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Non-nested configuration of algebraic limit cycles in quadratic systems ☆

J. Chavarriga, I.A. García *, J. Sorolla

Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69, 25001 Lleida, Spain

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Abstract

This work deals with algebraic limit cycles of planar polynomial differential systems of degree two. More concretely, we show among other facts that a quadratic vector field cannot possess two non-nested algebraic limit cycles contained in different irreducible invariant algebraic curves.

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1. Introduction and statement of the results

We consider here two-dimensional polynomial differential systems of the form

$$\dot{x} = \frac{dx}{dt} = P(x, y) = \sum_{i=0}^{2} P_i(x, y), \qquad \dot{y} = \frac{dy}{dt} = Q(x, y) = \sum_{i=0}^{2} Q_i(x, y), \tag{1}$$

in which $P, Q \in \mathbb{R}[x, y]$ are coprime polynomials where at least one of them has degree 2. Here, $\mathbb{R}[x, y]$ denotes, as usual, the ring of the polynomials in two variables with real coefficients

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* Corresponding author.
E-mail addresses: chava@eps.udl.es (J. Chavarriga), garcia@matematica.udl.es (I.A. García), jsorolla@matematica.udl.es (J. Sorolla).

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and P_i and Q_i denote homogeneous polynomials of degree i. In what follows, system (1) will simply be called *quadratic system*. Sometimes we shall associate to system (1) the vector field $\mathcal{X} = P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y$.

A point $(x_0, y_0) \in \mathbb{C}^2$ is called *finite critical point* of system (1) if $P(x_0, y_0) = Q(x_0, y_0) = 0$. Since P and Q are coprime, this implies that all such critical points are isolated. In the particular case $(x_0, y_0) \in \mathbb{R}^2$ we will call (x_0, y_0) a *real* finite critical point. Otherwise, we will call him *complex* finite critical point.

Let $D\mathcal{X}$ the Jacobian matrix associated to vector field $\mathcal{X}$. The critical point (x_0, y_0) of (1) is classified according to its associated eigenvalues $\lambda, \mu \in \mathbb{C}$, i.e., the eigenvalues of the matrix $D\mathcal{X}(x_0, y_0)$. In particular, if $\lambda\mu = 0$ then the critical point is called *degenerate*. Otherwise it is termed *non-degenerate*. If $D\mathcal{X}(x_0, y_0)$ has exactly one eigenvalue equal to zero then the critical point (x_0, y_0) is called *elementary degenerate*. Finally, if the Jacobian matrix $D\mathcal{X}(x_0, y_0)$ is not zero and it possesses two zero eigenvalues we say that (x_0, y_0) is a *nilpotent* point.

Let us observe that, as system (1) is real, if (x_0, y_0) is a real critical point of it with non-real associated eigenvalues λ and μ then $\mu = \bar{\lambda}$ where the bar denotes complex conjugation operation.

If the real polynomial $f \in \mathbb{R}[x, y]$ with $\deg f \geqslant 1$ satisfies the linear partial differential equation $\mathcal{X} f = kf$, i.e.,

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = kf, \tag{2}$$

for some real polynomial $k(x, y)$ with $\deg k \leqslant 1$, then we say $f = 0$ is a real *invariant algebraic curve* of system (1) and k is called its *cofactor*. We will show in this work the importance of invariant algebraic curves to understand the dynamics of system (1). This fact as been remarked by several authors, see, for instance, [16] and references therein for an exhaustive survey. In short, in most cases, the invariant algebraic curves of (1) are made of *graphics*, i.e., they are contained in a finite union of real singular points and oriented regular orbits connecting them.

Remark 1. It is clear from (2) that, given an invariant algebraic curve $f = 0$ with cofactor k, then all the finite critical points of system (1) verify either $f(x_0, y_0) = 0$ or $k(x_0, y_0) = 0$ or both above conditions. Moreover, since $f, k \in \mathbb{R}[x, y]$, if (x_0, y_0) is a complex critical point of (1) with $f(x_0, y_0) \neq 0$ then $k(x_0, y_0) = k(\bar{x}_0, \bar{y}_0) = 0$.

