO1I=1,N
      DO1J=1,N
      A(I,J)=-C(1)*Q1(I,J)+C(2)*Q2(I,J)+C(3)*Q3(I,J)+4*(C(4)*Q4(I,J)+
     1  C(5)*Q5(I,J))
    1 B(I,J)=-C(1)*Q1(I,J)-C(2)*Q2(I,J)-C(3)*Q3(I,J)+2*(C(4)*Q4(I,J)+
     1  C(5)*Q5(I,J))
      CALLQN(JL-1,N,0.)
      DO2I=1,N
      DO2J=1,N
      G(I,J)=2*C(1)*Q1(I,J)+ 2*(C(4)*Q4(I,J)+C(5)*Q5(I,J))
      CM(I,J)=C(1)*Q1(I,J)-C(2)*Q2(I,J)-C(3)*Q3(I,J)
    2 DM(I,J)=C(1)*Q1(I,J)+C(2)*Q2(I,J)+C(3)*Q3(I,J)+2*(C(4)*Q4(I,J)+
     1  C(5)*Q5(I,J))
      CALLQN(JL+1,N,0.)
      DO3I=1,N
      DO3J=1,N
      R(I,J)=2*C(1)*Q1(I,J)+2*(C(4)*Q4(I,J)+C(5)*Q5(I,J))
      DP(I,J)=C(1)*Q1(I,J)+C(2)*Q2(I,J)+C(3)*Q3(I,J)+2*(C(4)*Q4(I,J)+
     1  C(5)*Q5(I,J))
    3 CP(I,J)=C(1)*Q1(I,J)-C(2)*Q2(I,J)-C(3)*Q3(I,J)
      DO100I=1,N
      DO100J=1,N
      A(I,J)=A(I,J)*Z(J)*PI
      B(I,J)=B(I,J)*Z(J)*PI
      G(I,J)=G(I,J)*Z(J)*PI
      R(I,J)=R(I,J)*Z(J)*PI
      CM(I,J)=CM(I,J)*Z(J)*PI
      CP(I,J)=CP(I,J)*Z(J)*PI
      DM(I,J)=DM(I,J)*Z(J)*PI
  100 DP(I,J)=DP(I,J)*Z(J)*PI
C
      DO101I=1,N
      DO101J=1,N
      H(I,J)=P(I)/PM(I)*A(I,J)*P(J)
```

```
101 E(I,J)=P(I)*B(I,J)*P(J)/PM(J)
    CALLMATX(CM,H,F,N)
    CALLMATX(F,DP,R23,N)
    DO102I=1,N
    DO102J=1,N
    JN=J+N
102 G(I,JN)=PJL/PM(I)*(P(I)*(A(I,J)-B(I,J))*P(J)+CM(I,J)*P(J)*P(J)
   1)-P(I)*P(I)*DP(I,J)+R23(I,J)-F(I,J))/Z(JN)*(-1.0)
    CALLMATX(E,DM,R23,N)
    DO103I=1,N
    DO103J=1,N
103 E(I,J)=CM(I,J)/PM(J)
    CALLMATX(E,DM,H,N)
    DO105I=1,N
    DO105J=1,N
105 G(I,J)=1/PM(I)*(45.2976*(B(I,J)-H(I,J))+OJL*(CM(I,J)*P(J)*P(J)+
   1 P(I)*A(I,J)*P(J)-F(I,J))+PJL*(P(I)*P(I)*DM(I,J)+P(I)*B(I,J)*P(J)
   2 -R23(I,J)))/Z(J)*(-1.0)
    DO104I=1,N
    DO104J=1,N
104 F(I,J)=P(I)/PM(I)*A(I,J)*P(J)
    CALLMATX(CP,F,H,N)
    DO106I=1,N
    DO106J=1,N
    IN=N+I
106 G(IN,J)=OJL/PM(I)*(P(I)*(A(I,J)-B(I,J))*P(J)+CP(I,J)*P(J)*P(J)
   1 -P(I)*P(I)*DM(I,J)+R23(I,J)-H(I,J))/Z(J)*(-1.0)
    DO107I=1,N
    DO107J=1,N
