Functions with no Antiderivative

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Abstract

This is a project in an introductory abstract algebra course. In this paper, I present notions that help us understand why some real-valued functions of real variable do not possess an antiderivative. When I took calculus II, my professor mentioned in a lecture that $\int e^{-x^2}\,dx$ canot be integrated and that it has been proven. When I looked up the proof on the internet, I could not understand the math behind it. Because of this project, I am several steps closer to completely understand it. Beware that this is a basic exposure to elementary differential Galois theory and, consequently, Liouville's theorem (our main theorem) will not be proved. The purpose of this paper is to demonstrate that e^{-x^2} does not have an antiderivative, i.e., there is not a real-valued function whose derivative is e^{-x^2}.

Introduction

In an average calculus class, it is customary to work with familiar functions $f : \mathbb{R} \to \mathbb{R}$ whose antiderivatives or derivatives can be computed *nicely*. We sometimes encountered functions whose integral or derivative was hard to find; however, we could always eventually find it. Nevertheless, it turns out that some functions that we saw in calculus do not have an antiderivative (such as $e^{\pm x^2}$, $\frac{\sin x}{x}$, and $\frac{1}{\ln x}$), and that is why we were never asked to find them. Not all functions possess an antiderivative, and the elementary calculus does not have the tools to prove this fact.

Since our main interest is the study of the existence of integrals of functions, by *nicely* we mean that the integral (or antiderivative) of f is another familiar function. For example, the functions $\cos x$, $\frac{1}{x^2+1}$, $x^x(\ln x + 1)$, and $\operatorname{sech}^2 x$ have *nice* antiderivatives (in their respective domains of x) because $\sin x$, $\arctan x$, x^x, and $\tanh x$ are functions that we are also familiar with. Note that the functions we are going to be dealing with have antiderivatives that are *finite sums* of functions. This means that, for example, we will think of the integral of $\cos x$ as $\sin x$ rather than $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ because its McLaurin series expansion is an infinite sum of powers of x, which are familiar functions nonetheless. Also note that we are using the words *antiderivative* and *integral* indistinctly, even though we know that if a function g' has an antiderivative g, then it has infinitely many more antiderivatives of the form $g + C$ where C is a constant. Moreover, the antiderivatives of a function are the result of an indefinite integral not just an integral. We will not pay attention to these semantic subtleties in this paper. Additionally, beware that we are not going to deal with analysis concepts such as differentiability or Riemann-integrability, but rather with the notion of existence of antiderivatives.

If we could come up with a set R, which contains all of these familiar functions, with certain properties (such as closure under a derivative operator), then we could certainly determine whether a function possesses an antiderivative by realizing that this integral cannot be expressed as a finite sum of familiar functions that belong to this set R. Surprisingly (at least to me), it turns out that we can determine whether a function has an antiderivate with the aid of abstract algebra. This is why we begin the next section by introducing the concept of a differential ring.

Basic Notions

Definition 1. *A set R is a differential ring if R is itself a commutative ring of characteristic zero and possesses a map $\delta : R \to R$ where $\delta(r) = r'$. This map δ satisfies the following two properties*

$$(r+s)' = r' + s'$$
$$(rs)' = r's + rs'$$

for all $r, s \in R$. We call δ a derivation on R.

Note that δ is an additive group homomorphism between R and itself, which ensures that the for an element $r \in R$ we also have $r' \in R$. The homomorphism property does not hold for multiplication, and this multiplication property of a differential ring with its derivation is also known as the *product rule* in calculus. Furthermore, a differential field is a differential ring that is a field.

Theorem 1. *Let R be a differential ring where 0 and 1 are the zero element and unity respectively, then the following hold*

(i) $1' = 0$

(ii) $(r^n)' = nr^{n-1}r'$ *for $r \in R$ and $n \in \mathbb{Z}^+$.*

Proof.

(i) Since $1' = (1 \cdot 1)' = 1 \cdot 1' + 1' \cdot 1 = 2(1 \cdot 1') = 2(1')$, it follows that $1'$ must be 0.

