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Differential Algebra and Liouvillian first integrals of foliations

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ABSTRACT

Intuitively, a complex Liouvillian function is one that is obtained from complex rational functions by a finite process of integrations, exponentiations and algebraic operations. In the framework of ordinary differential equations the study of equations admitting Liouvillian solutions is related to the study of ordinary differential equations that can be integrated by the use of elementary functions, that is, functions appearing in the Differential Calculus. A more precise and geometrical approach to this problem naturally leads us to consider the theory of foliations. This paper is devoted to the study of foliations that admit a Liouvillian first integral. We study holomorphic foliations (of dimension or codimension one) that admit a Liouvillian first integral. We extend results of Singer (1992) [20] related to Camacho and Scárdua (2001) [4], to foliations on compact manifolds, Stein manifolds, codimension-one projective foliations and germs of foliations as well.

Introduction

This paper is based on the original work of Singer [20] on the existence of Liouvillian first integrals for systems of polynomial complex ordinary differential equations. Roughly speaking, a Liouvillian function is one that is obtained from polynomials by a finite process of integrations, exponentiations and algebraic operations. In the complex setting, the notion of Liouvillian function allows us to ask for differential equations that have a first integral of a more general type than the holomorphic or meromorphic one. Due to some geometrical and analytical aspects, this question is more properly addressed in the framework of holomorphic foliations. The problem of deciding whether a germ of holomorphic singular codimension-one foliation has a holomorphic first integral is studied in [13] where one finds a topological characterization. In the algebraic case one has the so called Theorem of Darboux that assures the existence of a rational first integral for a foliation on the complex projective plane admitting infinitely many algebraic leaves [10]. This result has been extended to codimension-one foliations on compact manifolds admitting infinitely many compact analytic invariant hypersurfaces [1,7]. A theorem assuring the existence of a meromorphic first integral for a proper parabolic foliation with isolated singularities on a Stein Surface is proved in [22]. In this same paper it is given an example of a germ of singular foliation which is topologically conjugated to one that has a meromorphic first integral but that does not admit a meromorphic first integral. However, this example admits a first integral of Liouvillian type. Thus we can ask.

Problem. Is it true that a germ of holomorphic foliation in dimension two which is topologically conjugated to one having a meromorphic first integral has a Liouvillian first integral?

This question is answered in a negative way in Section 11. In [20] it is proved that if a polynomial foliation on $\mathbb{C}^2$ has a non algebraic solution satisfying a Liouvillian relation then the foliation has a Liouvillian first integral of a very simple form (from the Differential Algebra viewpoint). The classification of such foliations is studied, in terms of hypotheses on the singular set, in [4]. A more general study, related to the existence of an affine transverse structure is given in [17,18].

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The aim of this paper is to give a geometrical approach to the work of Singer and to extend the results of [20] to foliations (mostly foliations by curves) on more general ambient spaces (Stein manifolds, compact complex manifolds) and to germs of singular foliations. We also extend these results to codimension-one foliations on the n-dimensional complex projective space $\mathbb{C}P^n$. This paper is organized as follows: Section 1 contains the basic language of Differential Algebra and the concepts of Liouvillian extension and of Liouvillian function that we will adopt. In Section 2 we introduce the concept of Liouvillian functions on a complex manifold that will be used in the extension to compact manifolds and to Stein manifolds as well as for germs of foliations. Section 3 is dedicated to the notion of holomorphic foliation with singularities and to the notion of affine transverse structure and its first relation with the existence of Liouvillian first integral. Section 4 consists of the basic examples of projective foliations admitting Liouvillian first integrals and of the so called *Suzuki's example* which is an example of a germ topologically conjugate to a germ admitting a meromorphic first integral but without meromorphic first integral. We study this example and show it admits a Liouvillian first integral. In Section 11 we show however that this is not a general fact. Next Section 5 begins with some basic examples of foliations admitting a Liouvillian first integral. The next part is dedicated to the proof of a slight generalization of a basic proposition found in [20] assuring the existence of a Liouvillian first integral for a foliation by curves (on a certain manifold) which has a solution satisfying a Liouvillian relation but not an analytic relation. In the last part of Section 5 we give an extension of the results of [20] for codimension-one foliations on $\mathbb{C}P^n$. Next in Section 7 one finds a main tool in the obtention of Liouvillian first integrals from the existence of a suitable Liouvillian solution. Section 8 contains the extension of Singer's results to foliations of codimension-one on projective spaces. Sections 9 and 10 are dedicated to the study of germs of (dimension-one and codimension-one) foliations at the origin $0 \in \mathbb{C}^n$. In the dimension-one case it is possible to extend [20] but in the codimension-one case there exists an interesting alternative due to the difficulty of stating a Theorem of Darboux [10] to these foliations but which is partially overcame by the use of the Weierstrass polynomials. In the end of this section we study the space of germs of singular foliations on $(\mathbb{C}^2, 0)$ which have a Liouvillian first integral. We prove in Section 11 that there are germs which are topologically conjugated to germs of foliations admitting a meromorphic first integral but which do not have a Liouvillian first integral. Section 12 is concerned with the extensions of [20] to foliations on Stein manifolds and Section 13 regards these extensions to foliations on compact manifolds. Finally, in Section 14 we give a local description of the nondegenerate germs of holomorphic differential one-forms, in dimension two, which admit a Liouvillian integrating factor of a generic type and use this description to study the codimension of this subspace in the total space.

1. Rudiments of Differential Algebra

Let us first introduce some basic material on Differential Algebra.

1.1. Derivations and differential fields

In what follows we refer to [11,15,20].

Let R be a ring (commutative with unit $1 \in R$). A *derivation* of R is a map $\delta\colon R \to R$ satisfying:

(i) $\delta(a+b) = \delta(a) + \delta(b)$.
(ii) $\delta(a \cdot b) = a \cdot \delta b + b \cdot \delta a$.

A *differential field* is a couple $(k, \triangle)$ where k is a field and $\triangle = \{\delta_i\}_{i \in I}$ is a set of derivations of k. We shall consider only *commutative* differential fields, that is the derivations $\delta_i \in \triangle$ commute $\delta_i \circ \delta_j = \delta_j \circ \delta_i, \forall\, i, j \in I$. The *constants* of $(k, \triangle)$ are the elements $c \in k$ such that $\delta_i c = 0, \forall\, i \in I$, they form a subfield $c(k, \triangle)$ of k.

A map $h\colon (k, \triangle) \to (k', \triangle')$ between two differential fields is said to be a *differential map* if: (i) The exists a map $\tau\colon \triangle \to \triangle'$. (ii) We have $h \circ \delta = \tau(\delta) \circ h, \forall\ \delta \in \triangle$.

A *differential extension* of $(k, \triangle)$ is a differential field $(\tilde{k}, \tilde{\triangle})$ where $\tilde{k}$ is an extension of k and each derivation $\tilde{\delta} \in \tilde{\triangle}$ induces by restriction an element $\delta \in \triangle$ and conversely each element $\delta \in \triangle$ extends to an element $\tilde{\delta} \in \tilde{\triangle}$. Thus it is natural to think of $\tilde{\triangle}$ as $\triangle$ extended to $\tilde{k}$ and write $(\tilde{k}, \triangle)$ in the place of $(\tilde{k}, \tilde{\triangle})$.

The following example gives an extension procedure which will be useful.

Example 1.1 (*Adjunction of a Variable*)**.** Let $\delta\colon k \to k$ be a derivation of the field k and let t be any transcendent element over k. Then δ extends in a natural way to a derivation $\tilde{\delta}\colon k(t) \to k(t)$. In fact, given any $p(t) \in k[t]$, say, $p(t) = \sum_{j=0}^{n} a_j t^j$ we define

$$\tilde{\delta}(p(t)) = \sum_{j=0}^{n} \delta(a_j) t^j + \sum_{j=0}^{n} j \cdot t^{j-1} \cdot a_j.$$

Then we can extend $\tilde{\delta}$ to the field $k(t)$ (notice that since t is not algebraic over k we have $\tilde{\delta}(p(t)) \neq 0, \quad \forall\, p(t) \in k[T]$). Let now $(k, \triangle)$ be a differential field and let t be any variable over k. The above procedure gives a differential extension $(\tilde{k}, \tilde{\triangle}), \tilde{\triangle} = \{\tilde{\delta}, \delta \in \triangle\}$, of $(k, \triangle)$.

1.2. Liouvillian extensions

Let $(k, \triangle)$ be a differential field.

Definition 1.2 (*Liouvillian Extension*). A differential extension $(k(t), \tilde{\triangle})$ of $(k, \triangle)$ is of type:

(i) *adjunction of an integral* if $\tilde{\delta}t \in k, \quad \forall \tilde{\delta} \in \tilde{\triangle}$.
(ii) *adjunction of the exponential of an integral* if $\frac{\tilde{\delta}t}{t} \in k, \forall \tilde{\delta} \in \tilde{\triangle}$.

A *Liouvillian extension of* $(k, \triangle)$ is a differential extension $(K, \tilde{\triangle})$ of $(k, \triangle)$ for which there exists a tower of differential extensions:

$$k = k_0 \subset k_1 \subset \cdots \subset k_m = K$$

such that $k_{i+1}/k_i = k_i(t_i)/k_i$ is either an algebraic extension or it is of the type adjunction of an integral or adjunction of the exponential of an integral.

The following example shows how are the *algebraic* differential extensions of a differential field of characteristic zero.

Example 1.3 (*Algebraic Extensions*). Let $(k, \triangle)$ be a differential field, where k is a field of characteristic zero, and let $k(t)/k$ be an *algebraic* field extension. Given any derivation τ of k and any polynomial $Q(T) = \sum_{j=0}^{r} q_j T^j \in k[T]$ we define

$$Q^{\tau}(T) := \sum_{j=0}^{r} \tau(q_j) T^j \in k[T]$$

and, as usual, $Q'(T) := \sum_{j=1}^{r} j q_j T^{j-1}$. Let $P(T) = \sum_{j=0}^{r} a_j T^j$ be the minimal polynomial of t over k with $a_r = 1$. Then we have $t^r + a_{r-1}t^{r-1} + \cdots + a_0 = 0$. Suppose $\tilde{\delta}$ is a derivation of $\tilde{k} = k(t)$ *over* k, that is, $\delta = \tilde{\delta}\big|_k$ is a derivation of k. Then we have $0 = \tilde{\delta}(0) = \tilde{\delta}(t^r + a_{r-1}t^{r-1} + \cdots + a_0)$ so that, since $\delta(1) = 0$,

$$\tilde{\delta}t = -\frac{t^{r-1}\delta(a_{r-1}) + \cdots + \delta(a_0)}{rt^{r-1} + (r-1)a_{r-1}t^{r-2} + \cdots + a_1} = -\frac{P^{\delta}(t)}{P'(t)}.$$

Notice that, since k is a field of zero characteristic, actually we have $P'(t) \neq 0$ so that $\frac{P^{\delta}(t)}{P'(t)} \in k(t)$. This shows how we can extend any derivation δ of a differential field $(k, \triangle)$ of characteristic zero to any algebraic extension $\tilde{k}/k$ of k, obtaining a differential extension.

The next example is the main tool in the study of Liouvillian functions on a compact complex manifold.

Example 1.4 (*Function Fields*). Let k be a field, an extension K/k is called a *function field of n variables* if the are n elements $x_1, \ldots, x_n \in K$ which are *transcendent* (that is, not algebraic) over k and algebraically independent over k, such that $K/k(x_1, \ldots, x_n)$ is a finite extension. The number n is called the *transcendency degree* of K/k. The elements $x_1, \ldots, x_n$ are called *separating variables* if $K/k(x_1, \ldots, x_n)$ is a finite and separable extension so that by the Primitive Element Theorem there is an element $y \in K$ such that $K = k(x_1, \ldots, x_n, y)$. We begin with the natural (partial) derivations

$$\frac{\partial}{\partial x_j} : k(x_1, \ldots, x_n) \longrightarrow k(x_1, \ldots, x_n), \quad j = 1, \ldots, n,$$

constructed as in Example 1.1, that is,

$$\frac{\partial}{\partial x_j}\left(\sum_{i_1,\ldots,i_n} a_{i_1\ldots i_n} x_1^{i_1} \ldots x_n^{i_n}\right) = \sum_{i_1,\ldots,i_n} i_j \cdot a_{i_1\ldots i_n} x_1^{i_1} x_2^{i_2} \ldots x_{j-1}^{i_{j-1}} x_j^{i_j - 1} x_{j+1}^{i_{j+1}} \ldots x_n^{i_n}$$

in $k[x_1, \ldots, x_n]$ and $\frac{\partial}{\partial x_j}$ is extended in the obvious way to the quotient field $k(x_1, \ldots, x_n)$. Let now $f(x_1, \ldots, x_n, y) = \sum_{i=1}^{r} a_i(x_1, \ldots, x_n) y^i = 0$ be the minimal equation of y over $k(x_1, \ldots, x_n)$. Then the polynomial $g(y) : = f(x_1, \ldots, x_n, y) \in k(x_1, \ldots, x_n)[y]$ is irreducible so that $g'(y) \neq 0$ and then $\frac{\partial f}{\partial y}(x_1, \ldots, x_n, y) \neq 0$. Let $D : K \longrightarrow K$ be any *k-derivation* (that is, $D\big|_k = 0$). Since $D(x_j^i) = i x_j^{i-1} \cdot D(x_j)$ it follows that

$$0 = D(f(x_1, \ldots, x_n, y)) = D\left(\sum_{i=1}^{r} a_i(x_1, \ldots, x_n) y^i\right) \tag{1}$$

$$= \sum_{i=1}^{r} D(a_i(x_1, \ldots, x_n)) y^i + \sum_{i=1}^{r} a_i(x_1, \ldots, x_n) i \cdot y^{i-1} \cdot D(y). \tag{2}$$

By the Chain-Rule we have $D(a_i(x_1,\dots,x_n)) = \sum_{j=1}^n \frac{\partial a_i}{\partial x_j}(x_1,\dots,x_n)\cdot D(x_j)$ so that from Eq. (2) above we have

$$0 = \sum_{i=1}^r \sum_{j=1}^n \frac{\partial a_i}{\partial x_j}(x_1,\dots,x_n)\cdot D(x_j) y^i + \sum_{i=1}^r i a_i(x_1,\dots,x_n) y^{i-1}\cdot D(y) \quad (3)$$

$$= \sum_{j=1}^n \sum_{i=1}^r \frac{\partial a_i}{\partial x_j}(x_1,\dots,x_n) y^i \cdot D(x_j) + \sum_{i=1}^r i a_i(x_1,\dots,x_n) y^{i-1}\cdot D(y) \quad (4)$$

$$= \sum_{j=1}^n \frac{\partial f}{\partial x_j}(x_1,\dots,x_n)\cdot D(x_j) + \frac{\partial f}{\partial y}(x_1,\dots,x_n)\cdot D(y) \quad (5)$$

and then

$$D(y) = -\frac{\sum_{j=1}^n \frac{\partial f}{\partial x_j}(x_1,\dots,x_n)\cdot D(x_j)}{\frac{\partial f}{\partial y}(x_1,\dots,x_n,y)}$$

which is a well-defined relation because $\frac{\partial f}{\partial y}(x_1,\dots,x_n,y) \neq 0$ as we have observed above. Given any $Z \in K$ we can write $Z = \sum_{i=0}^{r-1} b_i(x_1,\dots,x_n) y^i$ with $b_i \in k(x_1,\dots,x_n)$ so that, if we write $x = (x_1,\dots,x_n)$, then

$$D(z) = \sum_{i=0}^{r-1} D(b_i(x))\cdot y^i + i\cdot b_i(x) y^{i-1}\cdot D(y),$$

but $D(b_i(x)) = \sum_{j=1}^n \frac{\partial b_i}{\partial x_j}(x)\cdot D(x_j)$ so that

$$D(z) = \sum_{i=0}^{r-1}\left(\sum_{j=1}^n \frac{\partial b_i}{\partial x_j}(x)\cdot D(x_j)\cdot y^i + i b_i(x) y^{i-1} D(y)\right) \quad (6)$$

$$= \sum_{j=1}^n \left(\sum_{i=0}^{r-1} \frac{\partial b_i}{\partial x_j}(x) y^i\right)\cdot D(x_j) + \sum_{i=0}^{r-1} i b_i(x) y^{i-1}\cdot D(y) \quad (7)$$

$$= \sum_{j=1}^n \left[\sum_{i=0}^{r-1}\left(\frac{\partial b_i}{\partial x_j}(x) y^i - i b_i(x) y^{i-1}\right)\cdot \frac{\partial f}{\partial x_j}(x,y) \Big/ \frac{\partial f}{\partial x}(x,y)\right] D(x_j) \quad (8)$$

$$= \sum_{j=1}^n \left[\sum_{i=0}^{r-1} \frac{\partial b_i}{\partial x_j}(x) y^i + i b_i(x) y^{i-1}\cdot \frac{\partial y}{\partial x_j}\right]\cdot D(x_j) \quad (9)$$

$$= \sum_{j=1}^n \left[\sum_{i=0}^{r-1} \frac{\partial}{\partial x_j}(b_i(x) y^i)\right] D(x_j) = \sum_{j=1}^n \frac{\partial}{\partial x_j}\left(\sum_{i=0}^{r-1} b_i(x) y^i\right)\cdot D(x_j) \quad (10)$$

$$= \sum_{j=1}^n \frac{\partial}{\partial x_j}(x)\cdot D(x_j). \quad (11)$$

Therefore we can write $D = \sum_{j=1}^n D(x_j)\cdot \frac{\partial}{\partial x_j}$, $D(x_j) \in K$. Summarizing we have:

Proposition 1.5. *Let K/k be a function field of n variables so that there exist $x_1,\dots,x_n$ separating variables. Then the canonical k- derivations $\frac{\partial}{\partial x_j}: k(x_1,\dots,x_n) \to k(x_1,\dots,x_n)$ extend uniquely to k-derivations $K \to K$. Moreover the k-derivations $K \to K$ are of the form $D = \sum_{j=1}^n h_j \frac{\partial}{\partial x_j}$, where $h_j \in K$. In particular, the set of k-derivations on K is isomorphic to K^n as a vector space over K.*

1.3. Two basic tools from Differential Algebra

In order to study the Liouvillian extensions of a differential field we shall use some results found in [20,15]. A basic tool is the following proposition:

Proposition 1.6 ([11,20]). *Let $(k, \{\delta_1,\delta_2\})$ be a differential field of characteristic zero. Let $P, Q \in k$ be such that the derivation $D = P\delta_1 + Q\delta_2$ satisfies $c(k,\{\delta_1,\delta_2\}) = c(k, D)$. If there exists a Liouvillian extension $(K,\{\delta_1,\delta_2\})$ of $(k,\{\delta_1,\delta_2\})$ such that $c(K,\{\delta_1,\delta_2\})$ is a proper subset of $c(K, D)$, then there exist elements $U, V \in k$ such that:*

(i) $PU + QV = -(\delta_1 P + \delta_2 Q)$,
(ii) $\delta_2 U - \delta_1 V = 0$.