A function $V : U \subset \mathbb{R}^2 \to \mathbb{R}$ of class C^1 in U that satisfy the linear partial differential equation $\mathcal{X}V = V \operatorname{div} \mathcal{X}$ is called an *inverse integrating factor* of system (1) in U. Here $\operatorname{div} \mathcal{X}$ means the divergence of the associated vector field $\mathcal{X} = P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y$ to system (1). Observe that the set $\Sigma := \{(x, y) \in U\colon\ V(x, y) = 0\}$ is formed by orbits of system (1). Additionally, $1/V$ defines an integrating factor of (1) which allows to compute a first integral of (1) in $U \backslash \Sigma$. Moreover, every compact α- or ω-limit set on U of system (1) that contains a regular point of (1) lies in the zero set of V provided V is an analytic function on U, see [11]. In particular if γ is *limit cycle* of (1) in U then $\gamma \subset \Sigma$ even for V of class C^1 in U. We recall that a limit cycle is an isolated periodic non-constant orbits of (1).

In 1900 Hilbert [12] in the second part of his 16th problem proposed to find the maximum number of limit cycles that can occur in polynomial planar vector fields of given fixed degree, and also to study their distribution in the plane. This problem remains open even for the simplest case, i.e., for quadratic vector fields.

The object of our study will be limit cycles. More precisely we shall study *algebraic limit cycles* which are limit cycles contained in an oval of some real invariant algebraic curve.

Concretely, the open question that we think about is the following one: *Can a quadratic system possess more than one algebraic limit cycle?*

Of course, if system (1) has more than one limit cycle, then they can be distributed in many different ways. Assuming system (1) possesses two algebraic limit cycles γ_i with $i = 1, 2$, two situations distinguished algebraically are presented. Either the two limit cycles are contained in a unique irreducible invariant algebraic curve or there are two different irreducible invariant algebraic curves $f_i(x, y) = 0$ with $i = 1, 2$, such that each one of them contains only one limit cycle. Here, irreducibility is taken over the real field. In this work we will concentrate on the second case. But one still has two cases with different topology respect to the configuration of limit cycles: either the two algebraic limit cycles are nested or not. The main result of this work is the following one.

Theorem 1. *A quadratic system* (1) *cannot possess two non-nested algebraic limit cycles contained in different irreducible invariant algebraic curves.*

It is well known (see [18]) that in a given quadratic system at most two singularities are surrounded by limit cycles and that these singularities necessarily are foci. We say that limit cycles of system (1) have (p, q)-distribution if it possesses p nested limit cycles surrounding one focus and q nested limit cycles surrounding another different focus. In [14], Z. Pingguang proves that limit cycles of a quadratic system with two foci must be $(1, i)$-distribution ($i = 0, 1, \ldots$).

Corollary 2. *If a quadratic system* (1) *with two foci possesses r limit cycles $C_1, \ldots, C_r$ $(r > 1)$ surrounding the same focus and at least one of them is algebraic, i.e., $C_1, \ldots, C_s$ $(1 \leqslant s \leqslant r)$ are algebraic, then there exists another limit cycle $C^\star$ surrounding the other focus. Moreover, either*

(i) *$C^\star$ is a non-algebraic limit cycle or,*
(ii) *$C^\star$ is an algebraic limit cycle and the algebraic limit cycles C_i and $C^\star$ are contained in the same irreducible invariant algebraic curve for some $i = 1, \ldots, s$.*

In the next section, we present the necessary concepts on quadratic systems, projective differential equations, formal differential equations and known results. Next, some preliminary technical results are proved. Using them, we get the proof of Theorem 1 in the last section.

2. Background

We will state some well-known results on differential equations that we shall use later on to prove the main results.

2.1. Configuration of singular points in quadratic systems

The following theorem establishes the coexistence of different type of real singular points in a quadratic system. A simple proof can be found in Kukles and Casanova [13] or Coppel [8] but the property was previously stated by Berlinskiĭ [1].

Theorem 3. *Suppose that there are four real different critical points of a quadratic. If the quadrilateral with vertices these points is convex then the opposite critical points are saddles and the other two are antisaddles* (*nodes, foci or centers*)*. But if the quadrilateral is not convex then* *either the three exterior vertices are saddles and the interior vertex an antisaddle or the exterior vertices are antisaddles and the interior vertex is a saddle.*

2.2. Rational closed 1-forms

Let $\mathbb{C}(x,y)$ be the quotient field of the ring $\mathbb{C}[x,y]$. A rational closed 1-form ω over $\mathbb{C}(x,y)$ is given by $\omega = A(x,y)\,dx + B(x,y)\,dy$ with rational $A, B \in \mathbb{C}(x,y)$ such that $\partial A/\partial y = \partial B/\partial y$. In [15, Lemma 2, p. 205] the next result is proved.