(ii) We will induct on n. For $n = 1$, the result is trivial. Assume the claim holds for $n = k \geq 1$, so $(r^k)' = kr^{k-1}r'$. Consider now

$$(r^{k+1})' = (rr^k)' = r(r^k)' + r'r^k = rkr^{k-1}r' + r'r^k = kr^kr' + r'r^k = (k+1)r^kr'.$$

Therefore, the result follows by mathematical induction. □

Example 1. The polynomial ring that we are all familiar with $\mathbb{R}[x]$ is a differential ring where the derivation $\delta = \frac{d}{dx}$ is our usual derivative operator on $p(x) \in \mathbb{R}[x]$ from calculus. Note that any differential ring R has the trivial derivation $\delta_0 : R \to \{0_R\}$.

We have seen that $\mathbb{R}[x]$ is not a field but an integral domain. We will present a field that contains $\mathbb{R}[x]$ as a subring after the next definition.

Definition 2. *Let F be a field, then $F(x)$ is the field of rational polynomial functions of one indeterminate on F (also called the fraction field of $F[x]$).*

Example 2. Note that $\mathbb{R}[x]$ is a subring of the field $\mathbb{R}(x)$, for $\mathbb{R}(x)$ contains all polynomials of the form $\frac{p(x)}{q(x)}$ where $p(x), q(x) \in \mathbb{R}[x]$ and $q(x) \neq 0$, i.e., the zero polynomial in $\mathbb{R}[x]$. Hence, if we use the same derivation δ from $\mathbb{R}[x]$, it is easy to check that the derivative of an element in $\mathbb{R}(x)$ is also in $\mathbb{R}(x)$.

Example 3. Since $\mathbb{C}$ is an extension field of $\mathbb{R}$, we have that $\mathbb{C}(x)$ is a field.

Theorem 2. *Let R be a differential field, then if $r, s \in R$ and $s \neq 0_R$ (so it makes sense to speak of $1/s$) we have $\left(\frac{r}{s}\right)' = \frac{r's - rs'}{s^2}$.*

Proof. Since $s \neq 0_R$, then $s\frac{1}{s} = 1_R$. Hence, by Theorem 1 (i) we have $(s\frac{1}{s})' = 1_R' = 0_R$, so $s'\frac{1}{s} + s(\frac{1}{s})' = 0_R$. Thus, $(\frac{1}{s})' = -\frac{s'}{s^2}$. Finally, consider $\left(\frac{r}{s}\right)' = r(\frac{1}{s})' + r'(\frac{1}{s}) = -\frac{rs'}{s^2} + r'(\frac{1}{s}) = \frac{r's - rs'}{s^2}$, as desired. □

Note that Theorem 1 (ii) and Theorem 2 are also known as the *power rule* and the *quotient rule*, respectively, in calculus.

Before we begin to formalize what we initially referred to as *familiar* function, we make clear that the functions that we are going to deal with here are complex-valued functions of real variable $f : \mathbb{R} \to \mathbb{C}$. This will strengthen our theory because if we incorporate complex numbers in the outputs of our functions, we can express any familiar function $f(x)$ in terms of exponentials $e^{g(x)}$ and natural logarithms $\ln h(x)$, where $g(x), h(x) \in \mathbb{C}(x)$, as well as other combinations of roots and ratios between polynomials in $\mathbb{C}(x)$. From now on we will denote the complex unit by $i = \sqrt{-1}$.

Example 4. Recall Euler's identity $e^{ix} = \cos x + i \sin x$. With this, we can express the sine and cosine functions as follows

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

Note that we can represent all six trigonometric functions like this by using trigonometric identities such as $\tan x = \frac{\sin x}{\cos x}$, etc.