This proposition is used to prove that a foliation on $\mathbb{C}P^2$ having a Liouvillian first integral must have a Liouvillian first integral of a very simple form (cf. Theorem 5.1). Another basic tool in our approach is the following theorem due to Rosenlicht (see Theorem 1 in [15] page 488 and Corollary 1 page 489):

Theorem 1.7 (*Rosenlicht, [15]*)**.** *Let $(k, \triangle)$ be a differential field of characteristic zero and let $(K, \tilde{\triangle})$ be a differential extension of $(k, \triangle)$ with the following properties:*

(i) $c(K, \tilde{\triangle}) = c(k, \triangle)$.
(ii) *There are $u_1, \dots, u_n, v_1, \dots, v_n \in M$ with $u_i \neq 0$ and $\frac{\tilde{\delta} u_i}{u_i} + \tilde{\delta} v_i \in k, \forall \tilde{\delta} \in \tilde{\triangle}$.*
(iii) *The transcendency degree of $k(u_1, \dots, u_n, v_1, \dots, v_n)/k$ is smaller than n.*

Then:

(a) *There exists a nontrivial linear combination of $v_1, \dots, v_n$ with coefficients in $c(k, \triangle)$ which is algebraic over k.*
(b) *There exists a product $\prod_{i=1}^{n} u_i^{n_i}$, $n_i \in \mathbb{Z}$, with not all the n_i's zero, which is algebraic over k.*

We shall use this theorem in the following particular form:

Corollary 1.8 (*[15]*)**.** *Let $(k, \triangle)$ be a differential field of characteristic zero, let $(K, \tilde{\triangle})$ be a differential extension of $(k, \triangle)$ and $u, v \in K$ such that:*

(i) *$u \neq 0$ and $c(K, \tilde{\triangle}) = c(k, \triangle) = k$.*
(ii) *$\frac{\tilde{\delta} u}{u} + \tilde{\delta} v \in k, \forall \tilde{\delta} \in \tilde{\triangle}$.*
(iii) *u and v are algebraic over k.*

Then:

(a) *There exists $v_0 \in k$ such that $v - v_0 \in k$.*
(b) *There exists $r \in \mathbb{N} \setminus 0$ such that $u^r \in k$.*

2. Liouvillian functions on complex manifolds

In this section we introduce the notion of Liouvillian function in a complex manifold, according to the Differential Algebra framework and to our purposes. We begin by recalling the notion of Liouvillian function on the n-dimensional complex projective space $\mathbb{C}P^n$ as in [20]:

Definition 2.1 (*Liouvillian Function on Projective Spaces*)**.** A *Liouvillian function* on $\mathbb{C}P^n$ is an element f of a Liouvillian extension $(K, \hat{\triangle})$ of the differential field $(\mu_n, \{\frac{\partial}{\partial y_j}, j = 1, \dots, n\})$ where $\mu_n = \mathbb{C}(x_1, \dots, x_n)$ is the field of rational functions $\frac{P(x_1,\dots,x_n)}{Q(x_1,\dots,x_n)}$, $P, Q \in \mathbb{C}[x_1, \dots, x_n]$ in the variables $x_1, \dots, x_n$ and $\frac{\partial}{\partial y_j} : \mu_n \to \mu_n$ are the usual partial derivatives, $j = 1, \dots, n$.
It is clear that:

(i) $(\mu_n, \{\frac{\partial}{\partial y_j}\}_{j=1}^n)$ is a commutative differential field;
(ii) The field of constants $c(\mu_n, \{\frac{\partial}{\partial y_j}\}_{j=1}^n) = \mathbb{C}$;
(iii) Given any Liouvillian extension $(K, \tilde{\triangle})$ of $(\mu_n, \{\frac{\partial}{\partial y_j}\}_{j=1}^n)$, any element $f \in K$ defines an *analytic* function over some open dense (Zariski) subset $U_f \subset \mathbb{C}P^n$.

Given a holomorphic foliation $\mathcal{F}$ on $\mathbb{C}P^n$, a Liouvillian function F on $\mathbb{C}P^n$ is a *first integral* for $\mathcal{F}$ if given any open subset $U \subset \mathbb{C}P^n$ where F is analytic and given any leaf $\mathcal{L}$ of $\mathcal{F}$ intersecting U, the restriction $F\big|_U$ is constant on the connected components of $\mathcal{L} \cap U$.

Clearly the notion of Liouvillian function introduced above can be also defined for functions on $\mathbb{C}^n$ or on connected open subsets of $\mathbb{C}^n$; but we will give a more general notion:

Definition 2.2 (*Liouvillian Function*)**.** Let M be a connected complex manifold. Denote by $\mu(M)$ the field of meromorphic functions on M. A commutative set of derivations $\triangle = \{\delta_j\}_{j=1}^n$ of $\mu(M)$ is said to be *basic* if:

(i) $\dim M \geq n$;
(ii) $c(\mu(M), \triangle) = \mathbb{C}$;
(iii) Any derivation of $\mu(M)$ is locally a (meromorphic) linear combination of the derivations δ_j of $\triangle$.

Under these assumptions any element belonging to a Liouvillian extension of $(\mu(M), \{\delta_j\}_{j=1}^n)$ is called a *Liouvillian function* on M.

Example 2.3 (*Compact Manifolds*)**.** Let M be a *compact complex connected manifold*. Then the field $\mu(M)$ of meromorphic functions on M is a finitely generated algebraic function field with transcendency degree $\leq \dim M$. Hence there are $t_1, \dots, t_r \in \mu(M)$ such that $\mu(M)/\mathbb{C}(t_1, \dots, t_r)$ is finite, $t_1, \dots, t_r$ are not algebraic over $\mathbb{C}$ and $r \leq n = \dim M$. If we choose $t_1, \dots, t_r$ algebraically independent over $\mathbb{C}$ then $t_1, \dots, t_r$ are separating variables of $\mu(M)$ as defined in Example 1.4, thus according to Proposition 1.5 the canonical derivations $\frac{\partial}{\partial t_j} : \mathbb{C}(t_1, \dots, t_r) \to \mathbb{C}(t_1, \dots, t_r)$ extend naturally to $\mathbb{C}$- derivations $\frac{\partial}{\partial t_j} : \mu(M) \to \mu(M)$, moreover any $\mathbb{C}$-derivation $D : \mu(M) \to \mu(M)$ is of the form $D = \sum_{j=1}^{r} h_j \frac{\partial}{\partial t_j}$ where $h_j \in \mu(M)$ so that we have an isomorphism

$$\{\mathbb{C} - \text{ derivations } \mu(M) \to \mu(M)\} \cong \mu(M)^r$$

as $\mathbb{C}$-vector spaces, where $x = r$ transcendency degree of $\mu(M)$ over $\mathbb{C}$. This shows that $\{\frac{\partial}{\partial t}\}_{j=1}^{m}$ is basic (see Example 1.4). The number $r =$ transcendency degree of $\mu(M)$ is called the *algebraic dimension* of M.

Example 2.4 (*Projective Manifolds*)**.** Let M be a smooth *algebraic projective variety*. Then M is a Zariski's closed subset of some projective space, say, $M = \{p \in \mathbb{C}P^n : f_j(p) = 0, j = 1, \dots, r\}$ where $f_1, \dots, f_r$ are homogeneous polynomials. A *rational function* on M is a function $f : M \to \overline{\mathbb{C}}$ obtained as the restriction of a rational function P/Q on $\mathbb{C}P^n$. It is well-known that each meromorphic function on M is indeed the restriction of a rational function, i.e.,

$$\mu(M) = \{\text{rational functions on } M\}.$$

Thus, using the natural partial derivatives, $\frac{\partial}{\partial y_j} : \mathbb{C}(y_1, \dots, y_n) \to \mathbb{C}(y_1, \dots, y_n) \quad j = 1, \dots, n$ we can exhibit a basic set of derivations $\{\frac{\partial}{\partial y_j}\}_{j=1}^{n}$ for $\mu(M)$. Thus we can define the notion of Liouvillian function on M. Any Liouvillian function f on M defines an *analytic* function on some dense open subset of M.

Example 2.5 (*Stein Manifolds*)**.** Let M be a connected *Stein Manifold* of dimension n ([9]). Given any point $p \in M$ there are globally defined holomorphic functions $f_1, \dots, f_n \in \mathcal{O}(M)$ such that $(df_1 \wedge \cdots \wedge df_n)(p) \neq 0$. As usual let us denote by $\mu(M)$ the field of meromorphic functions on M. Let us show how we can define a basic set of derivations of $\mu(M)$, say $\{\delta_j\}_{j=1}^{n}$. Let $\delta_j : \mu(M) \to \mu(M)$ be defined in the following way: given any $f \in \mu(M)$ the differential df is a well-defined meromorphic one-form on M (holomorphic if f is holomorphic).

Claim 2.6. *There are meromorphic functions $\alpha_j : M \to \overline{\mathbb{C}}$ such that $df = \sum_{j=1}^{n} \alpha_j df_j$ and the α_j's are uniquely determined by this formula.*

Proof of the claim. Suppose that we have $df = \sum_{j=1}^{n} \alpha_j df_j$ then $df \wedge df_{j_0} = \sum_{j=1}^{n} \alpha_j df_j \wedge df_{j_0}, \forall j_0 \in \{1, \dots, n\}$. For simplicity we will assume that $n = 2$. Then the equation above shows that $\alpha_1 = \frac{df \wedge df_2}{df_1 \wedge df_2}$ and $\alpha_2 = -\frac{df \wedge df_1}{df_1 \wedge df_2}$. This shows that α_1 and α_2 are uniquely determined. Now to prove their existence we only need to define α_1 and α_2 as above to obtain a meromorphic one-form $\beta = \alpha_1 df_1 + \alpha_2 df_2$ such that $\theta = \alpha - \beta$ satisfies $\theta \wedge df_1 = 0 = \theta \wedge df_2$. But since $(df_1 \wedge df_2)(p) \neq 0$ it follows that $\theta \equiv 0$ in a neighborhood of p and therefore $\theta \equiv 0$ in M, that is, $\alpha = \sum_{j=1}^{n} \alpha_j df_j$. This proves the claim. □

Thus we can write $df = \sum_{j=1}^{n} \frac{\partial f}{\partial f_j} df_j$ replacing α_j in the notation by $\frac{\partial f}{\partial f_j} \in \mu(M)$. Define $\delta_j(f) = \frac{\partial f}{\partial f_j}, j = 1, \dots, n$. The following is straightforward:

Claim 2.7. $\{\delta_j : \mu(M) \to \mu(M)\}$ *defines a basic set of derivations of* $\mu(M)$.

Hence, again we can set the notion of Liouvillian function on M.

An interesting particular case is the following: Let M be a compact connected complex manifold such $M \setminus \Lambda$ is a Stein manifold for some codimension-one analytic subset $\Lambda \subset M$. Using the above procedure we can introduce the notion of Liouvillian function on M by considering Liouvillian extensions of the differential field $(\mu(M)|_{M \setminus \Lambda}, \{\delta_j\}_{j=1}^{n})$. For instance, we can take M as a projective space or affine space and $\Lambda \subset M$ as an algebraic hypersurface. We claim that the manifold $M \setminus \Lambda$ is a Stein manifold (cf. [5]).

Indeed, if Λ is given in homogeneous coordinates $[z] = [z_0 : \cdots : z_n]$ by an irreducible homogeneous polynomial $f(z) = 0$ of degree d then the function $\psi : M \setminus \Lambda \to \mathbb{R}$ defined by

$$f([z]) = \log\left(\frac{\left(\sum_{j=0}^{n} |z_j|^2\right)^d}{|f(z)|^2}\right)$$

where $z = (z_0, \dots, z_n) \neq 0$, is well-defined and is in fact a plurisubharmonic exhaustion of $M \setminus \Lambda$. By Levi's Extension theorem ([21]) we conclude that $M \setminus \Lambda$ is Stein.

3. Holomorphic foliations with singularities

In this paper we shall refer to *singular holomorphic foliations* on a complex manifold M of dimension $n \geq 2$. By this we mean a pair $\mathcal{F} = (\mathcal{F}', \text{sing}(\mathcal{F}))$ where $\text{sing}(\mathcal{F}) \subset M$ is a codimension ≥ 2 analytic subset of M and $\mathcal{F}'$ is a holomorphic foliation (without singularities) in the usual sense in the open manifold $M' = M \backslash \text{sing}(\mathcal{F}) \subset M$. The *leaves* of $\mathcal{F}$ are defined as the leaves of the foliation $\mathcal{F}'$. The set $\text{sing}(\mathcal{F})$ is called the *singular set* of $\mathcal{F}$. In the one-dimensional case there is an open cover $\{U_j\}_{j \in J}$ of M such that on each U_j it is defined a holomorphic vector field X_j such that: if $U_i \cap U_j \neq \emptyset$ then $X_i\big|_{U_i \cap U_j} = g_{ij} X_j\big|_{U_i \cap U_j}$ for some non-vanishing holomorphic function g_{ij} in $U_i \cap U_j$. The leaves of the restriction $\mathcal{F}\big|_{U_j}$ are the nonsingular orbits of X_j in U_j while we have $\text{sing}(\mathcal{F}) \cap U_j = \text{sing}(X_j)$. In the codimension-one case there are one-forms ω_j in the open sets U_j such that ω_j is integrable and for each nonempty intersection $U_i \cap U_j \neq \emptyset$ we have $\omega_i\big|_{U_i \cap U_j} = g_{ij}\omega_j\big|_{U_i \cap U_j}$. The leaves of $\mathcal{F}\big|_{U_j}$ are the nonsingular integral manifolds of the distribution $\text{Ker}(\omega_j)$ and the singular set is given by $\text{sing}(\mathcal{F}) \cap U_j = \text{sing}(\omega_j)$.

3.1. Transversely affine foliations

Let $\mathcal{F}$ be a codimension-one holomorphic foliation with singularities on M. The corresponding nonsingular foliation $\mathcal{F}'$ can be defined by a covering of M' by open subsets U_i, $i \in I$, and distinguished mappings $f_i \colon U_i \to \mathbb{C}$, i.e. each f_i is a holomorphic submersion and the leaves of $\mathcal{F}'\big|_{U_i}$ are the connected components of the level surfaces $f_i^{-1}(x)$, $x \in \mathbb{C}$. Whenever $U_i \cap U_j \neq \phi$ we have $f_i = f_{ij} \circ f_j$ for some local biholomorphism $f_{ij} \colon f_j(U_i \cap U_j) \subset \mathbb{C} \to f_i(U_i \cap U_j) \subset \mathbb{C}$. If $U_i \cap U_j \cap U_k \neq \phi$ then we have in the common domain the cocycle condition $f_{ij} \circ f_{jk} = f_{ik}$. The *transverse structure* of $\mathcal{F}$ in M is defined by the pseudogroup $\{f_{ij}\}_{i,j \in I}$ so that $\mathcal{F}$ has a "simple" transverse structure if this pseudogroup is "simple" for some choice. The correct meaning of the expression "simple" above is given by the notion of transversely homogeneous foliation [8] where the local biholomorphisms f_{ij} are restrictions of elements of a Lie group action on an homogeneous space. The foliation $\mathcal{F}$ is called *transversely affine* on M if we can choose the local submersions $f_j \colon U_j \to \mathbb{C}$ such that they are related by affine relations: $f_i = a_{ij} f_j + b_{ij}$ for some constants $a_{ij}, b_{ij} \in \mathbb{C}$.

The problem of deciding whether there exist affine transverse structures for a given foliation is equivalent to a problem on differential forms, see [8] for the case of real non-singular foliations:

Proposition 3.1 (*[17]*). *Let $\mathcal{F}$, M be as above. The possible transverse affine structures for $\mathcal{F}$ in M are classified by the collections (Ω_i, η_i) of differential 1-forms defined in the open sets $U_i \subset M$ such that:* (i) *(Ω_i, U_i) is like above;* (ii) *η_i is holomorphic, closed and $d\Omega_i = \eta_i \wedge \Omega_i$;* (iii) *In each $U_i \cap U_j \neq \phi$ we have $\eta_i = \eta_j + \frac{df_{ij}}{f_{ij}}$. Furthermore two such collections (Ω_i, η_i) and (Ω_i', η_i') define the same transverse affine structure for $\mathcal{F}$ in M if and only if $\Omega_i' = f_i \Omega_i$ and $\eta_i' = \eta_i + \frac{df_i}{f_i}$ for some $f_i \in \mathcal{O}(U_i)^*$.*

This proposition is concerned with the case of a foliation given by a collection of *holomorphic* one-forms. Nevertheless, often in our applications it is useful to consider foliations given by a globally defined integrable *meromorphic* one-form. For this case we have:

Proposition 3.2 (*[17,18]*). *Let Ω be an integrable meromorphic 1-form which defines $\mathcal{F}$ outside the polar divisor $(\Omega)_\infty$. The foliation $\mathcal{F}$ is transversely affine in M if and only if there exists a 1-form η in M satisfying: η is meromorphic, closed, $d\Omega = \eta \wedge \Omega$, $(\eta)_\infty = (\Omega)_\infty$ and Res $\eta = -(\text{order of } (\Omega)_\infty\big|_L)$ for each irreducible component L of $(\Omega)_\infty$, and $(\eta)_\infty$ has order one. Furthermore, two pairs (Ω, η) and (Ω', η') define the same affine structure for $\mathcal{F}$ in M if and only if there exists a meromorphic map $g \colon M \to \overline{\mathbb{C}}$ satisfying $\Omega' = g\Omega$ and $\eta' = \eta + \frac{dg}{g}$.*

A closed one-form η as above is called a closed *logarithmic derivative* of Ω. As a corollary of the above propositions we obtain:

Proposition 3.3. *Let $\mathcal{F}$ be a codimension-one singular holomorphic foliation on a complex manifold M. Assume that there is a meromorphic integrable one-form Ω on M which defines $\mathcal{F}$ off the polar set of Ω. If $\mathcal{F}$ is transversely affine on M then $\mathcal{F}$ admits a Liouvillian first integral. Indeed, $\mathcal{F}$ admits a Liouvillian first integral of the form $F = \int \frac{1}{h}\Omega$ where $h = \int \eta$ is the primitive of a closed logarithmic derivative of Ω.*

In the next section we shall investigate the converse of this proposition.

4. Some examples of foliations with Liouvillian first integral

In [4] codimension-one projective foliations (with generic singularities) admitting Liouvillian first integrals are classified as logarithmic foliations or pull-back of suitable Riccati foliations (called *Bernoulli foliations*). Let us recall these examples:

Example 4.1 (*Logarithmic (Darboux Type)* Foliations). Let M be a complex manifold, $f_j \colon M \to \overline{\mathbb{C}}$ meromorphic functions and $\lambda_j \in \mathbb{C}^*$ complex numbers, $j = 1, \ldots, r$. The meromorphic integrable one-form $\Omega = \prod_{j=1}^r f_j \cdot \sum_{i=1}^r \lambda_i \frac{df_i}{f_i}$ defines a *logarithmic (Darboux type) foliation* $\mathcal{F} = \mathcal{F}(\Omega)$ on M. The foliation $\mathcal{F}$ has $f = \prod_{j=1}^r f_j^{\lambda_j}$ as a *multiform* first integral. We remark that $f = \exp\left(\sum_{j=1}^r \lambda_j \log f_j\right)$ so that f is a Liouvillian function on M.

Example 4.2 (*Bernoulli Foliations*)**.** A *Bernoulli foliation* of degree k on $\mathbb{C}^2$ is an algebraic foliation $\mathcal{F}$ given by $\Omega = 0$ where $\Omega = p(x)dy - (y^k a(x) - yb(x))dx$ where $a(x)$, $b(x)$ and $p(x)$ are polynomials and $(x, y) \in \mathbb{C}^2 \subset \mathbb{C}P^2$ in some affine chart. We prove the existence of a first integral for $\mathcal{F}$ of Liouvillian type. First we observe that $\mathcal{F}$ can be given by

$$(k-1)\frac{\Omega}{py^k} = (k-1)\frac{dy}{y^k} - (k-1)\left[\frac{a(x)}{p(x)} - \frac{b(x)}{p(x)y^{k-1}}\right]dx = 0.$$

Let now $f(x)$ be such that $\frac{f'(x)}{f(x)} = (k-1)\frac{b(x)}{p(x)}$ and let $g(x)$ be such that $g'(x) = -\frac{a(x)}{p(x)f(x)} \cdot (k-1)$. Then $\mathcal{F}$ can be given by

$$(k-1)\frac{dy}{y^k} - (k-1)\frac{a(x)}{p(x)}dx + \frac{f'(x)}{y^{k-1}f(x)}\,dx = 0.$$

Therefore $F(x, y) = g(x) - \frac{1}{f(x)y^{k-1}}$ defines a first integral for $\mathcal{F}$ which is clearly of Liouvillian type.

Example 4.3 (*Suzuki's Example, [22]*)**.** Consider the *germ of singular foliation* (see also Definition 9.3) $\underline{\mathcal{F}}$ at the origin $0 \in \mathbb{C}^2$ given by: $\Omega = 0$ where $\Omega = (y^3 + y^2 - xy)dx - (2xy^2 + xy - x^2)dy$. The germ $\underline{\mathcal{F}}$ has the Liouvillian first integral $f(x, y) = \frac{x}{y}\exp[\frac{y(y+1)}{x}]$ and the following remarkable properties:

(i) $\underline{\mathcal{F}}$ is $\underline{\mu - \text{simple}}$, that is, it is a dicritical germ which is desingularized with only one blow-up and the resulting foliation has no singularities on the exceptional divisor, it is transverse to this projective line everywhere except for (a unique) point of tangency (see [12]).

Therefore it follows that:

(i)′ Every leaf of $\underline{\mathcal{F}}$ is a separatrix and therefore is given by some equation $(f = 0)$ where $f \in \mathcal{O}_2$.

(ii) $\underline{\mathcal{F}}$ does not admit a meromorphic first integral in any neighborhood of the origin $0 \in \mathbb{C}^2$ (see [6] for a proof).