    E(I,J)=P(I)*B(I,J)*P(J)/PM(J)
107 F(I,J)=CP(I,J)/PM(J)
    CALLMATX(F,DP,R23,N)
    CALLMATX(F,DP,E,N)
    DO108I=1,N
    DO108J=1,N
    IN=N+I
    JN=N+J
108 G(IN,JN)=1/PM(I)*(45.2976*(R(I,J)-E(I,J))+PJL*(CP(I,J)*P(J)*P(J)
   1 +P(I)*A(I,J)*P(J)-H(I,J))+OJL*(P(I)*P(I)*DP(I,J)+P(I)*B(I,J)*P(J)
   2 -R23(I,J)))/Z(JN)*(-1.0)
C
    DO18IT=1,KA
    PRINT 209,   EN(IT)
    PZ=SQRT(.024918*EN(IT)+.0000343*EN(IT)**2)
    PRINT 203,   PZ
    J=0
 21 J=J+1
    IF(P(J)-PZ    )21,21,22
 22 IF((P(J-1)+P(J))/2-PZ    )24,24,25
 24 M=J
```

```
      GOTO26
   25 M=J-1
   26 PRINT 201,
      PRINT 205
      MN=M+N
C
      DO120IMS=1,2
      IF(IMS-1)11,11,12
   11 DO13I=1,NN
      DO13J=1,NN
   13 R(I,J)=G(I,J)*P(J)/P(I)
      GOTO17
   12 DO10I=1,NN
      DO10J=1,NN
   10 R(I,J)=G(J,I)*P(J)/P(I)
      SMJ=(JL+1.0)/JL
      SNJ=1.0/SMJ
      DO300I=1,N
      DO300J=1,N
      JN=J+N
  300 R(I,JN)=R(I,JN)*SMJ
      DO301I=1,N
      DO301J=1,N
      IN=I+N
  301 R(IN,J)=R(IN,J)*SNJ
   17 DO39I=1,NN
      CALLINTERP(R(I,M-1),R(I,M),R(I,M+1),P(M-1),P(M),P(M+1),PZ,
     1R(I,NN+1),AY,BY,CY)
   39 CALLINTERP(R(I,MN-1),R(I,MN),R(I,MN+1),P(M-1),P(M),P(M+1),PZ,
     1R(I,NN+2),AY,BY,CY)
      CALLINTERP(R(M-1,NN+1),R(M,NN+1), R(M+1,NN+1),P(M-1),P(M),
     1 P(M+1),PZ,DG,AY,BY,CY)
      CALLINTERP(R(M-1,NN+2),R(M,NN+2), R(M+1,NN+2),P(M-1),P(M),
     1 P(M+1),PZ,DH,AY,BY,CY)
      CALLINTERP(R(MN-1,NN+1),R(MN,NN+1),R(MN+1,NN+1),P(M-1),P(M),
     1 P(M+1),PZ,DE,AY,BY,CY)
      CALLINTERP(R(MN-1,NN+2),R(MN,NN+2),R(MN+1,NN+2),P(M-1),P(M),
     1 P(M+1),PZ,DF,AY,BY,CY)
      DO47I=1,NN
      DO47J=1,N
   47 R(I,J)=(R(I,NN+1)-R(I,J))*Z(J)
      DO41I=1,NN
      DO41J=NP,NN
   41 R(I,J)=(R(I,NN+2)-R(I,J))*Z(J)
      DO42I=1,NN
      DO43J=1,NN
   43 R(I,J)=R(I,J)/(PZ*PZ-P(J)*P(J))
   42 R(I,I)=1.0+R(I,I)
C
      CALLSIMLEQ(R,NN,NN+2,IRR)
```

```
      PRINT206,IRR
      DO44I=1,N
      F1(I)=R(I,NN+1)
      F2(I)= R(I,NN+2)