In the case of other functions, such as $\arctan x$, we will need to first accept that any function $f : \mathbb{R} \to \mathbb{C}$ can be expressed as $f(x) = u(x) + iv(x)$ where $u, v : \mathbb{R} \to \mathbb{R}$, and that the derivative of $f(x)$ is $f'(x) = u'(x) + iv'(x)$ if it exists. Hence, we find that $\mathbb{C}(x)$ is a differential ring. Furthermore, we accept that the same reasoning applies to the antiderivative of f.

Example 5. We know that $\arctan x = \int_0^x \frac{1}{1+t^2}\, dt$. Splitting $1 + t^2 = (1 + it)(1 - it)$ in $\mathbb{C}$ and integrating [1] we find that $\arctan x = \frac{i}{2}[\ln(1 - ix) - \ln(1 + ix)]$. We can represent all six inverse trigonometric identities like this by using identities such as $\arcsin x = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$, $\arccos x = \arctan\left(\sqrt{\frac{1}{x^2} - 1}\right)$, $\operatorname{arcsec} x = \arccos \frac{1}{x}$, etc.

Notice that all hyperbolic and inverse hyperbolic functions are already defined in terms of exponentials and natural logarithms, so we should not worry about them.

We would like to remark that all the functions mentioned are well-defined in their respective domains, and we will assume this for all functions encountered in this paper. Having said all of this, we introduce the following definition.

[1] In this type of integration, we assume that the antiderivative of $1/z$ is $\ln z$ not $\ln |z|$ since z can be complex.

Definition 3. *Let R be a differential field and let E be a differential field extension of R with a common derivation δ. If for some $f \in E$ and $g \in R$ with $g \neq 0_R$ we have*

$$\delta(f) = \frac{\delta(g)}{g}$$

then we say f is a logarithm of g. Additionally, we say that g is an exponential of f.

There are two thinks to remark now. First, note that Definition 3 also works if $R = E$. Secondly, notice that the only differential rings we know so far are $\mathbb{R}(x)$ and $\mathbb{C}(x)$ and we cannot find elements that are logarithms or exponentials of other elements in these two differential rings. We have not yet built a differential field whose elements are functions other than rational polynomial functions; however, we are one step closer to this. Assume for now that the functions in the next examples belong to some differential rings and that the derivation is the familiar one $\delta = \frac{d}{dx}$.

Example 6. The most intuitive example would be $\ln x$, which is a logarithm of x since $(\ln x)' = \frac{x'}{x} = \frac{1}{x}$. Also, x is the exponential of $\ln x$. This can be verified by noting $x = e^{\ln x}$.

Example 7. A more interesting example is $\arctan x$, which is a logarithm of $\frac{1}{2i}\left(\frac{x-i}{x+i}\right)$.

In Example 5, we showed that $\arctan x$ can be expressed in terms of logarithms of functions in $\mathbb{C}(x)$ by integrating it. Notice that if we use Definition 3 instead, then we avoid issues like the one in the footnote in page 3. We now enhance our theory with the following definition.

Definition 4. *A meromorphic function is a function defined on an open interval I of the real numbers whose values are also real numbers or $\pm\infty$ with the property that sufficiently close to any $x_0 \in I$ the function is given by a convergent Taylor series in $x - x_0$.*

This definition says that all functions in $\mathbb{R}[x]$ are well-defined in some interval. This means that the functions in $\mathbb{R}(x)$ and $\mathbb{C}(x)$ are also well-defined in some interval. Moreover, $\ln f(x)$ and $e^{f(x)}$ for $f(x) \in \mathbb{C}(x)$ are well-defined in some interval. Hence, we are ensured that any root of any linear combination of terms from $\mathbb{C}(x)$ and its logarithms and exponentials is well-defined in some interval. Definition 4 is important because that is why we know these objects, which are elements of differential rings such as $\mathbb{C}(x)$, are genuine functions. In other words, this definition allows us to go on with our theory.
Before we move on to the next section, let us introduce some new fields through examples.

