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Self-similarity of Siegel disks and Hausdorff dimension of Julia sets

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Abstract

Let $f(z) = e^{2\pi i\theta}z + z^2$, where θ is an irrational number of bounded type. According to Siegel, f is linearizable on a disk containing the origin. In this paper we show:

- the Hausdorff dimension of the Julia set $J(f)$ is strictly less than two; and
- if θ is a quadratic irrational (such as the golden mean), then the Siegel disk for f is self-similar about the critical point.

In the latter case, we also show the rescaled first-return maps converge exponentially fast to a system of commuting branched coverings of the complex plane.

Contents

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1 Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be the quadratic polynomial

$$f(z) = e^{2\pi i\theta} z + z^2,$$

where θ has continued fraction expansion

$$\theta = [a_1, a_2, a_3, \cdots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

Assume θ is an irrational of *bounded type*, meaning $\sup a_i < \infty$. Then by a classical result of Siegel, the map f is conformally conjugate to a rotation near the origin [Sie]. The maximal domain D on which this conjugacy takes place is the *Siegel disk* for f; it is the component of the interior of the filled Julia set $K(f)$ containing the origin.

Let $c_0 = -e^{2\pi i\theta}/2$ denote the critical point of f, let $c_i = f^i(c_0)$, and let

$$P(f) \;=\; \overline{\{c_1, c_2, c_3, \ldots\}}$$

be the *postcritical set*. By work of Herman and Świątek, $\partial D = P(f)$ and ∂D is a quasicircle passing through the critical point.

The behavior of f is dominanted by the transition between linear and nonlinear dynamics that occurs at the boundary of the Siegel disk. In this paper we study that transition and prove:

Theorem 1.1 *The Julia set $J(f)$ has Hausdorff dimension strictly less than two.*

Theorem 1.2 *If θ is a quadratic irrational, then the boundary of the Siegel disk for f is self-similar about the critical point.*

Here is a more precise statement of the second Theorem. Suppose θ is a quadratic irrational; its continued fraction expansion is then preperiodic, satisfying $a_{n+s} = a_n$ for all $n \gg 0$. For simplicity assume the period s is even. Then we find the periodicity of $\langle a_n \rangle$ allows one to construct a symmetry of f, namely a homeomorphism $\psi : \mathbb{C} \to \mathbb{C}$ satisfying

$$\psi \circ f^{q_{ns}}(z) = f^{q_{n(s+1)}} \circ \psi(z)$$

for $n \gg 0$ and z near c_0, where $p_n/q_n = [a_1, \cdots, a_n]$. We show ψ stabilizes the Siegel disk and fixes the critical point; its complex derivative $\psi'(c_0)$ exists; we have $0 < |\psi'(c_0)| < 1$; and ψ behaves like a $C^{1+\alpha}$ diffeomorphism near the critical point.

Since $\psi'(c_0)$ is contracting, all features of ∂D repeat at every scale, and the Siegel disk exhibits a self-similar shape (§7). For example, the successive closest returns of the critical orbit $\langle c_{q_n} \rangle$ shadow geometric series converging to c_0.

The dynamical system generated by f is also self-similar: when suitably rescaled, the maps $\langle f^{q_{ns+i}}(z) \rangle$ converge exponentially fast to a self-similar family $\langle g_i(z) \rangle$ of commuting holomorphic branched coverings of the plane (§8). Conceptually this means f is attracted to a fixed point of renormalization.

For θ of bounded type, we also prove a form of *universality* holds (§5):

Theorem 1.3 *Let $g : U \to V$ be a quadratic-like map with a fixed point of multiplier $e^{2\pi i\theta}$. Then there is a conjugacy between f and g which is $C^{1+\alpha}$ and conformal on the boundary of the Siegel disk.*

This result shows the fine-scale behavior of $f|\partial D$ depends only on the combinatorics of f, not on its explicit analytical form. For example, the Hausdorff dimension of the boundary of the Siegel disk for $g_a(z) = e^{2\pi i\theta}z + z^2 + az^3$ is *constant* for small values of a.

In the course of the discussion, we also find (§4):

1. The Julia set of f is *shallow* (or porous): there is an $\epsilon > 0$ such that any r-ball in $\mathbb{C}$ contains an ϵr-ball disjoint from $J(f)$. On the other hand:

2. The critical point is a Lebesgue density point of the filled Julia set; in fact,

$$\text{area}(B(c_0, r) - K(f)) = O(r^{2+\alpha})$$

   for some $\alpha > 0$.

3. The dynamical system $(\mathcal{F}(f), J(f))$ is *uniformly twisting*: at every scale near every point in the Julia set, one can find a degree two branched covering $h : U \to V$ with bounded geometry, contained in the full dynamics $\langle f^{-i} \circ f^j \rangle$.

The golden mean. All these assertions apply to the *golden mean* Siegel disk, where $\theta = (\sqrt{5}-1)/2 = [1, 1, 1, \cdots]$; see Figure 1. The critical point is in the center of the picture, which depicts the filled Julia set in gray; the quasicircle bounding the Siegel disk is outlined in black at the lower left.

We also show the boundary of the golden mean Siegel disk does not spiral about the critical point (Corollary 7.5). This means any continuous branch of $\arg(z - c_0)$ defined along $\partial D - \{c_0\}$ is bounded. The same result holds whenever the continued fraction expansion of θ has odd period.

Empirically, the golden mean Siegel disk has the stronger property that D contains a triangle with one vertex at c_0.

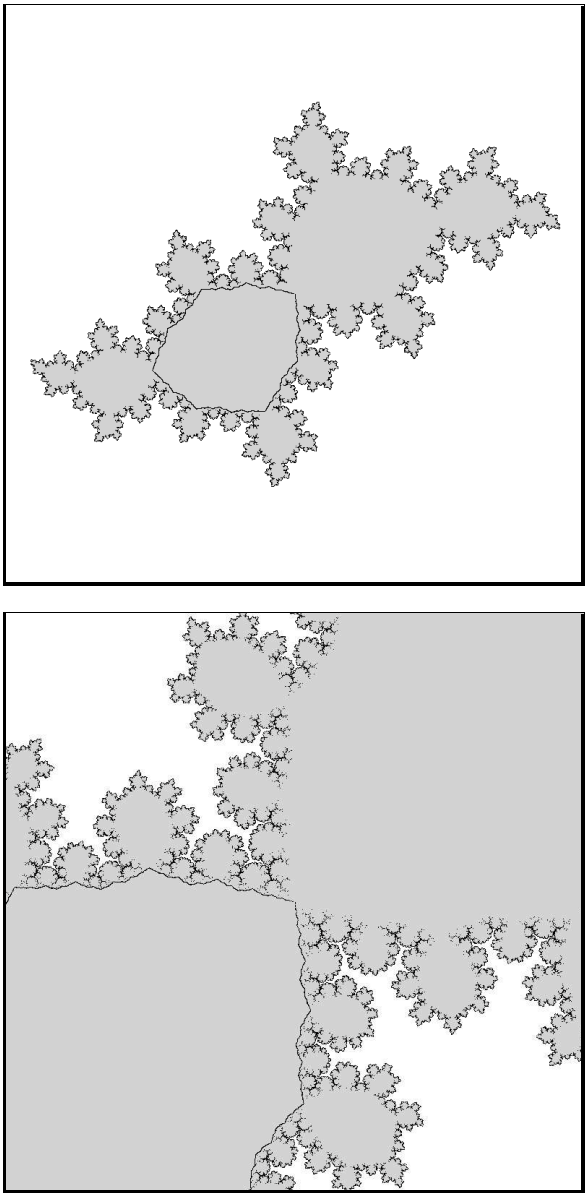

Figure 1. The golden mean Siegel disk and blowup around the critical point.

Sources of rigidity. Convergence of the renormalized dynamics for the golden mean Siegel disk and its cousins may at first sight appear unrelated to considerations such as the dimension of the Julia set. In fact a principal goal of this paper is to give an explanation for convergence of renormalization directly in terms of the geometry of the filled Julia set $K(f)$ and its boundary $J(f)$. Here is a sketch of the argument.

Since ∂D is a quasicircle, there is definite space inside $K(f)$ near every $z \in \partial D$. The dynamics propagates this space throughout the Julia set, and we conclude that $\dim J(f) < 2$ (Theorem 1.1).

The picture is enhanced by the observation that the critical point (indeed every $z \in \partial D$) is a Lebesgue density point of $K(f)$. To see this, note that points just outside the Siegel disk must visit the critical point many times before they can escape to infinity. On each visit, there is a definite chance of landing in the pre-image D' of D. Thus a random point close to ∂D has a high probability of eventually landing in D', so the density of $K(f)$ tends to one. (The study of f near ∂D is facilitated by working in the *linear coordinate system*, a quasiconformal chart in which D is the unit disk, $f|D$ is a rigid rotation, and the iterates of f are uniformly quasiregular.)

Next we discuss universality. Consider a quadratic-like map $g(z)$ with a fixed point of multiplier $e^{2\pi i\theta}$. By a basic result of Douady and Hubbard, $g = \phi \circ f \circ \phi^{-1}$, where ϕ is *conformal* on $K(f)$. Measure-theoretically, the filled Julia set is sufficiently dense near $z \in \partial D$ that the conformal behavior of ϕ dominates; we conclude $\phi'(z)$ exists and $\phi|\partial D$ is $C^{1+\alpha}$ for some $\alpha > 0$.

A similar argument provides smoothness of the conjugating map ψ between two renormalizations of f. Thus ∂D is self-similar when θ is a quadratic irrational (Theorem 1.2). To construct these renormalizations, which are variants of de Faria's commuting pairs [dF], we use Petersen's result on local connectivity of $J(f)$ [Pet].

Finally we describe the limiting dynamics. By smoothness of ψ and a harmonic measure argument, one finds for $i \in \mathbb{Z}$, the rescaled first return maps $f^{q_{ns+i}}$ converge exponentially fast as $n \to \infty$. Their limit $g_i : W_i \to \mathbb{C}$ is an infinite-sheeted branched covering of the plane, with ∂W_i as its natural boundary; moreover, g_i can be expressed as a countable union of proper maps. We have $g_{i+s} = A \circ g_i \circ A^{-1}$, where A is the linearization of ψ; thus the limiting dynamical system is a fixed point of renormalization. By universality, the same limit arises starting with any quadratic-like map with a fixed point of multiplier $e^{2\pi i\theta}$.

Notes and references. Various forms of the universality and self-similarity results we establish here were conjectured and observed numerically more than a decade ago in the physics literature [MN], [Wid]. Petersen proved that the Lebesgue area of $J(f)$ is zero and $J(f)$ is locally connected when θ has bounded type [Pet]. On the other hand, if $\theta = [a_1, a_2, \cdots]$ and $a_n \to \infty$ sufficiently fast, then there is no Siegel disk, $J(f)$ is not locally connected and Shishikura shows the Hausdorff dimension of $J(f)$ is two [Shi]. The regime of unbounded type still presents many mysteries.

This paper is a sequel to [Mc4], which develops the theme of rigidity and inflexibility in conformal dynamics and hyperbolic geometry. Most of our results for the golden mean Siegel disk have parallels for the Feigenbaum polynomial $f(z) = z^2 + c_F$ (the unique real quadratic polynomial such that $f^n(z)$ is renormalizable for $n = 1, 2, 4, 8, \ldots$). These parallel results are established in [Mc4] and are summarized in Table 2.

Parallels	
Infinitely renormalizable map $f(z) = z^2 + c, \quad c \in \mathbb{R}$	Siegel linearizable map $f(z) = e^{2\pi i\theta} z + z^2, \quad \theta \in \mathbb{R}/\mathbb{Z}$
Tuning invariant	Continued fraction
$c = s_1 * s_2 * s_3 \cdots$	$\theta = [a_1, a_2, \cdots]$
Bounded combinatorics	Bounded type
$P(f) =$ quasi-Cantor set	$P(f) =$ quasi-circle
Quadratic-like map	Holomorphic pair
Feigenbaum polynomial	Golden mean polynomial
$(\mathbb{Z}_2, x + 1)$	$(S^1, x + \theta)$
f^n, $n = 1, 2, 4, 8, 16, \ldots$	f^n, $n = 1, 2, 3, 5, 8, 13, \ldots$
$(\mathcal{F}(f), J(f))$ is uniformly twisting	
The critical point of f is a deep point of $K(f)$	
Conjugacies are $C^{1+\alpha}$ on $P(f)$	

Table 2.

A *critical circle map* is a smooth homeomorphism $f : S^1 \to S^1$ with a single critical point. The self-similarity of Siegel disks is also related to convergence of renormalization for critical circle maps, studied in [dF] and [dFdM].

I would like to thank de Faria, de Melo and Petersen for useful conversations about their work, and Petersen and the referee for helpful suggestions and corrections.

2 Rotations of the circle

Let $S^1 = \mathbb{R}/\mathbb{Z}$, let $\theta \in (0, 1)$ be an irrational number with continued fraction expansion $\theta = [a_1, a_2, \cdots]$, and let $F : S^1 \to S^1$ be the rotation $F(x) = x + \theta$. In this section we will briefly summarize some relations between the dynamics of F and the continued fraction

expansion of θ. See [HW, Ch. X] for a detailed treatment of continued fractions.

For $x \in S^1$ let $\{x\}$ denote the unique real number representing x in $(-1/2, 1/2]$, and let $\|x\| = |\{x\}|$ denote the distance from x to 0.

Let

$$\frac{p_n}{q_n} = [a_1, \cdots, a_n]$$

denote the rational approximations to θ (in lowest terms) obtained by truncating its continued fraction expansion.

The points $q_n\theta$, $n = 1, 2, \ldots$ are successive closest returns of the origin to itself under rotation by θ. That is,

$$\theta > \|q_1\theta\| > \|q_2\theta\| > \|q_3\theta\| > \cdots$$

and the q_n's can be defined inductively as the smallest integers so the above inequalities hold.

The closest returns occur alternately to the right and left of the origin. In fact, $(-1)^n\{q_n\theta\} > 0$, and the fractional part is given by

$$\{q_n\theta\} = q_n\theta - p_n. \tag{2.1}$$

The a_n are related to q_n and p_n by the recursive formulas

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2}, \\ q_n &= a_n q_{n-1} + q_{n-2}, \end{aligned} \tag{2.2}$$

where $p_{-1}/q_{-1} = 1/0$ and $p_0/q_0 = 0/1$. Since $a_n \geq 1$, this recursion shows q_n grows at least as fast the Fibonacci sequence $1, 2, 3, 5, 8, \ldots$, and therefore $q_n \to \infty$ at least as fast as the nth power of the golden mean.

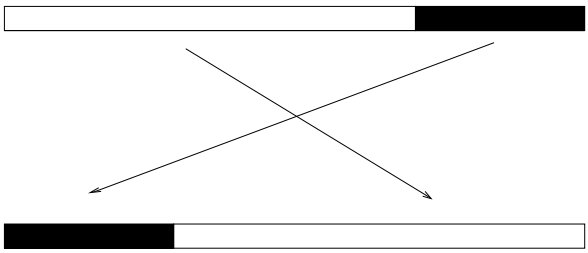

Figure 3. Renormalized rotation.

First return maps. For $x, x' \in S^1$ with $x \neq 1/2 + x'$, let $[x, x']$ denote the shorter interval bounded by these points. (Note that $[x, x'] = [x', x]$.)

Let $I_n = [q_n\theta, q_{n+1}\theta]$ be the interval bounded by two successive closest returns to the origin. Note that 0 lies in I_n. Let us consider the dynamics induced by the *first return map* on I_n; that is, for $x \in I_n$ we consider the least $i > 0$ such that $F^i(x) \in I_n$. The result is the *renormalized* dynamical system generated by the two transformations:

$$\begin{array}{lrcll} F^{q_n}: & [0, q_{n+1}\theta] & \to & [q_n\theta, x_n], & \text{and} \\ F^{q_{n+1}}: & [q_n\theta, 0] & \to & [x_n,\ q_{n+1}\theta] & \end{array} \tag{2.3}$$

where $x_n = (q_n + q_{n+1})\theta$. This pair of transformations is an interval exchange map; see Figure 3.