      NI=N+I
      G1(I)=R(NI,NN+1)
   44 G2(I)=R(NI,NN+2)
      CALLINTERP(F1(M-1),F1(M),F1(M+1),P(M-1),P(M),P(M+1),PZ,FD1,
     1AY,BY,CY)
      CALLINTERP(F2(M-1),F2(M),F2(M+1),P(M-1),P(M),P(M+1),PZ,FD2,
     1AY,BY,CY)
      CALLINTERP(G1(M-1),G1(M),G1(M+1),P(M-1),P(M),P(M+1),PZ,GD1,
     1AY,BY,CY)
      CALLINTERP(G2(M-1),G2(M),G2(M+1),P(M-1),P(M),P(M+1),PZ,GD2,
     1AY,BY,CY)
      FC1=SENT(F1,PZ,M,N,NS)
      FC2=SENT(F2,PZ,M,N,NS)
      GC1=SENT(G1,PZ,M,N,NS)
      GC2=SENT(G2,PZ,M,N,NS)
      PRINT 200,    DG,DH,DE,DF
      PRINT 200,    FC1,FC2,GC1,GC2,FD1,FD2,GD1,GD2
      EM=SQRTF(45.2976+PZ*PZ)
      CALLQN(JL,1,PZ)
      Q1M=Q1(1,1)
      Q2M=Q2(1,1)
      Q3M=Q3(1,1)
      Q4M=Q4(1,1)
      Q5M=Q5(1,1)
      CALLQN(JL-1,1,PZ)
      Q1MM=Q1(1,1)
      Q2MM=Q2(1,1)
      Q3MM=Q3(1,1)
      Q4MM=Q4(1,1)
      Q5MM=Q5(1,1)
      CALLQN( JL+1,1,PZ)
      Q1MP=Q1(1,1)
      Q2MP=Q2(1,1)
      Q3MP=Q3(1,1)
      Q4MP=Q4(1,1)
      Q5MP=Q5(1,1)
      DDG=1/EM*(C(1)*((2*45.2976+PZ*PZ)*Q1MM-PZ*PZ*Q1M)+1./( 2.*JL+1.)
     1   *C(2)*(Q2MM-Q2M)*PZ*PZ+2*       C(4)  *((45.2976+PZ*PZ*PJL)*
     2   Q4MM+PZ*PZ*(3.*JL+1.)/(2.*JL+1.)*Q4M)+1./(2.*JL+1)*C(3)*(Q3MM-Q3
     3   M)*PZ*PZ+2*C(5)*((45.2976+PZ*PZ*PJL)*Q5MM+PZ*PZ*(3.*JL+1)
     4   /(2.*JL+1)*Q5M))*(-PI)
      DDH=-RJL*PZ*PZ/EM*(C(2)*(Q2MM-2*Q2M+Q2MP)+2*C(4)*(Q4MM-2*Q4M+Q4MP)
     1   +C(3)*(Q3MM-2*Q3M+Q3MP)+2*C(5)*(Q5MM-2*Q5M+Q5MP))*PI
      DDF=1/EM*(C(1)*((2*45.2976+PZ*PZ)*Q1MP-PZ*PZ*Q1M)+1./(2.*JL+1.)*
     1   C(2)*(Q2M-Q2MP)*PZ*PZ+2*       C(4)  *((45.2976+0JL*PZ*PZ)*Q4MP
     2   +(3.*JL+2.)/(2.*JL+1.)*PZ*PZ*Q4M)+1./(2.*JL+1)*C(3)*(Q3M-Q3MP)
```

```
3   *PZ*PZ+2*C(5)*((45.2976+QJL*PZ*PZ)*QSMP+(3.*JL+1)/(2.*JL+1)
4   *PZ*PZ*QSM))*(-PI)
    BET=(1-GC2)*(1-FC1)-FC2*GC1
    ZF1(IMS)=((1-GC2)*FD1+GC1*FD2)/BET
    ZF2(IMS)=(FC2*FD1+(1-FC1)*FD2)/BET
    ZG1(IMS)=((1-GC2)*GD1+GC1*GD2)/BET