Example 8. We know that $\mathbb{C}(x)$ is a field with indeterminate x. We can then construct an extension field $\mathbb{C}(x)(y)$ by introducing a new indeterminate y. For example, some elements in $\mathbb{C}(x)(y)$ are $\frac{3ix}{i+x} - xy^2$ and $\frac{1+(ix^2-1)y+(x^6+x+1)y^3}{\frac{1}{xi}-iy}$. Note that $\mathbb{C}(x)(y)$ is isomorphic to $\mathbb{C}(x,y)$. We will generalize this in the next theorem.

Theorem 3. *Let F be a field and $x_1, x_2, \ldots, x_n$ be indeterminates. Then, the extension field $F(x_1)(x_2)\cdots(x_n)$ is isomorphic to $F(x_1, x_2, \ldots, x_n)$.*

Proof. It suffices to show that $F(x_1)(x_2) \cong F(x_1, x_2)$ since we may repeat the argument setting $F_{n-1}(x_n) = F(x_1, x_2, \cdots, x_{n-1})(x_n)$. Define the map $\iota : F(x_1)(x_2) \to F(x_1, x_2)$ by the identity mapping. It is clear that ι is a well-defined ring homomorphism and a one-to-one correspondence, therefore, the result follows. $\square$

Example 9. Let $K = \mathbb{C}(x)$ and note it is a differential field with the usual calculus derivation. Now consider the extension field $K(\sin x, \cos x) = \mathbb{C}(x, \sin x, \cos x)$, i.e., the field generated by x, $\sin x$, and $\cos x$. We know $\mathbb{C}(x, \sin x, \cos x)$ is a field because it is isomorphic to $\mathbb{C}(x, y, z)$ and we are treating $\sin x$ and $\cos x$ as the indeterminates y and z respectively. The elements $\mathbb{C}(x, \sin x, \cos x)$ are of the form

$$\frac{p(x, \sin x, \cos x)}{q(x, \sin x, \cos x)} = \frac{\sum_{i,j,k} a_{ijk} x^i \sin^j x \cos^k x}{\sum_{p,q,r} b_{pqr} x^p \sin^q x \cos^r x}$$

where $a_{ijk}, b_{pqr} \in \mathbb{C}$ and the denominator does not equal to the zero of the field. An example of an element in $\mathbb{C}(x, \sin x, \cos x)$ is $\frac{x + i\sin^3 x \cos x + x^3 \cos^2 x}{-i - x \sin x \cos^5 x}$. It is important to see that $\mathbb{C}(x, \sin x, \cos x)$ is a differential field since it is closed under differentiation (this will be proved in the next section). Finally, note that $\mathbb{C}(x, \sin x, \cos x) = \mathbb{C}(x, e^{ix})$ because of the identities in Example 4. In general, if $f_1, f_2, \cdots, f_n$ are meromorphic functions, then the field $\mathbb{C}(x, f_1, f_2, \cdots, f_n)$ consists meromorphic functions.

Elementary Fields

So far, we only know three types of differential fields, i.e., $\mathbb{R}(x), \mathbb{C}(x)$, and $\mathbb{C}(x, e^{ix})$. In this section we will learn how to construct as many differential fields as we want.

Definition 5. *Let $f_1, f_2, \cdots, f_n$ be meromorphic functions. A field of meromorphic functions K is an elementary field if $K = \mathbb{C}(x, f_1, f_2, \cdots, f_n)$ and only one of the following hold*

(i) each f_j is a logarithm or an exponential of an element in $K_{j-1} = \mathbb{C}(x, f_1, f_2, \cdots, f_{j-1})$, or

(ii) each f_j is algebraic over K_{j-1}.

Moreover, functions from an elementary field are called elementary functions.

The above definition tells us that we can construct an elementary field from $\mathbb{C}(x)$ in finitely many steps by adjoining special meromorphic functions f_j in the same manner we adjoin the indeterminate y to $\mathbb{C}(x)$ to form $\mathbb{C}(x, y)$. Hence, we have the following tower of fields consisting of elementary fields

$$K = \mathbb{C}(x, f_1, f_2, \cdots, f_{n-1}, f_n) \supset \mathbb{C}(x, f_1, f_2, \cdots, f_{n-1}) \supset \cdots \supset \mathbb{C}(x, f_1) \supset \mathbb{C}(x)$$

for functions f_j satisfying properties (i) and (ii) in our last definition.