The orbit of every point $x \in S^1$ is dense, so x eventually enters I_n. How long does it take to do so? We claim

$$F^i(x) = x + i\theta \in I_n \quad \text{for some } i,\ 0 \le i < q_{n+1}.$$

To see this, first choose $j < 0$ such that $F^j(x) \in I_n$. The generators of the renormalized dynamics $\langle F^{q_n}, F^{q_{n+1}} \rangle$ map I_n onto itself, so we may apply them one at a time to replace j by $j + q_n$ or $j + q_{n+1}$ while keeping $F^j(x) \in I_n$. When j first becomes nonnegative it lies in the range $[0, q_{n+1})$ as claimed.

Now let J_n be the interval centered in I_n with length $|J_n| = |I_n| - \|q_{n+1}\theta\|$. In other words, J_n excludes the outer halves of the two short black intervals appearing in Figure 3.

We claim

$$F^i(x) = x + i\theta \in J_n \quad \text{for some } i,\ 0 \le i < q_n + q_{n+1}.$$

To find $F^i(x)$ with this property, begin by choosing j with $0 < j < q_{n+1}$ and $F^j(x) \in I_n$. If $F^j(x) \in J_n$ we are done. Otherwise $F^j(x)$ lies near one of the endpoints of I_n. If $F^j(x)$ is closer to $q_{n+1}\theta$, then $F^{j+q_n}(x) \in J_n$, and since

$$j + q_n < q_{n+1} + q_n$$

we are done. Finally if $F^j(x)$ is closer to $q_n\theta$, then both $x + (j + q_{n+1})\theta$ and $x + (j - q_n)\theta$ lie in J_n; the first of these works if $j < q_n$, and the second if $j \ge q_n$.

Univalent branches. We will eventually apply these results to the quadratic polynomial f, using the fact that (F, S^1) is topologically conjugate to f on the boundary of the Siegel disk. In this application $0 \in S^1$ will correspond to the critical point of f, and we will be interested in constructing a univalent restriction of f^i sending x into I_n. Now the critical values of f^i correspond to $\theta, 2\theta, \ldots i\theta$; since $i < q_n + q_{n+1}$, there are no critical values in the interior of the interval I_n, and therefore $f^{-i}|\operatorname{int}(I_n)$ has univalent branches. The fact that $f^i(x)$ lands in the strictly smaller interval J_n provides Koebe space for the inverse univalent map.

Bounded type. The orbits of a bounded type rotation are especially evenly distributed. The following are equivalent characterizations of an irrational θ of bounded type:

1. The partial quotients a_n of θ are bounded.

2. The number θ is Diophantine of exponent two; i.e. there is a $C > 0$ such that

$$\left|\theta - \frac{p}{q}\right| > \frac{C}{q^2}$$

for every rational p/q.

3. The denominators q_n increase no faster than a geometric series; that is, $q_n < C^n$ for some $C > 1$.

4. Adjacent closest returns of $F^i(0)$ are approximately evenly spaced; that is,

$$\|q_n\theta\| > \|q_{n+1}\theta\| > C\|q_n\theta\|$$

for some $C > 0$.

5. For any $k > 0$, the orbit $F^i(0)$, $i = 1, \ldots, k$ cuts the circle into intervals of approximately the same size; i.e. the ratio of the longest to the shortest is bounded above independent of k.

6. We have $|J_n|/|I_n| > C > 0$ for all n.

The constants appearing above can be chosen to depend only on $\sup a_n$.

Preperiodic continued fractions. Finally we describe the self-similar behavior of rotations where the continued fraction of θ is preperiodic.

Theorem 2.1 *Let $\theta = [a_1, a_2, \cdots]$, where $a_{n+s} = a_n$ for $n \geq N$. Then there is a $\gamma \in (0, 1)$ such that*

$$\{q_{n+s}\theta\} \;=\; (-1)^s\gamma\,\{q_n\theta\}$$

for $n \geq N$.

Proof. For any continued fraction, one has

$$A_n = \begin{pmatrix} a_n & -1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & -1 \\ -1 & 0 \end{pmatrix} \;=\; \begin{pmatrix} q_n & -p_n \\ -q_{n-1} & p_{n-1} \end{pmatrix}$$

by (2.2). In addition, it is easy to prove by induction that

$$A_n \begin{pmatrix} \theta \\ 1 \end{pmatrix} \;=\; (-1)^n\theta_1\theta_2\cdots\theta_n \begin{pmatrix} \theta_{n+1} \\ 1 \end{pmatrix},$$

where $\theta_i = [a_i, a_{i+1}, a_{i+2}, \cdots]$.

Now suppose the continued fraction for θ is periodic with period s. Then

$$A_{n+s}\begin{pmatrix}\theta\\1\end{pmatrix} = A_n A_s \begin{pmatrix}\theta\\1\end{pmatrix} = (-1)^s \gamma A_n \begin{pmatrix}\theta\\1\end{pmatrix}, \tag{2.4}$$

where $\gamma = \theta_1\theta_2\cdots\theta_s$. Equivalently,

$$\begin{pmatrix} q_{n+s} & -p_{n+s} \\ -q_{n+s-1} & p_{n+s-1}\end{pmatrix}\begin{pmatrix}\theta\\1\end{pmatrix} \;=\; (-1)^s\gamma \begin{pmatrix} q_n & -p_n \\ -q_{n-1} & p_{n-1}\end{pmatrix}\begin{pmatrix}\theta\\1\end{pmatrix}.$$

By (2.1), $\{q_n\theta\} = q_n\theta - p_n$, so the Theorem follows by multiplying through above.

The preperiodic case is similar. ∎

Example. Let $\theta = [1,1,1,\cdots] = (\sqrt{5}-1)/2$ be the golden mean. Then $\gamma = \theta$, $\langle q_1, q_2, \ldots\rangle = \langle 1,2,3,5,8,13,\ldots\rangle$, and we have $\{q_n\theta\} = (-1)^n\theta^{n+1}$. In this example (or any other with period one) the closest returns occur along a geometric series.

To obtain a more complete dynamical picture, for $n \geq 3$ let $r_n = q_{n-1} - q_n$, $L_n = [r_{n+1}\theta, r_n\theta]$, and define the *extended* renormalization $\mathcal{R}_n(F)$ to be the pair of mappings

$$\begin{array}{lccl} F^{q_n}: & [0, r_n\theta] & \to & [q_n\theta, q_{n-1}\theta] \subset L_n \quad \text{and} \\ F^{q_{n+1}}: & [r_{n+1}\theta, 0] & \to & [q_n\theta, q_{n+1}\theta] \subset L_n. \end{array} \tag{2.5}$$

Then $L_n \supset I_n$ and the maps above extend the interval exchange dynamics defined by (2.3).

The choice of the domains for F^{q_n} and $F^{q_{n+1}}$ is dictated by branching considerations, again under the topological conjugacy between F and the quadratic polynomial f acting on the boundary of its Siegel disk. If $x = 0$ corresponds to the critical point of f, then the critical values of f^i on the boundary of the Siegel disk correspond to $\{\theta, 2\theta, \ldots, i\theta\} \subset S^1$. Thus the interval $[q_n\theta, q_{n-1}\theta]$ comprising the range of F^{q_n} above is the maximal interval containing $x = 0$ and with no critical values in its interior. Similarly, the domain $[0, r_n\theta]$ is a maximal interval whose interior is disjoint from the critical points of F^{q_n}, and the same considerations dictate the domain and range of $F^{q_{n+1}}$.

Corollary 2.2 *The similarity $x \mapsto (-1)^s\gamma x$ sends L_n to L_{n+s} and conjugates $\mathcal{R}_n(F)$ to $\mathcal{R}_{n+s}(F)$, for all $n \geq \max(N, 3)$.*

Proof. This follows from the Theorem and the fact that $x \mapsto \alpha x$ conjugates rotation by t to rotation by αt. ∎

The functional equation. The renormalization theory has a nice description on the universal cover of the circle. Let $T \cong \mathbb{R}$ be the group of translations of the real line, let $V = \mathrm{Hom}(\mathbb{Z}^2, T) \cong \mathbb{R}^2$ and let $\rho_\theta \in V$ be given by $\rho_\theta(i,j) = i\theta + j$. Then ρ_θ represents the combined dynamics of the rotation $x \mapsto x + \theta$ and the deck transformation $x \mapsto x + 1$ acting on the universal cover of $S^1 = \mathbb{R}/\mathbb{Z}$.

Let $G \cong \mathbb{R}^*$ be the group of similarities $x \mapsto \alpha x$, and let $\overline{V} = V//G \cong \mathbb{RP}^1$ be the quotient of the nontrivial representations under conjugation by G. Then $GL_2(\mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}^2)$ acts on $\overline{V}$ by $[\rho] \mapsto [\rho \circ A^{-1}]$; this agrees with its usual action on $\mathbb{RP}^1$ by fractional linear transformations.

A representation $[\rho]$ is fixed by A if and only if there is a similarity $\alpha \in G$ such that

$$\rho \circ A^{-1} \;\; = \;\; \alpha\rho\alpha^{-1}.$$

This *functional equation* says the renormalized dynamical system $\rho \circ A^{-1}$ is just a rescaling of the original dynamical system ρ. It can be compared to the Cvitanović-Feigenbaum equation

$$F^p(z) = \alpha^{-1} F(\alpha z)$$

for unimodal maps, and the fixed point equation

$$\rho \circ \psi_*^{-1} = \alpha\rho\alpha^{-1}$$

for a hyperbolic 3-manifold which fibers over the circle, discussed in [Mc4].

Theorem 2.3 *Let θ be irrational. Then:*

$[\rho_\theta]$ is fixed by an automorphism $A \neq \mathrm{id}$ in $GL_2(\mathbb{Z})$ $\iff$
θ is a quadratic irrational $\iff$
the continued fraction expansion of θ is preperiodic.

Proof. If $A[\rho_\theta] = [\rho_\theta]$, then $(\theta, 1)$ is an eigenvector for A, so θ is quadratic over $\mathbb{Q}$. A quadratic irrational has a preperiodic continued fraction [HW, §10.12], and by (2.4) a preperiodic continued fraction expansion gives a representation fixed by an automorphism. ∎

3 Visiting the critical point

We now turn to the study of the polynomial $f(z) = e^{2\pi i\theta} z + z^2$, where $\theta = [a_1, a_2, a_3, \cdots]$ is an irrational of bounded type.

The usual notations $A = O(B)$ and $A \asymp B$ mean $A < CB$ and $B/C < A < CB$, for an implicit constant C. In this section, the implicit constants depend only on θ. (In fact, a detailed examination of the proofs shows the bounds only depend on $\sup a_n$.)

To study f we will use the Koebe distortion theorem, the Poincaré metric and quasiconformal maps. See [Ah1], [Ah2], [LV] and [Mc3, Ch. 2] for background on these methods.

Metrics. We will use $d(x, y)$ to denote the Euclidean metric in the plane, and $d_U(x, y)$ for the hyperbolic metric on a region $U \subset \widehat{\mathbb{C}}$.

The Siegel disk. The filled Julia set $K(f)$ is defined by

$$K(f) = \{z \in \mathbb{C} \,:\, \text{the forward orbit } f(z),\, f^2(z),\, f^3(z), \ldots \text{ is bounded}\};$$

it is bounded by the Julia set $J(f)$. The Siegel disk D of f is the component of the interior of $K(f)$ containing the origin. The critical point c_0 lies in ∂D and $\partial D = P(f)$.

The central result which brings quasiconformal methods into play is:

Theorem 3.1 (Herman-Świątek) *The Siegel disk of f is bounded by a quasicircle.*

See [Dou], [Her], [Sw].

For a pointed disk $(U, u) \subset \mathbb{C}$, let

$$\begin{aligned} \text{in-radius}(U, u) &= \sup\{r \,:\, B(u, r) \subset U\}, \qquad \text{and} \\ \text{out-radius}(U, u) &= \inf\{r \,:\, B(u, r) \supset U\}. \end{aligned}$$

The main result we will establish in this section is:

Theorem 3.2 (Nearby critical visits) *For every $z \in J(f)$ and $r > 0$, there is a univalent map between pointed disks of the form*

$$f^i : (U, y) \to (V, c_0) \qquad\qquad (i \geq 0)$$

such that in-radius$(U, y) \asymp r$, *and* $|y - z| = O(r)$.

Thus the geometry of $J(f)$ near the critical point is replicated with bounded distortion everywhere in the Julia set.

The linear coordinate. By the Herman-Świątek Theorem, the Riemann mapping from $(D, 0)$ to the unit disk $(\Delta, 0)$ extends to a quasiconformal mapping $\Phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$. Let us normalize this mapping so that $\Phi(c_0) = 1$. It is useful to work in both the coordinate system z where f is conformal, and the coordinate system $w = \Phi(z)$, where $f|D$ is linear. Since Φ is quasiconformal, many properties can be passed from one coordinate system to the other with a bounded change in constants.

For clarity we will denote the domain and range of Φ by $\mathbb{C}_z$ and $\mathbb{C}_w$ respectively. We refer to w as the *linear coordinate*. In the w coordinate system, D is the unit disk, and $f : \mathbb{C}_w \to \mathbb{C}_w$ is a map whose iterates f^i are uniformly quasiregular.

We identify the circle $S^1 = \mathbb{R}/\mathbb{Z}$ with the unit circle in $\mathbb{C}_w$, via the map $t \mapsto \exp(2\pi i t)$. For two points $x, x' \in S^1$ we continue to use the notation $[x, x'] \subset S^1$ to denote the shorter arc containing them.

Define $F : \mathbb{C}_w \to \mathbb{C}_w$ by $F(w) = e^{2\pi i\theta} w$. Clearly $F^i = f^i$ on D. We will see that F^i and f^i tend to be close near ∂D as well. To prove this, it is convenient to introduce some hyperbolic geometry.

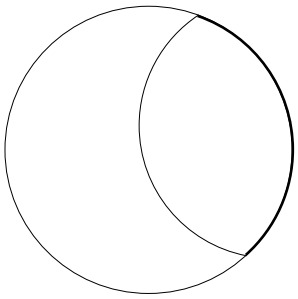

Figure 4. The region $H_\alpha(I)$.

Let $\mathbb{H}$ denote the hyperbolic plane, bounded by the circle S^1_∞. For any closed interval $I \subset S^1_\infty$, let $H_\alpha(I) \subset \mathbb{H}$ be the region bounded by I and a circular arc with endpoints ∂I, meeting S^1_∞ in angle α (see Figure 4.) For example, $H_{\pi/2}(I)$ is the half-space formed by the convex hull of I.

Lemma 3.3 *Let $g : H_\alpha(I) \to \mathbb{H}$ be a K-quasiconformal map which extends continuously to the identity on I. Then for all $x \in H_\beta(I)$, $\beta < \alpha$, we have*

$$d_{\mathbb{H}}(g(x), x) < C$$

where C depends only on K, β and α.

Proof. The image of ∂H_β in $\mathbb{H}$ is a quasiarc, converging radially to S^1_∞ because $\beta < \alpha$. This implies $g|H_\beta$ can be extended to a K'-quasiconformal map $h : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ with $h|S^1_\infty = \mathrm{id}$. Such a mapping moves points only a bounded hyperbolic distance. ∎

Approximate rotations. Let

$$\Omega = \mathbb{C}_z - \overline{D}$$

denote the exterior of the Siegel disk in the plane. Let us say $f^i : U \to V$ is an *approximate rotation* if U and V are disks in $\mathbb{C}_z$, f^i is univalent and

$$d_\Omega(f^i(x), F^i(x)) \;\; = \;\; O(1)$$

for all $x \in \Omega \cap U$. Of course $f^i(x) = F^i(x)$ for $x \notin \Omega$, so this condition says f^i nearly matches up with the rotation F^i throughout U.