Performing a blow-up $(y = tx)$ at the origin $0 \in \mathbb{C}^2$ we obtain the foliation

$$\tilde{\mathcal{F}} : t^3 dx + (2xt^2 + t - 1)dt = 0$$

given by the vector field

$$\dot{x} = 2xt^2 + t - 1, \quad \dot{t} = t^3.$$

The initial foliation has the Liouvillian first integral $f = \frac{x}{y}\exp(\frac{y(y+1)}{x})$ and therefore the foliation above has the Liouvillian first integral $f(x, t) = \frac{1}{t}\,e^{t(xt+1)}$. Restricting this function to the projective line $(x = 0)$ we obtain $f(0, t) = \frac{1}{t}\,e^{\frac{1}{t}}$ which is a Liouvillian function on $\overline{\mathbb{C}}$. The map $\sigma : (\overline{\mathbb{C}}, 1) \to (\overline{\mathbb{C}}, 1)$ defined by mapping the point $p \in (\overline{\mathbb{C}}, 1)$ onto the other intersection point of the leaf L_p of $\tilde{\mathcal{F}}$ though p with the projective line, is (because of the order-2 tangency) a germ of involution on $(\overline{\mathbb{C}}, 1)$. This germ is given by the relation $f(0, t) \circ \sigma = f(0, t)$, that is, $\frac{1}{t}\,e^t = \frac{1}{\sigma(t)}\,e^{\sigma(t)}$. This defines $\sigma(t)$ as a nonalgebraic Liouvillian function on $\overline{\mathbb{C}}$ and according to what we will observe in Section 9 this is enough to conclude that $\underline{\mathcal{F}}$ does not admit a nontrivial meromorphic first integral.

5. Projective foliations having Liouvillian first integrals

5.1. Existence of elementary first integrals (cf. Singer [20])

In this paragraph we refer strongly to [20]. We aim is to prove that a projective foliation in the complex plane admitting a Liouvillian first integral must admit a first integral of a very simple form, called *elementary*. Using this we are able to conclude that most foliations on $\mathbb{C}P^2$ admit no Liouvillian first integral. We recall that a function h (of Liouvillian type) is an *integrating factor* for a one-form Ω if $\frac{1}{h}\Omega$ is a closed one-form. The main part of the following theorem is found in [20].

Theorem 5.1 (*M. Singer, [20]*)**.** *Let $\mathcal{F}$ be a foliation on $\mathbb{C}P^2$ given in some affine space $\mathbb{C}^2 \subset \mathbb{C}P^2$ by the polynomial one-form $\Omega = Pdy - Qdx$. Then the following conditions are equivalent:*

(1) *$\mathcal{F}$ admits a Liouvillian first integral.*
(2) *Ω has an integrating factor of the form $h = \exp\int\eta$ where η is a closed rational one-form.*
(3) *There exists a rational one-form η satisfying:*
  (i) *$d\eta = 0$;*
  (ii) *$d\Omega = \eta \wedge \Omega$.*

Proof of Theorem 5.1 (*cf. [20]*)**.** Let us first prove that (2) and (3) are equivalent. Given a one-form Ω and a function h of Liouvillian type, a straightforward computation shows that we have $d(\frac{1}{h}\Omega) = 0$ if and only if $d\Omega = \frac{dh}{h} \wedge \Omega$. Given a closed rational one-form η such that $d\Omega = \eta \wedge \Omega$, if we write $h = \exp\int\eta$ then we obtain a Liouvillian function which is an integrating factor for Ω. Conversely, given a Liouvillian integrating factor h for Ω of the form $h \exp\int\eta$ where η is a closed rational one-form then we have $d\Omega = \eta \wedge \Omega$. Now, if Ω admits an integrating factor as in (2) then clearly $F = \int(\frac{1}{h}\Omega)$ is a Liouvillian first integral for the foliation defined by Ω. Let us now prove that (1) implies (3). If $\mathcal{F}$ admits a rational first integral then there is nothing to do: indeed, if R is a rational first integral then $\Omega = HdR$ for some rational function H and therefore $\eta = \frac{dH}{H}$ satisfies $d\Omega = \eta \wedge \Omega$ and $d\eta = 0$. Let us therefore assume that $\mathcal{F}$ admits no rational first integral. Let $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ be a polynomial vector field defining $\mathcal{F}$ in an affine space $\mathbb{C}^2 \subset \mathbb{C}P^2$. We define $k := \mu_2 = \mathbb{C}(x, y)$, $\delta_1 = \frac{\partial}{\partial x}$, $\delta_2 = \frac{\partial}{\partial y}$ as in Definition 2.1 and take a Liouvillian tower $k(X) = k_0 \subset k_1 \subset \cdots \subset k_m = K$ of $(k(X), \{\delta_1, \delta_2\})$ containing a Liouvillian first integral F of $\mathcal{F}$ as in Definition 1.2.

Claim 5.2. *The hypotheses of Proposition 1.6 are satisfied for* k, $\{\delta_1, \delta_2\}$, K *and* $D := P\delta_1 + Q\delta_2 = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ *as above.*

Proof of the claim. In fact, given any $R \in \mathbb{C}(x, y)$ such that $DR = PR_x + QR_y = 0$ we have that $R \in \mathbb{C}$ because by hypothesis the foliation $\mathcal{F}$ given by $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ does not admit a rational nonconstant first integral. This shows that $c(k, \{\delta_1, \delta_2\}) = \mathbb{C} = c(k, \{D\})$. It is also clear that $c(k, \{\delta_1, \delta_2\}) = c(K, \{\delta_1, \delta_2\})$. Since by hypothesis there is a Liouvillian first integral $F \in K$ for $\mathcal{F}$, we conclude that there exists an element $F \in K$ such that $F \notin \mathbb{C}$ but $DF = 0$, that is, $F \in c(K, \{D\}) \setminus c(K, \{\delta_1, \delta_2\})$ so that $c(K, \{\delta_1, \delta_2\})$ is a proper subset of $c(K, \{D\})$. This proves the claim. □

Using now Proposition 1.6 we conclude that there exist $U, V \in k = \mathbb{C}(x, y)$ such that

$$PU + QV = -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \quad \text{and} \quad \frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} = 0.$$

Thus $\eta = -(Udx + Vdy)$ is a closed rational one-form which satisfies $d\Omega = \eta \wedge \Omega$ where $\Omega = Pdy - Qdx$ defines $\mathcal{F}$ on $\mathbb{C}^2$. This proves that (1) implies (3) in Theorem 5.1. To prove that (3) implies (1) it is enough to observe that if we write $H = \exp\int\eta$ then $d\Omega = \eta \wedge \Omega$ implies that $d(\frac{1}{H}\Omega) = 0$, so that $F = \int\frac{1}{H}\Omega$ is a first integral for $\mathcal{F}$. □

We shall refer to a first integral of the form $F = \int\frac{1}{H}\Omega$ where $H = \exp\int\eta$ as above, as *elementary*.

Remark 5.3 (*Integrating Factors and the Integration Lemma*)**.** The following will be useful in understanding the geometrical consequences of Theorem 5.1.

(i) Given a one-form Ω, and a function h put $\eta = \frac{dh}{h}$. Then, η is closed and as we have already mentioned, an easy computation shows that $d\Omega = \eta \wedge \Omega$ if and only if h is an *integrating factor* for Ω, i.e., $d(\frac{1}{h}\Omega) = 0$.

(ii) Consider now be a closed rational one-form η on $\mathbb{C}P^2$. According to [6] and [17] we can write

$$\eta|_{\mathbb{C}^2} = \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j} + d\left(\frac{g}{\prod_{j=1}^{r} f_j^{n_j-1}}\right),$$

for some $\lambda_j \in \mathbb{C} n_j \in \mathbb{N}$ and for some irreducible polynomials f_j and g on $\mathbb{C}^2$. The polar divisor of η is given by $(\eta)_\infty \cap \mathbb{C}^2 = \bigcup_{j=1}^{r}(f_j = 0)$. We have $\eta = \frac{dh}{h}$ for the Liouvillian function $h = \exp\int\eta$ that is, h is given by $h = \prod_{j=1}^{r} f_j^{\lambda_j} \cdot \exp(\frac{g}{\prod_{j=1}^{r} f_j^{n_j-1}})$ on $\mathbb{C}^2$.

5.2. Liouvillian first integrals and affine transverse structures

The following theorem motivates the more geometric approach we use in this paper.

Theorem 5.4. *Let $\mathcal{F}$ be a codimension-one foliation on $\mathbb{C}P(n)$ admitting a Liouvillian first integral. Then $\mathcal{F}$ is transversely affine outside some algebraic invariant subset of codimension one.*

The first step in the proof of Theorem 5.4 is the following:

Lemma 5.5. *Let ω be a meromorphic 1-form, in an open subset $U \subset \mathbb{C}^n$, $n \geq 2$. Denote by X the union of the irreducible components of the zero set and polar set of ω. Assume that there exists a meromorphic 1-form η in U, which satisfies $d\omega = \eta \wedge \omega$, and which is closed. Then ω is integrable and the poles of η outside X are invariant by the foliation defined by ω.*

Proof. We have $\omega \wedge d\omega = \omega \wedge \eta \wedge \omega = 0$, so that ω is integrable. First we remark the following:

Claim 5.6. *Let X be as above. Then, the polar divisor $(\eta)_\infty$ of η has order one along (each irreducible component of) X, and consists of the union of X and an invariant divisor of $\mathcal{F}$. Moreover, the residue of η along any noninvariant irreducible component Λ of X is equal to either – (the order of the poles of ω along Λ), or (the order of the zero set of ω along Λ).*

Proof of the claim. For sake of simplicity, assume that $n = 2$. Given a generic point $p \in X$ we may choose a small neighborhood $U \supset V \ni p$ and local coordinates $(x, y) \in V$ so that $X \cap V = \Lambda \cap V = \{y = 0\}$ and $\omega\big|_V = y^k g dx$, for some holomorphic non-vanishing function $g \in \mathcal{O}(V)^*$, $k \in \mathbb{Z}$ (recall that Λ is non invariant). We write $\eta\big|_V = \lambda\frac{dy}{y} + \frac{dg}{g} + dR$ for some meromorphic function R in V and some $\lambda \in \mathbb{C}$. Let also $\eta_1 := k\frac{dy}{y} + \frac{dg}{g}$. Then the difference $\eta - \eta_1$ satisfies $(\eta - \eta_1) \wedge \omega = d\omega - d\omega = 0$, so that $\eta - \eta_1 = hdx$ for some meromorphic function h in V. Since $d\eta = d\eta_1 = 0$ we have $h = h(x)$ and therefore $(\lambda - k)\frac{dy}{y} + dR = h(x)dx$. This implies $\lambda = k$ and $dR = h(x)dx$. Therefore, $\eta\big|_V = k\frac{dy}{y} + \frac{dg}{g} + dR(x)$. This proves the claim. □

Suppose by contradiction that some component $\Gamma \subset (\eta)_\infty \setminus X$ is not invariant. At a generic point $q \in \Gamma$ we have that $\omega(q) \neq 0$, and Γ is smooth at q. Therefore, there are local coordinates $(x, y) \in V \subset U$, centered at q and such that $\omega\big|_V = gdx$ and $\Gamma \cap V = \{y = 0\}$, where $g \in \mathcal{O}^*(V)$. We can also assume that $\Gamma \cap V = (\eta)_\infty \cap V$. We have that $d\omega = \frac{dg}{g} \wedge \omega$, so that $\eta - \frac{dg}{g} = h.dx$ for some meromorphic function h in V. Since η is closed we have $dh \wedge dx = 0$, so that $h = h(x)$. Therefore it follows that $\eta\big|_V = \frac{dg}{g} + h(x)dx$, and it has polar set transverse to Γ, contradiction. This concludes the proof of Lemma 5.5. □

Proof of Theorem 5.4. Choose a polynomial 1-form Ω with codimension ≥ 2 two singular set and which defines $\mathcal{F}$ in $\mathbb{C}^n \subset \mathbb{C}P(n)$, where $\mathbb{C}P(n)\backslash\mathbb{C}^n$ is generically transverse to $\mathcal{F}$. Since $\mathcal{F}$ admits a Liouvillian first integral it follows from Theorem 5.1 that there exists there is an elementary first integral and therefore a closed rational 1-form η which is a logarithmic derivative of Ω. According to Proposition 3.2 this implies that $\mathcal{F}$ is transversely affine outside the polar set $(\eta)_\infty$. It remains to see that $(\eta)_\infty$ is invariant, what has been proved in Lemma 5.5. This proves Theorem 5.4. □

Corollary 5.7. *Let $\mathcal{F}$ be a codimension-one foliation on $\mathbb{C}P(n)$. If $\mathcal{F}$ admits a Liouvillian first integral then it has some algebraic leaf.*

Proof. Let us fix an affine system $\mathbb{C}^n \subset \mathbb{C}P(n)$ such that $\mathbb{C}P(n)\backslash\mathbb{C}^n$ is generically transverse to $\mathcal{F}$ and contains no singularity of $\mathcal{F}$. Given ω a polynomial 1-form with singular set of codimension ≥ 2, defining $\mathcal{F}$ in $\mathbb{C}^n \subset \mathbb{C}P(n)$, we take η given by Singer's theorem [16]. If η is polynomial then we have $\eta = dP$ for some polynomial and $\frac{\omega}{\exp(P)}$ is closed and entire. In particular there exists an entire first integral for $\mathcal{F}$ in $\mathbb{C}^n$. This implies that the leaves of $\mathcal{F}$ are closed in $\mathbb{C}^n$. Now, since $\mathbb{C}P(n)\backslash\mathbb{C}^n$ is generically transverse to $\mathcal{F}$ and contains no singularity of $\mathcal{F}$, we conclude that the leaves of $\mathcal{F}$ are closed in $\mathbb{C}P(n)$. Therefore it follows that the closure of each leaf of $\mathcal{F}$ is an algebraic curve. This implies, by a theorem of Darboux [7], that $\mathcal{F}$ has a rational first integral. Now we assume that η has poles in $\mathbb{C}^n$. In this case we only have to apply Lemma 5.5 above. □

Using Jouanolou's theorem ([10]) on the generality of projective foliations without algebraic leaves we obtain:

Corollary 5.8. *There exists an open dense subset of* $\mathrm{Fol}(\mathbb{C}P^2)$*, whose elements are foliations which do not admit a Liouvillian first integral.*

Nevertheless, we can go a little bit further in this direction as follows. Take a polynomial one-form $\Omega = Pdy - Qdx$ on $\mathbb{C}^2$ with $\gcd(P, Q) = 1$. Consider the rational one-form $\eta(\Omega) = \frac{P_x}{P}dx + \frac{Q_y}{Q}dy$. We have $\eta(\Omega)$ rational and $d\Omega = \eta(\Omega) \wedge \Omega$ as it is easy to check. Given any rational one-form η on $\mathbb{C}P^2$ satisfying $d\Omega = \eta \wedge \Omega$ we have $\eta = \eta(\Omega) + h\Omega$ for some rational function h on $\mathbb{C}P^2$. Conversely if h is any rational function then $\eta = \eta(\Omega) + h\Omega$ is rational and satisfies $d\Omega = \eta \wedge \Omega$. Finally, the one-form $\eta = \eta(\Omega) + h\Omega$ is closed if, and only if, we have

$$0 = d(\eta(\Omega) + h\Omega) = d\left(\frac{P_x}{P}dx + \frac{Q_y}{Q}dy + h(Pdy - Qdx)\right)$$

that is,

$$0 = \left[-\left(\frac{P_x}{P}\right)_y + \left(\frac{Q_y}{Q}\right)_x + h_xP + h_yQ + h(P_x + Q_y)\right]$$

that is equivalent to

$$(QQ_{xy} - Q_yQ_x)P^2 - (PP_{xy} - P_yP_x)Q^2 + P^2Q^2[(P_x + Q_y)h + (Ph_x + Qh_y)] = 0.$$

Let us write $h = f/g$ where f and g are polynomials on $\mathbb{C}^2$ with $< f, g >:= \gcd\{f, g\} = 1$. Then the last equation is equivalent to

$$0 = g^2[(QQ_{xy} - Q_xQ_y)P^2 - (PP_{xy} - P_xP_y)Q^2] + P^2Q^2[fg(P_x + Q_y) + P(f_xg - fg_x) + Q(f_yg - fg_y)],$$

which is an algebraic polynomial equation on (the coefficients of) f, g. Let us denote by $\mathrm{Fol}(\mathbb{C}P^2)$ the algebraic space of (dimension-one holomorphic) foliations on $\mathbb{C}P^2$. Define $\Phi : \mathbb{C}[x, y] \times \mathbb{C}[x, y] \times \mathrm{Fol}(\mathbb{C}P^2) \to \mathbb{C}[x, y]$ in the following way:

Given $(f, g, \mathcal{F}) \in \mathbb{C}[x, y] \times \mathbb{C}[x, y] \times \mathrm{Fol}(\mathbb{C}P^2)$ choose a polynomial one-form $\Omega = Pdy - Qdx$ that defines $\mathcal{F}$ on $\mathbb{C}^2$, with $\gcd(P, Q) = 1$ and define $\Phi(f, g, \mathcal{F})$ as the right hand side of the last equation above. Let

$$\pi_3 : \mathbb{C}[x, y] \times \mathbb{C}[x, y] \times \mathrm{Fol}(\mathbb{C}P^2) \to \mathrm{Fol}(\mathbb{C}P^2)$$

be the third coordinate projection.

Proposition 5.9. *The subset $\zeta \subset \mathrm{Fol}(\mathbb{C}P^2)$ formed by the foliations which admit a Liouvillian first integral coincides with $\pi_3(\Phi^{-1}(0))$ and therefore ζ is a Zariski's closed subset of $\mathrm{Fol}(\mathbb{C}P^2)$; in particular we obtain: there exists an open dense – Zariski open – subset of $\mathrm{Fol}(\mathbb{C}P^2)$, whose elements are foliations which do not admit a Liouvillian first integral.*

6. Classification of foliations with Liouvillian first integrals: generic case

We shall now reobtain the classification of projective foliations admitting a Liouvillian first integral, in the case of generic singularities that we pass to describe.

Resolution of singularities

Let $\mathcal{F}$ be a holomorphic singular codimension-one foliation with isolated singularities on a compact two dimensional complex manifold M^2. Let $\Lambda \subset M$ be an analytic invariant curve. A theorem of Seidenberg [19] gives a *resolution* of the singular points of $\mathcal{F}$ on Λ.

Theorem 6.1. *There is a finite sequence of blow-ups at the points of $\mathrm{sing}(\mathcal{F})$ such that their composition gives a proper holomorphic map $\pi : \widetilde{M} \to M$ a complex compact 2-manifold $\widetilde{M}$ and a foliation $\tilde{\mathcal{F}} = \pi^*\mathcal{F}$ with isolated singularities such that:*

(i) *$\pi^{-1}(\mathrm{sing}(\mathcal{F})) = \bigcup_{j=1}^{k} P_j$ is a finite connected union of complex projective lines with normal crossings and $\pi : \widetilde{M} \setminus \bigcup_{j=1}^{k} P_j \to M \setminus \mathrm{sing}(\mathcal{F})$ is a biholomorphism (P_o is the closure of $\pi^{-1}(\Lambda \setminus \mathrm{sing}(\mathcal{F}))$ on $\widetilde{M}$ and P_j is isomorphic to the Riemann sphere for $j \geq 1$);*

(ii) *At any singularity $p \in \bigcup_{j=1}^{k} P_j$ of $\mathcal{F}^*$ there is a local chart (x, y) such that $x(p) = y(p) = 0$ and $\mathcal{F}^*$ is given by one of the one-forms:* (ii.1) $xdy - \lambda y\, dx + \mathbf{h}.o.t, \lambda \notin \mathbb{Q}_+$ *(non-degenerate case);* (ii.2) $x^{p+1}\, dy + y(1 + \lambda x^p)dx + (h.o.t)dx, p \geq 1$ *(saddle-node case).*

In case (ii.1) we say that p is *resonant* if $\lambda \in \mathbb{Q}_-$. Let $p \in \mathrm{sing}(\mathcal{F})$, be a singular point of $\mathcal{F}$, by the Separatrix Theorem [3] the foliation $\mathcal{F}$ admits at least one *separatrix* through p; if the number of these separatrices is finite the singularity is called *non-dicritical*. This fact is equivalent to the fact that all the projective lines P_j belonging to $\pi^{-1}(p)$ are tangent to $\tilde{\mathcal{F}}$. The foliation $\tilde{\mathcal{F}}$ is called the *resolution* of the foliation $\mathcal{F}$.