Example 10. We see that $\mathbb{C}(x, e^{ix})$ is an elementary field since e^{ix} is an exponential of $ix \in \mathbb{C}(x)$. Likewise, $\mathbb{C}(x, \ln x)$ and $\mathbb{C}(x, e^{\frac{x^2}{i+x}})$ are also elementary fields. These two examples satisfy condition (i) in Definition 5. We will see next the meaning of condition (ii).

Example 11. We know that $\sqrt{x} \notin \mathbb{C}(x)$. However, consider the polynomial ring with indeterminate T denoted by $\mathbb{C}(x)[T]$ and note that $T^2 - x \in \mathbb{C}(x)[T]$. Since $\sqrt{x}$ is a root of $T^2 - x$, we have that $\sqrt{x}$ is algebraic over $\mathbb{C}(x)$. Hence, $\mathbb{C}(x, \sqrt{x})$ is an elementary field, which is an example of condition (ii) in Definition 5.

Example 12. The function $f(x) = \sqrt[3]{\ln x + \cos\left(\frac{x}{x^2+i}\right)}$ is an elementary function of

$$\mathbb{C}(x, \ln x, e^{\frac{x}{x^2+i}}, f(x)) \supset \mathbb{C}(x, \ln x, e^{\frac{x}{x^2+i}}) \supset \mathbb{C}(x, \ln x) \supset \mathbb{C}(x).$$

Example 13. The function $f(x) = x^x$ is an elementary function of $\mathbb{C}(x, \ln x, e^{x \ln x})$.

Just as we mentioned in the introduction, we are interested in sets of functions that allow us to determine whether one of its elements can be integrated in terms of other elements in the set. We are ready now to present our next theorem, which allows us to construct sets of functions that are closed under differentiation, that is, differential fields of functions. From now on, we assume the derivation δ is the usual calculus differentiation $\frac{d}{dx}$. This convention serves our purposes since all meromorphic functions have a derivative in some interval, and this derivative is another meromorphic function.

Theorem 4. *An elementary field K is closed under the derivation $\delta(f) = f'$ where $f \in K$. This means, elementary fields are differential fields.*

Proof. Let $K = \mathbb{C}(x, f_1, \cdots, f_n)$ for $n \geq 0$. We will prove the theorem by induction on n. For $n = 0$, we have $K = \mathbb{C}(x)$ which we know is a differential field, so it is closed under the derivation δ. Assume the claim is true for $K_0 = \mathbb{C}(x, f_1, \cdots, f_{n-1})$ with $n - 1 \geq 0$, that is, assume K_0 is closed under differentiation. Thus, we want to prove that $K_0(f_n) = \mathbb{C}(x, f_1, \cdots, f_{n-1}, f_n)$ is also closed under differentiation [2]. Let f_n be a logarithm of an element in K_0, an exponential of an element in K_0, or algebraic over K_0. First, we prove that $f_n' \in K_0(f_n)$ by cases.

Case 1 If f_n is a logarithm of $g \in K_0$, then $f_n' = g'/g$. Since $K_0 \subset K_0(f_n)$ and $g'/g \in K_0$ by hypothesis, then $g'/g \in K_0(f_n)$. Hence, it follows that $f_n' \in K_0(f_n)$.

Case 2 If f_n is an exponential of $g \in K_0$, then $g' = f_n'/f_n$ so $f_n' = f_n g'$. Since $g' \in K_0$ by hypothesis, then $f_n' \in K_0(f_n)$.