Theorem 3.4 (Approximate rotations) *For any $r > 0$ and $z \in P(f)$, there is an approximate rotation*

$$f^i : (U, y) \to (V, c_0) \quad (i \geq 0)$$

such that $[y, z] \subset U$, $\mathrm{mod}(U - [y, z]) > C > 0$ and $\text{in-radius}(U, z) \asymp r$.

Remarks.

1. The proof will also show $f^j|U$ is an approximate rotation for $j = 1, 2, \ldots, i$.

2. The notation $\mathrm{mod}(U - [y, z])$ denotes the conformal modulus of the annulus between the arc $[y, z] \subset P(f)$ and ∂U. The lower bound C for this modulus depends only on θ. It follows that the hyperbolic distance $d_U(y, z) = O(1)$.

3. One also has $\text{in-radius}(U, y) \asymp r$.

4. The Theorem easily implies a slightly stronger statement, namely that U and V can be chosen as K-quasidisks, $K = O(1)$, with $\text{out-radius}(U, z) \asymp r$. To see this, replace U by a large ball about y in the hyperbolic metric on U, and apply the Koebe distortion theorem.

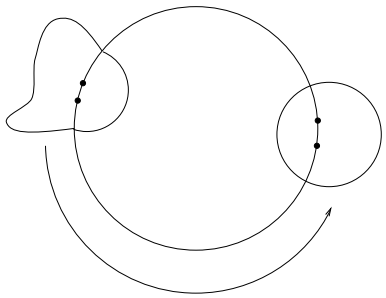

Figure 5. An approximate rotation in the linear coordinate w.

Proof. By general properties of quasiconformal mappings applied to the change of coordinates $\Phi : \mathbb{C}_z \to \mathbb{C}_w$, it is sufficient to prove the Theorem in the linear coordinate w. For

example, a quasiconformal map $\mathbb{H} \to \mathbb{H}$ is uniformly continuous in the hyperbolic metric, so it suffices to verify the approximate isometry condition $d_\Omega(f^i(x), F^i(x)) = O(1)$ using the hyperbolic metric coming from the w-coordinate. Similarly, if there are approximate isometries at every scale r in the w-plane, then the same is true in the z-plane.

Letting $w = \Phi(z)$, we now proceed to prove the Theorem at every scale r about w. In this coordinate system, $f|S^1 = F|S^1$ is rotation by angle θ. Thus the results of §2 apply to the dynamics of f on S^1.

The case $r \geq 1$ can be handled trivially by taking $y = c_0$, $U = V = B(w, 10r)$ and $i = 0$ (so $f^i = F^i = \mathrm{id}$.) So assume $0 < r < 1$.

Choose n such that $\|q_n\theta\| \asymp r$; this is possible because θ has bounded type. For convenience, also choose n large enough that $\|q_n\theta\| < 1/8$. Next choose the least $i \geq 0$ such that

$$f^i(w) \in J_n \subset I_n = [q_n\theta, q_{n+1}\theta] \subset S^1.$$

The critical point of f is at angle zero on S^1, so $c_0 \in J_n$.

As we saw in §2, f^i has no critical values in $\mathrm{int}(I_n)$ because $i < q_n + q_{n+1}$. Let V be the round disk orthogonal to S^1 and with $V \cap S^1 = \mathrm{int}(I_n)$. Then there is a unique univalent branch

$$f^{-i} : V \to U$$

such that $f^{-i}(f^i(w)) = w$. See Figure 5.

Now consider the region $H_\alpha(I_n) \subset \Omega$ determined by the hyperbolic geometry of $\Omega = \mathbb{C}_w - \overline{\Delta}$. (This region is constructed by lifting I_n to an interval on the boundary of the universal cover of Ω, forming $H_\alpha(\widetilde{I_n})$ there and projecting down. We choose α small enough that the projection is an embedding.) One can find universal angles $\pi/2 < \beta < \alpha < \pi$ such that

$$V \cap \Omega \subset H_\beta(I_n) \subset H_\alpha(I_n).$$

Indeed, when $|I_n|$ is small, we have $V \subset H_{\pi/2+\epsilon}(I_n)$.

Since the iterates of f are uniformly quasiregular in the w-coordinate, the map f^{-i} is K-quasiconformal on $H_\alpha(I_n)$ with $K = O(1)$. Furthermore, F^i is conformal and $F^i \circ f^{-i} = \mathrm{id}$ on I_n. Therefore Lemma 3.3 shows $d_\Omega(F^i(x), f^i(x)) = O(1)$ for all $x \in U \cap \Omega$. In other words, $f^i : U \to V$ is an approximate isometry. By the same reasoning, $f^j|U$ is an approximate isometry for $0 < j < i$.

Finally, setting $y = f^{-i}(c_0)$, we show the annulus $U - [w, y]$ has definite modulus, and in-radius$(U, w) \asymp r$. To prove these statements, recall that J_n is shorter than I_n by a definite factor (because θ has bounded type). Thus the annulus $V - J_n$ has definite modulus. But

$$\mathrm{mod}(U - [w, y]) \;\; \geq \;\; K^{-1}\,\mathrm{mod}(V - [f^i(w), c_0]) \;\; \geq \;\; K^{-1}\,\mathrm{mod}(V - J_n),$$

so $U - [w, y]$ has definite modulus as well. Similarly, in-radius$(V, f^i(w)) \asymp |I_n| \asymp r$. From the quasiconformality of f^{-i} and the fact that $f|S^1$ is an isometry, we conclude

that in-radius$(U, w) \asymp r$ as well. ∎

The hyperbolic geometry of Ω. By the *thick part* of Ω we will mean the region where the injectivity radius of the hyperbolic metric exceeds some small constant $\epsilon_0 > 0$. Since Ω is a punctured disk, its thin part is simply a standard horoball neighborhood of the cusp at $z = \infty$. We choose ϵ_0 small enough that $J(f)$ is contained in the thick part.

The hyperbolic metric ρ_Ω in the thick part is comparable to the $1/d$ metric (cf. [Mc3, Thm 2.3]); that is,

$$\rho_\Omega(z)|dz| \asymp \frac{|dz|}{d(z, \partial\Omega)}.$$

The pre-Siegel disk. Let ι be the involution of the z-plane given by $\iota(z) = 2c_0 - z$. This rotation about the critical point exchanges the two sheets of f; that is, $f(\iota(z)) = f(z)$.

Let $D' = \iota(D)$ be the *pre-Siegel disk*. Then $f(D') = D$ and $\overline{D} \cap \overline{D'} = \{c_0\}$.

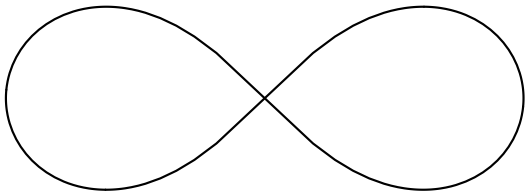

Figure 6. A lemniscate models $\partial(D \cup D')$.

The two quasidisks D and D' meet "radially" at c_0. More precisely, there is a quasiconformal homeomorphism of the plane sending $D \cup D'$ to the region bounded by the lemniscate $L = \{z : |z^2 - 1| = 1\}$ (see Figure 6). Indeed, the critical value of f lies in the quasicircle ∂D, so $f^{-1}(\partial D) = \partial(D \cup D')$ is quasiconformally homeomorphic to the square-root of a circle passing through the origin, which is L. Thus with respect to the hyperbolic metric on Ω, the disk D' is contained in a bounded neighborhood of a geodesic ray terminating at the critical point c_0.

With this background in mind we can now prove:

Proposition 3.5 *For any x in the thick part of Ω, there is an embedded hyperbolic ball B centered at y, and an $i \geq 0$, such that*

$$f^i : (B, y) \to (B', c_0)$$

is univalent, $\operatorname{diam}_\Omega B \asymp 1$ *and* $d_\Omega(x, y) = O(1)$.

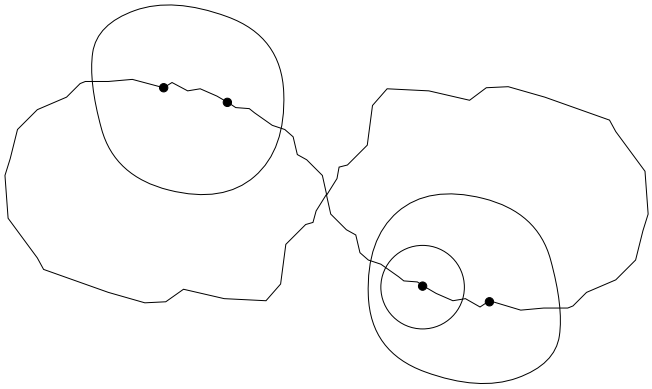

Figure 7. Univalent maps near points in the boundary of the pre-Siegel disk D'.

Proof. In brief, B is constructed as follows. Apply the dynamics to rotate x until it is close to the pre-Siegel disk; then find a nearby point in $\partial D'$. After one iterate, the nearby point maps to ∂D, and after a suitable further iterate it rotates into a ball B' containing the critical point. The pre-image of B' near x is the desired ball B.

For the detailed proof, we first suppose $x \in \partial D'$. Let $x' = \iota(x) \in \partial D = P(f)$. For any $r > 0$, Theorem 3.4 (and the remarks following) provide a univalent map $f^i : (U', y') \to (V, c_0)$ such that $d_{U'}(x', y') = O(1)$ and in-radius$(U', y') \asymp$ out-radius$(U', y') \asymp r$. With suitable r we obtain in-radius$(U', y') \asymp d(y', D')$ and out-radius$(U', y') < d(y', D')$. In particular, U' and D' are disjoint.

Now let $(U, y) = (\iota(U'), \iota(y'))$. Then (U, y) also maps to (V, c_0) under f^i, and $U \cap D = \emptyset$, so $U \subset \Omega$. Since ι is an isometry,

$$\text{in-radius}(U, y) \asymp d(y, D) = d(y, \partial\Omega).$$

But the hyperbolic metric on Ω is comparable to $|dz|/d(z, \partial\Omega)$, so there is a hyperbolic ball B of definite radius centered at y and contained in U (see Figure 7); by the Schwarz lemma,

$$d_\Omega(x, y) \le d_U(x, y) = d_{U'}(x', y') = O(1),$$

completing the proof of the Proposition for $x \in \partial D'$.

The general case of an arbitrary x in the thick part of Ω is conveniently visualized in the linear coordinate.

For $w \in S^1$, let $\gamma(w) = \{rw \,:\, 1 < r < \infty\} \subset \mathbb{C}_w$ denote the ray joining w to infinity. With respect to the hyperbolic metric on $\Omega = \{w \,:\, |w| > 1\}$, this ray is a geodesic. Since D' tends to c_0 radially, there is a C such that

$$D' \subset K = \{w \in \Omega \,:\, d_\Omega(w, \gamma(c_0)) < C\}.$$

By the first part of the proof, we may also choose C so the cone K is uniformly filled with hyperbolic balls (B, y) of definite radius mapping univalently over the critical point. See Figure 8.

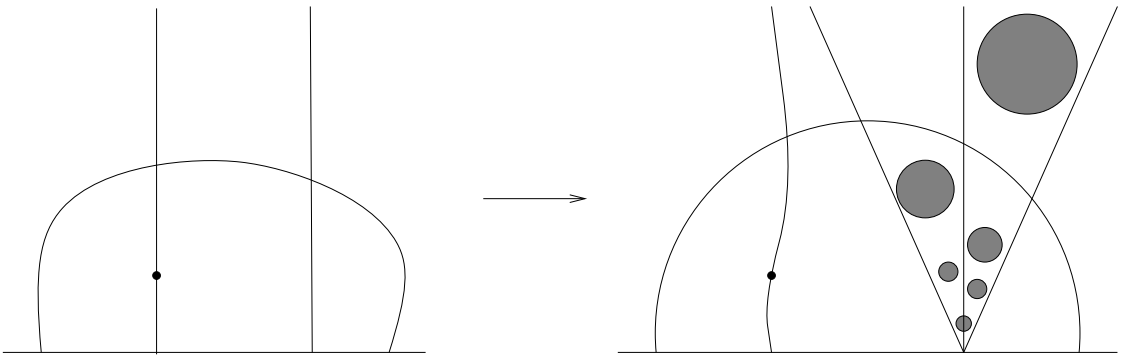

Figure 8. Pulling back from the cone at the critical point.

Given x in the thick part of Ω, choose $w \in S^1 = \partial D$ so $x \in \gamma(w)$. By Theorem 3.4 there is a $p \in S^1$ and an approximate rotation

$$f^i : (U, p) \to (V, c_0)$$

such that $2|x - w| <$ in-radius(U, w) and $|w - p| = O(|w - x|)$. In particular, $d_\Omega(x, \gamma(p)) = O(1)$. But $F^i(\gamma(p)) = \gamma(c_0)$, and since $f^i|U$ is an approximate isometry, we also have $d_\Omega(f^i(x), \gamma(c_0)) = O(1)$. Thus $f^i(x)$ is near the cone K, so there is a hyperbolic ball $(B_0, y_0) \subset K \cap V$ of definite radius with $d_\Omega(x, B_0) = O(1)$, such that $f^j : (B_0, y_0) \to (B_0', c_0)$ is univalent for some $j > 0$. Then $f^{-i}(B_0)$ contains a hyperbolic ball B of definite radius centered at $y = f^{-i}(y_0)$, we have $d_\Omega(x, B) = O(1)$, and

$$f^{i+j} : (B, y) \to (B', c_0)$$

is the desired univalent mapping. ∎

Expansion in the Julia set. To establish Theorem 3.2, we need to find disks mapping univalently over the critical point at every scale near every point in the Julia set. To populate small scales, we will use expansion relative to the hyperbolic metric on Ω.

Quite generally, for any polynomial map $f : \mathbb{C} \to \mathbb{C}$ with $|P(f)| > 1$, we have $\|f'(x)\| \geq 1$ with respect to the hyperbolic metric on $\mathbb{C} - P(f)$ (so long as $f(x) \notin P(f)$). Moreover, if $x \in J(f)$, then either the forward orbit of x lands in $P(f)$ or $\|f^i(x)\| \to \infty$. These

statements follow from the Schwarz lemma, the forward invariance of $P(f)$ and the fact that $J(f) \subset \overline{\bigcup f^{-i}(P(f))}$. See [Mc3, §3.2].

In the case at hand, $\mathbb{C} - P(f) = \Omega \cup D$. Of course $f|D$ is an isometry, but f expands the hyperbolic metric on Ω. Note too that $J(f) \subset P(f) \cup \Omega$, so the orbit of a point in the Julia set can exit Ω only by landing in the boundary of the Siegel disk.

For a vector v in the tangent space to a point $z \in \mathbb{C}$, we let $|v|$ and $\ell(v)$ denote the lengths of v in the Euclidean metric and in the hyperbolic metric on $\mathbb{C} - P(f)$ respectively. We set $\ell(v) = \infty$ if $z \in P(f)$.

We may now complete the:

Proof of Theorem 3.2 (Nearby critical visits). Let $z_0 = z \in J(f)$ and $r > 0$ be a given point in the Julia set and a given scale. Let $v_0 \in T_z\mathbb{C}$ be a vector with $|v_0| = r$, let $z_i = f^i(z_0)$ and let $v_i \in T_{z_i}\mathbb{C}$ be the image of v_0 under $(f^i)'$. Then

$$\ell(v_0) \leq \ell(v_1) \leq \ell(v_2) \leq \cdots \quad \text{and } \ell(v_i) \to \infty.$$

We wish to find a univalent map $f^i : (U, y) \to (V, c_0)$ with in-radius$(U, y) \asymp r$ and $|y - z| = O(r)$. To do so, we distinguish three cases.