Finally, as a consequence of Theorem I in [17] we obtain:

Theorem 6.2. *Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{C}P^2$ having a Liouvillian first integral. Assume that the singularities of $\mathcal{F}$ on $\mathbb{C}P^2$ are all nondicritical and produce only nondegenerate singularities after the resolution process by blow-ups. Suppose also that for each singularity $p \in \mathrm{sing}(\mathcal{F})$ its resolution produces at least one singularity of nonresonant type. Then $\mathcal{F}$ is a logarithmic (Darboux type) foliation.*

Proof. Let $\mathcal{F}$ be a foliation on $\mathbb{C}P^2$ admitting a Liouvillian first integral. According to Theorem 5.4 the foliation $\mathcal{F}$ is transversely affine off some invariant algebraic subset $\Lambda \subset \mathbb{C}P^2$ of dimension one. Because of the hypotheses on the singular set $\mathrm{sing}(\mathcal{F})$ we may apply Theorem I in [17] and conclude that $\mathcal{F}$ is a logarithmic foliation. □

7. Liouvillian solutions and Liouvillian first integrals (cf. Singer, [20])

In this section we keep on following the exposition in [20]. Some arguments however are slightly modified according to our context.

Let M be a complex manifold such that it is possible to define Liouvillian functions on M (cf. Definition 2.2). Let $\mathcal{F}$ be a singular holomorphic foliation on M ($\mathcal{F}$ has codimension $q,\ q \geq 1$).

Definition 7.1 (*Liouvillian First Integral*)**.** A Liouvillian function F on M is a *Liouvillian first integral for $\mathcal{F}$* if given any open set $U \subset M$ where F is analytic and given any leaf $\mathcal{L}$ of $\mathcal{F}$ intersecting U we have that $F\big|_{\mathcal{L}}$ is constant on each connected component of $\mathcal{L} \cap U$.

Example 7.2. Consider a foliation $\mathcal{F}$ by curves on $\mathbb{C}P^n$ given in an affine chart $\mathbb{C}^n$ by the polynomial vector field $X = \sum_{j=1}^{n} X_j \cdot \frac{\partial}{\partial y_j}$. Consider the derivation $D_X : \mathbb{C}(y_1, \ldots, y_n) \to \mathbb{C}(y_1, \ldots, y_n)$ defined by $D_X(f) = \sum_{i=1}^{n} X_j \cdot \frac{\partial f}{\partial y_j}$. Denote by $k_X = k(X)$ the field generated over $\mathbb{C}(y_1, \ldots, y_n)$ by the the coefficients X_j of X and by all its partial derivatives of all orders. It is clear that the derivations $\{\frac{\partial}{\partial y_j}\}$ extend naturally to $k(X)$. Since X is polynomial we have $k(X) = \mathbb{C}(y_1, \ldots, y_n)$ but this notation will be useful and we obtain a differential extension $(k(X), \{\frac{\partial}{\partial y_j}\})$ of $(\mathbb{C}(y_1, \ldots, y_n), \{\frac{\partial}{\partial y_j}\})$. It is also clear that the derivation D_X extends to a derivation on $k(X)$ as well as on any differential extension of this field. Therefore given any Liouvillian function F on $\mathbb{C}P^n$ we can define F to be a Liouvillian first integral of $\mathcal{F}$ if, and only if $DF = 0$ on any open subset $\mathcal{U} \subset \mathbb{C}P^n$ where F is analytic.

Now we will consider the general case: Let M be a complex manifold and $\mu(M)$ the field of meromorphic functions on M. Let $\triangle = \{\frac{\partial}{\partial y_j}\}_{j=1}^{m}$ be a basic set of derivations of $\mu(M)$ and let $X = \sum_{j=1}^{n} X_j \frac{\partial}{\partial y_j}$ be a meromorphic vector field on M. Consider

$k(X)$ and D_X to be defined as above in the natural way. The notion of *admissible solution* to be introduced below is motivated by Proposition **2.1** (i) in [20], which is a main tool that we wish to generalize. We shall give a proof of this in Proposition 7.7.

Definition 7.3 (*Admissible Solution, Liouvillian Relation*)**.** Let $y(z) = (y_1(z), \ldots, y_n(z))$ be a solution of $\mathcal{F}$ defined and *analytic* in some domain $V \subset \mathbb{C}$. We shall say that the solution $(y(z), V)$ is an *admissible solution* for $\mathcal{F}$ if each element $g \in k(X)$ is analytic non-vanishing in some dense open subset of $y(V)$. Under these conditions we say that the solution (y, V) *satisfies a Liouvillian relation* if there exists a Liouvillian function F on M – which will be necessarily holomorphic in some open dense subset of M but which may not a priori intersect $y(V)$ – such that F is analytic in some open set U intersecting the trace $y(V)$ and satisfying $F \circ y = 0$ in V. We also say that any solution – admissible or not – of $\mathcal{F}$, say, (y, V) satisfies an *analytic (algebraic* if M is an algebraic manifold) relation if there exists a nonconstant meromorphic (rational if M is algebraic projective) function P on M such that $P \circ y = 0$ on V.

Remark 7.4 (*Admissible Solutions*)**.** Given a meromorphic function f on M, the indeterminacy set of f is an analytic subset $\sigma(f) \subset M$ of codimension ≥ 2. The polar set $(f)_\infty$ and the set of zeroes $(f = 0)$ are analytic subsets of M, each either empty or of codimension one. Given a solution $y(z) = (y_1(z), \ldots, y_n(z))$ of a foliation $\mathcal{F}$ as above, defined and analytic in some domain $V \subset \mathbb{C}$ if this solution is not an admissible solution then there is a meromorphic function g in M which is identically zero on $y(V)$ or is not analytic on an open dense subset $y(V)$. In the first case, we conclude that $y(z)$ satisfies an analytic relation. In the second case, two are the possibilities. (i) It may occur that $y(V) \subset \sigma(f)$, and in this case we must have $\dim M \geq 3$ (otherwise the indeterminacy set has dimension zero). (ii) g is meromorphic but not analytic in any open dense subset of $y(V)$. Then clearly $y(V) \not\subset \sigma(g)$ (so that $\sigma(g) \cap y(V)$ has codimension ≥ 1 in $\sigma(g)$) and $y(V) \subset (g)_\infty$. Therefore $f = 1/g$ is analytic and vanishes identically on an open dense subset of $y(V)$, more precisely on $y(V) \setminus (y(V) \cap \sigma(g))$. The conclusion is that *either the solution $y(z)$ is admissible or $y(V)$ is contained in the indeterminacy set of some meromorphic function g, so that $y(V)$ is contained in a codimension ≥ 2 analytic subset of M. This occurs only if* $\dim M \geq 3$ *and, in case* $\dim M = 3$*, this implies that $y(V)$ is an analytic subset of M*. Nevertheless, this last condition, even in the case where $\dim M = 3$, does not imply that $y(V)$ satisfies an analytic relation. The reason is that a meromorphic function on M is not necessarily a quotient of holomorphic functions on M. On the other hand, on a projective space or, more generally, on a projective manifold as well as on a Stein manifold M with $H^2(M, \mathbb{Z}) = 0$, any meromorphic function is a quotient of holomorphic functions, so that in such a manifold, either a solution is admissible or it satisfies an analytic relation. As observed in [20] if $X = \sum_{j=1}^n X_j \frac{\partial}{\partial y_j}$ is a polynomial vector field, i.e., the coefficients X_j are polynomials in the variables $y_1, \ldots, y_n$ then a solution $y(z) = (y_1(z), \ldots, y_n(z))$ is admissible with respect to the differential field $(\mathbb{C}(X), \{\frac{\partial}{\partial y_j}\})$ provided that $y_1(z), \ldots, y_n(z)$ are algebraically independent over $\mathbb{C}$ or, in other words, the solution $y(z)$ does not satisfy an algebraic relation. Notice that, however, it can satisfy an analytic relation. Finally, assume now that $X_n = 1$ and the other coefficients X_j of X are polynomials whose coefficients are liouvillian functions on some variable y_n. Put $k_0 = \mathbb{C}(X)$ the differential field generated by these coefficients with respect to the derivation $\frac{partial}{\partial y_n}$ and let $k = k_0(y_1, \ldots, y_{n-1})$. Then a solution $y(z) = (y_1(z), \ldots, y_n(z))$ is admissible provided that the $y_j(z)$ are algebraically independent over k_0. Notice that since $X_n = 1$ we can assume that $y_n(z) = z$.

Lemma 7.5. *Let $\mathcal{F}$ be a dimension-one foliation on $\mathbb{C}P^n$ given by the vector field X as in* Example 7.2 *above and let (y, V) be any solution of $\mathcal{F}$ analytic on $V \subset \mathbb{C}$.*

(i) *If $n = 2$ and if (y, V) satisfies an algebraic relation then the leaf L of $\mathcal{F}$ which contains $y(V)$ is an algebraic leaf, that is, the closure $\overline{L}$ is an algebraic curve on $\mathbb{C}P^2$ invariant by $\mathcal{F}$.*
(ii) *If (y, V) does not satisfy an algebraic relation (what in the case $n = 2$ means that (y, V) is not contained in an algebraic leaf of $\mathcal{F}$) then (y, V) is an admissible solution of $\mathcal{F}$.*
(iii) *If $n = 2$ and if (y, V) does not satisfy an algebraic relation then $\mathcal{F}$ is the unique dimension-one foliation on $\mathbb{C}P^n$ which has (y, V) as a solution.*

Proof. (i) By hypothesis there exists a nonconstant polynomial $P(y_1, y_2)$ such that $P(y_1(z), y_2(z)) = 0, \forall z \in V$ where $z \mapsto (y_1(z), y_2(z)),\ z \in V$ is the solution of $\mathcal{F}$ that we are considering. Denote by L the leaf of $\mathcal{F}$ which contains this solution. Then L and the algebraic curve: $P(x_1, x_2) = 0$ coincide in an open set so that by the Identity Principle [9] they coincide globally on $\mathbb{C}P^2$.

(ii) In fact any polynomial $P(y, \ldots, y_n)$ is nonvanishing in a nonempty dense open subset of $y(V)$ so that any rational function $R = \frac{P(y_1,\ldots,y_n)}{Q(y_1,\ldots,y_n)}$ is holomorphic nonvanishing on a dense open set of $y(V)$.

(iii) Suppose $(x(z), y(z)),\ z \in V$ is a common solution of the foliations $\mathcal{F}$ and $\mathcal{F}_1$ on $\mathbb{C}P^2$ say: $\mathcal{F}$ is given by $\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}$ and $\mathcal{F}_1$ by $\frac{dy}{dx} = \frac{P_1(x,y)}{Q_1(x,y)}$ where P, Q and P_1, Q_1 are relatively prime polynomials. Then we have

$$\frac{P(x(z), y(z)}{Q(x(z), y(z))} = \frac{dy/dz}{dx/dz} = \frac{P_1(x(z), y(z))}{Q_1(x(z), y(z))}$$

so that $(PQ_1 - P_1Q)(x(z), y(z)) = 0$. By hypothesis we must have $PQ_1 - P_1Q \equiv 0$ so that $PQ_1 = P_1Q$. It follows that $\mathcal{F} = \mathcal{F}_1$ on $\mathbb{C}P^2$. $\square$

According to Lemma 7.5 above, a Liouvillian nonalgebraic solution completely determines the foliation on $\mathbb{C}P^2$. This is a good motivation for the following result found in [20].

Theorem 7.6 (*Singer, [20]*)**.** *Let $\mathcal{F}$ be a dimension-one foliation on $\mathbb{C}P^n$ having a solution (y, V) which satisfies a Liouvillian relation but which is not contained in an algebraic leaf of $\mathcal{F}$. Then $\mathcal{F}$ admits a Liouvillian first integral.*

In the case $n = 2$ it follows from Theorem 5.1 that $\mathcal{F}$ admits a Liouvillian first integral of an *elementary form*.

Theorem 7.6 is a straightforward consequence of the following key-proposition (see [20] Proposition **2.1**):

Proposition 7.7 (*Key-proposition, [20]*)**.** *Let $\triangle = \{\frac{\partial}{\partial y_j}\}_{j=1}^n$ be a basic set of derivations of $\mu(M)$ where M is a complex manifold and let $X = \sum_{j=1}^n X_j \frac{\partial}{\partial y_j}$ by any meromorphic vector field on M. Denote by $\mathcal{F}$ the dimension-one foliation defined on M by X and suppose $\mathcal{F}$ has an admissible solution (y, V) which satisfies a Liouvillian relation. Then there exists a tower of Liouvillian extensions $k(X) = k_0 \subset k_1 \subset \cdots \subset k_m = K$ of $(k(X), \{\frac{\partial}{\partial y_j}\}_{j=1}^n)$ as in Definition 2.1 such that:*

- (i) *There exist a subdomain $V_\tau \subset V$ and an open set $U \subset M$ such that $y(V_\tau) \subset U$, each function $F \in K$ defines a meromorphic function on U and,*
- (ii) *each element $f \in k_{m-1}$ satisfies: f is analytic non-vanishing in some open dense subset of $y(V_\tau)$,*
- (iii) *the map $k_{m-1} \in f \overset{\varphi}{\longmapsto} f \circ y|_{V_\tau}$ defines an isomorphism from k_{m-1} onto a differential field of functions of one variable L, meromorphic on V_τ with derivation $d\colon L \to L$ defined in the obvious way and we have from the Chain-Rule $\frac{d}{dz}(\varphi(f)) = \sum_{j=1}^n \varphi(\frac{\partial f}{\partial y_j}) \cdot \varphi(X_y)$.*
- (iv) *In particular, there exists $f \in K$ which is a (nonconstant) Liouvillian first integral for $\mathcal{F}$.*

Proof of Proposition 7.7. Let us consider $\mathcal{C}$ the collection of 3-uples (V_τ, U, F), $V_\tau \subset V$ a subdomain, $U \subset M$ an open subset with $y(V_\tau) \subset U$, F a Liouvillian function on M, analytic and not identically null on U such that $F \circ y|_{V_\tau} \equiv 0$. Since (y, V) satisfies a Liouvillian relation, $\mathcal{C} \neq \emptyset$. Let us fix $(V_{\tau_0}, U_0, F_0) \in \mathcal{C}$ such that F_0 belongs to a defining tower (as in Definition 1.2) of minimal length, say, $k(X) = k_0 \subset k_1 \subset \cdots \subset k_m = K \ni F_0$. Then clearly (y, V_{τ_0}) is an admissible solution for $\mathcal{F}$ which satisfies a Liouvillian relation, therefore we may replace the solution (y, V) by this last so that we will write $V = V_{\tau_0}$, $U = U_0$ and $F = F_0$.

Claim 7.8. *We have $m > 0$.*

In fact if $m = 0$ then $K = k(X)$ so that $F \in k(X)$ and since $F \circ y|_V \equiv 0$ it follows that (y, V) is not an admissible solution for $\mathcal{F}$ which gives a contradiction. Thus $m \geq 1$ proving the claim. We will prove (i) and (ii) for each k_j by induction on j. Let us so suppose that there exist, a subdomain $V_{\tau_{j-1}} \subset V$ and an open set $U_{j-1} \subset U$ with $y(V_{\tau_{j-1}}) \subset U_{j-1}$ such that each $f \in k_{j-1}$ is meromorphic on U_{j-1}, analytic and not identically zero on an open dense subset of $y(V_{\tau_{j-1}})$ and furthermore, $k_{j-1} =$ quotient field of $(k_{j-1} \cap \mathcal{O}(U_{j-1}))$. We write $k_j = k_{j-1}(t)$ and have three cases:

1st case. $\frac{\partial t}{\partial y_i} \in k_{j-1}, \forall\, i$ and t is not algebraic over k_{j-1}. By the induction hypothesis for each $\underline{i}$ there exists an open set $U^i \subset U_{j-1}$ such that $\frac{\partial t}{\partial y_i}$ is analytic in $U^i \subset U_{j-1}$ such that $\frac{\partial t}{\partial y_i}$ is analytic in U^i and $U^i \cap y(V_{\tau_j-1})$ is dense in $y(V_{\tau_{j-1}})$. Define $U = \bigcap_{i=1}^n U^i \neq \phi$, $U \cap y(V_{\tau_{j-1}})$ is dense in $y(V_{\tau_{j-1}})$. Since t is analytic in some dense open subset of M, there exists $p \in U$ such that t is analytic in p and we have a small open ball $B_p \ni p$, around p, with $B_p \subset U$ and also $B_p \cap y(V_{\tau_{j-1}}) \neq \phi$.

Claim 7.9. *We can define t to be analytic in B_p.*

Indeed, $\frac{\partial t}{\partial y_i}$ is holomorphic in $B_p, \forall\, i$ and we also have $\frac{\partial^2 t}{\partial y_i \partial y_\ell} = \frac{\partial^2 t}{\partial y_\ell \partial y_i}$, $\forall\, i, \ell$ therefore $\alpha = \sum_{i=1}^n \frac{\partial t}{\partial y_i} dy_i$ defines an analytic closed one-form on B_p so that $\alpha = d\zeta$ for some analytic function ζ on B_p. Clearly we have $t = \zeta + \lambda$ for some constant $\lambda \in \mathbb{C}$. This shows that t is analytic on B_p with $B_p \cap y(V_{\tau_{j-1}}) \neq \phi$. Choose now a smaller subdomain $V_{\tau_j} \subset V_{\tau_{j-1}}$ such that $y(V_\tau) \subset B_p$ and define $U_j = B_p$. By the minimality of m we have $t \equiv 0$ on $y(V_{\tau_j})$. Clearly (i) holds for this choice of U_j and V_{τ_j} for k_j. Now we will prove (ii) for k_j. In fact, let be given $p(t) = a_0 t + a_1 t + \cdots + a_r t^r$ $p(t) \in k_{j-1}[t] \subset k_j$; then $p(t)$ is meromorphic on U_j and since t is not algebraic over $k_{j-1} p(t)$ is not identically zero on U_j. It is also clear that $p(t)$ is analytic on some dense open subset of $y(V_{\tau_j})$. Since m is minimal it follows that: $i < m \Rightarrow p(t)$ is not identically zero on any open set of $y(V_{\tau_j})$ and therefore given any $f = q(t)/p(t)$, $p(t), q(t) \in k_{j-1}[t]$, then f is meromorphic on U_j and from what we have observed above f is meromorphic on U^j and f is analytic in some dense open subset of $y(V_{\tau_j})$ being not identically null if $j < m$. This shows that any element $f \in k_j = k_{j-1}(t)$ satisfies (ii). Finally, since t is analytic on U_j it follows that $k_j =$ quotient field of $(k_j \cap \mathcal{A}(U_j))$, where $\mathcal{A}(U_j) = \{$analytic functions on $U_j\} = \mathcal{O}(U_j)$.

2nd case. $\frac{\partial t}{\partial y_i}/t \in k_{j-1}, \forall\, i$ and t is not algebraic over k_{j-1}. Let $s_i = \frac{\partial t}{\partial y}$ then $\frac{\partial s_i}{\partial y_\ell} = \frac{\partial s_\ell}{\partial y_i}, \forall\, i, \ell$

Claim 7.10. *There is a Liouvillian function v such that $\frac{\partial v}{\partial y_i} = s_i, \forall\, i$.*

Indeed again $\alpha = \sum_i s_i dy_i$ is a closed one-form so that $\alpha = dv$ for some function v so that $\frac{\partial v}{\partial y_i} = s_i, \forall i$. Define now $\tilde{t} = \exp(v)$ then we can replace t by $\tilde{t}$ and argue as above to prove this case.

3rd case. t is algebraic over k_{j-1}. Let $P(T) = T^r + a_1 T^{r-1} + \cdots + a_r$ be the minimum polynomial of t over k_{j-1}. By the induction hypothesis there exists an open subset $U_P \subset \mathbb{C}P^m$ such that a_k is analytic on U_P, $\forall k$ and $U_P \cap V_{\tau_{j-1}}$ is dense in $V_{\tau_{j-1}}$. Let D be the discriminant of $P(T)$, then D is a polynomial in the a_k's with coefficients which are complex numbers and thus $D \in k_{j-1}$, D is analytic on U_P. Since $P(T)$ is irreducible $D \not\equiv 0$ on U_P and since $D \in k_{j-1}$ it follows by the induction hypothesis that $D \not\equiv 0$ on $y(V_{\tau_{j-1}})$ and therefore there exists $V_{\tau_j} \subset U_j$, $U_j =$ some component of $\{\bar{y} \in U_P \mid D(\bar{y}) \neq 0\}$, which intersects $y(V_{\tau_{j-1}})$ and t is analytic on U_j. Any $f \in k_j$ is of the form $f = b_0 + \cdots + b_r t^r$, with $b_k \in k_{j-1}$ (recall that t is algebraic over k_{j-1}, of degree $\underline{r}$). Arguing as above we can then conclude that (i) and (ii) hold for k_j. Now we consider the map

$$\varphi\colon k_{m-1} \ni f \mapsto f \circ y|_{V_\tau} \in L, \quad V_\tau = V_{\tau_{m-1}}$$

where L is defined in the obvious way. By the Chain-Rule we have

$$\frac{d}{dz}\,(\varphi(f)) = \frac{d}{dz}(f \circ y) \tag{12}$$

$$= \sum_i \frac{\partial f}{\partial y_i}(y) \cdot \frac{dy_i}{dz} = \sum_i \varphi\left(\frac{\partial f}{\partial y_i}\right) \cdot y_i'(z) \tag{13}$$

$$= \sum_i \varphi\left(\frac{\partial f}{\partial y_i}\right) \cdot X_i(y) = \sum_i \varphi\left(\frac{\partial f}{\partial y_i}\right) \cdot \varphi(X_i). \tag{14}$$

Conditions (i), (ii) obtained for k_{m-1} show that φ is a well-defined injective homomorphism and therefore is an isomorphism of k_{m-1} onto L which is a differential field of meromorphic functions of one variable. This proves (iii). Now it remains to prove (iv): Let $K = k_{m-1}(t)$ and let $f \in K$ defining the Liouvillian relation which (y, V) satisfies.