Case 3 If f_n is algebraic over K_0, then there exists a polynomial $P(T) \in K_0[T]$ such that $P(f_n) = 0$. Let $P(T) = a_m T^m + a_{m-1} T^{m-1} + \cdots + a_1 T + a_0$ where $a_i \in K_0$. Thus,

$$P(f_n) = a_m f_n^m + a_{m-1} f_n^{m-1} + \cdots + a_1 f_n + a_0 = 0.$$

Differentiating both sides using Definition 1, noting $0' = 0$, and applying Theorem 1 (ii) several times, we obtain

$$(a_m' f_n^m + a_{m-1}' f_n^{m-1} + \cdots + a_1' f_n + a_0') + f_n'[m a_m f_n^{m-1} + (m-1) a_{m-1} f_n^{m-2} + \cdots + a_1] = 0,$$

hence,

$$f_n' = -\frac{a_m' f_n^m + a_{m-1}' f_n^{m-1} + \cdots + a_1' f_n + a_0'}{m a_m f_n^{m-1} + (m-1) a_{m-1} f_n^{m-2} + \cdots + a_1}.$$

The reason we know the denominator above is not the zero is simple. We know that the leading coefficient of $P(T)$ is not zero because it has a root, i.e., $a_m \neq 0$. Hence, since K_0 has characteristic zero by Definition 1, the leading coefficient of $P'(T)$ is not zero, i.e., $m a_m \neq 0$ which means the denominator is not zero. Finally, clearly we note that $f_n \in K_0(f_n)$.

[2] In this sentence, the equal sign actually means 'isomorphic to' by Theorem 3. However, we treat those two fields as if they were equal for the sake of our proof.

Now that we know $f_n' \in K_0(f_n)$, we show that for any $f \in K_0(f_n)$, we have $f' \in K_0(f_n)$. Let $\sum_{k=0}^{s} c_k T^k, \sum_{k=0}^{t} d_k T^k \in K_0[T]$, hence we know that any element $f \in K_0(f_n)$ has the form $f = \frac{\sum_{k=0}^{s} c_k f_n^k}{\sum_{k=0}^{t} d_k f_n^k}$ with the denominator nonzero. Consider

$$\begin{aligned} f' &= \left(\frac{\sum_{k=0}^{s} c_k f_n^k}{\sum_{k=0}^{t} d_k f_n^k} \right)' \\ &= \frac{\left(\sum_{k=0}^{s} c_k f_n^k\right)' \sum_{k=0}^{t} d_k f_n^k - \left(\sum_{k=0}^{t} d_k f_n^k\right)' \sum_{k=0}^{s} c_k f_n^k}{\left(\sum_{k=0}^{t} d_k f_n^k\right)^2} \end{aligned}$$

by Theorem 2. Therefore,

$$f' = \frac{\left[c_0' + \sum_{k=1}^{s}(c_k' f_n^k + k c_k f_n^{k-1} f_n')\right] \sum_{k=0}^{t} d_k f_n^k - \left[d_0' + \sum_{k=1}^{t}(d_k' f_n^k + k d_k f_n^{k-1} f_n')\right] \sum_{k=0}^{s} c_k f_n^k}{\left(\sum_{k=0}^{t} d_k f_n^k\right)^2}$$

which shows $f' \in K_0(f_n)$ since $c_k, c_k', d_k, d_k', f_n^k, f_n' \in K_0(f_n)$ for all corresponding k. □

We have proved a big result in our theory since we are now able to create any differential field of meromorphic functions we want. Note that we have finally formalized what we meant by *familiar* function in the beginning of the paper because our familiar functions are functions that belong to some elementary field. Before we proceed to present our main theorem, we will introduce our last definition which formalizes what we meant by having a function that has a *nice* antiderivative.

Definition 6. *A meromorphic function f can be integrated in elementary terms if there exists g in some elementary field such that $f = g'$.*

Example 14. The meromorphic function $f(x) = xe^x$ can be integrated in elementary terms since $g(x) = xe^x - e^x$ is in the elementary field $\mathbb{C}(x, e^x)$ and $f = g'$. Note that in Definition 6, g can be in *any* elementary field and is not restricted to f.