I. $\ell(v_0) \geq 1$. (We include the case $\ell(v_0) = \infty$, which arises when $z_0 \in P(f)$.) In this case, there is a point $z' \in P(f)$ with $|z - z'| = O(r)$, since the hyperbolic metric on Ω at z is comparable to the $1/d$-metric. Applying Theorem 3.4, we obtain a univalent map $f^i : (U, y) \to (V, c_0)$ with in-radius$(U, y) \asymp r$ and $|y - z'| = O(r)$. Then $|z - y| = O(r)$ and we are done.

II. There exists a j such that $\ell(v_{j-1}) \leq \epsilon$ but $\ell(v_j) \geq 1$. Here $\epsilon > 0$ depends only on θ; its value is chosen below.

By case I there is a univalent map $f^i : (U', y') \to (V, c_0)$ with in-radius$(U', y') \asymp |v_j|$ and $|y' - z_j| = O(|v_j|)$. Replacing U' with a ball centered at y, we may also assume $\text{diam}(U') \asymp |v_j|$.

We claim (U', y') can be pulled back by f^j to yield the desired disk (U, y) for z_0. To see this, first note that the condition

$$\epsilon > \ell(v_{j-1}) \asymp \frac{|v_{j-1}|}{d(z_{j-1}, P(f))}$$

implies there is a ball $B_{j-1} = B(z_{j-1}, s)$, disjoint from the postcritical set $P(f)$, with $s \asymp |v_{j-1}|/\epsilon$. Then $f|B_{j-1}$ is univalent. By the Koebe 1/4-theorem, $B_j = f(B_{j-1})$ contains a ball of radius comparable to $|v_j|/\epsilon$ centered at z_j. Thus we may choose ϵ sufficiently small (depending only on θ) such that U' is contained well within B_j.

Let $f^{-j} : B_j \to B_0$ be the univalent inverse branch of f^j sending z_j to z_0. This map exists because B_{j-1} is disjoint from $P(f)$, and because $f : B_{j-1} \to B_j$ is univalent. Let $(U, y) = f^{-j}(U', y')$. By the Koebe distortion theorem, f^{-j} is approximately a similarity

with factor $|v_0|/|v_j|$ near z_j. Since U' is contained well within B_j, we have in-radius$(U,y) \asymp |v_0|$, $|z_0 - y| = O(|v_0|)$, and $f^{i+j} : (U,y) \to (V,c_0)$ is univalent. This completes the proof of case II.

III. There exists a j such that $\epsilon < \ell(v_j) < 1$. By Proposition 3.5, there is a centered hyperbolic ball $(B,y') \subset \Omega$ with $\mathrm{diam}_\Omega(B) \asymp 1$ and $d_\Omega(z_j,y') = O(1)$, such that $f^i : (B,y') \to (B',c_0)$ is univalent. Let γ be a minimal length geodesic segment joining z_j to B. Then $\gamma \cup B$ is simply-connected and disjoint from $P(f)$, so it supports a unique branch of f^{-j} sending z_j back to z_0.

Let $(U,y) = f^{-j}(B,y')$; then $f^{i+j} : (U,y) \to (V,c_0)$ is univalent. To complete the proof of this case, we just need to check in-radius$(U,y) \asymp |v_0|$ and $|z-y| = O(|v_0|)$. Equivalently, in the hyperbolic metric on Ω, it suffices to check that $d_\Omega(z,y) = O(\ell(v_0))$ and in-radius$_\Omega(U,y) \asymp \ell(v_0)$.

Now $(f^{-j})'(z_j)$ sends the vector v_j with $\ell(v_j) \asymp 1$ to the vector v_0. Since $\mathrm{diam}_\Omega(\gamma \cup B) = O(1)$, and the Julia set is contained in the thick part of Ω, the map f^{-j} is approximately a contraction by the constant factor $\ell(v_0)$ on $\gamma \cup B$ (cf. [Mc3, Thm. 3.8]). But the bounds provided for (B,y) by Proposition 3.5 then imply

$$\begin{aligned} \text{in-radius}_\Omega(U,y) &\asymp \ell(v_0) \cdot \text{in-radius}_\Omega(B,y') \asymp \ell(v_0) \quad \text{and} \\ d_\Omega(z,y) &\asymp \ell(v_0) \cdot d_\Omega(z_j,y') = O(\ell(v_0)) \end{aligned}$$

as desired.

End of the proof. Since $\ell(v_j) \to \infty$, one of the three cases above must hold, and the proof of the Theorem is complete. ∎

The argument above follows the proof that there are "small Julia sets everywhere" for certain infinitely renormalizable quadratic polynomials, given in [Mc4, §8.3].

4 Geometry of the Julia set

In this section we prove the dimension of the Julia set is less than two, the critical point is a deep point of $K(f)$, and the full dynamics of f is uniformly twisting on $J(f)$. A more complete development of the ideas of shallow and deep points, geometric limits of dynamical systems and uniform twisting appears in [Mc4].

Dimension. The *Hausdorff dimension* of a compact metric space X, denoted H. dim(X), is the infimum of those $\delta > 0$ such that X can be covered by balls $B(x_i,r_i)$ with $\sum r_i^\delta$ arbitrarily small. The (upper) *box dimension* is given by

$$\text{box-dim}(X) = \limsup_{r \to 0} \frac{\log N(X,r)}{\log(1/r)},$$

where $N(X,r)$ is the minimum number of r-balls required to cover X. Let us identify the Riemann sphere $\widehat{\mathbb{C}}$ with the sphere at infinity S^2_∞ for hyperbolic space $\mathbb{H}^3$. A closed set $\Lambda \subset S^2_\infty$ is *shallow* if every point in its hyperbolic convex hull $K \subset \mathbb{H}^3$ is within a uniformly bounded distance of ∂K. (Such sets are also called *porous*.)

A compact set $\Lambda \subset \mathbb{C}$ is shallow if and only if for any $z \in \Lambda$ and $r < 1$, there is a ball B disjoint from Λ with $\operatorname{diam} B \asymp r$ and $d(z, B) = O(r)$. For example, any quasicircle is shallow. The inequality $\mathrm{H.dim}(X) \leq \text{box-dim}(X)$ holds for any metric space, and it is easy to see that $\text{box-dim}(\Lambda) < 2$ if $\Lambda \subset S^2_\infty$ is shallow.

Theorem 4.1 (Dimension of J) *The Julia set $J(f)$ is shallow, and thus*

$$\mathrm{H.dim}(J(f)) \leq \text{box-dim}(J(f)) < 2.$$

Proof. Since the Siegel disk D is a quasidisk, it fills definite space at every scale around the critical point $c_0 \in \partial D$. More precisely, for any $s < 1$ there is a ball $B \subset B(c_0, s) \cap D$ with $\operatorname{diam} B \asymp s$.

Now consider a point $z \in J(f)$ and a scale $r < 1$. By Theorem 3.2, there is a univalent map

$$f^i : (U, y) \to (V, c_0)$$

with $|y - z| = O(r)$ and $\text{in-radius}(U, y) \asymp r$. Let $s = \text{in-radius}(c_0, V)$. Choose a ball $B \subset B(c_0, s) \cap D$ with $\operatorname{diam} B \asymp s$. By the Koebe distortion theorem, $f^{-i}(B)$ contains a ball B' with $\operatorname{diam} B' \asymp r$ and $d(y, B') = O(r)$. Since $f^i(B') \subset D$, this ball is disjoint from the Julia set, and since $d(z, B') \leq d(z, y) + d(y, B') = O(r)$, we have shown the Julia set is shallow.

As remarked above, the dimension bounds follow. ∎

Deep points. Let $\Lambda \subset S^2_\infty$ be a compact set with convex hull $K \subset \mathbb{H}^3$. A point $x \in \Lambda \subset S^2_\infty$ is a *deep point* of Λ if there is a geodesic ray $\gamma : [0, \infty) \to K$, converging to x and parameterized by hyperbolic arclength, such that

$$d(\gamma(s), \partial K) \geq \epsilon s > 0$$

for all $s > 0$.

One can easily check that $x \in \Lambda \subset \mathbb{C}$ is deep if and only if there is a $\delta > 0$ such that for all $r < 1$,

$$B(y, s) \subset B(x, r) - \Lambda \implies s = O(r^{1+\delta}).$$

For example, a shallow set has no deep points.

By the classification of stable regions, every point in the interior of $K(f)$ lands in the Siegel disk D under forward iteration. Thus D acts like an attractor. Let

$$K_\epsilon(f) = \{z \in \mathbb{C} \,:\, d(f^n(z), D) \leq \epsilon \text{ for all } n \geq 0\}.$$

Clearly $K_\epsilon(f) \subset K(f)$; it consists of points whose orbits stay near the Siegel disk.

Theorem 4.2 (Deep points) *The critical point c_0 is a deep point of $K(f)$. More generally, z is a deep point of $K_\epsilon(f)$ for any $z \in P(f)$ and any $\epsilon > 0$.*

A point $x \in \Lambda \subset \mathbb{C}$ is a *measurable* deep point if for some $\delta > 0$ we have

$$\text{area}(B(x,r) - \Lambda) \;\; = \;\; O(r^{2+\delta})$$

for all $r > 0$. This condition means the density of Λ in $B(x,r)$ tends to 1 exponentially fast (it is bounded below by $1 - Cr^\delta$).

The next result also appears in [Mc4, Prop. 2.24].

Proposition 4.3 *If x is a deep point of Λ and $\partial\Lambda$ is shallow, then x is a measurable deep point of Λ.*

Proof. Consider a square S of size r centered at x. Since x is a deep point, there is an $\alpha > 0$ such that every point of $X = (S - \Lambda)$ lies within distance $r^{1+\alpha}$ of $\partial\Lambda$. On the other hand, since $\partial\Lambda$ is shallow, there is a $\delta < 2$ such that if we cut S into subsquares of size $\asymp r^{1+\alpha}$, whose total number is $O(r^{-2\alpha})$, it only takes $O(r^{-\delta\alpha})$ of them to cover $X \cap S$. Thus the area of $X \cap S$ is $O(r^{2+(2-\delta)\alpha})$, and x is a measurable deep point. ∎

Since $J(f) = \partial K(f)$ is shallow, we may deduce:

Corollary 4.4 *A random z near the critical point has a high probability of lying in $K(f)$. More precisely, there is a $\delta > 0$ such that for small r, we have*

$$\text{area}(B(c_0, r) - K(f)) = O(r^{2+\delta}).$$

More generally one sees:

Corollary 4.5 *Every $z \in P(f)$ is a measurable deep point of $K_\epsilon(f)$.*

Proof. Choose a smoothly bounded disk A with $\overline{D}$ in its interior, lying within an ϵ-neighborhood of D and outside the part of $\mathbb{C} - J(f)$ where the hyperbolic injectivity radius is less than one. Let $K_A(f) = \bigcap f^{-n}(A)$. Then $K_\delta(f) \subset K_A(f) \subset K_\epsilon(f)$ for some $\delta > 0$. Since z is a deep point of $K_\delta(f)$, to prove the Corollary it suffices to show $\partial K_A(f)$ is shallow.

Consider a point $w \in \partial K_A(f)$ and a scale $r < 1$. We must find a ball disjoint from $\partial K_A(f)$, within distance r of w and with radius $\asymp r$.

By shallowness of the Julia set, there is such a ball $B(z_1, r_1)$ disjoint from $J(f)$. If $B(z_1, r_1/2)$ is disjoint from $\partial K_A(f)$ we are done; otherwise, there is a $B(z_2, r_1/2) \subset B(z_1, r_1)$ with $z_2 \in \partial K_A(f) - J(f)$. Then $f^n(z_2) \in \partial A$ for some n. The ball $U = B(z_2, r_1/10)$ has diameter less than one in the hyperbolic metric on $\mathbb{C} - J(f)$, so $f^n|U$ is injective. Since ∂A is smooth, $f^n(U) - A$ contains a ball B with $\operatorname{diam} B \asymp \operatorname{diam} f^n(U)$, and by Koebe $f^{-n}(B) \subset U$ contains a ball $B(z_3, r_3)$ disjoint from $K_A(f)$ with $r_3 \asymp r$. ∎

Escape from the Siegel disk. To prove c_0 is a deep point of $K(f)$, we will analyze the orbit of a point z that starts near c_0 but escapes to infinity. While the orbit is near D, it shadows a rotating orbit until it comes close to the critical point. As it passes the critical point, the orbit can move much closer to D, but it can only move farther away by a bounded factor. Thus before $f^i(z)$ can escape from the influence of D, it must make many visits to the critical point. It turns out each visit results in definite expansion with respect to the hyperbolic metric on $\Omega = \mathbb{C} - \overline{D}$. On the last visit, there is a point y' in $\partial D' = \iota(\partial D)$ at a bounded hyperbolic distance from $f^i(z)$. Pulling this point back by f^{-i}, we obtain a point $y \in K(f)$ very close to z. This point limits the radius of a ball $B(z,s) \subset B(c_0,r) - K(f)$, and shows c_0 is deep.

To measure the rate of escape from the influence of Siegel disk D, we will work in the linear coordinate w. Let $L(w) = \log d(w, D)$; that is,

$$L(w) \;=\; \begin{cases} \log(|w|-1) & |w|>1 \\ -\infty & \text{otherwise.} \end{cases}$$

If $w \in U$, w is near D and $f^i|U$ is an approximate rotation (in the sense of §3), then

$$L(f^i(w)) = L(w) + O(1);$$

while for *all* w near D we still have the weaker statement

$$L(f(w)) \le L(w) + O(1). \tag{4.1}$$

To see 4.1, first note that near the critical point (recalling c_1 is the critical value),

$$|f(w) - c_1| \asymp |w - c_0|,$$

because f is quasiregular and $f|S^1$ is an isometry. (The branching behavior of f is similar to that of the map $(r,\theta) \mapsto (r, 2\theta)$ in polar coordinates.) If $d(w,D) \ll d(w,c_0)$, then f is an approximate rotation at w, so $L(f(w)) = L(w) + O(1)$; otherwise $d(f(w),D) \le d(f(w),c_1) \asymp d(w,c_0) \asymp d(w,D)$, which gives (4.1).

Proof of Theorem 4.2 (Deep points). We will first prove that every point of $P(f)$ is a deep point of $K(f)$, deferring the case of $K_\epsilon(f)$ to the end.

We will actually show there is an $\alpha > 0$ so for all z in the thick part of Ω, we have

$$d_\Omega(z, K(f)) = O(d(z,D)^\alpha). \tag{4.2}$$

Since the Poincaré metric on Ω is comparable to the $1/d$ metric, this bound implies $d(z,K(f)) = O(d(z,P(f))^{1+\alpha})$, from which it follows easily that every point in the post-critical set is (uniformly) deep in $K(f)$.

Let us also remark that

$$f : (\mathbb{C} - \overline{D \cap D'}) \to (\mathbb{C} - \overline{D}) = \Omega$$

is a covering map, hence an isometry between the respective hyperbolic metrics. The inclusion between domain and range is a contraction, by the Schwarz lemma. The amount of contraction is bounded in terms of $d_\Omega(z, D')$. Thus the expansion of f, measured by $\|f'(z)\|$ using the hyperbolic metric of Ω at both z and $f(z)$, satisfies

$$\|f'(z)\| > C(d) > 1,$$

where $d = d_\Omega(z, D')$. Compare [Mc4, Prop. 4.4.2].