Lemma 4. *If ω is a closed rational differential 1-form over $\mathbb{C}(x,y)$ then there exist polynomials $f_i, f, g \in \mathbb{C}[x,y]$ and constants $\lambda_i \in \mathbb{C}$ for $i = 1, \dots, m$, such that*

$$\omega = \sum_{i=1}^{m} \lambda_i \frac{df_i}{f_i} + d\left(\frac{g}{f}\right). \tag{3}$$

Corollary 5. *Assume that a polynomial system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ with $P, Q \in \mathbb{C}[x,y]$ possesses a rational inverse integrating factor $V \in \mathbb{C}(x,y)$. Then it has a generalized Darboux first integral.*

Proof. We associate to system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ the rational 1-form $\omega = A(x,y)\,dx + B(x,y)\,dy$ with $A = Q/V \in \mathbb{C}(x,y)$ and $B = -P/V \in \mathbb{C}(x,y)$. Since V is an inverse integrating factor of the system it is clear that ω is closed. Therefore, using Lemma 4, we can write ω as in (3). Hence, integrating we have that

$$\bar{H} = \sum_{i=1}^{m} \lambda_i \log f_i + \frac{g}{f},$$

verifies $\partial \bar{H}/\partial x = A$ and $\partial \bar{H}/\partial y = B$, i.e., $\bar{H}$ is a first integral of the system. Finally, taking exponentials, we have that $H = \exp(\bar{H})$ is a generalized Darboux first integral of the form

$$H = \exp\left(\frac{g}{f}\right) \prod_{i=1}^{m} f_i^{\lambda_i},$$

as stated in the corollary. □

We emphasize that the proof of Lemma 4 (and therefore the proof of Corollary 5) is constructive. Moreover, these same ideas with almost identical proof are used in the main result of [6]. In fact, Theorem 2 of that paper looks different, but the proof essentially gives our Corollary 5. Additionally, there is a different point of view of the subject in [20] where it is proved that a necessary and sufficient condition for a first integral

$$H = \int_{(x_0,y_0)}^{(x,y)} \omega$$

of $\omega = Q/V\,dx - P/V\,dy$ to be generalized Darboux is that the extended monodromy group of H is abelian.

2.3. Projective differential equations

Following Darboux [9], we will also extend the differential equation

$$P(x,y)\,dy - Q(x,y)\,dx = 0 \tag{4}$$

with $P(x,y) = \sum_{k=0}^{m} P_k(x,y)$ and $Q(x,y) = \sum_{k=0}^{m} Q_k(x,y)$ polynomials of degree m to the complex projective plane $\mathbb{C}P^2$ in order to compactify. In particular, we can study the nature of the critical points at infinity (points where the homogeneous polynomial $xQ_m(x,y) - yP_m(x,y)$ vanishes) and the behavior of the solutions through them. Now we shortly describe the process of immersing into $\mathbb{C}P^2$ a differential equation (4) defined in the affine plane as well as the inverse path, i.e., the restriction to the projective plane to an affine chart by means of the so called *local coordinates*, see also, for instance, [3].

A differential equation of degree m defined in $\mathbb{C}P^2$ is given by

$$\mathcal{P}(X,Y,Z)\,dX + \mathcal{Q}(X,Y,Z)\,dY + \mathcal{R}(X,Y,Z)\,dZ = 0$$

where $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are homogeneous polynomials of degree $m+1$ verifying $X\mathcal{P} + Y\mathcal{Q} + Z\mathcal{R} \equiv 0$. A point $p_0 = (X_0, Y_0, Z_0) \in \mathbb{C}P^2$ is a critical point of the former differential equation if $\mathcal{P}(X_0,Y_0,Z_0) = \mathcal{Q}(X_0,Y_0,Z_0) = \mathcal{R}(X_0,Y_0,Z_0) = 0$.

It is well known, see [9], that the above differential equation on $\mathbb{C}P^2$ is equivalent to the differential equation

$$(ZM - YN)\,dX + (XN - ZL)\,dY + (YL - XM)\,dZ = 0, \tag{5}$$

where $L(X,Y,Z)$, $M(X,Y,Z)$ and $N(X,Y,Z)$ are homogeneous polynomials of degree m. Additionally, we remark that L, M and N are not uniquely determined by $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$. On the contrary, these polynomials can be replaced by L', M' and N' where $L' = L + X\Lambda$, $M' = M + Y\Lambda$ and $N' = N + Z\Lambda$ where $\Lambda(X,Y,X)$ is any homogeneous polynomial of degree $m-1$ and Eq. (5) remains invariant. The critical points of (5) must satisfy the next system of equation

$$ZL = 0, \qquad ZM = 0, \qquad XM - YL = 0. \tag{6}$$

The projective curve $F(X,Y,Z) = 0$ with F an homogeneous polynomial of degree n is an invariant algebraic curve of (5) if $\tilde{\mathcal{X}}F = KF$ for some homogeneous polynomial $K(X,Y,Z)$ of degree $m-1$ called *cofactor*. Here we define the vector field $\tilde{\mathcal{X}} := L\partial/\partial X + M\partial/\partial Y + N\partial/\partial Z$. In this context, the cofactor K associated to projective invariant algebraic curve $F = 0$ is not uniquely defined due to the former commented invariance of (5). In fact, an easy application of Euler theorem on homogeneous functions shows that when we change (L,M,N) by (L',M',N') then the cofactor changes to $\tilde{K} = K + n\Lambda$, see [3], for example.