Claim 7.11. *t is not algebraic over k_{m-1}.*

Proof of Claim 7.11. Suppose on the contrary that t is algebraic over k_{m-1}, then $F \in k_{m-1}(t)$ is algebraic over k_{m-1} and then we can take $P(Y) = a_0 + \cdots + a_r Y^r$ its minimum polynomial over k_{m-1}. Since $a_0 \in k_{m-1}$ is non identically zero it follows that a_0 is analytic and not identically zero on some open dense subset of $y(V)$. But $P(F) = 0$ and $F \circ y|_V \equiv 0$ so that $a_0 \circ y|_V = 0$, contradiction. □

Therefore we have two cases to consider:

1st Case: $\frac{\partial t}{\partial y_i} \in k_{m-1}, \forall i = 1, \ldots, n$ and

2nd Case: $\frac{\partial t}{\partial y_i}/t \in k_{m-1}, \forall i = 1, \ldots, n.$

Multiplying F by suitable elements of $k_{m-1}[t]$ we may assume that $F \in k_{m-1}[t]$. The isomorphism $\varphi\colon k_{m-1} \to L$ extends therefore in a natural way to a homomorphism $\tilde{\varphi}\colon k_{m-1}[t] \to \bar{L}$ where $\bar{L}$ denotes the algebraic closure of L: In fact we remark that since $F \circ y|_V = 0$ it follows that $\varphi(F) = 0$ but since $F \in K_{m-1}[t]$ we have that $t \circ y$, that is, $\tilde{\varphi}(t)$ is algebraic over L. Since t is not algebraic over k_{m-1} it follows from Example 1.1 that φ extends to a homomorphism $k_{m-1}[t] \to \bar{L}$.

1st Case: In this case we have that

$$\frac{d}{dz}(\tilde{\varphi}(t)) = \sum_{i=1}^n \left(\frac{\partial t}{\partial y_i} \circ y\right) \cdot \frac{dy_i}{dz} = \sum_{i=1}^n \varphi\left(\frac{\partial t}{\partial y_i}\right) \cdot \varphi(X_i)$$

belongs to L. We also have that $\tilde{\varphi}(t)$ is algebraic over L so that according to Corollary 1.8 we have $\tilde{\varphi}(t) - u \in \mathbb{C}$ for some $u \in L$. Choose now $\alpha \in k_{m-1}$ such that $\tilde{\varphi}(\alpha) = u$, then

$$0 = \frac{d}{dz}(\tilde{\varphi}(t) - u) = \frac{d}{dz}(\tilde{\varphi}(t - \alpha)) = \tilde{\varphi}\left(\sum_{i=1}^n \frac{\partial}{\partial y_i}\right) = \tilde{\varphi}(D_X(t - \alpha)).$$

Since $t - \alpha \in k_{m-1}$ and since $\tilde{\varphi}$ is an isomorphism on k_{m-1} it follows that $D(t - \alpha) = 0$ now, if $\frac{\partial}{\partial y_i}(t - \alpha) = 0, \forall i$ then $t - \alpha \in \mathbb{C}$ and then $t = \alpha + (t - \alpha) \in \mathbb{C} + k_{m-1}$ so that t is algebraic over k_{m-1}, a contradiction. Therefore $t - \alpha$ is not a constant.

2nd Case: $\frac{\partial t}{\partial y_i}/t \in k_{m-1}, \forall i.$

Claim 7.12. $\tilde{\varphi}(t) \neq 0.$

Proof. In fact, since $\frac{\partial t}{\partial y_i}/t$ in analytic non identically zero on some open set of $y(V)$ it follows that t is no identically zero on some open set of $y(V)$ so that $\tilde{\varphi}(t) = t \circ y|_V$ is not zero. □

Now

$$\frac{d}{dz}(\tilde{\varphi}(t))/\tilde{\varphi}(t) = \frac{\sum_i \tilde{\varphi}\left(\frac{\partial t}{\partial y_i}\right) \cdot \tilde{\varphi}(X_i)}{\tilde{\varphi}(t)}$$

but since $\frac{\partial t}{\partial y_i}/t \in k_{m-1}$, $\forall i$ it follows that $\frac{d}{dz}(\tilde{\varphi}(t))/\tilde{\varphi}(t) \in L$. Again we remark that $\tilde{\varphi}(t)$ is algebraic over L so that according to Corollary 1.8 we have $\tilde{\varphi}(t^r) \in L$ for some $r \in \mathbb{N}$. Choose now $\beta \in k_{m-1}$ such that $\tilde{\varphi}(t^r/\beta) = 1$ and define $G = t^r/\beta$. Then $G \in k_{m-1}$ is such that $\frac{DG}{G} \in k_{m-1}$: In fact one can use the fact that $G \in k_{m-1}$ and that D induces a derivation on any differential extension of $k(X)$. We also have $\tilde{\varphi}(DG/G) = 0$ (because $\tilde{\varphi}(G) = 1$) and since $DG/G \in k_{m-1}$ where $\tilde{\varphi}$ is an isomorphism it follows that $\frac{DG}{G} = 0$ so that $DG = 0$. If $G \in \mathbb{C}$ then $\frac{t^r}{\beta} \in \mathbb{C}$ and since $\beta \in k_{m-1}$ it follows that t is algebraic over k_{m-1}, a contradiction. Therefore $G = t^r/\beta$ is not a constant. This ends the proof of Proposition 7.7. □

Remark 7.13. As it is clear from the proof above, the notion of *admissible solution* plays an important role in the search of extensions of Theorem 7.6 to more general situations.

8. Codimension-one projective foliations having a Liouvillian first integral

Proposition 7.7 above deals with the case of foliations by curves, i.e., dimension one foliations. Now we are concerned with the codimension-one case. We shall begin with the case of foliations on $\mathbb{C}P^n$. In few words a *singular codimension-one foliation on* $\mathbb{C}P^n$, consists of a pair $\mathcal{F} = (\mathcal{F}', S)$ where $S \subset \mathbb{C}P^n$ is a codimension ≥ 2 analytic subset and $\mathcal{F}'$ is a nonsingular codimension-one holomorphic foliation on $\mathbb{C}P^n \setminus S$ which cannot be extended as a regular foliation to any point of S. We call S the *singular set* of $\mathcal{F}$ and write $\text{sing}(\mathcal{F}) = S$. The notions of *Liouvillian first integral* (Definition 1.2), and solution that satisfies a Liouvillian or analytic relation are extended in a natural way to these foliations. The notion of *admissible solution* (Definition 7.3) may require some attention. Nevertheless, we recall that for dimension-one foliations in projective spaces, a solution is admissible provided that it does not satisfy an algebraic relation. This clearly generalizes to the codimension-one case with the same idea of proof (a rational function is holomorphic and dense in the complement of a Zariski's subset, except if this set is contained in its polar set or set of zeroes). Thus we shall work with solutions which do not satisfy an algebraic relation and this will be enough for our purposes in this section.

Let us denote by $\text{Fol}(\mathbb{C}P^n)$ the space of codimension-one foliations on $\mathbb{C}P^n$. An element $\mathcal{F} \in \text{Fol}(\mathbb{C}P^n)$ is defined in homogeneous coordinates $[x_0 : x_1 : \cdots : x_n] \in \mathbb{C}^{n+1}$, by an homogeneous integrable one-form $\omega = \sum A_j(x_0, \ldots, x_n)dx_j$, where the A_j are polynomials of a same degree r, $\gcd(A_0, \ldots, A_n) = 1$, and satisfying the condition $\omega \cdot \vec{R} = 0$ for the radial vector field $\vec{R} = \sum_{j=0}^n \frac{\partial}{\partial x_j}$. The integrability condition is $\omega \wedge d\omega = 0$. The one-form ω is unique up to multiplication by nonzero constants. Thus, $\text{Fol}(\mathbb{C}P^n)$ is is an algebraic projective variety and, as a generalization of the results of [20], we shall prove the following:

Theorem 8.1. *Let $\mathcal{F}$ be a codimension-one foliation on $\mathbb{C}P^n$.*

1. *$\mathcal{F}$ admits a Liouvillian first integral if and only if it admits a solution which satisfies a Liouvillian relation but not an algebraic relation.*
2. *$\mathcal{F}$ admits a Liouvillian first integral if, and only if, $\mathcal{F}$ is given in some affine space by a polynomial one-form $\Omega = \sum_{j=1}^n P_j dy_j$ which has an integrating factor of the form $h = \prod_{j=1}^r f_j^{\lambda_j} \cdot \exp\left(\frac{g}{\prod_{j=1}^r f_j^{n_j-1}}\right)$, $\lambda_j \in \mathbb{C}$, $n_j \in \mathbb{N}$, for polynomials f_j, g.*

In particular we shall obtain the following version of Proposition 5.9:

Proposition 8.2. *The set $\mathcal{T}$ of codimension-one foliations on $\mathbb{C}P^n$ which admit Liouvillian first integrals is a proper Zariski's closed subset of* $\text{Fol}(\mathbb{C}P^n)$*. In particular, there exists a Zariski's (dense) open subset of* $\text{Fol}(\mathbb{C}P^n)$*, whose elements are foliations which do not admit a Liouvillian first integral.*

In order to prove these facts we must state a basic lemma which will allow us to use the dimension two case. First we recall that a linearly embedded q-plane $\mathbb{C}P^q \subset \mathbb{C}P^n$ *is in general position with respect to* $\mathcal{F}$ if:

(a) $\mathbb{C}P^q$ is not contained in any leaf of $\mathcal{F}$.
(b) $\mathbb{C}P^q$ is transverse to any smooth strata of $\text{sing}(\mathcal{F})$.
(c) Outside $\mathbb{C}P^q \cap \text{sing}(\mathcal{F})$ the set of points of tangency of $\mathcal{F}$ with $\mathbb{C}P^q$ has codimension $\geq q-1$ in $\mathbb{C}P^q$.

Lemma 8.3. *Concerning Liouvillian functions on $\mathbb{C}P^n$ we have:*

(i) *Let F be a Liouvillian function on $\mathbb{C}P^n$ and let $\mathbb{C}P^2 \subset \mathbb{C}P^n$ be any linearly embedded 2-dimensional plane. Then there exists a Liouvillian function f on $\mathbb{C}P^2$ such that $f = F|_{\mathbb{C}P^2}$.*
(ii) *Conversely, given a codimension-one foliation $\mathcal{F}$ on $\mathbb{C}P^n$ and a linearly embedded 2-plane $\mathbb{C}P^2 \subset \mathbb{C}P^n$ in general position with respect to $\mathcal{F}$ then any Liouvillian first integral f for the induced foliation $\mathcal{F}^* = \mathcal{F}|_{\mathbb{C}P^2}$ (induced by the inclusion map $i \colon \mathbb{C}P^2 \to \mathbb{C}P^n$, $\mathcal{F}^* = i^*(\mathcal{F})$) extends to a Liouvillian first integral F for $\mathcal{F}$ on $\mathbb{C}P^n$.*

(iii) *Let $\mathcal{F}$ be a codimension-one foliation on $\mathbb{C}P^n$ and let L be a leaf of $\mathcal{F}$ such that given any linearly embedded 2-plane $\mathbb{C}P^2 \subset \mathbb{C}P^n$, in general position with respect to $\mathcal{F}$ we have that $L|_{\mathbb{C}P^2}$ satisfies an algebraic relation on $\mathbb{C}P^2$, then the closure $\overline{L}$ is algebraic codimension-one subset of $\mathbb{C}P^n$.*

(iv) *The set of all q-planes $\mathbb{C}P^q \subset \mathbb{C}P^n$ in general position with respect to $\mathcal{F}$ is an open and dense subset of the Grassmanian of q-planes in $\mathbb{C}P^n$.*

Proof. Item (i) is immediate. Item (ii) is proved in [4]. Item (iii) is an easy consequence of the fact that the condition in (iii) implies that $\overline{L}$ is an analytic subset of $\mathbb{C}P^n$ and of Chow's Theorem which states that analytic subsets of projective spaces are algebraic [9]. Finally, (iv) is the content Lemma 10 of Section 6 of [5]. □

Now we consider a holomorphic codimension-one foliation $\mathcal{F}$ on $M = \mathbb{C}P(n)$, $n \geq 3$, $S \subset M$ an invariant codimension-one irreducible divisor. We need the following extension result for adapted logarithmic derivatives.

Lemma 8.4. *If $\mathcal{F}\big|_N$ admits a logarithmic derivative adapted to $S \cap N$ for some generic section $N \subset M$ then the same holds for $\mathcal{F}$ and S on M.*

Proof. By induction arguments it is enough to prove the case where $\dim N = \dim M - 1$. Now we recall that fixed ω defined on M and defining $\mathcal{F}$, for each regular point $q \in N \setminus \operatorname{sing} \mathcal{F}$, we have $\omega = gdf$ for some meromorphic function g and some holomorphic function f, both defined in a neighborhood U of q in M. Furthermore in $U \cap N$ we have $\eta = a\frac{df}{f} + \frac{dg}{g} + d\varphi$, for some holomorphic function $\varphi : U \cap N \to \mathbb{C}$, and therefore using a local flow box for $\mathcal{F}$ around q we can extend φ and η to U, φ is extended as a holomorphic first integral for $\mathcal{F}$ in U. Two such extensions η and η' are related by $\eta - \eta' = h\omega$, where h is holomorphic in $U \cap U'$ and $d(h\omega) = 0$. Therefore $h\omega = d\psi$ for some holomorphic first integral ψ of $\mathcal{F}$ in $U \cap U'$, and since $(\eta - \eta')\big|_{U \cap U' \cap N} \equiv 0$ it follows that ψ is constant in $U \cap U' \cap N$, and since it is a transverse section to $\mathcal{F}\big|_{U \cap U'}$, it follows that $d\psi \equiv 0$ and therefore $\eta \equiv \eta'$ in $U \cap U'$. Thus we can extend η to a neighborhood of $N \setminus (\operatorname{sing} \mathcal{F} \cap N)$ and then using Hartogs' extension theorem we can extend η to a neighborhood of N (recall that $\operatorname{sing} \mathcal{F}$ has codimension-2) and then using the fact that $M \setminus N$ is a Stein manifold and the compactness of M, we can use Levi's extension theorem to extend η to M and clearly it is adapted to S on M. □

Proof of Theorem 8.1. Let $\mathcal{F}$ be a codimension-one foliation on $\mathbb{C}P^n$ having a solution that satisfies a Liouvillian relation but not an algebraic relation. Let L be the leaf of $\mathcal{F}$ that contains this solution. By Lemma 8.3 given a linearly embedded projective plane $\mathbb{E} \subset \mathbb{C}P^n$ in general position with respect to $\mathcal{F}$, the induced foliation $\mathcal{F}_1 = \mathcal{F}\big|_{\mathbb{E}}$ exhibits a leaf $L_1 \subset L \cap \mathbb{E}$ that satisfies a Liouvillian relation but no algebraic relation. This implies by Theorem 7.6 that $\mathcal{F}_1$ admits a Liouvillian first integral and again by Lemma 8.3 the foliation $\mathcal{F}$ admits a Liouvillian first integral. This proves the nontrivial part of (1) in Theorem 8.1. Assume now that $\mathcal{F}$ admits a Liouvillian first integral. Choose a polynomial one-form Ω on $\mathbb{C}^n \subset \mathbb{C}P^n$, which is integrable and defines $\mathcal{F}$ in $\mathbb{C}^n$. Put $\Omega_1 := \Omega\big|_{\mathbb{C}^n \cap \mathbb{E}}$. Notice that $\mathbb{C}^n \cap \mathbb{E}$ is an affine space $E_0 \cong \mathbb{C}^2$ embedded in $\mathbb{E}$. The one-form Ω_1 is polynomial in E_0 and defines $\mathcal{F}$ in this affine space. By Lemma 8.3 $\mathcal{F}_1$ admits a Liouvillian first integral. Therefore, according to Proposition 1.6 there is a closed rational one-form η_0 on $\mathbb{E}$ which defines $\mathcal{F}$. Now, extension Lemma 8.4 assures that η_0 extends to a closed rational one-form on $\mathbb{C}P^n$ also satisfying the condition $d\Omega = \eta \wedge \Omega$. This proves the nontrivial part of (2). □

9. Dimension-one foliations having a Liouvillian first integral

Let us denote by $\mathcal{O}_n$ *the ring of germs of holomorphic functions of n-variables at the origin* $0 \in \mathbb{C}^n$. It is well-known that $\mathcal{O}_n$ is a Noetherian local ring and an integral domain; whose elements are defined as equivalence classes of pairs (f, V) where V is an open neighborhood of the origin and $f \in \mathcal{O}(V)$ and we have $(f, V) \sim (g, W)$ if, and only if, there exists an open set $0 \in U \subset V \cap W$ such that $f|_U = g|_U$ (see [9]). Thus, given any germ $\mathbf{f} \in \mathcal{O}_n$ there exists an open neighborhood $V \ni 0$ and a holomorphic function $f \in \mathcal{O}(V)$ such that $\mathbf{f} = [(f, V)]$, that is, $\mathbf{f}$ corresponds to the class defined by (f, V) in $\mathcal{O}_n$. We say that (f, V) is a *representative* of $\mathbf{f}$. Since $\mathcal{O}_n$ is an integral domain we can define its quotient field μ_n called *the field of germs of meromorphic functions of n-variables* at the origin. The *evaluation map* $E : \mathcal{O}_n \to \mathbb{C}$, $\mathbf{f} \mapsto f(0)$, (f, V) is a representative of $\mathbf{f}$ is a well defined ring homomorphism. The elements of the Kernel $E^{-1}(0)$ are called *nonunits*. In μ_n we have a natural basic set of derivations $\Delta_n = \left\{\frac{\partial}{\partial y_j}\right\}_{j=1}^{n}$ whose elements $\frac{\partial}{\partial y_j} : \mu_n \to \mu_n$, $j = 1, \ldots, n$, are defined in the obvious way: given $\frac{\mathbf{f}}{\mathbf{g}} \in \mu_n$ take (f, V) and (g, V) representatives of $\mathbf{f}$ and $\mathbf{g}$ respectively. Define $\frac{\partial}{\partial y_j}(\mathbf{f}/\mathbf{g}) = [(\frac{\partial}{\partial y_j}(f/g), V)]$; it is clear that this definition makes sense and that $\mathbf{f}/\mathbf{g} \in c(\mu_n, \{\frac{\partial}{\partial y_j}\}) \Leftrightarrow \frac{\partial}{\partial y_j}(f/g) = 0$ in a neighborhood of $0 \in \mathbb{C}^n$, $\forall j \Leftrightarrow f/g$ is constant in a neighborhood of $0 \in \mathbb{C}^n \Leftrightarrow \mathbf{f}/\mathbf{g} \in \mathbb{C}$. Therefore $c(\mu_n, \Delta_n) = \mathbb{C}$.

Definition 9.1. A field of germs of Liouvillian functions of n-variables at the origin is a Liouvillian extension $(\mathbb{M}, \tilde{\Delta}_n)$ of (μ_n, Δ_n). The elements $\mathbf{f} \in \mathbb{M}$ are called *germs of Liouvillian functions of n-variables* at the origin.

Now we ask about the relation between germs of Liouvillian functions of n-variables and Liouvillian functions of n-variables on open neighborhoods of the origin $0 \in \mathbb{C}^n$. Let $(\mathbb{K}, \delta_n')$ be a field of germs of Liouvillian functions of n-variables with defining tower $\mu_n = \mathbf{k}_0 \subset \mathbf{k}_1 \subset \cdots \mathbf{k}_m = \mathbb{K}$ where $\mathbf{k}_{j+1} = \mathbf{k}_j(\mathbb{T}_{j+1}), j = 0, \ldots, m-1$. Given any element $\mathbf{r} \in \mathbf{k}_1 = \mathbf{k}_0(\mathbb{T}_1) = \mu_n(\mathbb{T}_1)$ we can write $\mathbf{r} = \frac{\Sigma \mathbf{a}_j \cdot \mathbb{T}_1^j}{\Sigma \mathbf{b}_i \cdot \mathbb{T}_1^i}$ where $\mathbf{a}_j, \mathbf{b}_i \in \mu_n$. We have three different cases:

1st Case: $\frac{\partial \mathbb{T}_1}{\partial y_j} \in \mu_n, \forall j$

2nd Case: $\frac{\partial \mathbb{T}_1}{\partial y_j} / \mathbb{T}_1 \in \mu_n, \forall j$

3rd Case: $\mathbb{T}_1$ is algebraic over μ_n.