We now turn to the proof of (4.2). Let $z_0 = z$, and $z_j = f^j(z_0)$ for $j > 0$. Applying Theorem 3.4 to the point p in $P(f)$ nearest z_0, we obtain an approximate rotation $f^{i_1} : U \to V$ such that $z_0 \in U$ and $d_\Omega(z_{i_1}, \partial D') = O(1)$. (Here we use the fact that D' converges radially to c_0). Then $\|f'(z_{i_1})\| > \eta > 1$, where η depends only on the rotation number θ. On the other hand, $|L(z_j) - L(z_0)| = O(1)$ for $0 \le j \le i_1$, and by (4.1) we have

$$L(z_{i_1+1}) \;\; \le \;\; L(z_0) + O(1).$$

Now repeat the argument starting with z_{i_1+1}, to obtain a second visit z_{i_2} within a bounded distance of D'. Continuing in this way, we construct a sequence of visits z_{i_k} along which $L(z_{i_k})$ increases at most linearly in k. The construction terminates when the orbit gets a definite distance from D, say when $L(z_{i_k}) > 0$. Because of the linear increase of $L(z_{i_k})$, the sequence is defined at least for $k = 1, 2, \ldots, N$ where $N \asymp -L(z_0)$. But $L(z_0) \asymp \log d(z_0, D)$ (by Hölder continuity of the quasiconformal map $\Phi : \mathbb{C}_z \to \mathbb{C}_w$), and thus

$$N \asymp \log \left(\frac{1}{d(z_0, D)} \right). \tag{4.3}$$

Since there are $N - 1$ visits past D', we have $\|f^{i_N}(z_0)\| \ge \eta^{N-1}$. Join z_{i_N} to a point $y' \in \partial D'$ by an arc γ' of hyperbolic length $O(1)$. Then $y' \in K(f)$, and since γ' is disjoint from $P(f)$ it lifts under f^{-i_N} to an arc γ joining z_0 to $y \in K(f)$ with hyperbolic length $O(\eta^{-N})$. Thus by (4.3),

$$d_\Omega(z_0, K(f)) \le d_\Omega(z_0, y) = O(\eta^{-N}) = O(d(z_0, D)^\alpha)$$

for some $\alpha > 0$ depending only on θ. This completes the proof that each point in the postcritical set is a deep point of $K(f)$.

To handle the case of $K_\epsilon(f)$, we modify the construction so it terminates when $L(z_{i_k}) > M(\epsilon)$. Then $L(z_j) < M(\epsilon) + O(1)$ for $j = 0, \ldots, i_N$. For suitable choice of $M(\epsilon)$, this condition insures z_j lies well-within an ϵ-neighborhood of D. Now each $y_j = f^j(y)$, $j = 0, \ldots, i_N$, is a bounded hyperbolic distance from z_j, so it too is no farther than ϵ from D. Finally $f^{i_N}(y) = y' \in \partial D'$, so $d(f^j(y), D) = 0)$ for $j > i_N$. Thus $y \in K_\epsilon(f)$ and we have shown every point of $P(f)$ is also a deep point of $K_\epsilon(f)$. ∎

To conclude this section we will prove:

Theorem 4.6 (Uniform twisting) *The holomorphic dynamical system $(\mathcal{F}(f), J(f))$ is uniformly twisting.*

This Theorem illustrates one of the parallels summarized in Table 2. Although the result is not used in the sequel, we include it because the same result holds for any infinitely renormalizable real quadratic map $f(z) = z^2 + c$ with bounded combinatorics [Mc4, §9.4].

A dynamical system $(\mathcal{F}, \Lambda)$ is *uniformly twisting* if there is robust nonlinearity at every scale about every point in Λ. This concept is naturally formulated in terms of geometric limits; a complete development appears in [Mc4, §9]. For the purpose of the present discussion, we will simply state a sufficient criterion for uniform twisting.

Let $\mathcal{H}$ denote the space of all holomorphic maps $g : U \to \widehat{\mathbb{C}}$ such that U is an open set of $\widehat{\mathbb{C}}$. We give $\mathcal{H}$ the following (non-Hausdorff) topology: $g_n : U_n \to \widehat{\mathbb{C}}$ tends to $g : U \to \widehat{\mathbb{C}}$ if for every compact $K \subset U$, we have $K \subset U_n$ for all n sufficiently large and $g_n \to g$ uniformly on K.

Given a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, let

$$\mathcal{F}(f) \;=\; \{g : U \to \widehat{\mathbb{C}} \;:\; \text{for some } i, j \geq 0,\ f^i \circ g = f^j\} \subset \mathcal{H}.$$

Proposition 4.7 *Let $\Lambda \subset \mathbb{C}$ be compact. Suppose for any sequence*

$$A_n(z) = \alpha_n(z - \beta_n)$$

with $\alpha_n \to \infty$ and $\beta_n \in \Lambda$, there exist $g_n \in \mathcal{F}(f)$ such that after passing to a subsequence,

$$A_n \circ g_n \circ A_n^{-1} \to h \in \mathcal{H},$$

where $h : U \to V$ is a nonconstant map with a critical point.

Then $(\mathcal{F}(f), \Lambda)$ is uniformly twisting.

This criterion reduces the proof of uniform twisting to:

Proposition 4.8 *For every $x \in J(f)$ and $r < 1$, there is a proper degree two map $g \in \mathcal{F}(f)$,*

$$g : (U, u) \to (V, v)$$

such that U and V are disks, $g'(u) = 0$, in-radius$(U, u) \asymp$ in-radius$(V, v) \asymp r$, *$|x - u| = O(r)$ and $|x - v| = O(r)$.*

Proof. First assume $x = c_0$. Then g can be taken to be a first return map near the critical point. To make this precise, we prove the result in the linear coordinate w. Choose n such that $\|q_n\theta\| \asymp r$. Let V be the round disk orthogonal to S^1, meeting the circle in the interior of the arc $[q_n\theta, q_{n+1}\theta]$, and let $i = q_n + q_{n+1}$. Then f^i has a unique critical value $v \in V$, with angular coordinate $(q_n + q_{n+1})\theta$. Since θ has bounded type, in-radius$(V, v) \asymp r$. Let U be the unique component of $f^{-i}(V)$ containing c_0. Then

$$g = f^i : (U, c_0) \to (V, v)$$

is proper of degree two, and it is easy to see in-radius$(U, c_0) \asymp r$, so the result is established for $x = c_0$. Moreover, we can shrink U and V slightly so their diameters are comparable to r.

To treat the case of an arbitrary $z \in J(f)$, we use Theorem 3.2 to find a univalent map $f^j : (U, u) \to (V, c_0)$ at scale r near z. Pulling back a map $\widetilde{g} = f^i$ which works at a scale $s \asymp$ in-radius(V, c_0) near the critical point, we obtain a map $g = f^{-j} \circ f^{i+j}$ which works at scale r near z, completing the proof. ∎

Proof of Theorem 4.6 (Uniform twisting). Given $A_n(z) = \alpha_n(z - \beta_n)$ as in Proposition 4.7, apply the preceding proposition with $x = \beta_n$ and $r = 1/\alpha_n$ to obtain a sequence of maps $g_n \in \mathcal{F}(f)$ such that

$$h_n \;=\; A_n \circ g_n \circ A_n^{-1} \;:\; (U_n, u_n) \to (V_n, v_n)$$

is proper of degree two, $h_n'(u_n) = 0$, in-radius$(U, u_n) \asymp$ in-radius$(V_n, v_n) \asymp 1$, $|u_n| = O(1)$ and $|v_n| = O(1)$. In the Carathéodory topology, a subsequence h_{n_k} converges to a degree two proper map $h : (U, u) \to (V, v)$ with $h'(u) = 0$ [Mc3, Thm 5.6], and the same subsequence tends to h in $\mathcal{H}$. ∎

5 Universality

A *polynomial-like map* $g : U \to V$ is a proper holomorphic map between disks in $\mathbb{C}$, such that $\overline{U}$ is a compact subset of V. A *quadratic-like map* is a polynomial-like map of degree two.

The filled Julia set $K(g)$ is the set of points which never escape under iteration; it is defined by

$$K(g) = \bigcap_1^\infty g^{-i}(V).$$

Similarly, the postcritical set $P(g)$ is the closure of the forward orbit of the critical points of g.

Let $g_i : U_i \to V_i$, $i = 1, 2$ be polynomial-like maps of the same degree. A *hybrid conjugacy* is a quasiconformal map ϕ between neighborhoods of $K(g_1)$ and $K(g_2)$, conjugating g_1 to g_2, with $\overline{\partial}\phi = 0$ a.e. on $K(g_1)$. We say g_1 and g_2 are *hybrid equivalent* if such a conjugacy exists.

A fundamental result of Douady and Hubbard states that every polynomial-like map g is hybrid equivalent to a polynomial of the same degree. If $K(g)$ is connected then f is unique up to conformal conjugacy [DH1, Thm. 1].

Theorem 5.1 (Universality) *Let θ be an irrational of bounded type, and let g and h be quadratic-like maps with fixed points of multiplier* $\exp(2\pi i\theta)$. *Then:*

1. *there is a hybrid conjugacy ϕ between g and h;*
2. *the complex derivative $\phi'(z)$ exists for all $z \in P(g)$; and*
3. *there are $M, \alpha, \delta > 0$ so for all $z \in P(g)$ and all t with $|t| < \delta$, we have:*

$$\left| \frac{\phi(z+t) - \phi(z)}{t} - \phi'(z) \right| \leq M|t|^{\alpha}. \tag{5.1}$$

By the Whitney extension theorem [St, §VI.2.3] we have:

Corollary 5.2 *The derivative $\phi'(z)$ is Hölder continuous on $P(g)$, and $\phi|P(g)$ extends to a continuous, $C^{1+\alpha}$ function on the whole plane.*

Corollary 5.3 *The Hausdorff dimension of $P(g)$ is equal to that of $P(f)$, where $f(z) = e^{2\pi i\theta}z + z^2$.*

Proof. By (5.1) a hybrid conjugacy from f to g is bi-Lipschitz on the postcritical set, and such maps preserve dimension. ∎

We now proceed to the proof of Theorem 5.1.

The quadratic-like maps g and h are each hybrid equivalent to a quadratic polynomial. It is not hard to see that a hybrid conjugacy preserves the multiplier of any indifferent cycle; thus g and h are hybrid equivalent to $f(z) = e^{2\pi i\theta}z + z^2$, and hence to each other.

For convenience we let $\phi : \mathbb{C} \to \mathbb{C}$ denote a quasiconformal map of the whole plane which restricts to a hybrid conjugacy from g to h. Let us say ϕ is *$C^{1+\alpha}$-conformal at z* if $\phi'(z)$ exists and

$$\phi(z+t) = \phi(z) + \phi'(z) \cdot t + O(|t|^{1+\alpha}).$$

Thus (5.1) says ϕ is *uniformly* $C^{1+\alpha}$-conformal on $P(f)$.

The following result appears in [Mc4, §2.6]:

Theorem 5.4 (Boundary conformality) *Let $\phi : \mathbb{C} \to \mathbb{C}$ be a quasiconformal map with $\overline{\partial}\phi = 0$ on a measurable set Ω, and let x be a measurable deep point of Ω. Then ϕ is $C^{1+\alpha}$-conformal at x.*

Corollary 5.5 *The hybrid conjugacy ϕ is $C^{1+\alpha}$ at the critical point $c_0(g)$.*

Proof. By Corollary 4.4, the critical point $c_0(f)$ is a measurable deep point of the interior of $K(f)$ (since $J(f)$ has measure zero). This deepness property is preserved by quasiconformal maps (using e.g. the estimates of [Ast]), so $c_0(g)$ is a measurable deep point of $\Omega = \operatorname{int} K(g)$. But ϕ is conformal on Ω, so ϕ is $C^{1+\alpha}$ at $c_0(g)$. ∎

One way to now complete the proof of Theorem 5.1 is to observe that all points in $P(f)$ are measurable deep points of $K(f)$, with uniform bounds, so ϕ is *uniformly* $C^{1+\alpha}$ on $P(f)$, by an effective version of Theorem 5.4. We will proceed differently, however, to illustrate directly that smoothness along the post-critical set is inherited from smoothness at the critical point.

Distortion of triangles. Let

$$D_t(z) = \frac{\phi(z+t) - \phi(z)}{t},$$

and define

$$N_t(z) \;=\; \sup_{\frac{1}{2}<|t/s|<2} \left| \frac{D_t(z)}{D_s(s)} - 1 \right|.$$

By a calculus argument (controlling $\log D_t(z)$ with a geometric series) one has:

Lemma 5.6 *Suppose $N_t(z) < C|t|^\alpha$ for all z in a compact set K and all t sufficiently small. Then ϕ is uniformly $C^{1+\alpha}$ on K.*

Now think of an ordered triple of points $T = (a, b, c) \subset \mathbb{C}$ as a triangle, with shape determined (up to similarity) by the ratio $\sigma(T) = (b-a)/(c-a)$. Then

$$N_t(z) \;\leq\; 2 \sup_T |\sigma(\phi(T)) - \sigma(T)|,$$

where the sup is over all triangles $T = (z, z+t, z+s)$ with $|\sigma(T)|$ in $[1/2, 2]$. So to show ϕ is uniformly $C^{1+\alpha}$ on $P(f)$, we need only check that

$$|\sigma(\phi(T)) - \sigma(T)| = O(|t|^\alpha) \tag{5.2}$$

when $z \in P(f)$ and t is sufficiently small.

To obtain this estimate, we will use the dynamics to move T and $\phi(T)$ close to the critical point. By the Koebe distortion theorem, the shapes of these triangles are almost

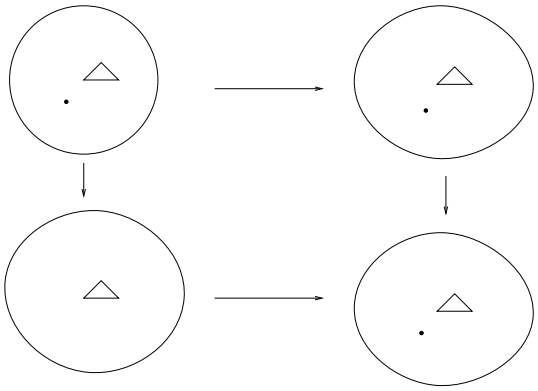

Figure 9. Nearly similar triangles.

preserved if the dynamics is univalent on a large neighborhood. On the other hand, ϕ is smooth at the critical point, so the shape is nearly preserved there as well. See Figure 9.

Completion of the proof of Theorem 5.1 (Universality). By Corollary 5.5, there is a $\beta > 0$ such that ϕ is $C^{1+\beta}$-conformal at the critical point $c_0(g)$.

Consider a small $\epsilon > 0$, whose value is to be chosen shortly. Since g is hybrid equivalent to a quadratic polynomial, Theorem 3.4 provides an approximate rotation

$$g^i : (U, y) \to (V, c_0(g))$$

with $d_U(y, z) = O(1)$ and in-radius$(U, z) \asymp |t|^{1-\epsilon}$. For small t, U contains a large neighborhood of T.