    ZG2(IMS)=(FC2*GD1+(1-FC1)*GD2)/BET
    GG=ZF1(IMS)
    PRINT203,ZF1(IMS),ZF2(IMS),ZG1(IMS),ZG2(IMS)
    PRINT 200,   DDG,DDH,DDF
120 PRINT 205
    AS=3.1415927/(2*PZ)
    OF1=ZF1(1)*AS $ OF2=ZF2(1)*AS $ OG1=ZG1(1)*AS $ OG2=ZG2(1)*AS
    DF1=ZF1(2)*AS $ DF2=ZF2(2)*AS $ DG1=ZG1(2)*AS $ DG2=ZG2(2)*AS
    OA=1.0+OG1*OF2-OG2*OF1
    DA=1.0+DG1*DF2-DG2*DF1
    OB=OG2+OF1
    DB=DG2+DF1
    OX=OG1*OF2-OG2*OF1
    DX=DG1*DF2-DG2*DF1
    RD=(OX*(2+OX)+OB*OB)*(1+DX*(2+DX)+DB*DB)+(DX*(2+DX)+DB*OB)
    QJLP=1.0+PJL*PZ*PZ/45.2976
    QJL=1.0+QJL*PZ*PZ/45.2976
    QJ=PJL*QJL*PZ*PZ/(EM*EM)
    JLP=JL+1
    DGG=JL*DG2-JLP*DG1
    DFF=JL*DF2-JLP*DF1
    XW2=QJ*((OX+(OG1+OG2)*OB)*(JL*DA+DGG*DB)+(JL*DX+DGG*DB)
1   +((OG1+OG2)*OA-DB)*(DGG*DA-JL*DB))
    XW1=QJ*(QJLP*((OX+(OG1+OG2)*OB)*(JLP*DA-DFF*DB)+(JLP*DX-DFF*DB)
1   -((OG1+OG2)*OA-OB)*(DFF*DA+JLP*DB))+QJL*(-(OX+(OF1+OF2)*OB)*
2   (JL*DA+DGG*DB)-(JL*DX+DGG*DB)+(-(OF1+OF2)*OA+OB)*
3   (DGG*DA-JL*DB)))
    XW0=QJ*(-(OX+(OF1+OF2)*OB)*(JLP*DA-DFF*DB)-(JLP*DX-DFF*DB)
1   -(-(OF1+OF2)*OA+OB)*(DFF*DA+JLP*DB))
    XA=QJLP*(JL*((OX+OG2*OB)*(DA+DG2*DB)+DX+DG2*DB
1   +(OG2*OA-OB)*(DG2*DA-DB))-XW2)
    XB=-QJLP*JL*((OA+OG2*OB)*DB+(OG2*OA-OB)*DA)*DF2
1   -QJL*JL*((DA+DG2*DB)*OB+(DG2*DA-DB)*OA)*DF2-XW1
    XC=QJL*(JL*DF2*DF2*(OB*DB+OA*DA)-XW0)
    XAD=QJLP*(JLP*OG1*OG1*(OB*OB+OA*DA)+XW2)
    XBD=-QJLP*JLP*(OB*(DA+DF1*DB)+OA*(DF1*DA-DB))*OG1
1   -QJL*JLP*((OA+DF1*OB)*DB+(DF1*OA-OB)*DA)*OG1+XW1
    XCD=QJL*(JLP*((OX+DF1*OB)*(DA+DF1*DB)+(DX+DF1*DB)+(DF1*OA-OB)*
1   (DF1*DA-DB))+XW0)
    XO=JL*QJLP*(1.0-QJ)
    XE=-QJL*PJL*PZ*PZ/45.2976
    XF=QJ*JLP*QJL
    XDD=QJ*JL*QJLP
    XEO=-XE
```

```
      XFD=JLP*QJL*(1.0-QJ)
      DA=XA*XDD-XAD*XD
      X3=XB*XDD-XBD*XD+XA*XFD-XAD*XE
      X2=XA*XFD-XAD*XF+XB*XFD-XBD*XE+XC*XDD-XCD*XD
      X1=XB*XFD-XBD*XF+XC*XED-XCD*XE
      X0=XC*XFD-XCD*XF
      X4=1. $X3=X3/DA  $ X2=X2/DA  $ X1=X1/DA $ X0=X0/DA
      I=0
      F1(2)=X2/11.