Example 15. The function $f(x) = \frac{1}{1+x^2}$ can be integrated in elementary terms since $g(x) = \arctan x$ is in the elementary field $\mathbb{C}(x, \ln(\frac{1-ix}{1+ix}))$ by Example 5 and $f = g'$.

Because we are working with complex-valued functions, our theory of integration in elementary terms is stronger than if we worked with only real-valued functions. In fact, it turns out that all of our theory that we have constructed in this paper would be invalid if we only worked in $\mathbb{R}$ since we would not be able to build fields containing, for example, $\arctan x$ or $\sin x$ in terms of other elementary functions, such as logarithmic and exponential functions.

Liouville's Theorem

Here, we present our star theorem without a proof since it involves concepts from differential Galois theory, a subject in mathematics that is beyond the scope of this article.

Theorem 5. *(Liouville) Let f be an elementary function and let K be any elementary field containing f. Then f can be integrated in elementary terms within some elementary extension field of K if and only if there exist nonzero $c_1, \cdots, c_n \in \mathbb{C}$, nonzero $g_1, \cdots, g_n \in K$, and an element $h \in K$ such that*

$$f = \sum_{j=1}^{n} c_j \frac{g_j'}{g_j} + h'.$$

Proof. Omitted. □

Since the purpose of the article is to show that e^{-x^2} does not have an antiderivative, i.e. it does not integrate in elementary terms, then we will use the following corollary which serves us for a particular family of functions.

Theorem 6. *(Corollary to Liouville's theorem) Let $K = \mathbb{C}(x, e^{g(x)})$ be an elementary field such that $g(x) \in \mathbb{C}(x)$. Then, an element in K of the form $f(x)e^{g(x)}$, where $f(x), g(x) \in \mathbb{C}(x)$, can be integrated in elementary terms within some extension field of K if and only if there exists $r(x) \in \mathbb{C}(x)$ such that $r'(x) + g'(x)r(x) = f(x)$.*

Proof. ($\Rightarrow$) Suppose $f(x)e^{g(x)}$ can be integrated in elementary terms in some field extension of K. Then by Theorem 5, there exist nonzero $c_1, \cdots, c_n \in \mathbb{C}$, nonzero $g_1(x), \cdots, g_n(x) \in K$, and an element $h(x) \in K$ such that

$$f(x)e^{g(x)} = \sum_{j=1}^{n} c_j \frac{g_j'(x)}{g_j(x)} + h'(x).$$

Note that all $g_j(x)$ and $h(x)$ are of the form $\frac{P(e^{g(x)})}{Q(e^{g(x)})}$, where $P(T), Q(T) \in \mathbb{C}(x)[T]$ and $Q(T) \neq 0$. Hence, for each j we have

$$\frac{g_j'(x)}{g_j(x)} = \frac{[P(e^{g(x)})/Q(e^{g(x)})]'}{P(e^{g(x)})/Q(e^{g(x)})} = \frac{[P(e^{g(x)})]'}{P(e^{g(x)})} - \frac{[Q(e^{g(x)})]'}{Q(e^{g(x)})}$$

by Theorem 2 and dividing. Notice that the only polynomials in $\mathbb{C}(x)[e^{g(x)}]$ that make $[P(e^{g(x)})]'$ a multiple of $P(e^{g(x)})$ are monic. Similarly, $e^{g(x)}$ is the only possible monic irreducible factor in a denominator of the partial fraction decomposition expansion of $h(x)$. Thus, $\sum_{j=1}^{n} c_j \frac{g_j'(x)}{g_j(x)} + h'(x)$ can be expressed in the form $[\sum_{j=-t}^{t} k_j(x)(e^{g(x)})^j]'$ for $k_j(x) \in \mathbb{C}(x)$, which is its partial fraction decomposition. Finally, after differentiating the terms in the sum

$$f(x)e^{g(x)} = \sum_{j=-t}^{t} [(k_j'(x) + jg'(x)k_j(x))(e^{g(x)})^j].$$

By matching the coefficients of $e^{g(x)}$ in both sides, we obtain $f(x) = k_1'(x) + g'(x)k_1(x)$. Let $r(x) = k_1(x)$, so the result follows.