Let $T' = g^i(T)$. Since $\operatorname{diam}(T)/d(T, \partial U) \asymp |t|^\epsilon$, the Koebe distortion theorem shows

$$|\sigma(T') - \sigma(T)| = O(|t|^\epsilon).$$

Let $T'' = \phi(T')$. Since ϕ is $C^{1+\beta}$ at $c_0(g)$, we have

$$|\sigma(T'') - \sigma(T')| = O\left(\frac{(d(T', c_0(g)) + \operatorname{diam}(T'))^{1+\beta}}{\operatorname{diam}(T')}\right).$$

Because $g|P(g)$ is quasiconformally conjugate to a rotation, there is a $\gamma > 0$ such that $\operatorname{diam}(T') = O(\operatorname{diam}(T)^\gamma) = O(|t|^\gamma)$. Since $|t|^\epsilon d(T, y) = O(\operatorname{diam}(T))$, Koebe again implies $|t|^\epsilon d(T', c_0(g)) = O(\operatorname{diam}(T'))$. Combining these inequalities and choosing ϵ suitably small, we obtain

$$|\sigma(T'') - \sigma(T')| = O(|t|^{\beta\gamma - \epsilon(1+\beta)}) = O(|t|^{\beta\gamma/2}).$$

Finally consider $\phi(T) = h^{-i}(T'')$. Since ϕ is K-quasiconformal, from $\mathrm{diam}(T) = O(|t|^{\epsilon} d(T, \partial U))$ we deduce $\mathrm{diam}(\phi(T)) = O(|t|^{\eta} d(\phi(T), \partial\phi(U)))$, where $\eta = \epsilon/K$. Applying Koebe once more to the univalent map $h^i : \phi(U) \to \phi(V)$, we conclude

$$|\sigma(\phi(T)) - \sigma(T'')| = O(|t|^{\eta}).$$

All three bounds on the change in the similarity invariant are in terms of powers of $|t|$, so we obtain an $\alpha > 0$ such that $|\sigma(\phi(T)) - \sigma(T)| = O(|t|^{\alpha})$. This establishes (5.2) and thereby completes the proof of Theorem 5.1. ∎

6 Renormalization

In §2 we constructed renormalizations of rigid rotations on the circle. Since $f|\partial D$ is topologically conjugate to a rotation, it too admits such renormalizations.

In this section we show the renormalizations of $f|\partial D$ can be complexified to yield *holomorphic pairs* $(f^{q_n} : U_1 \to V,\ f^{q_{n+1}} : U_2 \to V)$. These renormalizations are variants of the *holomorphic commuting pairs* introduced in [dF]. We then use the results of §4 to study the Lebesgue density of the escape loci for these renormalized dynamical systems.

Holomorphic pairs. Let $g_i : U_i \to V$, $i = 1, 2$, be univalent maps between quasidisks in $\mathbb{C}$ with $U_i \subset V$. Each g_i extends to a homeomorphism $\overline{U_i} \to \overline{V}$ which we denote by the same letter.

We say (g_1, g_2) is a *holomorphic pair* if (as in Figure 10),

- $V - \overline{U_1 \cup U_2}$ is also a quasidisk;
- $\overline{U_i} \cap \partial V = I_i$ is an arc;
- $g_i(I_i) \subset I_1 \cup I_2$, for $i = 1, 2$; and
- $\overline{U_1} \cap \overline{U_2} = \{c\}$ is a single point.

A holomorphic pair is designed to complexify the *boundary dynamics* of (g_1, g_2) on $I_1 \cup I_2$. Note that the interval $I_1 \cup I_2$ inherits an orientation from V.

Let us say a map between sets in $\mathbb{C}$ is *quasisymmetric* if it extends to a quasiconformal homeomorphism of the plane. We next show boundary conjugacies can be promoted to the complex domain.

Theorem 6.1 *Let $\phi : (I_1 \cup I_2) \to (I_1' \cup I_2')$ be an orientation-preserving quasisymmetric map conjugating the boundary dynamics of one holomorphic pair (g_1, g_2) to that of another (g_1', g_2'). Then ϕ extends to a quasiconformal conjugacy $\overline{\phi} : \overline{V} \to \overline{V'}$.*

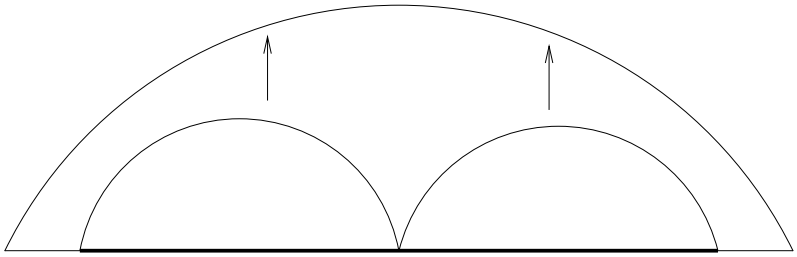

Figure 10. A holomorphic pair.

Proof. Choose any extension of ϕ to a quasisymmetric map $\partial V \to \partial V'$. Define ϕ on ∂U_i by $\phi(x) = (g_i')^{-1} \circ \phi \circ g_i(x)$. This definition is consistent with the values already specified on $\partial U_i \cap \partial V = I_i$ by the conjugacy condition on ϕ. An orientation-preserving quasisymmetric map between the boundaries of quasidisks extends to the interior, so we may further prolong ϕ to a quasiconformal map $\phi_0 : \overline{V} \to \overline{V'}$ which is a conjugacy on $\partial U_1 \cup \partial U_2$.

Define $\phi_1 : \overline{V} \to \overline{V'}$ by

$$\phi_1(x) = \begin{cases} (g_i')^{-1} \circ \phi_0 \circ g_i(x) & \text{if } x \in U_i, \\ \phi_0(x) & \text{otherwise.} \end{cases}$$

Then ϕ_1 is quasiconformal, and it agrees with ϕ_0 on ∂V. Therefore ϕ_1 is isotopic to ϕ_0 *rel* the ideal boundary of V.

The map $(g_1 \cup g_2) : (U_1 \cup U_2) \to V$ is a holomorphic covering, and in the language of the Appendix, (ϕ_0, ϕ_1) is a combinatorial conjugacy between $(g_1 \cup g_2)$ and $(g_1' \cup g_2')$. By Theorem A.1, there is a quasiconformal conjugacy $\overline{\phi} : \overline{V} \to \overline{V'}$ with $\overline{\phi} = \phi_0 = \phi$ on ∂V. ∎

Siegel renormalization. Consider once again the quadratic polynomial $f(z) = e^{2\pi i\theta} z + z^2$, where θ has bounded type and $p_n/q_n \to \theta$ are its continued fraction approximants. Let c_{-i} be the unique point in the boundary of the Siegel disk D such that $f^i(c_{-i}) = c_0$.

We say f is (q_n, q_{n+1})-*renormalizable* if there is a holomorphic pair of the form

$$(f^{q_n} : U_1 \to V,\ f^{q_{n+1}} : U_2 \to V)$$

such that

- $V \cap D = \emptyset$,
- $\overline{U_1} \cap \overline{U_2} = c_0$, the critical point of f, and
- $I_1 = \overline{U_1} \cap \partial V = [c_{r_n}, c_0]$, while $I_2 = \overline{U_2} \cap \partial V = [c_0, c_{r_{n+1}}]$.

(As before, $[a,b] \subset \partial D$ denotes the interval of shorter length in the linear coordinate, and $r_n = q_{n-1} - q_n$.) The boundary dynamics of $(f^{q_n}, f^{q_{n+1}})$ on $I_1 \cup I_2$ agrees, in the linear coordinate, with the extended renormalization $\mathcal{R}_n(F)$ discussed at the end of §2.

Theorem 6.2 (Siegel renormalizations) *The map $f(z) = e^{2\pi i\theta}z + z^2$ is (q_n, q_{n+1})-renormalizable for every $n \geq 3$.*

These renormalizations are conveniently constructed using external rays. To apply this method we need the following result from [Pet]:

Theorem 6.3 (Petersen) *The Julia set $J(f)$ is locally connected.*

By local connectivity, the Riemann mapping

$$\omega : (\mathbb{C} - \overline{\Delta}) \to (\mathbb{C} - K(f)),$$

normalized so $\omega(z^2) = f(\omega(z))$, extends continuously to a semiconjugacy $S^1 \to J(f)$. The *external rays* R_t and *equipotentials* C_s are defined to be the images of the rays $\arg(z) = 2\pi t$ and the circles $|z| = \exp(2\pi s)$ under ω. (The term *equipotential* comes from the fact that C_s is a level set for the Green's function of $K(f)$.)

Proof of Theorem 6.2 (Siegel renormalizations). Let $J_n = [c_0, c_{r_n}] \subset \partial D$. The critical value c_1 is disjoint from $\overline{\bigcup_{n\geq 3} J_n}$.

Consider the annulus A between the boundary of the Siegel disk and a fixed equipotential C_S, $S > 0$. Let R_t be the external ray landing at the critical value c_1. Let V be a quasidisk obtained by removing from A a slight thickening of R_t. The thickening need only be small enough that ∂V is disjoint from every external ray landing in $\overline{\bigcup J_n}$.

The critical point c_0 lies in ∂V, but V itself is disjoint from the postcritical set. Thus every component of $f^{-i}(V)$ is a quasidisk mapping univalently to V, and there is one component touching each element of $f^{-i}(c_0)$.

Let U_n be the component of $f^{-q_n}(V)$ with $c_{-q_n} \in \partial U_n$. The critical values of f^i are $\{c_1, c_2, \ldots c_i\}$, so $[c_{q_n}, c_{q_{n-1}}]$ is the maximal interval around c_0 in ∂D with no critical values of f^{q_n} in its interior. We have $f^{q_n}(J_n) = [c_{q_n}, c_{q_{n-1}}]$ (by the definition of r_n), so $\overline{U_n} \cap \partial D = J_n$. Since $U_n \subset f^{-q_n}(A - R_t)$, U_n lies between a pair of external rays landing in J_n, and within the equipotential C_s, $s = 2^{-q_n}S$. Therefore $\partial U_n \cap \partial V = \partial U_n \cap \partial D = J_n$.

By similar reasoning, $\overline{U_n}$ and $\overline{U_{n+1}}$ meet only at the critical point c_0. It is also easy to check that $V - \overline{U_n \cup U_{n+1}}$ is a quasidisk; for example, ∂U_n and ∂U_{n+1} meet nontangentially at the critical point because the quasidisk $D' = \iota(D)$ comes between them. Thus the holomorphic pair

$$(f^{q_n} : U_n \to V,\ f^{q_{n+1}} : U_{n+1} \to V)$$

provides the desired renormalization of f. ∎

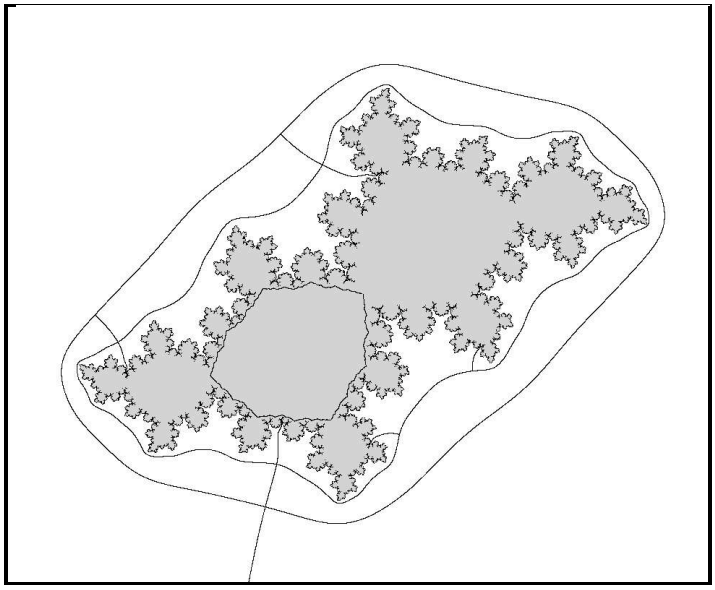

Figure 11. Renormalization of the golden mean Siegel disk.

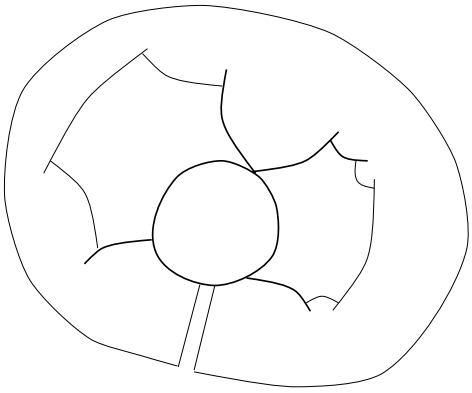

Figure 12. Schematic of the renormalization.

Example. A $(3,5)$-renormalization of the golden mean Siegel disk is depicted in Figure 11. The external ray R_t landing at the critical value c_1 enters at the bottom edge of the picture; it crosses the equipotentials $C_{2^{-3}S}$ and $C_{2^{-5}S}$, where $\exp(2\pi S) = 2$. (The equipotential C_S is outside the frame.) Approximations to U_3 and U_5 appear to the left and right of the critical point respectively. The boundary of each approximate U_n consists of external rays (landing at certain preimages of the critical point *not* on the boundary of the Siegel disk), several arcs running along preimages of ∂D, and the interval J_n in ∂D. To obtain the actual renormalization, one must cut along slight thickenings of each external ray, as suggested in Figure 12.

Escape loci. Let (g_1, g_2) be a holomorphic pair. Let $U = U_1 \cup U_2$ and define

$$g = (g_1 \cup g_2) : U \to V.$$

By analogy with a polynomial-like map, we define the *filled Julia set* of (g_1, g_2) by

$$K(g_1, g_2) \;=\; \bigcap_{n>0} g^{-n}(U).$$

If (g_1, g_2) is a renormalization of f, then $K(g_1, g_2) \subset J(f)$, because under iteration these points stay bound, but they never reach the Siegel disk D. In particular, the filled Julia set of a renormalization is nowhere dense and of measure zero.

It is also useful to include points which escape from U but do so by landing within distance ϵ of the "critical point" $\{c\} = \overline{U_1} \cap \overline{U_2}$. To this end we define

$$K_\epsilon(g_1, g_2) \;=\; K(g_1, g_2) \cup \bigcup_{n>0} g^{-n}(B(c, \epsilon) - U).$$

Proposition 6.4 *For any renormalization*

$$(f^{q_n} : U_n \to V,\ f^{q_{n+1}} : U_{n+1} \to V)$$

of f, the critical point c_0 lies in the interior of $\overline{D} \cup \overline{D'} \cup U_1 \cup U_2$.

Proof. Since f^{q_n} and $f^{q_{n+1}}$ each map c_0 into the interior of the arc $\partial V \cap \partial D$, a neighborhood of c in $\partial(U_1 \cup U_2)$ is contained in $\partial(D \cup D')$. ∎

Corollary 6.5 *For all $\epsilon > 0$ sufficiently small, there is a $\delta > 0$ such that*

$$K_\delta(f) \cap (U_1 \cup U_2) \;\subset\; K_\epsilon(f^{q_n}, f^{q_{n+1}}) \;\subset\; K(f).$$

Proof. Let $U = U_1 \cup U_2$. For ϵ small enough, the part of $V - U$ within distance ϵ of $c = c_0$ lies in the pre-Siegel disk $\overline{D'}$. Thus points which escape from U by landing in $B(c, \epsilon)$ must also lie in $K(f)$.

Now consider a point $z \in K_\delta(f) \cap U$ that escapes under iteration of the renormalized map, by landing at $z' \in V - U$. By assumption z remains close to ∂D, and thus close to $I_1 \cup I_2$, until the moment it escapes. Thus z' is close to $f^{q_n}(I_1) \cup f^{q_{n+1}}(I_2) = J$. But J is contained in the interior of $I_1 \cup I_2$, so $\overline{V - U} \cap J = \{c\}$, and thus for δ sufficiently small we must have $z' \in B(c, \epsilon)$. ∎

Corollary 6.6 *The critical point c_0 is a measurable deep point of $D \cup D' \cup K_\epsilon(f^{q_n}, f^{q_{n+1}})$ for every $\epsilon > 0$.*

Proof. By Corollary 4.5, c_0 is a measurable deep point of $K_\delta(f)$. ∎

7 Self-similarity

In this section we show the Siegel disk of $f(z) = e^{2\pi i\theta}z + z^2$ is self-similar when θ is a quadratic irrational.