      GOTO160
  161 F1(2)=-X2*21./11.
  160 F1(1)=0
      F1(3)=20
      F1(4)=X2/11.
      F1(5)=1.0E-6
      I=I+1
  138 CALLROOT(F1)
      IF(F1(1))135,132,133
  135 GOTO(161,131)I
  131 PRINT210, X4,X3,X2,X1,X0
      GOTO145
  132 DA=F1(2)
      GOTO137
  133 F1(3)=F1(2)+X0/F1(2)+(X3-(X1-X3*F1(2))/(X0/F1(2)-F1(2)))*
     1  (X1-X3*F1(2))/(X0/F1(2)-F1(2))-X2
      GOTO138
  137 DA=X0/DA
      OB=(X1-X3*DA)/(DA-DA) $ DB=X3-OB
      DX=OB*OB-4*DA
      DX=DB*DB-4*DA
      IF(DX.GT.0.AND.DX.GT.0)141,142
  141 DG1=(-DB+SQRT(DX))/2.
      DF1=(-DB-SQRT(DX))/2.
      PRINT200,DG1,DF1
      DG2=(-DB+SQRT(DX))/2.
      DF2=(-DB-SQRT(DX))/2.
      PRINT200,DG2,DF2
      X1=FT(DG1,XD,XE,XF,XDD,XED,XFD)
      X2=FT(DF1,XD,XE,XF,XDD,XED,XFD)
      X3=FT(DG2,XD,XE,XF,XDD,XED,XFD)
      X4=FT(DF2,XD,XE,XF,XDD,XED,XFD)
      PRINT200,X1,X2,X3,X4
      AY=ABS(1.-X2*X1) $ BY=ABS(1.-X3*X1) $ CY=ABS(1.-X4*X1)
      DY=ABS(1.-X3*X2) $ EY=ABS(1.-X4*X2) $ FY=ABS(1.-X4*X3)
      IF(AY.LT.BY)400,404
  400 IF(AY.LT.CY)401,408
  401 IF(AY.LT.DY)402,411
  402 IF(AY.LT.EY)403,413
  403 IF(AY.LT.FY)414,419
  404 IF(BY.LT.CY)405,408
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```
405 IF(BY.LT.DY)406,411
406 IF(BY.LT.FY)407,413
407 IF(BY.LT.FY)415,419
408 IF(CY.LT.DY)409,411
409 IF(CY.LT.EY)410,413
410 IF(CY.LT.FY)416,419
411 IF(DY.LT.EY)412,413
412 IF(DY.LT.FY)417,419
413 IF(EY.LT.FY)418,419
414 DG2=DF1 $ GOTO420
415 X2=X3 $ GOTO420
416 X2=X4 $ DG2=DF2 $ GOTO420
417 X1=X2 $ X2=X3 $ DG1=DF1 $ GOTO420
418 X1=X2 $ X2=X4 $ DG1=DF1 $ DG2=DF2 $ GOTO420
419 X1=X3 $ X2=X4 $ DG1=DG2 $ DG2=DF2
420 PRINT200,X1,X2,DG1,DG2
    X0=ABS((X1-1/X2)/X1)*100.