The projective differential equation (5) becomes equation of type (4) when taking local coordinates in the chart associated to $Z \neq 0$. To do this, let us consider a point $p_0 \in \mathbb{C}P^2$ in homogeneous coordinates $p_0 = (X_0 : Y_0 : Z_0)$ with $Z_0 \neq 0$. We define the *local coordinates* in p_0 as $x = X/Z$, $y = Y/Z$. Hence, in local coordinates we have $p_0 = (x_0, y_0)$ where

$x_0 = X_0/Z_0$ and $y_0 = Y_0/Z_0$. We say that Eq. (4) is the differential equation (5) at the local chart at p_0 where $P(x,y) := L(x,y,1) - xN(x,y,1)$ and $Q(x,y) := M(x,y,1) - yN(x,y,1)$. It is easy to show that if $F(X,Y,Z) = 0$ is an invariant algebraic curve of (5) with associated cofactor $K(X,Y,Z)$ then $f(x,y) = 0$ is an invariant algebraic curve of (4) with cofactor $k(x,y) = \tilde{K}(x,y,1) - nN(x,y,1)$. Moreover, if p is a critical point of (5) then (x_0, y_0) is a critical point of (4).

The inverse process consists on extending a differential equation (4) defined in the affine plane to $\mathbb{C}P^2$. To get it we make the change to homogeneous coordinates $x = X/Z$ and $y = Y/Z$. Substituting into (4) we obtain

$$L(Y\,dZ - Z\,dY) + M(Z\,dX - X\,dZ) = 0 \tag{7}$$

which is a projective differential equation (5) with $N \equiv 0$. Here we have defined $L(X,Y,Z) := Z^m P(X/Z, Y/Z)$ and $M(X,Y,Z) := Z^m Q(X/Z, Y/Z)$ which are homogeneous polynomials of degree m. Each point (x,y) in the affine plane is transformed into the point $(x : y : 1)$ of the projective plane. The complement in $\mathbb{C}P^2$ of the above points, i.e., the set of points $(X : Y : 0) \in \mathbb{C}P^2$ is called the *line at infinity*. The critical points of (7) that belong to the line at infinity are called *infinite critical points*.

An invariant algebraic curve $f(x,y) = 0$ of degree n of system (4) with cofactor $k(x,y)$ defines an invariant algebraic curve $F(X,Y,Z) = 0$ of (5) with cofactor $K(X,Y,Z)$ where $F(X,Y,Z) = Z^n f(X/Z, Y/Z)$ and $K(X,Y,Z) = Z^{m-1} K(X/Z, Y/Z)$.

The next theorem, proved in [5], provides sufficient conditions in order to have a quadratic system with all its limit cycles algebraic.

Theorem 6. *Let $f(x,y) = 0$ be a real invariant algebraic curve of degree larger than one of a real quadratic system* (1). *Let k be the cofactor of $f = 0$. Suppose that there are two points $p_1, p_2 \in \mathbb{C}P^2$ such that $L(p_i) = M(p_i) = K(p_i) = 0$ for $i = 1, 2$, where $L = Z^2 P(X/Z, Y/Z)$, $M = Z^2 Q(X/Z, Y/Z)$ and $K = Zk(X/Z, Y/Z)$. Then all the limit cycles of* (1) *must be algebraic and contained into $f(x,y) = 0$.*

2.4. Formal differential equations

In this section we summarize some definitions and results about formal differential equations and their solutions, that we shall use later on. For more details and proofs about these results see Seidenberg [17]. Walcher in [19] states also similar results with some precisions.

We consider the field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$). We denote by $\mathbb{K}[[x,y]]$ the ring of formal power series. A *unit* is an invertible element of this ring. In particular, if $U(x,y) = \sum_{i,j=0}^{\infty} u_{ij} x^i y^j$ is a unit then $u_{00} \neq 0$.