($\Leftarrow$) Suppose there exists such $r(x) \in \mathbb{C}(x)$, hence, $f(x)e^{g(x)} = (r(x)e^{g(x)})'$, so it can be integrated in elementary terms since $r(x)e^{g(x)} \in \mathbb{C}(x, e^{g(x)})$. □

The proof of Theorem 5 and the details of Theorem 6 that we did not justify may be found among the cited sources (Churchill, Conrad, and Rosenlicht) in great detail. We have obtained a strong result since we know that if e^{-x^2} can be integrated in elementary terms, then this antiderivative must be of the form $p(x)e^{-x^2}$ for some rational polynomial $p(x) \in \mathbb{C}(x)$ just like in Example 14, where the antiderivative of $xe^x \in \mathbb{C}(x, e^x)$ is $(x-1)e^x \in \mathbb{C}(x, e^x)$.

Example 16. The function e^{-x^2} cannot be integrated in elementary terms.

Proof. Suppose e^{-x^2} can be integrated in elementary terms. Then letting $f(x) = 1$ and $g(x) = -x^2$ there exists $r(x) \in \mathbb{C}(x)$ such that $r'(x) - 2xr(x) = 1$ by Theorem 6. Let $r(x) = p(x)/q(x)$ where $p(x)$ and $q(x)$ are relatively prime elements of $\mathbb{C}[x]$ with $q(x) \neq 0$. Hence, by Theorem 2 we obtain

$$\frac{p'(x)q(x) - p(x)q'(x)}{q^2(x)} - 2x\frac{p(x)}{q(x)} = 1,$$

so

$$p'(x)q(x) - p(x)q'(x) - 2xp(x)q(x) = q^2(x),$$

and after reordering we get

$$q(x)[p'(x) - 2xp(x) - q(x)] = p(x)q'(x).$$

Since $q(x)$ does not divide $p(x)$, it follows that $q(x)$ divides $q'(x)$, so $q(x)$ is a nonzero constant $c \in \mathbb{C}$. Hence, we have

$$p'(x) - 2xp(x) = c.$$

The degree of the left-hand side is at least 1 or it is the zero polynomial, whereas the right-hand side is a nonzero constant. We have arrived at a contradiction, therefore, e^{-x^2} cannot be integrated in elementary terms. □

Note that if we attempted to find $r(x)$ in $r'(x) - 2xr(x) = 1$ by using our tools for solving differential equations, we would obtain $r(x) = \int e^{-x^2}\,dx$, which is not helpful. This is why we need to develop the theory in this paper to demonstrate such antiderivative does not exist.

Exercises

1. Give an elementary field of the form $\mathbb{C}(x, f_1, \cdots, f_n)$ which contains the following functions.

   (a) x^π.

   (b) $\frac{\sqrt{2\pi}x^2 - 3x\ln x}{\sqrt{e^x - \sin(x/(x^3-7))}}$.

   Also, indicate the tower of fields in which the elementary field you have given is the biggest extension (Hint: See Example 12).

2. Prove that for any $p(x) \in \mathbb{C}(x)$, $p(x)$ divides $p'(x)$ if and only if $p(x) \in \mathbb{C}$. We used this fact in the proof of Example 16.

3. Prove that the function $\int \frac{dx}{\ln x}$, which is an important analysis tool in number theory, cannot be integrated in elementary terms (Hint: make a substitution and work as in Example 16).

4. Even though some functions do not possess an antiderivative, we can compute particular definite integrals of these functions. Find the given definite integrals of the following functions which lack an antiderivative.

   (a) $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ (Hint: Let $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$ and consider $I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$, then change to polar coordinates using multivariable calculus).

   (b) $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$ (Hint: Consider the function $F(x,y) = e^{-xy}\sin x$ and integrate it over the entire plane $\mathbb{R}^2$ in two different ways).

References

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3. Goetz P. Why Certain Integrals Are "Impossible." Web.

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