    PRINT220,X0
220 FORMAT(20X, F6.1,21H  PERCENT  DIFFERENCE)
    OA=(DG1*DG1*XA+DG1*XB+XC)/(DG1*DG1*XD+DG1*XE+XF)
    OB=(DG2*DG2*XA+DG2*XB+XC)/(DG2*DG2*XD+DG2*XE+XF)
    IF(RD.GT.OA.AND.RD.GT.OB)180,181
181 PRINT221,RD,OA,OB
221 FORMAT (16H   ALPHA  UNREAL,3E16.5)
    GOTO18
180 DF1=SQRT((RD-OB)/(1+OB))
    DF2=SQRT((RD-OA)/(1+OA))
    PRINT205
    PRINT200,DF1,DF2,X1,X2
    DG=DDG $ DH=DDH $ DF=DDF
    IF(DH)290,145,290
290 X1=(DG-DF+SQRT((DG-DF)*(DG-DF)+4*DH*DH))/(2*DH) $ X2=-1/X1
    OX=(DG+DF+SQRT((DG-DF)*(DG-DF)+4*DH*DH))*AS*(-0.5)
    OA=(DG+DF-SQRT((DG-DF)*(DG-DF)+4*DH*DH))*AS*(-0.5)
    PRINT200,OX,OA,X1
    GOTO145
142 PRINT211
    PRINT200,OX,OB,OA,OX,OB,OA
145 CONTINUE
 18 PRINT 205
    STOP
    END
```

```
      SUBROUTINEQFN(Q,J,T,N,XX)
      COMMONT,UMASS,P
      DIMENSIONT(40),UMASS(5),P(40),Q(20,20)
      JT=J+1
      IF(XX)10,10,11
   11 L=1
      M=1
      R=P(1)
      P(1)=XX
      GOTO12
   10 DO52L=1,N
      DO52M=1,L
   12 Z=(P(L)*P(L)+P(M)*P(M)+UMASS(T)**2)/(2.0*P(L)*P(M))
      IF(Z-10.0)70,70,80
   70 Q(L,M)=0.5*ALOG((Z+1)/(Z-1))
      GOTO(52,53,54,55,56,57)JT
   53 Q(L,M)=Z*Q(L,M)-1.0
      GOTO52
   54 Q(L,M)=0.5*(3*Z*Z-1)*Q(L,M)-1.5*Z
      GOTO52
   55 Q(L,M)=0.5*(5.0*Z**3-3*Z)*Q(L,M)-2.5*Z*Z+2.0/3.0
      GOTO52
   56 Q(L,M)=(35.0*Z**4-30.0*Z*Z+3.0)/8.0*Q(L,M)-35.0/8.0*Z**3+55.0/24.0
     1*Z
      GOTO52
   57 Q(L,M)=(63.*Z**5-70.*Z**3+15.*Z)/8.*Q(L,M)-63.*Z**4/8.+49.*Z*Z/8.
     1-3./15.0
      GOTO52
   80 GOTO(91,92,93,94,95,96)JT
   91 Q(L,M)=1/Z+1/(3*Z**3)
      GOTO52
   92 Q(L,M)=1/(3*Z**2)+1/(5*Z**4)
      GOTO52
   93 Q(L,M)=2/(15*Z**3)+4/(35*Z**5)
      GOTO52
   94 Q(L,M)=2/(35*Z**4)+4/(63*Z**6)
      GOTO52
   95 Q(L,M)=8./(315.*Z**5)+24./(693.*Z**7)
      GOTO52
   96 Q(L,M)=8./(693.*Z**6)+24./(1287.*Z**8)
   52 CONTINUE
      IF(XX)13,13,14
   13 DO60L=1,N
      DO60M=1,L
   60 Q(M,L)=Q(L,M)
      GOTO15
   14 P(1)=R
   15 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE ROOT(X)
      DIMENSION X(20)
      EQUIVALENCE(A,I),(B,J) ,(C,N)
      C=X(12)
      A=X(10) $ B=X(11)