Given an open subset $V \subset \mathbb{C}^n$ denote by $\mu(V)$ the field of meromorphic functions on V, constructed as in [9]. Using the fact that given any finite subset $\{\mathbf{f}_\alpha, \alpha \in A\} \subset \mu_n$ there exist an open set $0 \in V \subset \mathbb{C}^n$ and meromorphic functions m_α on V such that $m_\alpha = \mathbf{f}_\alpha / \mathbf{g}_\alpha$ where $\mathbf{f}_\alpha = [f_\alpha, V], \mathbf{g}_\alpha = [g_\alpha, V], \forall \alpha \in A$; one can prove that: There exists an open connected neighborhood V of $0 \in \mathbb{C}^n$ and a differential extension $\left(\mu(V)(t_1), \left\{\frac{\partial}{\partial y_n}\right\}\right)$ of $\left(\mu(V), \left\{\frac{\partial}{\partial y_n}\right\}\right)$ such that τ *is represented* by an element $\tau \in \mu(V)(t_1)$. This can be done for any finite subset of elements $\{\tau_\alpha, \alpha \in A\} \subset \mathbf{k}_1 = \mu_n(\mathbb{T}_1)$. Using an induction argument one can prove:

Proposition 9.2. *Let $(\mathbb{M}, \tilde{\Delta}_n)$ be any field of germs of Liouvillian functions of n-variables. Given any finite subset $X \subset \mathbb{M}$ there exists a Liouvillian extension $\left(K, \left\{\frac{\partial}{\partial y_j}\right\}\right)$ of $\left(\mu(V), \left\{\frac{\partial}{\partial y_j}\right\}\right)$, where V is some connected neighborhood of the origin in $\mathbb{C}^n$, such that each element $\tau \in X$ is represented by an element $\tau \in M$.*

Now we introduce some definitions in order to set up our main result. In what follows we will consider only dimension-one foliations:

Definition 9.3. A *germ of holomorphic singular foliation* on $(\mathbb{C}^n, 0)$ is an equivalence class $[(\mathcal{F}, \mathcal{U})]$ of pairs $(\mathcal{F}, \mathcal{U})$ where $\mathcal{U}$ is an open neighborhood of the origin and $\mathcal{F}$ is a holomorphic foliation on $\mathcal{U}$ with an isolated singularity at the origin and where we define $(\mathcal{F}, \mathcal{U}) \sim (\mathcal{F}', \mathcal{U}')$ if, and only if, $\mathcal{F}$ and $\mathcal{F}'$ coincide in some neighborhood $0 \in W \subset \mathcal{U} \cap \mathcal{U}'$. Clearly the space $\mathcal{F}_n^1 = \text{Fol}^1(\mathbb{C}^n, 0)$ of the germs of holomorphic singular dimension-one foliations on $(\mathbb{C}^n, 0)$ is isomorphic to the quotient of the space $\mathbb{X}_n$ of germs of singular holomorphic vector fields on $(\mathbb{C}^n, 0)$ by the equivalence relation: $\mathbb{X} \sim \mathbb{Y} \Leftrightarrow \mathbb{X} = \mathbf{f} \cdot \mathbb{Y}$ for some unit $\mathbf{f} \in \mathcal{O}_n$. In particular to any germ $\underline{\underline{\mathcal{F}}} \in \mathcal{F}_n^1$ it is associated a representative $(\mathcal{F}, \mathcal{U})$ which is defined by a holomorphic vector field X on $\mathcal{U}$ with an isolated singularity at the origin. Using this we say that $\underline{\underline{\mathcal{F}}}$ has a *Liouvillian first integral* if this holds for some representative $(\mathcal{F}, \mathcal{U})$ and we say that $\underline{\underline{\mathcal{F}}}$ *has a solution which satisfies an analytic relation* if some representative $(\mathcal{F}, \mathcal{U})$ has such a solution, that is, if there exists a solution (y, V) of $(\mathcal{F}, \mathcal{U})$ which *satisfies an analytic relation*, i.e., $f \circ y = 0$ for some $f \in \mathcal{O}(\mathcal{U})$. We also say that $\underline{\underline{\mathcal{F}}}$ *has a solution which satisfies a Liouvillian relation* if for some representative $(\mathcal{F}, \mathcal{U})$ there exists a solution (y, V) of $(\mathcal{F}, \mathcal{U})$ which satisfies a *Liouvillian relation*, i.e., $\tau \circ y = 0$ for some Liouvillian function τ on $\mathcal{U}$. Finally we say that a germ of Liouvillian function of n-variables is a *Liouvillian first integral for* $\underline{\underline{\mathcal{F}}}$ if it has a representative defined as a Liouvillian function on some small enough $\mathcal{U}$ which is a Liouvillian first integral for $\underline{\underline{\mathcal{F}}}$.

All these definitions make sense thanks to Proposition 9.2 and from what we have observed above. Using now (the same arguments used in the proof of) Proposition 7.7 one can prove:

Theorem 9.4. *Let $\underline{\underline{\mathcal{F}}}$ be a germ of dimension-one holomorphic singular foliation on $(\mathbb{C}^n, 0)$ which has a solution satisfying a Liouvillian relation but not an analytic relation. Then $\mathcal{F}$ has a Liouvillian first integral.*

Remark 9.5 (*Absolutely Dicritic versus Absolutely Analytic*)**.** Let us give an interpretation of the conditions under which we cannot apply Theorem 9.4 above. Let $\underline{\underline{\mathcal{F}}}$ be a germ of dimension-one holomorphic singular foliation on $(\mathbb{C}^n, 0)$ which has a solution satisfying a Liouvillian relation. For the case $n = 2$ we have two possibilities:
(1) $\underline{\underline{\mathcal{F}}}$ is *absolutely analytic*, that is, every leaf of (a representative of) $\underline{\underline{\mathcal{F}}}$ satisfies an analytic relation and in this case $\underline{\underline{\mathcal{F}}}$ does not exhibit an admissible solution (see Definition 7.3). Given a leaf of $\mathbf{f}$ we call it a *separatrix* if such a leaf accumulates at and only at the singularity. This implies that the closure of this leaf consists of the leaf and the singularity. Also, such a leaf is closed off the singularity and by a well-known theorem of Remmert and Stein [9] this implies that the closure of the leaf is an analytic subset of dimension one. *In dimension* two this is equivalent to say that the closure of the leaf is the zero set of a germ of holomorphic function, i.e., that the leaf satisfies an analytic relation. Thus, in the two dimensional case, if a foliation $\mathbf{f}$ has all its leaves separatrices then it is absolutely analytic. A foliation with infinitely many separatrices is called *dicritical*. A foliation for which all the leaves are separatrices will be called *absolutely dicritical*. An absolutely dicritical foliation is absolutely analytic and therefore admits no admissible solution. For an absolutely analytic foliation admitting a Liouvillian solution we cannot apply Proposition 7.7 in order to conclude the existence of a Liouvillian first integral. Even if the existence of a Liouvillian first integral is granted and the dimension is $n = 2$ we cannot apply Theorem 5.1 in order to conclude the existence of an elementary Liouvillian first integral.

(2) $\underset{=}{\mathcal{F}}$ is not absolutely analytic. In this case we can conclude that $\mathbf{w}$ has an integrating factor $\mathbf{h} = \exp \int \eta$ where η is meromorphic and closed.

Therefore we can state:

Theorem 9.6. *Let $\underset{=}{\mathcal{F}}$ be a non absolutely analytic germ of foliation on $(\mathbb{C}^2, 0)$ given by the germ of holomorphic one-form $\mathbf{w}$. Then $\underset{=}{\mathcal{F}}$ admits a Liouvillian first integral if, and only if, $\mathbf{w}$ has an integrating factor $\mathbf{h} = \prod_{j=1}^{r} \mathbf{f}_j^{\lambda_j} \exp\left(\frac{\mathbf{g}}{\prod_{j=1}^{r} \mathbf{f}_j^{n_j-1}}\right)$, $\mathbf{f}_j, \mathbf{g} \in \mathcal{O}_2$, $\lambda_j \in \mathbb{C}$, $n_j \in \mathbb{N}$. This occurs in particular if $\underset{=}{\mathcal{F}}$ has a Liouvillian non-analytic leaf (solution).*

A word should be said about this last theorem: It follows from Theorem 9.4 and Proposition 1.6 in a similar way to that used to prove Theorem 7.6; but this is possible because of the following local version of the Integration Lemma due to Cerveau–Mattei [6].

Lemma 9.7. *Let η be a closed germ of meromorphic one-form on $(\mathbb{C}^n, 0)$. Then there are $\mathbf{g}, \mathbf{f}_j \in \mathcal{O}_n$, $\lambda_j \in \mathbb{C}$, $n_j \in \mathbb{N}$ with $j = 1, \ldots, r$ such that $\eta = \sum_{j=1}^{r} \lambda_j \frac{d\mathbf{f}_j}{\mathbf{f}_j} + d\left(\frac{\mathbf{g}}{\prod_{j=1}^{r} \mathbf{f}_j^{n_j-1}}\right)$ so that $\mathbf{h} = \exp \int \eta = \prod_{j=1}^{r} \mathbf{f}_j^{\lambda_j} \cdot \exp\left(\frac{\mathbf{g}}{\prod_{j=1}^{r} \mathbf{f}_j^{n_j-1}}\right)$.*

Let us denote by $\mathcal{A}_2$ the set of absolutely analytic germs $\underset{=}{\mathcal{F}} \in \mathcal{F}_2^1 =: \mathcal{F}_2$. Using Theorem 9.6 and a procedure similar to the one used to prove Proposition 5.9 we obtain:

Proposition 9.8. *Denote by $\tau_2 = \{\underset{=}{\mathcal{F}} \in \mathcal{F}_2 : \underset{=}{\mathcal{F}}$ admits a Liouvillian first integral$\}$. The set $\tau_2 \setminus (\tau_2 \cap \mathcal{A}_2)$ is a codimension-one analytic subset of $\mathcal{F}_2 \setminus \mathcal{A}_2$ and in particular its complementary is an open dense subset of $\mathcal{F}_2 \setminus \mathcal{A}_2$.*

And also:

Proposition 9.9. *A germ of holomorphic singular non absolutely analytic foliation $\underset{=}{\mathcal{F}}$ on $(\mathbb{C}^2, 0)$ admits a Liouvillian first integral if, and only if, $\underset{=}{\mathcal{F}}$ is represented by some pair (w, U) where Ω is a holomorphic singular one-form in $U \ni 0$, $\Omega(p) = 0 \Leftrightarrow p = 0$, which has an integrating factor of the form $\mathbf{h} = \prod_{j=1}^{r} \mathbf{f}_j^{\lambda_j} \cdot \exp\left(\frac{\mathbf{g}}{\prod_{j=1}^{r} \mathbf{f}_j^{n_j-1}}\right)$, $\lambda_j \in \mathbb{C}$, $n_j \in \mathbb{N}$, $\mathbf{f}_j, \mathbf{g} \in \mathcal{O}_2$.*

The following example gives us a sufficient condition to have a non absolutely analytic germ of foliation:

Example 9.10. Let $\underset{=}{\mathcal{F}}$ be a germ of singular foliation on $(\mathbb{C}^2, 0)$ and suppose that the resolution $\underset{=}{\tilde{\mathcal{F}}}$ of $\underset{=}{\mathcal{F}}$ by the blow-up method (cf. [19,13]) exhibits a hyperbolic singularity. Then $\underset{=}{\mathcal{F}}$ has (infinitely many) nonanalytic solutions. In fact this is a straightforward consequence of the following claim.

Claim 9.11. *Let $\underset{=}{\mathcal{F}}$ be a germ of singular foliation on $(\mathbb{C}^2, 0)$ given in local coordinates $(x, y) \in (\mathbb{C}^2, 0)$ by $xdy - \lambda ydx = 0$, $\lambda \in \mathbb{C} \setminus \mathbb{Q}$. Then except for $(x = 0)$ and $(y = 0)$ all the solutions of $\underset{=}{\mathcal{F}}$ are nonanalytic.*

Proof. In fact the solutions of $\underset{=}{\mathcal{F}}$ are given by $x \mapsto (x, c \cdot x^\lambda)$, $c \in \mathbb{C}$ and $y \mapsto (c \cdot y^{1/\lambda}, y)$, $c \in \mathbb{C}$. Choose now any analytic function $f = \sum_{i,j \in \mathbb{N}} f_{ij} x^i y^j$ and suppose f is constant along some solution of $\underset{=}{\mathcal{F}}$ different from $(x = 0)$ and $(y = 0)$; say, given by $x \mapsto (x, cx^\lambda)$, $c \in \mathbb{C}^*$. Then we have

$$f(x, cx^\lambda) = \sum_{i,j \in \mathbb{N}} f_{ij} x^i \cdot c^j x^{\lambda j} = \sum_{i,j \in \mathbb{N}} c^j \cdot f_{ij} \cdot x^{i+\lambda j} \equiv a$$

for some $a \in \mathbb{C}$. We can assume that $a = 0$. Therefore $\sum_{i,j \in \mathbb{N}} c^j f_{ij} x^{i+\lambda j} = 0$ so that since $\lambda \notin \mathbb{Q}$ we have $c^j f_{ij} = 0$, $\forall (i, j) \neq (0, 0)$ and therefore $f_{ij} = 0, \forall (i, j) \neq (0, 0)$, a contradiction. □

10. Germs of singular codimension-one foliations

In what follows we will consider only codimension-one holomorphic foliations having singular set of codimension ≥ 2.

Definition 10.1. A *germ of (holomorphic singular codimension-one) foliation* on $(\mathbb{C}^n, 0)$ is an equivalence class $[(\mathcal{F}, \mathcal{U})]$ of pairs $(\mathcal{F}, \mathcal{U})$ where $\mathcal{U}$ is an open neighborhood of the origin and $\mathcal{F}$ is a holomorphic foliation on $\mathcal{U}$ given by a holomorphic integrable differential one-form Ω on $\mathcal{U}$, $\mathcal{F} : \Omega = 0$, having singular set $\{p \in \mathcal{U} : \Omega(p) = 0\} = \text{sing}(\Omega)$ of codimension ≥ 2. Two such pairs $(\mathcal{F}, \mathcal{U})$ and $(\mathcal{F}', \mathcal{U}')$ given by $\Omega = 0$ and $\Omega' = 0$ respectively are *equivalent* if, and only if, we have $\Omega \wedge \Omega' = 0$ in some neighborhood $0 \in W \subset \mathcal{U} \cap \mathcal{U}'$.

Since $\mathrm{sing}(\Omega)$ and $\mathrm{sing}(\Omega')$ have codimension ≥ 2 it follows from De Rham's Division Lemma [16] that this is equivalent to say that $\Omega' = f\Omega$ in some $0 \in W \subset \mathcal{U} \cap \mathcal{U}'$ for some $f \in \mathcal{O}(W)^*$. Therefore we have $\mathrm{sing}(\Omega) \cap W = \mathrm{sing}(\Omega') \cap W$ so that it is well-defined the *singular set of the germ* $\underline{\underline{\mathcal{F}}} = [(\mathcal{F}, \mathcal{U})]$ as the germ of codimension ≥ 2 analytic set $\mathrm{sing}(\underline{\underline{\mathcal{F}}})$ induced by $\mathrm{sing}(\Omega)$ in $0 \in \mathbb{C}^n$. The space of germs of holomorphic codimension-one foliations on $(\mathbb{C}^n, 0)$ will be denoted by $\mathcal{F}_n = \mathrm{Fol}(\mathbb{C}^n, 0)$. We say that an element $\underline{\underline{\mathcal{F}}} \in \mathcal{F}_n$ has $\frac{\mathbf{f}}{\mathbf{g}} \in \mu_n$ as a (germ of) *meromorphic first integral* if there is a representative $(\mathcal{F}, \mathcal{U})$ for $\underline{\underline{\mathcal{F}}}$ and a representative $f/g \in \mu(U)$ for $\mathbf{f}/\mathbf{g}$ which is a first integral for $(\mathcal{F}, \mathcal{U})$, that is, $d(f/g) \wedge \Omega = 0$ if $\mathcal{F} : \Omega = 0$ on $\mathcal{U}$. A leaf L of $(\mathcal{F}, \mathcal{U})$ is a *separatrix* if $\bar{L} \subset L \cup \mathrm{sing}(\mathcal{F})$. Since L is an analytic subset of $\mathcal{U} \setminus \mathrm{sing}(\mathcal{F})$ and since $\mathrm{sing}(\mathcal{F})$ has codimension ≥ 2 it follows from Remmert-Stein theorem [9] that the closure $\bar{L}$ of L is an analytic subset (of codimension-one) of $\mathcal{U}$. Therefore if Δ_L is a small enough polydisc around the origin $0 \in \Delta_L \subset \mathcal{U}$ then there exists a holomorphic function $f_L \in \mathcal{O}(\Delta_L) \setminus 0$ such that $L \cap \Delta = \{p \in \Delta | f_L(p) = 0\}$. However this can also happen for leaves of $\mathcal{F}$ which are not separatrices of $\underline{\underline{\mathcal{F}}}$. Applying Lemma 5.5 we obtain:

Lemma 10.2. *Let $\mathcal{F}$ be given in $\mathcal{U} \subset \mathbb{C}^n$ by the holomorphic one-form Ω with $\mathrm{sing}(\Omega)$ of codimension ≥ 2. Let $f : \mathcal{U} \to \mathbb{C}$ be any nonconstant reduced holomorphic function. Then the following conditions are equivalent:*

(i) *$f^{-1}(0)$ is invariant by $\mathcal{F}$;*
(ii) *$\Omega \wedge \frac{df}{f}$ is a holomorphic 2-form on $\mathcal{U}$.*

Proof. Denote by $P < TU$ the complex distribution $\mathrm{Ker}(\Omega)$ in U and by $\Lambda \subset U$ the variety $(f = 0)$. Since all objects involved are analytic we may consider the local case also at a generic (and therefore non-singular) point $p \in \Lambda^*$. In suitable local coordinates $(z_1, \dots, z_n) = (z_1, \dots, z_{n-1}, f)$ we have $p = 0$ and Λ given by $\{f = z_n = 0\}$. Also we may write $\Omega = \sum_{j=1}^n a_j\, dz_j$. Suppose Λ is P_Ω invariant. Then, since Λ is given by $\{z_n = 0\}$ we have $\Omega \cdot \frac{\partial}{\partial z_j}|_{\{z_n=0\}} = 0$ for all $j \in \{1, \dots, n-1\}$. In other words z_n divides a_j in $\mathbb{C}\{z_1, \dots, z_n\}$ for every $j \in \{1, \dots, n-1\}$ and therefore $\Omega = \sum_{j=1}^{n-1} z_n\, \tilde{a}_j\, dz_j + a_n\, dz_n$ for some holomorphic $\tilde{a}_1, \dots, \tilde{a}_{n-1} \in \mathbb{C}\{z_1, \dots, z_n\}$. Thus $\Omega \wedge \frac{dz_n}{z_n} = \sum_{j=1}^{n-1} \tilde{a}_j\, dz_j \wedge dz_n$ is holomorphic. Conversely, if $\Omega \wedge \frac{dz_n}{z_n}$ is holomorphic then $\Omega = \sum_{j=1}^{n-1} z_n\, \tilde{a}_j\, dz_j + a_n\, dz_n$ as above therefore $\Omega \cdot \frac{\partial}{\partial z_j}$ vanishes on $\{z_n = 0\}$ for every $j = 1, \dots, n-1$. Since $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_{n-1}}\}$ generate $T\Lambda$ in a neighborhood of 0 we obtain that $T\Lambda \subset P$ in a neighborhood of p. Therefore Λ is P-invariant. $\square$

We recall that a germ $\underline{\underline{\mathcal{F}}} \in \mathcal{F}_n$ is *absolutely dicritical* if every leaf of $\underline{\underline{\mathcal{F}}}$ is a separatrix. Another lemma, which is a consequence of what we have observed above, is the following (see Remark 9.5):

Lemma 10.3. *Let $\underline{\underline{\mathcal{F}}} \in \mathcal{F}_n$ be an absolutely dicritical germ of foliation. Then $\underline{\underline{\mathcal{F}}}$ is absolutely analytic.*

Now we will study a special class of absolutely analytic germs of foliations. In order to define this class we introduce the notion of Weierstrass polynomials. Let $(x_1, \dots, x_n)$ holomorphic coordinates on $(\mathbb{C}^n, 0)$.