A succinct description of this self-similarity can be formulated as follows. Suppose $\theta = [a_1, a_2, \cdots]$, where $a_{n+s} = a_n$ for all $n \geq N$. Let $F(x) = x + \theta$ be the standard rotation on $S^1 = \mathbb{R}/\mathbb{Z}$. By Corollary 2.2 there is a $\gamma \in (0, 1)$ such that near $x = 0$, the contraction $\widetilde{\psi}(x) = (-1)^s\gamma x$ conjugates F^{q_n} to $F^{q_{n+s}}$ for $n \gg 0$. The topological conjugacy from $\langle S^1, F\rangle$ to $\langle \partial D, f\rangle$, normalized to send $x = 0$ to the critical point, transports $\widetilde{\psi}$ to a quasi-symmetric mapping ψ defined on a neighborhood of c_0 in ∂D. Then we have:

Theorem 7.1 (Self-similarity) *The mapping ψ is $C^{1+\alpha}$-conformal or anticonformal at the critical point. That is, for all z in a neighborhood of c_0 in ∂D, we have*

$$\psi(z) \quad = \quad \begin{cases} c_0 + \lambda(z - c_0) + O(|z - c_0|^\alpha), & \text{for } s \text{ even, or} \\ c_0 + \lambda\overline{(z - c_0)} + O(|z - c_0|^\alpha), & \text{for } s \text{ odd,} \end{cases}$$

where $\alpha > 0$, $\lambda \in \mathbb{C}$ and $0 < |\lambda| < 1$.

Here is a more complete statement which extends ψ to a conjugacy between holomorphic pairs. By results of the preceding section, we may choose a sequence of renormalizations

$$(f^{q_n} : U_1^n \to V^n, \quad f^{q_{n+1}} : U_2^n \to V^n)$$

for all $n \geq 3$.

Theorem 7.2 *For any $n \geq \max(N, 3)$, there exists a homeomorphism*

$$\psi : \overline{V^n} \to \overline{V^{n+s}},$$

fixing the critical point c_0 and conjugating the (q_n, q_{n+1})-renormalization of f to its (q_{n+s}, q_{n+s+1})-renormalization. For even s, the conjugacy can be chosen so:

1. *ψ extends to a quasiconformal map on $\overline{V^n} \cup \overline{D}$, conformal on a neighborhood of c_0 in D;*
2. *the extended mapping is $C^{1+\alpha}$-conformal at the critical point; and*
3. *$\psi'(c_0)$ is independent of n and satisfies $0 < |\psi'(c_0)| < 1$.*

For odd s, ψ is orientation-reversing, and the same statements hold for $\overline{\psi(z)}$.

Proof. The idea is the same as that used in the proof of universality: a conjugacy between two successive renormalizations is smooth at the critical point because it is conformal on a set of high density.

We first assume s is even.

Let $U^n = U_1^n \cup U_2^n$, and let $L_n = \overline{U^n} \cap \partial D = [c_{r_n}, c_{r_{n+1}}]$. Consider the dynamics in the linear coordinate system $\mathbb{C}_w$. On the disk $\{w : |w| \leq 1\}$, we have $f(w) = F(w) = e^{2\pi i\theta} w$. The arc L_n is contained in the circle $|w| = 1$, and the boundary dynamics of $(f^{q_n}, f^{q_{n+1}})$ is the same as that of the extended renormalization $\mathcal{R}_n(F)$ discussed in §2. By Corollary 2.2, there is a $\gamma \in (0, 1)$ (depending only on θ) such that the map $\phi : L_n \to L_{n+s}$ given by $\phi(w) = w^\gamma$ conjugates $\mathcal{R}_n(F)$ to $\mathcal{R}_{n+s}(F)$. (Here ϕ is normalized to fix the critical point, so $\phi(1) = 1$.)

Since the linearizing map $\Phi : \mathbb{C}_z \to \mathbb{C}_w$ is quasiconformal, ϕ is quasisymmetric in the original coordinate $\mathbb{C}_z$. By Theorem 6.1, ϕ extends to a quasiconformal conjugacy $\phi_0 : \overline{V^n} \to \overline{V^{n+s}}$.

Next we modify this conjugacy so it is conformal on a neighborhood of the critical point in $\overline{D \cup D'}$. Since $\Phi | D$ is conformal, there is a quasiconformal map $\beta : D \to D$, prolonging ϕ_0 on $\partial V \cap \partial D$, such that $\beta(z)$ is conformal on a neighborhood of the critical point. Indeed, near c_0 we may simply take $\beta(z) = \Phi^{-1}(\Phi(z)^\gamma))$. Conjugating by the involution $\iota : D \to D'$, we obtain a map $\beta' : D' \to D'$ which is conformal near c_0 and agrees with ϕ_0 on a neighborhood of c_0 in $\partial(V^n - U^n)$. Thus we may modify ϕ_0 to a quasiconformal map $\phi_1 : \overline{V^n} \to \overline{V^{n+s}}$ such that

- $\phi_1(z) = \phi_0(z)$ on $\overline{U^n} \cup \partial V_n$; and
- $\phi_1(z) = \beta'(z)$ on a neighborhood of c_0 in $V^n - \overline{U^n}$.

The pair (ϕ_0, ϕ_1) is a combinatorial conjugacy between the two renormalizations, in the sense of the Appendix. By Theorem A.1, there is a quasiconformal conjugacy

$$\psi : \overline{V^n} \to \overline{V^{n+s}},$$

equal to ϕ_1 on $V^n - U^n$. In particular, ψ is *conformal* on $B(c_0, \epsilon) \cap (V - \overline{U})$ for some $\epsilon > 0$.

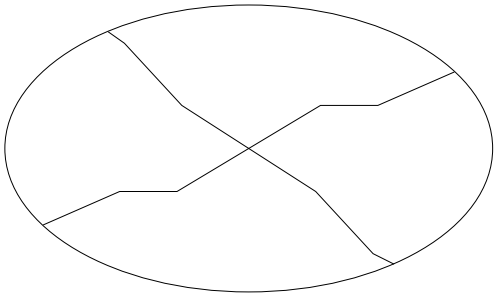

Figure 13. A neighborhood of the critical point.

The points of conformality are invariant under the dynamics, so ψ is also conformal a.e. on $K_\epsilon(f^{q_n}, f^{q_{n+1}})$. (Since $K(f^{q_n}, f^{q_{n+1}}) \subset J(f)$ has measure zero, almost every point in $K_\epsilon(f^{q_n}, f^{q_{n+1}})$ escapes through $B(c, \epsilon) - U^n$.)

The extended map

$$\xi \;=\; \psi \cup \beta \;:\; \overline{V^n} \cup D \;\to\; \overline{V^{n+s}} \cup D$$

is conformal on

$$E \;=\; K_\epsilon(f^{q_n}, f^{q_{n+1}}) \;\cup\; (B(c_0, \epsilon) \cap (D \cup D'))$$

(see Figure 13). By Corollary 6.6, c_0 is a measurable deep point of E, so ξ is $C^{1+\alpha}$-conformal at the critical point, by the Boundary Conformality Theorem 5.4.

Since in the linear coordinate, $\psi | S^1$ is a contraction by γ, $0 < \gamma < 1$, we have $0 < |\psi'(c_0)| < 1$. The germ of $\psi | \partial D$ at c_0 is independent of n, so $\psi'(c_0)$ is also independent of n.

Finally, when s is odd the map $\phi : L_n \to L_{n+s}$ conjugating $\mathcal{R}_n(F)$ to $\mathcal{R}_{n+s}(F)$ has the form $\phi(w) = \overline{w}^\gamma$. Thus we may promote $\overline{\phi(z)}$ to a quasiconformal conjugacy $\overline{\psi(z)}$ between $(\overline{f^{q_n}(z)}, \overline{f^{q_{n+1}}(z)})$ and the complex conjugate dynamical system $(\overline{f^{q_{n+s}}(\overline{z})}, \overline{f^{q_{n+s+1}}(\overline{z})})$. Then $\overline{\psi(z)}$ has the same properties enjoyed by $\psi(z)$ for s even. ∎

Let

$$A(z) = \begin{cases} c_0 + \dfrac{\partial \psi}{\partial z}(c_0)(z - c_0), & s \text{ even}, \\ c_0 + \dfrac{\partial \psi}{\partial \overline{z}}(c_0)\overline{(z - c_0)}, & s \text{ odd}, \end{cases} \tag{7.1}$$

denote the linearization of ψ at the critical point.

Corollary 7.3 *The blowups $A^{-n}(\partial D)$ of the boundary of the Siegel disk converge exponentially fast to an A-invariant quasicircle through ∞, in the Hausdorff metric on compact subsets of the sphere.*

Proof. Fix any conjugacy ψ as in the Theorem and let $\lambda = |\psi'(c_0)|$ or $|\overline{\psi}'(c_0)|$, depending on whether s is even or odd. Then $\lambda < 1$. For $n \gg 0$ choose a segment $I \subset \partial D$ of diameter about λ^n such that $A^{-n}I$ is the component of $A^{-n}(\partial D) \cap B(c_0, 1)$ passing through the critical point. Then $\psi(I) \subset \partial D$ and since ψ is $C^{1+\alpha}$, the Hausdorff distance d_H satisfies

$$d_H(A^{-n}I, A^{-(n+1)}\psi(I)) \;\leq\; \lambda^{-n} d_H(I, A^{-1}\psi(I)) \;=\; O(\lambda^{-n}(\operatorname{diam} I)^{1+\alpha}) \;=\; O(\lambda^{\alpha n}).$$

Thus the blowups of ∂D converge exponentially fast at scale 1 around the critical point. A similar argument holds on the sphere. ∎

Using the fact that $\psi(c_{q_n}) = c_{q_{n+s}}$ for all $n \gg 0$, it is also easy to verify:

Corollary 7.4 *There exist $d_0, \dots d_{s-1} \in \mathbb{C}$ such that the closest returns of the critical orbit satisfy*

$$c_{q_n} \;=\; A^m(d_i) + O(|c_0 - c_{q_n}|^{1+\alpha}),$$

where $n = ms + i$.

Spiraling. A Jordan curve C in the plane *spirals* about $c \in C$ if any continuous branch of $\arg(z - c)$ on $C - \{c\}$ is unbounded. The curve C does *not* spiral if and only if there exists an n such that one petal of $\sqrt[n]{C - c}$ lies in a half-plane.

Corollary 7.5 *If the period s is odd, then the boundary of the Siegel disk does not spiral about the critical point.*

Proof. It suffices to show the Hausdorff limit $X = \lim A^{-n}(\partial D)$ does not spiral about c_0. Pick any point p in the limiting disk $D_\infty = \operatorname{int} \lim A^{-n}(\overline{D})$, and let $\gamma \subset D_\infty$ be a path joining p to $A(p)$. Then

$$\delta = \gamma \cup A(\gamma) \cup A^2(\gamma) \cup \cdots$$

converges to c_0 without spiraling, since alternate segments wind in opposite directions. Since δ is disjoint from X, the latter does not spiral either. ∎

8 The limiting dynamics

In this section we describe more completely the limiting dynamical system obtained by rescaling the first return maps around the critical point. As in the preceding section, $f(z) = e^{2\pi i\theta}z + z^2$, where $\theta = [a_1, a_2, \ldots]$ and $a_i = a_{i+s}$ for $i \gg 0$.

Let b_i, $i \in \mathbb{Z}$ be the unique sequence with period s that agrees with a_i for $i \gg 0$. Let $\theta_i = [b_i, b_{i+1}, \cdots]$, and define t_i so $t_0 = \theta_1$ and $t_i = -\theta_{i+1}t_{i-1}$.

Let $A(z)$ be the linearization of the contracting self-similarity map given by (7.1). Let $D_\infty \subset \mathbb{C}$ denote limit of the rescaled Siegel disks; it is the unique quasidisk such that $A^{-n}(\overline{D}) \to \overline{D_\infty}$ in the Hausdorff topology.

Theorem 8.1 (Limiting dynamics) *There exists a sequence of analytic functions $g_i : W_i \to \mathbb{C}$, $i \in \mathbb{Z}$, such that*

1. *W_i is an open disk with $\overline{D_\infty} \subset W_i \subset \mathbb{C}$;*
2. *∂W_i is the natural boundary of g_i;*
3. *g_i is a σ-proper branched covering of the plane;*
4. *$g_i(z) = \lim_{n\to\infty} A^{-n} \circ f^{q_{ns+i}} \circ A^n(z)$;*
5. *on each compact subset of W_i, this limit converges uniformly and exponentially fast;*
6. *for all $z \in W_i$, $g_i(z) = g_{i-1}^{b_i} \circ g_{i-2}(z) = A^{-1} \circ g_{i+s} \circ A(z)$; and*
7. *there is a Riemann mapping $D_\infty \to \mathbb{H}$ which conjugates the mappings $\langle g_i(z)\rangle$ to the translations $\langle T_i(z) = z + t_i\rangle$.*

Remark on universality. By the results of §5, any quadratic-like map h with a fixed point of multiplier $e^{2\pi i\theta}$ is $C^{1+\alpha}$-conjugate to f on the boundary of its Siegel disk. Thus $P(h)$ is self-similar about $c_0(h)$ with the same rescaling factor as $P(f)$, Corollaries 7.3, 7.4 and 7.5 also hold for h, and the rescalings of h^{q_n} converge to the same dynamical system $\langle g_i\rangle$ as above.

Before embarking on the proof a few preliminaries are in order.

Branched coverings and σ-proper maps. Let $g : X \to Y$ be a holomorphic map between Riemann surfaces. Then g is:

- *proper*, if $g^{-1}(K)$ is compact for each compact $K \subset Y$;
- *σ-proper*, if each component of $g^{-1}(K)$ is compact for each compact $K \subset Y$; and
- a *branched covering*, if each $y \in Y$ has a neighborhood V such that $g : U \to V$ is proper for each component U of $g^{-1}(V)$.

It is not hard to show that each condition implies the next. Also, g is σ-proper if and only if X and Y can be expressed as increasing unions of subsurfaces X_i, Y_i such that $g : X_i \to Y_i$ is proper (hence the terminology). The trigonometric function $\sin : \mathbb{C} \to \mathbb{C}$ is a branched covering which is not σ-proper, because $\sin^{-1}([-1,1]) = \mathbb{R}$ is not compact. The maximal analytic continuation $\widetilde{F}$ of the Feigenbaum fixed-point, on the other hand, is a σ-proper branched covering of the plane (cf. [Mc4, §7]). The composition of two σ-proper maps is also σ-proper; this property fails for branched coverings.

Proposition 8.2 *Let $g : U \to \mathbb{C}$ be a branched covering. Then every $p \in \partial U$ is a singular point of g.*

Proof. Suppose to the contrary there exists a $p \in \partial U$ and an analytic function h extending g near p. Then the branched covering condition implies $g^{-1}(h(p))$ accumulates on p, so h is constant, which is impossible. ∎

Carathéodory convergence. A sequence of pointed open disks (U_n, u_n) in $\widehat{\mathbb{C}}$ converges to (U, u) in the *Carathéodory topology* if $u_n \to u$, and if for any Hausdorff limit $K = \lim(\widehat{\mathbb{C}} - U_{n_k})$, U is the component of $\widehat{\mathbb{C}} - K$ containing u.

Theorem 8.3 (Limits of proper maps) *Let*

$$g_n : (U_n, u_n) \to (V_n, v_n)$$

be a sequence of proper maps between pointed disks, with $\deg(g_n) \leq d$. *Suppose $u_n \to u$, g_n converges uniformly to a nonconstant limit on a neighborhood of u, and $(V_n, v_n) \to (V, v)$. Then (U_n, u_n) converges to a pointed disk (U, u), and g_n converges uniformly on compact subsets of U to a proper map*

$$g : (U, u) \to (V, v)$$

with $1 \leq \deg(g) \leq d$.