      IF(X(1))1,2,3
    2 IF(X(3))1,4,5
    4 X(1)=+1.
      I=1
      GO TO 1
    3 GO TO (6,6,6,7,8,9,10,11,21),I
    6 X(I+5)=X(3)
      X(I+7)=X(2)
      X(2)  =X(4)
      I=I+1
      IF( I .EQ. 3) 12, 1
   12 IF(X(3)*X(I+3)) 20,13,400
   13 IF(X(3))15,14
   14 X(1)=0
      GO TO 1
   15 IF(X(I+3))4,74
   74 X(2)=X(8)
      GO TO 14
   20 X(2)=X(8) -((X(8)-X(9))*X(6))/(X(6)-X(7))
      I=6
      GO TO 1
    9 IF(ABSF(X(3))-1E-09)14,14,16
   16 IF(ABSF(X(2) - X(8))-ABSF(X(5)*X(2)))14,14,17
   17 IF( X(3)*X(6)) 18,13,19
   18 X(9)=X(2)
                  X(7)=X(3) $ GOTO 20
   19 X(6)=X(3)
                 X(8)=X(2) $ X(7)=X(7)/2.  $  GOTO 20
  400 X(2)=X(8)*1.001
      I=4
      GO TO 1
    7 SLOPE=( X(3)-X(6))/.001*X(8)
          X(2)=X(8)-X(6)/SLOPE
      I=5
      GO TO 1
    8 IF(X(3)*X(6))100,14,30
  100 X(9)=X(2)
                  I=7  $ GO TO 1
   10 X(7)=X(3)
      GOTO20
    5 X(1)=1.
      A=8.0  $  C=X(3)
                N=X(3) $I=8 $ GOTO 1
   11 X(6)= X(3)
                   X(8)= X(2) $ IF( X(6))322,14
```

```
322 DO 321 J=1,N
    X(2)=X(8)+ J*X(4)   $ I=9  $ GO TO 1
 21 IF(X(6)*X(3))301,14,321
321 CONTINUE
 30 X(1)=-1.
    GOTO1
301 X(9)=X(2)
                X(4)= X(2)   $ X(2)=X(8) $ X(7)=X(3)   $ GO TO 20
  1 X(12)=C
            X(10)=A $X(11)=B $ RETURN
    END



    FUNCTIONFT(X,YD,XE,XF,XDD,XED,XFD)
    A=(X*X*XD+X*XF+XF)/(X*X*XDD+X*XED+XFD)
    IF(A)1,2,2
  1 PRINT212
    FT=0.0
    GOTO5
  2 FT=SQRT(A)
212 FORMAT(21H         M/N   IMAGINARY)
  5 RETURN
    END



    SUBROUTINEQN(K,N,XX)
    DIMENSIONZ(40),UMASS(5),P(40),Q1(20,20),Q2(20,20),Q3(20,20),
   1  Q4(20,20),Q5(20,20)
    COMMONZ,UMASS,P,Q1,Q2,Q3,Q4,Q5
    CALLQFN(Q1,K+1,N,XX)
    CALLQFN(Q2,K+2,N,XX)
    CALLQFN(Q3,K+3,N,XX)
    CALLQFN(Q4,K+4,N,XX)
    CALLQFN(Q5,K+5,N,XX)
    RETURN
    END



    SUBROUTINEMATX(A,B,C,  N)
    DIMENSIONA(20,20),B(20,20),C(20,20)
    DO2I=1,N
    DO2J=1,N
    C(J,I)=0.0
    DO2K=1,N
  2 C(J,I)=C(J,I)+A(J,K)*B(K,I)
    RETURN
    END
```

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