Definition 10.4. A *Weierstrass polynomial of degree $k \in \mathbb{N}$ in x_n* is an element $\mathbf{h} \in \mathcal{O}_n$ that is of the form $\mathbf{h} = x_n^k + \mathbf{a}_1 \cdot x_n^{k-1} + \cdots + \mathbf{a}_k$ with $\mathbf{a}_j \in \mathcal{O}_{n-1}$ a germ of holomorphic function depending only on the variables $(x_1, \dots, x_{n-1}) \in (\mathbb{C}^{n-1}, 0)$, and $\mathbf{a}_j \in \mathcal{O}_{n-1}$ is a *nonunit*, that is, $\mathbf{a}_j(0) = 0\ \forall j \in \{1, \dots, k\}$.

A holomorphic function $h \in \mathcal{O}(U)$, $U \ni 0$ in $\mathbb{C}^n$, $h \not\equiv 0$, is said to be *regular of order $k \in \mathbb{N}$* at the origin $0 \in \mathbb{C}^n$ if k is the less greater $\nu \in \mathbb{N}$ such that the ν-jet h_ν of h at the origin is not zero. We say that f is *x_j-regular of order k* at $0 \in \mathbb{C}^n$ if $x_j \mapsto h(0, \dots, x_j, \dots, 0)$ is regular of order k at the origin $0 \in \mathbb{C}^1$. Clearly any Weierstrass polynomial of degree k in the local variable x_n is regular of order k at the origin in this variable. The converse of this fact is the following:

Theorem 10.5 ([9], *Weierstrass Preparation Theorem*)**.** *Let $\mathbf{f} \in \mathcal{O}_n$ be a germ regular of order k in x_n. Then there exists a unique polynomial of Weierstrass of degree k in x_n, $h \in \mathcal{O}_{n-1}[x_n]$ such that $\mathbf{f} = \mathbf{u} \cdot \mathbf{h}$ for some unit $\mathbf{u} \in \mathcal{O}_n$.*

We recall that given an element $\mathbf{h} \in \mathcal{O}_n$ we can define the germ of analytic set $v(\mathbf{h})$ as the germ at the origin induced by the set $\{p \in U | h(p) = 0\}$ where (h, U) is any representative of $\mathbf{h}$. It is clear that given $\mathbf{h}$ and $\mathbf{f} \in \mathcal{O}_n$ we have $v(\mathbf{h}) = v(\mathbf{f})$ if, and only if, $\mathbf{h} = \mathbf{u} \cdot \mathbf{f}$ for some unit $\mathbf{u} \in \mathcal{O}_n$. In particular it follows from Theorem 10.5 that:

"*The germ of analytic set $v(\mathbf{f})$ defined by a germ $\mathbf{f} \in \mathcal{O}_n$ coincides with the variety $v(\mathbf{h})$ where $\mathbf{h}$ is a Weierstrass polynomial in some variable x_j.*"

We also have:

Theorem 10.6 ([9], *Weierstrass Division Theorem*)**.** *Let $\mathbf{h} \in \mathcal{O}_{n-1}[x_n]$ be a Weierstrass polynomial in x_n of degree k. Then any germ $\mathbf{f} \in \mathcal{O}_n$ can be written in a unique manner in the form $\mathbf{f} = \mathbf{g} \cdot \mathbf{h} + \mathbf{r}$ where $\mathbf{g} \in \mathcal{O}_n$ and $\mathbf{r} \in \mathcal{O}_{n-1}[x_n]$ is a Weierstrass polynomial of degree $< k$. Moreover if $\mathbf{f} \in \mathcal{O}_{n-1}[x_n]$ then $\mathbf{g} \in \mathcal{O}_{n-1}[x_n]$.*

Motivated by these facts we define:

Definition 10.7. A germ of holomorphic one-form $\mathbf{w}$ in $(\mathbb{C}^n, 0)$ is a *Weierstrass polynomial k-form in the variable x_n* if its coefficients are Weierstrass polynomials in x_n.

According Lemma 10.2 and Theorem 10.6 we have:

Lemma 10.8. *Let $\mathbf{w}$ be a germ of Weierstrass polynomial integrable one-form in the variable x_n, at the origin $0 \in \mathbb{C}^n$, and let $\mathbf{h} \in \mathcal{O}_{n-1}[x_n]$ be a Weierstrass polynomial of degree k in x_n. Then the following conditions are equivalent:*

(i) *$v(\mathbf{h})$ is a union of solutions of the germ of foliation $\underset{=}{\mathcal{F}} : \mathbf{w} = 0$*

(ii) *$\mathbf{w} \wedge \frac{d\mathbf{h}}{\mathbf{h}}$ is a Weierstrass polynomial 2-form.*

We shall use another fact:

Lemma 10.9. *Let $\mathbf{h}_j \in \mathcal{O}_{n-1}[x_n]$, $j = 1, \dots, r$, be Weierstrass polynomials satisfying:* (i) *$\mathbf{h}_j$ is irreducible, $\forall j$;* (ii) *$\mathbf{h}_i$ and $\mathbf{h}_j$ have no common factor (nonunit), $\forall i \neq j$.*

Then the map $\mathcal{O}^r_{n-1} \to \Lambda_2 \bigotimes_{\mathcal{O}_{n-1}} \mu_n$ defined by $(\lambda_j)^r_{j=1} \mapsto \sum_{j=1}^r \lambda_j \cdot \frac{d\mathbf{h}_j}{\mathbf{h}_j}$ is injective.

Proof. Suppose on the contrary that $\sum_{j=1}^r \lambda_j \frac{d\mathbf{f}_j}{\mathbf{f}_j} = 0$ for some $\mathbf{f}_j \in \mathcal{O}_n$ and some $\lambda_j \in \mathcal{O}_{n-1} \subset \mathcal{O}_n$. For reasons of simplicity we will assume that $n = 2$ and consider local coordinates $(x, y) \in (\mathbb{C}^2, 0)$ and $x \in (\mathbb{C}^1, 0) \subset (\mathbb{C}^2, 0)$. Let us choose an open bidisc $\Delta = \Delta_x \times \Delta_y$ centered at the origin and representatives $f_j \in \mathcal{O}(\Delta)$, $\lambda_j \in \mathcal{O}(\Delta_x) \subset \mathcal{O}(\Delta)$ for $\mathbf{f}_j$ and λ_j respectively. Thus we have

$$\sum_{j=1}^r \lambda_j(x) \frac{df_j}{f_j}(x, y) = 0 \quad \text{in} \quad \Delta.$$

Let us denote by X_j the hypersurface $f_j = 0$ in Δ.

Claim 10.10. $X_i \not\subset \bigcup_{j \neq i} X_j$.

Proof. In fact if $X_i \subset \bigcup_{j \neq i} X_j$ then since the germ $\mathbf{f}_j$ is irreducible it follows that (for Δ small enough) X_j is irreducible as an analytic set and therefore $X_i = X_j$ for some $j \neq i$ so that necessarily we have $\mathbf{f}_i = \mathbf{u}.\ \mathbf{f}_j$ for some unit $\mathbf{u} \in \mathcal{O}_n$ which is a contradiction. □

Since X_j is closed it follows from the claim above that we can obtain a ball B_i with $D_i = B_i \cap X_i \neq \emptyset$ and $B_i \cap (\bigcup_{j \neq i} X_j) = \emptyset$. Choose now any $a = (a_1, a_2) \in \mathbb{D}_i$ and let L be the vertical line segment through given in parametric coordinates by $\mathbb{D}_\varepsilon \ni t \mapsto (a_1, a_2 + t) = L(t)$ where $\mathbb{D}_\varepsilon$ is a small disc around the origin $0 \in \mathbb{C}^1$. Let $q_j(t) = f_j L(t)$ then from (2) we have

$$\sum_{j=1}^r \lambda_j(a_1) \cdot \frac{q_j'(t)}{q_j(t)} = 0, \quad t \in \mathbb{D}_\varepsilon.$$

But $\frac{q_j'(t)}{q_j(t)}$ is holomorphic in $\mathbb{D}_\varepsilon$ $\forall j \neq i$ and $\frac{q_i'(t)}{q_i(t)}$ has an order one pole at the origin $t = 0$ with residue equal to the multiplicity ν of the root $t = 0$ of $q_i(t)$. Therefore from (2) we obtain

$$\nu \cdot \lambda_i(a_1) = 0 \quad \text{so that} \quad \lambda_i(a_1) = 0.$$

Since $a \in D_i$ is arbitrary it follows that $\lambda_i = 0$ in a nonempty open subset of Δ_x and therefore $\lambda_i = 0$ in Δ_x. This finishes the proof. □

Now we recall that, according to a theorem of Darboux [10], a codimension foliation on $\mathbb{C}P^n$ having infinitely many algebraic solutions has a rational first integral. The notions introduced above allow us to state a kind of Darboux Theorem, as follows:

Theorem 10.11. *Let $\underset{=}{\mathcal{F}} \in \mathcal{F}_2$ be a germ of foliation given by a Weierstrass polynomial integrable one-form $\mathbf{w}$ in $(\mathbb{C}^2, 0)$. Suppose that $\underset{=}{\mathcal{F}}$ has infinitely many separatrices then there are Weierstrass polynomials $\mathbf{h}_\nu \in \mathcal{O}_1[y]$ $\nu = 1, \dots, r$, and linearly independent elements $(\lambda_\nu), (\mu_\nu) \in \mathcal{O}_1^r$ such that*

$$\mathbf{w} \wedge \sum_{\nu=1}^r \lambda_\nu \frac{d\mathbf{h}_\nu}{\mathbf{h}_\nu} = 0.$$

Proof. Since we are in a dimension 2 situation we will denote by (x, y) the local coordinates in $(\mathbb{C}^2, 0)$ and assume that $\mathbf{w}$ is a Weierstrass polynomial one-form in the variable y.

Let us denote by S the set of separatrices of $\underset{=}{\mathcal{F}}$ and by $\tilde{S} = S \cap \mathcal{O}_1[y]$ the set of separatrices of S which are defined as $v(\mathbf{h})$ where $h \in \mathcal{O}_n[y]$ is a Weierstrass polynomial. Using the remark after Theorem 10.5 we conclude that: *an element $\mathbf{s} \in S$ belongs to $\tilde{S}$ if and only if $\mathbf{s} = v(\mathbf{h})$ where $\mathbf{h} \in \mathcal{O}_2$ is regular of some order k in y and in this case we may assume that $\mathbf{h}$ is itself a Weierstrass polynomial in y.*

Claim 10.12. *$\tilde{S}$ is infinite, that is, there are infinitely many elements $\mathbf{s} = \upsilon(\mathbf{h}_s) \in S$ such that $\mathbf{h}_s$ is regular of some order in y.*

Proof. In fact if a germ $\mathbf{f} \in \mathcal{O}_2$ is not regular of any order in y then $f(0, y) = 0$ for some representative (f, V) of $\mathbf{f}$ and in particular $\mathbf{f} \in \mathcal{O}_1$, that is, $f = f(x)$. Since $f(0) = 0$ it follows that $x|\mathbf{f}$ in $\mathcal{O}_2$ so that since $\upsilon(\mathbf{f})$ is irreducible (because it is a separatrix of $\underline{\underline{\mathcal{F}}}$) it follows that $\upsilon(\mathbf{f}) = \upsilon(x) = y$-axis. Therefore except eventually for the y axis the separatrices of $\underline{\underline{\mathcal{F}}}$ are given by elements $\mathbf{h} \in \mathcal{O}_2$ regular of some order in y and this proves the claim. □

Now we define the $\mathcal{O}_1$-module $\mathcal{O}_1(\tilde{S})$ of all the sequences $(\lambda_s)_{s\in\tilde{S}}$ where $\lambda_s \in \mathcal{O}_1$ and $\lambda_s = 0$ except for a finite number of indexes $s \in \tilde{S}$. We denote by $\mathbf{r}_2$ the $\mathcal{O}_2$-module of germs of holomorphic 2-forms in $(\mathbb{C}^n, 0)$ and by $T : \mathcal{O}_1^{(\tilde{S})} \to \mathbf{r}_2$ the map $(\lambda_s) \mapsto \sum_s \lambda_s \frac{d\mathbf{h}_s}{\mathbf{h}_s} \wedge \mathbf{w}$ which is clearly a finite sum for each $(\lambda_s) \in \mathcal{O}_1^{(\tilde{S})}$. We recall that given any $s \in \tilde{S}$, $\mathbf{h}_s \in \mathcal{O}_1[y]$ is a Weierstrass polynomial such that $s = \upsilon(\mathbf{h}_s)$. According to Lemma 10.8 the map T is a well-defined homomorphism of $\mathcal{O}_1$-modules and the image $T(\mathcal{O}_1^{(\tilde{S})})$ consists of Weierstrass polynomials 2-forms and as it is easy to see they have a degree equal to degree of $\mathbf{w}$ in y minus 1; say, $k - 1$.

Claim 10.13. *T is not injective, in fact, we have $\dim_{\mathcal{O}_1} \operatorname{Ker} T \geq 2$.*

Proof. In fact since $\tilde{S}$ is infinite and since $\dim_{\mathcal{O}_1} \mathcal{O}_1^{(\tilde{S})}$ = cardinality of $\tilde{S}$ the claim follows from the fact that the image $T(\mathcal{O}_1^{(\tilde{S})}) \subset \mathbf{r}_2$ has a finite dimension (which depends only on k) over $\mathcal{O}_1$ (see also Lemma 10.9). □

Now given any element $(\lambda_s) \in \operatorname{Ker} T$ we have $\mathbf{w} \wedge \sum_s \lambda_s \frac{d\mathbf{h}_s}{\mathbf{h}_s} = 0$ and $\sum_s \lambda_s \frac{d\mathbf{h}_s}{\mathbf{h}_s} \neq 0$ where $\lambda_s \in \mathcal{O}_1$ is null for almost every $s \in \tilde{S}$. This ends the proof of Theorem 10.11. □

We attempt to the fact that Theorem 10.11 above may not be true (as stated) for the codimension-one case. In fact we cannot prove Claim 10.12 in the codimension-one case: for instance if $n = 3$ and the variables are (x, y, z) then the germs of the form $x\mathbf{f} + y\mathbf{g} \in \mathcal{O}_3$ are not regular of any order in z but can define infinitely many separatrices of a germ of foliation $\underline{\underline{\mathcal{F}}} \in \mathcal{F}_2$. However this difficult is overcame if we assume that the germ $\mathbf{w}$ is a Weierstrass polynomial in each variable:

Theorem 10.14. *Let $\underline{\underline{\mathcal{F}}}$ be a germ of codimension-one foliation given by a germ of integrable one-form $\mathbf{w}$ in $(\mathbb{C}^n, 0)$. Suppose:*

(i) *$\mathbf{w}$ is a Weierstrass polynomial in each variable x_j, $j = 1, \dots, n$.*
(ii) *$\underline{\underline{\mathcal{F}}}$ has infinitely many separatrices.*

Then there are Weierstrass polynomials, in some variable x_j, $\mathbf{h}_\nu \in \mathcal{O}_{n-1}[x_j]$, $\nu = 1, \dots, r$ and linearly independent elements (λ_ν), $(\mu_\nu) \in \mathcal{O}_{n-1}^r$ such that $\mathbf{w} \wedge \sum_{\nu=1}^r \lambda_\nu \frac{d\mathbf{h}_\nu}{\mathbf{h}_\nu} = 0$ and $\mathbf{w} \wedge \sum_{\nu=1}^r \mu_\nu \frac{d\mathbf{h}_\nu}{\mathbf{h}_\nu} = 0$.

11. Existence of absolutely dicritical germs of foliations without Liouvillian first integral

A germ of singular codimension-one foliation $\underline{\underline{\mathcal{F}}}$ on $(\mathbb{C}^n, 0)$ is said to be *absolutely dicritical* if all its leaves are separatrices, that is, $\underline{\underline{\mathcal{F}}}$ is represented by some pair $(\mathcal{F}, \mathcal{U})$ such that every leaf L of $\mathcal{F}$ accumulates only at the origin, i.e., $\bar{L} = L \cup \{0\}$. According to Remmert-Stein Theorem [9] this implies that $\bar{L}$ is an analytic subset of $\mathcal{U}$ and therefore there exists a holomorphic function $f \in \mathcal{O}_n$ such that $\bar{L}$ and $f^{-1}(0)$ coincide in a neighborhood of $0 \in \mathbb{C}^n$. Examples of these foliations are given by the μ-simple germs of foliations defined in Example 4.3 and studied in [12].

The most simple of these foliations are those of type $\mathcal{D}_0(1, 1)$ (according to the notation of [12]) which are foliations in dimension two that, after one single blow-up, are nonsingular foliations with only one point of tangency with the projective line and this is a order-2 tangency (see also [2] for further properties of such foliations). The map defined in a neighborhood of this point of tangency in the projective by mapping one point p on the other point of intersection of the leaf L_p, through p, with the projective; is a local biholomorphism fixing the tangent point and has order 2 (it is an involution). By performing a (single) translation or an inversion on $\overline{\mathbb{C}}$ we can consider that the tangent point is the origin and define a map $\mathcal{H} : \mathcal{D}_0(1, 1) \to \mathbb{B}_2$ where $\mathbb{B}_2$ is the subgroup of germs of biholomorphisms of $\mathbb{C}$ fixing the origin, generated by the involutions. We denote by $\mathcal{A}$ the equivalence relation defined by analytic conjugation in $\mathcal{D}_0(1, 1)$ and by $\mathcal{D}_0(1, 1)/\mathcal{A}$ the quotient space.

Theorem 11.1 (*[12]*). *The map $\mathcal{H}$ is onto and induces a bijective map*

$$\overline{\mathcal{H}} : \mathcal{D}_0(1, 1)/\mathcal{A} \longleftrightarrow \mathbb{B}_2/\mathbb{P}SL(2, \mathbb{C})$$

that is, two germs $\underline{\underline{\mathcal{F}}}$, $\underline{\underline{\mathcal{F}}}' \in \mathcal{D}_0(1, 1)$ are analytically conjugated if, and only if, the corresponding involutions are conjugated by a Möbius transformation $T \in \mathbb{P}SL(2, \mathbb{C})$.

This theorem shows that the problem of classifying the germs $\underline{\underline{\mathcal{F}}} \in \mathcal{D}_0(1, 1)$ which admit some type of first integral (meromorphic, Liouvillian) can be set up in terms of the involutions $\sigma \in \mathbb{B}_2$.

11.1. Construction of a non-Liouvillian absolutely dicritical germ of foliation in $(\mathbb{C}^2, 0)$

Let us now show how to construct an example of an absolutely dicritical foliation without Liouvillian first integral. We shall follow a construction proposed by Gabriel Calsamiglia. If $\underset{=}{\mathcal{F}}$ is an absolutely dicritical germ of foliation in $(\mathbb{C}^2, 0)$ and admits a Liouvillian first integral F we consider its restriction $\underset{=}{f}$ to the exceptional divisor, which is a Liouvillian function over $\mathbb{C}P^1$ having at most a countable number of singularities. Suppose $\underset{=}{\mathcal{F}}$ has only one tangent point with the exceptional divisor and consider the involution σ defined by it in a coordinate t centered at this point. In a neighborhood of $(0, 0) \in \mathbb{C}^2$ the Liouvillian function $G(X, Y) = f(X) - f(Y)$ defines σ implicitly: $G(t, \sigma(t)) = 0$. The set of singularities of G is contained in the set $\text{sing}(f) \times \text{sing}(f)$, and therefore the set of points where σ cannot be analytically continued is at most countable. To construct the example of a non-Liouvillian germ, we just need a germ of involution $\sigma \in \text{Diff}(\mathbb{C}^2, 0)$ having an uncountable set as natural boundary, since any germ of involution can be realized as the involution induced by a foliation at a tangent point with the exceptional divisor. Suppose $c : \mathbb{S}^1 \to \mathbb{C}$ is a $\mathcal{C}^1$ non-analytic Jordan curve bounding a topological disc $D \subset \mathbb{C}$ and such that the intersection of c and $-c$ occurs only transversely. Consider a conformal Riemann mapping $h : \mathbb{D} \to D$ which extends as a homeomorphism to the boundary, but does not extend analytically, since such an extension would imply the existence of points where c is analytic. Suppose $0 \in D$ and $h(0) = 0$ and define the germ $\sigma(z) = h^{-1}(-h(z))$. We can apply the result in [14] p. 628 to deduce that σ has a topological circle as natural boundary and by definition $\sigma^2 = \text{Id}$.