Proof. By [Mc3, Theorems 5.2 and 5.6], we can obtain convergence of (U_n, u_n) and g_n after passing to a subsequence. But any two subsequences have the same limit, by convergence near u. ∎

Harmonic measure and exponentially fast convergence. Let $E \subset X$ be a closed subset of a Riemann surface. The *harmonic measure* $\omega_E(z)$ is the probability that a Brownian path starting at z will land in E before exiting X. It is a harmonic function on $X - E$. If $\omega_E(z) > 0$ then E has *positive harmonic measure*. The "two-constant theorem" controls a bounded holomorphic function on X in terms of its behavior on E.

Theorem 8.4 *Let $f : X \to \mathbb{C}$ be an analytic function, with $|f(z)|$ bounded by M on X and by m on E. Then $|f(z)| \leq M^{1-\delta} m^{\delta}$ on $\{z : \omega_E(z) \geq \delta\}$.*

See [Ah2, Theorem 3-2].

Let us say $f_n \to f$ *exponentially fast on* E if there exist C and $\lambda < 1$ such that

$$\sup_E |f_n(z) - f(z)| < C\lambda^n.$$

Corollary 8.5 *Let f and $\langle f_n \rangle$ be analytic functions, uniformly bounded on compact subsets of X, and suppose $f_n \to f$ exponentially fast on a set $E \subset X$ of positive harmonic measure. Then $f_n \to f$ exponentially fast on any compact subset of X.*

Of course the C and λ depend on the compact set in question.

Proof of Theorem 8.1 (Limiting dynamics). To simplify notation, we will discuss the branched covering $g_0 : W_0 \to \mathbb{C}$ in detail, the case of g_i being analogous.

Let $\psi : \overline{D} \to \overline{D}$ be a self-similarity homeomorphism of the type provided by Theorem 7.2. That is, for $n \gg 0$, ψ conjugates f^{q_n} to $f^{q_{n+s}}$ on a neighborhood of c_0, and

$$\psi(z) = A(z) + O(|z - c_0|^{1+\alpha}). \tag{8.1}$$

We first show for $z \in D_\infty$,

$$\lim A^{-n} \circ f^{q_{ns}} \circ A^n(z)$$

exists, and the convergence is exponentially fast on compact subsets of D_∞. Let $D_n = A^{-n}(D)$ denote the nth dilate of the Siegel disk, and let

$$f_{0,n} \;=\; A^{-n} \circ f^{q_{ns}} \circ A^n \;:\; \overline{D_n} \to \overline{D_n}.$$

Then

$$\psi_n \;=\; A^{-n} \circ (A^{-1} \circ \psi) \circ A^n \;:\; \overline{D_n} \to \overline{D_{n+1}}$$

conjugates $f_{0,n}$ to $f_{0,n+1}$ on a definite neighborhood of c_0; on the other hand, ψ_n converges to the identity exponentially fast by (8.1). Thus $\langle f_{0,n}(z) \rangle$ converges exponentially fast on some compact ball $B \subset D_\infty$.

Now each $f_{0,n}$ is a conformal rotation of a quasidisk, so it can be extended to a K_0-quasiconformal map of the plane, where K_0 is independent of n. By convergence of $f_{0,n}|B$, the sequence $\langle f_{0,n} \rangle$ is bounded and equicontinuous on compact subsets of D_∞. By Corollary 8.5 above, $\langle f_{0,n} \rangle$ converges exponentially fast on compact subsets of D_∞.

To proceed further we need some additional notation. For $i \in \mathbb{Z}$, let

$$d_i \;=\; \lim A^{-n}(c_{q_{ns+i}}) \;\in\; \partial D_\infty$$

denote the limiting position of the rescaled closest returns of the critical point (cf. Corollary 7.4). Then $d_{i+s} = A(d_i)$, and $\{d_i : i \in \mathbb{Z}\}$ is a finite union of geometric sequences converging to the critical pont.

Just as the boundary of the Siegel disk divides $\mathbb{C}$ into two domains, D and Ω, the quasiarc ∂D_∞ divides $\mathbb{C}$ into two domains, D_∞ and Ω_∞. Mimicking the notation $[c_i, c_j] \subset \partial D$, we let $[d_i, d_j] \subset \partial D_\infty$ denote the unique quasiarc in ∂D_∞ with endpoints d_i and d_j.

We now extend the domain of convergence of $A^{-n} \circ f^{q_{ns}} \circ A^n$ beyond D_∞. Let $\Omega' = \Omega - S$ where S is an arc (e.g. an external ray) connecting c_1 to ∞ through Ω. By cutting along S we make Ω simply-connected. Consider the open disk

$$V_{k,n} \quad = \quad \text{int}\, A^{-n}(D \cup \Omega' \cup [c_{q_{ns-k}}, c_{q_{ns-k-1}}]),$$

obtained by joining D and Ω' together along an open arc in ∂D, then rescaling. Note that $V_{k,n} \supset D_n$. Picking basepoints $p_n \in D_n$ converging to $p \in D_\infty$, we have

$$(V_{k,n}, p_n) \to (V_k, p) \quad = \quad \text{int}(D_\infty \cup \Omega_\infty \cup [d_{-k}, d_{-k-1}])$$

in the Carathéodory topology. Note that the slit S moves off to infinity and disappears in the limit. There is a unique disk $U_{k,n} \supset D_n$ such that

$$f_{k,n} = A^{-n} \circ f^{q_{ns}} \circ A^n : U_{k,n} \to V_{k,n}$$

is a proper map. (The disk $U_{k,n}$ is a component of the preimage of $V_{k,n}$ under $A^{-n} \circ f^{q_{ns}} \circ A^n$.)

We claim that for fixed k, the degree of the proper map $f_{n,k}$ is eventually constant as $n \to \infty$. Indeed, the domain and range of $f_{k,n}$ are simply connected, so $\deg(f_{k,n})$ is one more than the number of its critical values. But the critical values of $f_{k,n}$ correspond to a subset of the critical values $\{c_1, c_2, \ldots, c_{q_{ns}}\}$ of $f^{q_{ns}}$, namely those lying in the interior of the arc $[c_{q_{ns-k}}, c_{q_{ns-k-1}}]$. Since the continued fraction expansion of θ is preperiodic, this subset has constant cardinality for all n sufficiently large, and thus $\deg(f_{k,n})$ is also eventually constant.

Now we already have convergence of $f_{k,n}|D_n$ as $n \to \infty$, so by Theorem 8.3 there is a pointed disk (U_k, p) and a proper map

$$f_k : (U_k, p) \to (V_k, p)$$

such that $f_{k,n} \to f_k$ uniformly on compact sets of U_k as $n \to \infty$. Each of these limits extends the domain of convergence of $A^{-n} \circ f^{q_{ns}} \circ A^n$.

Clearly $V_{k,n} \subset V_{k+1,n}$ and thus $V_k \subset V_{k+1}$. Similarly $U_k \subset U_{k+1}$ and $f_k = f_{k+1}|U_k$. Letting $W_0 = \bigcup U_k$, and noting $\mathbb{C} = \bigcup V_k$, we obtain a branched covering

$$g_0 : W_0 \to \mathbb{C}$$

by setting $g_0 = \bigcup f_k$. Since each f_k is proper, the map g_0 is σ-proper, and by Proposition 8.2) ∂W_0 is its natural boundary. Since $\overline{D_\infty} \subset \bigcup V_k$, we also have $\overline{D_\infty} \subset W_0$. Finally we have shown above that $A^{-n} \circ f^{q_{ns}} \circ A^n \to g_0$ exponentially fast on compact subsets of D_∞, so by Corollary 8.5 the convergence is also exponentially fast on compact subsets of W_0.

This completes the proof of statements (1) – (5) of Theorem 8.1. Statement (7) follows easily by a linear computation similar to Theorem 2.1.

To verify (6), we must show $g_{i-1}^{b_i} \circ g_{i-2}$ and $A^{-1} \circ g_{i+s} \circ A$ are defined on W_i. First note that

$$g_i = g_{i-1}^{b_i} \circ g_{i-2}$$

when restricted to D^∞. Since each g_i is σ-proper, the composition on the right defines a branched covering of the plane, as does the map on the left. A branched covering of the plane admits no analytic continuation, so the domains of these two mapping must agree, and thus equality holds on W_i.

The same principle establishes the rescaling equation $g_i = A^{-1} \circ g_{i+s} \circ A$. ∎

A Appendix: Promoting combinatorial equivalence to conjugacy

Let $X = \Delta/\Gamma$ be a hyperbolic Riemann surface, presented as the quotient of the unit disk by a Fuchsian group Γ. Let $\Omega \subset S^1$ be the complement of the limit set of Γ. Then $(\Delta \cup \Omega)/\Gamma$ is a surface with boundary Ω/Γ, the *ideal boundary* of X.

Any quasiconformal homeomorphism between hyperbolic surfaces extends to the ideal boundary.

Let $X_1 \subset X_0$ be a pair of complex 1-manifolds (perhaps disconnected) with each component hyperbolic, and let $f : X_1 \to X_0$ be a holomorphic covering map. A *combinatorial equivalence* from f to another such dynamical system $g : Y_1 \to Y_0$ is a pair of quasiconformal maps $\phi_i : X_0 \to Y_0$, $i = 0, 1$, such that

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_0 \\ {\scriptstyle \phi_1}\downarrow & & {\scriptstyle \phi_0}\downarrow \\ Y_1 & \xrightarrow{g} & Y_0 \end{array}$$

commutes, and ϕ_0 is isotopic to ϕ_1 *rel* the ideal boundary of X_0. If $\phi_0 = \phi_1$ then ϕ_0 is a *quasiconformal conjugacy*.

Theorem A.1 (Combinatorial promotion) *Any combinatorial equivalence (ϕ_0, ϕ_1) is isotopic,* rel *ideal boundary, to a quasiconformal conjugacy ψ. Moreover ψ can be chosen to agree with ϕ_1 on $X_0 - X_1$.*

Proof. The proof relies on some basic facts about natural extensions and quasiconformal isotopies treated in detail in [DE] and [EaM].

Using [EaM, Thm 1.3 and Prop 2.3], the initial isotopy $\phi : [0, 1] \times X_0 \to Y_0$ may be chosen to be uniformly continuous in the hyperbolic metric. That is, we can arrange that

$$d_{Y_0}(\phi_t(x), \phi_s(x)) \quad \leq \quad \delta(|s - t|) \tag{A.1}$$

where $\delta(r) \to 0$ as $r \to 0$, and distance is measured along the geodesic arc homotopic to the trace of the isotopy.

Roughly speaking, ϕ_1 gives a map on the "fundamental domain" $X_0 - X_1$ which is a conjugacy on the ideal boundary, so it can be pulled back repeatedly to obtain conjugacies on larger and larger regions. In the limit one obtains a conjugacy outside a region Ω mapped to itself by f. A canonical representative of the map on Ω will automatically commute with coverings, and give a conjugacy there.

More formally, let $X_n = f^{-n}(X_0)$ and $Y_n = g^{-n}(Y_0)$. Then there is a unique extension of ϕ_t to an isotopy defined for all $t \in [0, \infty)$ such that

$$\begin{array}{ccccc} \cdots X_2 & \xrightarrow{f} & X_1 & \xrightarrow{f} & X_0 \\ \downarrow{\scriptstyle \phi_{t+2}} & & \downarrow{\scriptstyle \phi_{t+1}} & & \downarrow{\scriptstyle \phi_t} \\ \cdots Y_2 & \xrightarrow{g} & Y_1 & \xrightarrow{g} & Y_0 \end{array}$$

commutes for $t \in [0, 1]$, and $\phi_t(x) = \phi_n(x)$ for $t > n$ and $x \notin X_n$. Here we use the fact that the hyperbolic metric on X_n blows up at ∂X_n to insure that $\phi_t(x)$ and $\phi_n(x)$ agree there for $t \in [n, n+1]$. Indeed, we have

$$d_{Y_n}(\phi_t(x), \phi_s(x)) \quad \leq \quad \delta(|s-t|)$$

for $x \in X_n$ and $s, t \geq n$, because the covering maps f and g are local hyperbolic isometries. Since inclusions are contracting, (A.1) also holds for all $s, t \geq 0$ and $x \in X_0$.

We have $K(\phi_n) \leq K(\phi_0)$ for all n, and all ϕ_n are in the same isotopy class, so after passing to a subsequence we can construct a limiting combinatorial equivalence

$$(\alpha_0, \alpha_1) \quad = \quad \lim\, (\phi_n, \phi_{n+1}).$$

Here we use (A.1) to obtain, along a further subsequence, a bounded isotopy $\alpha_t = \lim \phi_{t+n}$ connecting α_0 to α_1.

Let $\Omega = \operatorname{int} \bigcap X_n$. Then $\alpha_0(x) = \alpha_1(x)$ outside Ω. Moreover, α_t stabilizes Ω and $\alpha_t|\Omega$ is an isotopy *rel* ideal boundary. On each component of Ω, replace α_0 with the unique Douady-Earle map in the same isotopy class. (This map is obtained by lifting to the universal covers, and applying the Douady-Earle extension to the boundary values on S^1.) Since $f : \Omega \to \Omega$ is a holomorphic covering and the Douady-Earle extension is natural, these replacements are compatible with the dynamics and the resulting map ψ is a quasiconformal conjugacy. ∎

Remark. The dilatation of the conjugacy ψ is bounded in terms of $K(\phi_0)$. The only increase in dilatation comes by applying the Douady-Earle extension.

Examples.

1. Let f be a critically finite rational map on the sphere. Then $f : X_1 \to X_0$ is a covering map, where $X_0 = \widehat{\mathbb{C}} - P(f)$ and $X_1 = f^{-1}(X_0)$. In this setting, the notion combinatorial equivalence was introduced by Thurston, who proved that combinatorially equivalent rational maps are *conformally* conjugate, apart from certain Lattès examples [DH2], [Mc3, §B].

2. A similar principle holds when $P(f)$ is infinite. Namely, given a pair of isotopic quasiconformal maps $\phi_i : (\widehat{\mathbb{C}}, P(f)) \to (\widehat{\mathbb{C}}, P(g))$, $i = 1, 2$ with $g \circ \phi_1 = \phi_0 \circ f$, there is a quasiconformal conjugacy in the same isotopy class. This principle is used by Sullivan in his proof of rigidity of the Feigenbaum polynomial [Sul].

3. Consider a holomorphic family $f_\lambda(z)$ of rational maps such that the critical orbit relations are locally constant. By the theory of holomorphic motions, any two members of the family are combinatorially equivalent. Then they are also quasiconformally conjugate, by the Theorem. Compare [McS, Thm 7.1].

4. If X_0 is connected and $\overline{X_1}$ is a proper compact subset of X_0, then $f : X_1 \to X_0$ is an *expanding conformal dynamical system*. That is, $\|f'\| > c > 1$ in the hyperbolic metric on X_0. This expansion implies, for example, that $K(f) = \bigcap f^{-n}(X_0)$ is a compact nowhere dense set of Hausdorff dimension < 2. The Theorem provides structural stability for these mappings, since it is easy to construct a combinatorial equivalence from f to a small perturbation g of f.

5. The mapping $f(z) = z^2 + \lambda/z^3$, where λ is small but not zero, is described in [Mc2]. For a suitable annulus A centered at zero, we have $f^{-1}(A) = I \sqcup O \subset A$, where I and O are annuli covering A with degree 3 and 2 respectively. It is easy to construct a combinatorial equivalence from $f : (I \sqcup O) \to A$ to $g : (I' \sqcup O') \to A'$, where I', O' and A' are round annuli centered at $z = 0$, $g(z) = z^2$ in O' and $g(z) = \lambda/z^3$ in I'. Clearly $K(g)$ is a Cantor set of round circles, so $K(f)$ is a Cantor set of quasicircles.

   A rather different approach to this example appears in [Bea].

A prototype of Theorem A.1 appears in [Mc1, Proposition 8.1].

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