The following lemma will be useful:

Lemma 11.2. *Let $f(z)$ be a rational function on $\overline{\mathbb{C}}$ and let $\sigma \in Bih(\mathbb{C}, 0)$ be any germ of biholomorphism satisfying $f \circ \sigma = f$; then σ extends to $\overline{\mathbb{C}}$ as an algebraic function over $\mathbb{C}(z)$.*
(ii) *Let $f(z)$ be a Liouvillian function on $\overline{\mathbb{C}}$ and let $\sigma \in Bih(\mathbb{C}, 0)$ be such that $f \circ \sigma = f$; then σ satisfies a Liouvillian relation.*

Proof. Write $f(z) = P(z)/Q(z)$ where $P(z), Q(z)$ are polynomials. Then from $f \circ \sigma = f$ we obtain $\frac{P(\sigma(z))}{Q(\sigma(z))} = \frac{P(z)}{Q(z)}$, so that, $P(\sigma(z)) \cdot Q(z) - P(z) \cdot Q(\sigma(z)) = 0$. Define $p(Y) = Q(z) \cdot P(Y) - P(z) \cdot (Y)$, then $p(Y) \in \mathbb{C}(z)[Y]$ is such that $p(\sigma) = 0$ and this proves that σ extends in a natural way as an algebraic function over $\mathbb{C}(z)$.
(ii) In fact define $g(w) = f(w) - f$ then $g = 0$ is clearly a Liouvillian equation over $\mathbb{C}(z)$. □

Given a germ of foliation $\underset{=}{\mathcal{F}} \in \mathcal{D}_0(1, 1)$ we write $\underset{=}{\mathcal{F}} \in \mathcal{M}$ (respectively $\underset{=}{\mathcal{F}} \in \mathcal{L}$) if $\underset{=}{\mathcal{F}}$ admits a meromorphic first integral (respectively, a Liouvillian first integral). Also denote by $\mathcal{M}/\mathcal{A}$ and $\mathcal{L}/\mathcal{A}$ the corresponding quotient spaces by the above defined relation $\mathcal{A}$. Lemma 11.2 and the discussion in the beginning of this paragraph then give:

Theorem 11.3. *The map $\mathcal{H} : \mathcal{D}_0(1, 1) \to \mathbb{B}_2$ induces a bijection between the spaces $\mathcal{M}/\mathcal{A}$ and $\{[\sigma] \in \mathbb{B}_2/\mathbb{P}SL(2, \mathbb{C}) : \sigma$ is an algebraic function over $\mathbb{C}(z)\}$. Moreover we also have an injection*

$$\mathcal{L}/\mathcal{A} \longrightarrow \{[\sigma] \in \mathbb{B}_2/\mathbb{P}SL(2, \mathbb{C}) : \sigma \text{ satisfies a Liouvillian relation}\}.$$

In particular, there are elements $\underset{=}{\mathcal{F}} \in \mathcal{D}_0(1, 1)$ which do not admit a Liouvillian first integral.

12. Foliations on Stein manifolds having Liouvillian first integrals

The notion of Liouvillian function on M can be introduced in a natural way when M is a Stein manifold as $\mathbb{C}^2$ for instance (see Example 2.5). Using this we can generalize the results of [20] set up for foliations on $\mathbb{C}P^2$, to foliations on Stein 2-dimensional manifolds. We shall begin by proving basic lemmas for differential one-forms and foliations by curves on Stein manifolds:

Lemma 12.1. *Let $\mathcal{F}$ be a holomorphic foliation on a Stein manifold M^n.*

(i) *If $\mathcal{F}$ has dimension-one then $\mathcal{F}$ is given by a meromorphic vector field X on M.*
(ii) *If $\mathcal{F}$ has codimension-one then $\mathcal{F}$ is given by a meromorphic integrable differential one-form Ω on M.*

Proof. (i) Since M is a Stein manifold given any point $p \in M$ there exist $f_1, \ldots, f_n \in \mathcal{O}(M)$ with $(df_1 \wedge \cdots \wedge df_n)(p) \neq 0$. It is clear that the set $Y = \{q \in M/(df_1 \wedge \cdots \wedge df_n)(q) = 0\}$ is a codimension ≥ 1 analytic subset of M so that $M \setminus Y$ is connected (M is supposed to be connected). There exists an open cover $\bigcup_{i \in I} U_i = M$ by connected sets such that $\mathcal{F}|_{U_i}$ is given by a holomorphic vector field X_i on U_i and in each $U_i \cap U_j \neq \emptyset$ we have $X_i = f_{ij} \cdot X_j$ for some $f_{ij} \in \mathcal{O}(U_i \cap U_j)^*$. Given any point $q \in M \setminus Y$ we have $(df_1 \wedge \cdots \wedge df_n)(q) \neq 0$ so that $\frac{\partial}{\partial f_1}, \ldots, \frac{\partial}{\partial f_n}$ are holomorphic vector fields on M which are linearly independent on a neighborhood of p so that given any open set U_i we can write locally X_i on $U_i \setminus (U_i \cap Y)$ as a linear holomorphic combination of the $\frac{\partial}{\partial f_k}$ is but since $U_i \setminus (U_i \cap Y)$ is also connected this can be done in all $U_i \setminus (U_i \cap Y)$, say $X_i = \sum_{k=1}^{n} \alpha_k^i \frac{\partial}{\partial f_k}, \alpha_k \in \mathcal{O}(U_i \setminus (U_i \cap Y))$.

Claim 12.2. *The α_k^i's extend meromorphically to U_i.*

Proof. This is a straightforward consequence of Lemma 12.3 below. Thus we have extended the α_k^i's meromorphically to U_i. In each $U_i \cap U_j \neq \emptyset$ we have $X_i = f_{ij} \cdot X_j$ so that

$$\sum_k \alpha_k^i \frac{\partial}{\partial f_k} = f_{ij} \cdot \sum_k \alpha_k^j \frac{\partial}{\partial f_k},$$

that is,

$$\sum_k (\alpha_k^i - f_{ij} \cdot \alpha_k^j) \frac{\partial}{\partial f_k} = 0.$$

Since $\frac{\partial}{\partial f_1}, \ldots, \frac{\partial}{\partial f_n}$ are locally linearly independent outside a codimension-one analytic subset if follows that $\alpha_k^i - f_{ij} \cdot \alpha_k^j \equiv 0$ in $U_i \cap U_j$ so that $\alpha_k^i = f_{ij} \cdot \alpha_k^j$ in $U_i \cap U_j$. Thus we $\frac{\alpha_k^i}{\alpha_\ell^i} = \frac{\alpha_k^j}{\alpha_\ell^j}, \forall k, \ell$ and this allows us to define $\alpha_1, \ldots, \alpha_n$ meromorphic functions on M by setting $\alpha_k = \frac{\alpha_k^i}{\alpha_1^i}$ in each U_i so that we have

$$X_i = \sum_k \alpha_k^i \frac{\partial}{\partial f_k} = \sum_k \frac{\alpha_k^i}{\alpha_1} \frac{\partial}{\partial f_k} \Rightarrow X_i = (\alpha_1^i)^{-1} \cdot \sum_{k=1}^n \alpha_k \frac{\partial}{\partial f_k} = \alpha_1^i \cdot X$$

where $X = \sum_{k=1}^n \alpha_k \frac{\partial}{\partial f_k}$ is a meromorphic vector field on M. Clearly X also defines $\mathcal{F}$ on M. The proof of (ii) is similar to the proof of (i) and we shall omit it. $\square$

Lemma 12.3. *Let $X, Y_1, \ldots, Y_n$ be holomorphic vector fields on an open polydisc $\Delta = \Delta_1 \times \cdots \times \Delta_n \subset \mathbb{C}^n$. Suppose that:*

(i) *$Y_1, \ldots, Y_n$ are locally linearly independent on $\Delta \setminus (\Delta_1 \times 0 \cdots \times 0)$*
(ii) *$X = \sum_{j=1}^n \alpha_j Y_j$ in $\Delta \setminus \Delta_1$ for some holomorphic functions $\alpha_1, \ldots, \alpha_n$ on $\Delta \setminus \Delta_1$.*

Then $\alpha_1, \ldots, \alpha_n$ extend meromorphically to Δ.

Lemma 12.4. *Let $\Omega, \Omega_1, \ldots, \Omega_n$ be holomorphic integrable differential one-forms on an open polydisc $\Delta = \Delta_1 \times \cdots \times \Delta_n \subset \mathbb{C}^n$ such that $\Omega_1, \ldots, \Omega_n$ are locally linearly independent outside Δ_1. Then we can write $\Omega = \sum_{i=1}^n f_i \Omega_i$ for some meromorphic functions $f_1, \ldots, f_n$ on Δ, in a unique way.*

Proof. Given any point $p \in \Delta \setminus \Delta_1$ there exist local coordinates $(x_1, \ldots, x_n)$ centered at p such that $\Omega_j = dx_j$, $(j = 1, \ldots, n)$ (remark that $\Omega_1, \ldots, \Omega_n$ are integrable). Thus we can write $\Omega = \sum_{i=1}^n f_i \cdot dx_i = \sum_{i=1}^n f_i \cdot \Omega_i$ for some holomorphic $f_i(x_1, \ldots, x_n)$ defined in a neighborhood of p. We have clearly

$$\begin{pmatrix} \Omega \wedge \Omega_2 \wedge \cdots \wedge \Omega_n = f_1 \Omega_1 \wedge \cdots \wedge \Omega_n, \\ \Omega \wedge \Omega_1 \wedge \Omega_3 \wedge \cdots \wedge \Omega_n = -f_2 \Omega_1 \wedge \cdots \wedge \Omega_n, \\ \vdots \qquad \vdots \qquad \vdots \end{pmatrix}.$$

Therefore f_j is uniquely determined and extend in a natural way as a meromorphic function on Δ. $\square$

Using what we have observed above and Proposition 7.7 we can prove:

Theorem 12.5. *Let $\mathcal{F}$ be a foliation by curves on a Stein manifold M^n and suppose $\mathcal{F}$ has a solution (y, V) which satisfies a Liouvillian relation but does not satisfy an analytic relation. Then $\mathcal{F}$ has a Liouvillian first integral on M.*

Definition 12.6. A holomorphic singular foliation (of any dimension) on a complex manifold M is said to be *absolutely analytic* if each solution of $\mathcal{F}$ satisfies an analytic relation given by a meromorphic function on M.

Clearly an absolutely analytic codimension-one foliation $\mathcal{F}$ satisfies the following conditions:
(i) $\mathcal{F}$ is proper that is each leaf L of $\mathcal{F}$ is properly embedded
(ii) each leaf of $\mathcal{F}$ is an analytic subset of M
(iii) $\mathcal{F}$ does not admit a hyperbolic singularity.

Proposition 12.7. *Let $\mathcal{F}$ be a dimension-one foliation on a Stein manifold M^n. Suppose that each solution of $\mathcal{F}$ satisfies a Liouvillian relation. Then $\mathcal{F}$ is absolutely analytic or $\mathcal{F}$ has a Liouvillian first integral. In particular if $n = 2$ and $\mathcal{F}$ is not absolutely analytic then $\mathcal{F}$ has a Liouvillian first integral of the form $F = \int \frac{\Omega}{h}$ where Ω is a meromorphic one-form in M which defines $\mathcal{F}$ and h is an integrating factor for Ω of the form $h = \exp \int \eta$ where η is a closed meromorphic one-form on M.*

13. Foliations on compact complex manifolds having Liouvillian first integrals

As we have already observed the notion of Liouvillian function on a compact complex manifold M is naturally set up using the fact that its field of meromorphic functions is a finitely generated algebraic function field over $\mathbb{C}$. Therefore we can generalize the main results of Section 5 (from [20]) to this case. However we will have to suppose that the foliation by curves is given by a (global) meromorphic vector field on M. We begin with some basic notation: Let $x_1, \ldots, x_r, y \in \mu(M)$ be such that $\mu(M) = \mathbb{C}(x_1, \ldots, x_r, y)$ as in Example 2.3. A solution (y, V) of a foliation by curves $\mathcal{F}$ on M is called *an algebraic solution* if there is an *algebraic function* that is, an element $f \in \mathbb{C}(x_1, \ldots, x_r)$ such that $f \circ y = 0$ in V. We say that (y, V) *satisfies a Liouvillian relation* if there is Liouvillian function f on M analytic in some neighborhood of $y(V)$ which satisfies $f \circ y = 0$ in V.

Theorem 13.1. *Let $\mathcal{F}$ be a holomorphic singular foliation by curves on a compact complex manifold M given by a meromorphic vector field X on M. Suppose $\mathcal{F}$ has a nonalgebraic solution which satisfies a Liouvillian relation, then $\mathcal{F}$ has a Liouvillian first integral.*

Theorem 13.2. *Let $\mathcal{F}$ be a holomorphic singular foliation on a compact complex surface M^2 given by a meromorphic vector field X on M and a meromorphic one-form Ω on M. Then $\mathcal{F}$ has a Liouvillian first integral if, and only if Ω admits an integrating factor of the form $h = \exp \int \eta$ where η is a closed meromorphic one-form on M.*

Corollary 13.3. *The set τ of foliations by curves on a compact 2-dimensional complex manifold M^2 admitting Liouvillian first integrals is a Zariski's closed subset of the space $\mathrm{Fol}(M)$ of foliations by curves on M. In particular there exists an open dense subset $\mathcal{U} \subset \mathrm{Fol}(M)$ whose elements do not admit a Liouvillian first integral.*

In order to prove Theorem 13.2 we need the following version of Darboux's Theorem [1,7]:

Proposition 13.4. *Let $\mathcal{F}$ be a codimension-one holomorphic singular foliation on a compact manifold M given by a meromorphic integrable one-form Ω on M. Then $\mathcal{F}$ has a meromorphic first integral on M if, and only if, $\mathcal{F}$ has infinitely many algebraic leaves.*

Let us write $\mu(M) = \mathbb{C}(x_1, \ldots, x_r, y)$ in the usual notation. Given a meromorphic one-form Ω defining $\mathcal{F}$ we write $\Omega = \sum_{j=1}^{r} A_j dx_j, A_j \in \mathbb{C}(x_1, \ldots, x_r, y)$ (see Proposition 1.5). There are "polynomials" $D_j \in \mathbb{C}[x_1, \ldots, x_r, y]$ such that $\Omega' = (\prod_{j=1}^{r} D_j).\Omega$ is a "polynomial" integrable one-form on M also describing the foliation $\mathcal{F}$. Therefore we will assume that the coefficients A_j of Ω are elements of $\mathbb{C}[x_1, \ldots, x_r, y]$. Now we have the following lemma:

Lemma 13.5. *The following conditions are equivalent for a polynomial $P \in \mathbb{C}[x_1, \ldots, x_r, y]$:*

(i) *$\{P = 0\} \subset M$ is invariant by $\mathcal{F}$*
(ii) *$\Omega \wedge \frac{dP}{P}$ is a "polynomial" 2-form on M, that is, its coefficients belong to $\mathbb{C}[x_1, \ldots, x_r, y]$.*

Lemma 13.5 is proved just like Lemma 10.2 (see also the proof of Lemma **3.5.3** of [10]). The proof of Proposition 13.4 goes as the proof of Theorem 7.6. □

Once Proposition 13.4 is proved one can proceed as in Section 5 and prove Theorems 13.1 and 13.2, and also Corollary 13.3.

14. Local canonical forms of certain foliations with Liouvillian first integrals

According to what we have done above, in several situations, the existence of a Liouvillian first integral for a foliation $\mathcal{F} : \Omega = 0$ is equivalent to the existence of a *Liouvillian integrating factor*, that is, a meromorphic closed one-form η satisfying $d\Omega = \eta \wedge \Omega$ so that if we define $h = \exp \int \eta$ then $\eta = \frac{dh}{h}$ and therefore $d\Omega = \eta \wedge \Omega$ is equivalent to $d(\frac{\Omega}{h}) = 0$. When h is meromorphic the form Ω has a well-known general form [6]. Here we regard the problem of describing the possible general form for Ω when h is *multiform* and *generic*, say, $h = \prod f_j^{\mu_j}$ where f_j is a holomorphic function and μ_j is a complex number $\forall j$; such that:

(i) $(f_i = 0)$ is transverse to $(f_j = 0)$, $\forall i \neq j$.
(ii) If $\sum_j d_j \cdot \mu_j = 0$ with $d_j \in \mathbb{Q}$ then $d_j = 0$, $\forall j$.

We will also assume by simplicity that the ambient is a 2-dimensional complex manifold. Thus it is enough to consider the following situation: Let Ω be a holomorphic one-form defined in an open bidisc Δ centered at the origin of $\mathbb{C}^2$ and let $(x, y) \in \Delta$ be holomorphic coordinates in Δ centered at the origin. Let also $\lambda, \mu \in \mathbb{C}$ with $\lambda/\mu \in \mathbb{C} \setminus \mathbb{Q}$ be given and suppose that $\Omega(p) = 0, p \in \Delta \Leftrightarrow p = 0$.

Proposition 14.1. *Under the conditions above we have the following as equivalent conditions:*

(i) $d(\frac{\Omega}{x^\lambda y^\mu}) = 0$.
(ii) *If $\Omega = \sum_{\nu=}^{\infty} \Omega_\nu$ where Ω_ν is the ν-jet of Ω in the coordinates (x, y) then $d(\frac{\Omega_\nu}{x^\lambda y^\mu}) = 0, \forall \nu \geq 0$.*

(iii) *Ω is of the form $\Omega = f\frac{dy}{y} + f_{\lambda,\mu}\frac{dx}{x}$ for some $f \in \mathcal{O}(\Delta)$ where $f_{\lambda,\mu}$ is defined as*

$$f_{\lambda,\mu} = \sum_{\nu=1}^{\infty}\left(\sum_{j=0}^{\nu}\frac{\lambda-(\nu+1-j)}{\mu-j+1}\cdot a_j\cdot x^{\nu-j}\cdot y^j\right)$$

for $f = \sum_{\nu=1}^{\infty}\left(\sum_{j=0}^{\infty} a_j x^{\nu-j}y^j\right)$.

Proof. (i) $\Leftrightarrow$ (ii): Write $\Omega = \sum_{\nu=0}^{\infty}\Omega_\nu$ as in (ii). Then we have:

$$0 = d\left(\frac{\Omega}{x^\lambda y^\mu}\right) \Leftrightarrow d(\Omega) = \left(\lambda\frac{dx}{x}+\mu\frac{dy}{y}\right)\wedge\Omega \Leftrightarrow \sum_{\nu=0}^{\infty} d\Omega_\nu = \sum_{\nu=0}^{\infty}\left(\lambda\frac{dx}{x}+\mu\frac{dy}{y}\right)\wedge\Omega_\nu$$
$$\Leftrightarrow d\Omega_\nu = \left(\lambda\frac{dx}{x}+\mu\frac{dy}{y}\right)\wedge\Omega_\nu, \quad \forall\nu$$

where the last equivalence follows from the fact that $(\lambda\frac{dx}{x}+\mu\frac{dy}{y})\wedge\Omega_\nu$ is homogeneous of degree $\nu-1, \forall\nu>0$. Therefore

$$d\left(\frac{\Omega}{x^\lambda y^\mu}\right) = 0 \Leftrightarrow d\left(\frac{\Omega_\nu}{x^\lambda y^\mu}\right) = 0, \quad \forall\nu > 0.$$

(ii) $\Leftrightarrow$ (iii): Write $\frac{\Omega_{\nu+1}}{xy} = f_\nu(x,y)\frac{dy}{y} + g_\nu(x,y)\frac{dx}{x}$ for $\nu \geq 0$ where $f_\nu(x,y) = \sum_{j=0}^{\nu} a_j\cdot x^{\nu-j}\cdot y^j$ and $g_\nu(x,y) = \sum_{j=0}^{\nu}\overline{a}_j\cdot x^{\nu-j}\cdot y^j$, $a_j, \overline{a}_j \in \mathbb{C}$. Then an easy computation shows that (ii) equivalent to

$$\overline{a}_j = \frac{\lambda-(\nu+1-j)}{\mu-j+1}\cdot a_j, \quad \forall j = 0,\ldots,\nu.$$

This shows the proposition. □

Proposition 14.1 provides several examples of foliations which admit a Liouvillian first integral. In fact we have:

Corollary 14.2. *There is a canonical bijective correspondence*

$$\left\{\begin{pmatrix}\text{holomorphic one-forms } \Omega \text{ in } \Delta\\ \text{satisfying } d(\frac{\Omega}{x^\lambda y^\mu})=0\end{pmatrix}\right\} \longleftrightarrow \mathcal{O}(\Delta)$$
$$\Omega = xy\left(f\frac{dy}{y}+f_{\lambda,\mu}\frac{dx}{x}\right) \longleftrightarrow f.$$

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