Let $F(x,y)$ be an irreducible non-unit of $\mathbb{K}[[x,y]]$ such that $F(x,y) \not\equiv 0$. An *analytic branch centered at* $(0,0)$ is the equivalence class of F under the equivalence $F \sim G$ if $F = U \cdot G$ with U unit. We note that here the adjective *analytic* does not mean the convergence of the power series. On the other hand, $F(0,0) = 0$ because $F(x,y)$ is non-unit.

Given a representative of an analytic branch $F(x,y)$ centered at the origin, there are power series $x(t) = \sum_{i=1}^{\infty} x_i t^i$ and $y(t) = \sum_{i=1}^{\infty} y_i t^i$, with $x_i, y_i \in \mathbb{K}$, not both identically null, such that $F(x(t), y(t)) = 0$. Such a pair is called a *branch expansion* of the analytic branch. Note that $x(0) = 0$ and $y(0) = 0$. Given a branch expansion $x(t), y(t)$, there is an irreducible non-unit

$F(x, y) \not\equiv 0$ in $\mathbb{K}[[x, y]]$, uniquely determined up to a unit factor, such that $F(x(t), y(t)) = 0$. $F(x, y) = 0$ is called the *equation of the branch*.

Consider the formal differential equation

$$P(x, y)\,dy - Q(x, y)\,dx = 0, \tag{8}$$

where $P(x, y), Q(x, y) \in \mathbb{K}[[x, y]]$. For a formal power series

$$F(x, y) = \sum_{i,j=0}^{\infty} f_{ij} x^i y^j$$

we define $\partial F(x, y)/\partial x$ as the formal power series $\sum_{i=1,j=0}^{\infty} i f_{ij} x^{i-1} y^j$. Analogously, we define $\partial F(x, y)/\partial y$.

By a *solution* of the formal differential equation (8) we mean an analytic branch $(x(t), y(t))$, centered at the origin satisfying Eq. (8). More explicitly, if $F(x, y) = 0$ is the equation of the branch of the solution $(x(t), y(t))$ one has

$$P(x, y)\frac{\partial F}{\partial x} + Q(x, y)\frac{\partial F}{\partial y} = K(x, y)F(x, y), \tag{9}$$

for some $K \in \mathbb{K}[[x, y]]$. Conversely, every irreducible $F \in \mathbb{K}[[x, y]]$ with $F \not\equiv 0$ satisfying (9) for some $K \in \mathbb{K}[[x, y]]$, yields a solution of Eq. (8).

A branch $x(t) = \sum_{i=1}^{\infty} x_i t^i$ and $y(t) = \sum_{i=1}^{\infty} y_i t^i$, with $x_i, y_i \in \mathbb{K}$, centered at $(0, 0)$, is called *linear* if x_1 or y_1 is not zero. We shall use the following results also from [17].

Theorem 7. *Let the origin* $(0, 0)$ *be a critical point of the formal system* $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, *where* $P, Q \in \mathbb{C}[[x, y]]$, *with associated eigenvalues* $\lambda, \mu \in \mathbb{C}$. *In the following the dots denote higher order terms.*

(1) *Let* $(0, 0)$ *be a* non-degenerate *critical point associated to the formal differential system*

$$\dot{x} = \lambda x + \cdots, \qquad \dot{y} = \mu y + \cdots, \tag{10}$$

where $\lambda\mu \neq 0$. *If* $\lambda \neq \mu$ *then every formal solution of* (10) *at the origin has a horizontal or vertical tangent. Moreover,*

- (i) *If* $\lambda/\mu \notin \mathbb{Q}^+$ *then* (10) *has exactly two formal solutions at the origin* $F_i(x, y) = 0$ *with* $i = 1, 2$. *They are linear branches with horizontal and vertical tangent respectively, i.e.,* $F_1(x, y) = x + \cdots$, $F_2(x, y) = y + \cdots$.
- (ii) *If* $\lambda/\mu \in \mathbb{Q}^+$ *then the following holds.*
  - (a) *If* $\lambda = \mu$ *then, for each direction there exists only one linear branch formal solution at the origin.*
  - (b) *If* $\lambda/\mu \neq 1$ *(with* $\lambda/\mu > 1$*) then there is one unique linear branch formal solution at the origin with horizontal tangent* $F(x, y) = y + \cdots$. *The other formal solutions at the origin, if they exists, have vertical tangent, i.e., are of the form* $F(x, y) = x^s + \cdots$ *with* $s \in \mathbb{N}\backslash\{0\}$.

(b1) *If* $\lambda/\mu \in \mathbb{N}$ *then either there are no formal solutions at the origin with vertical tangent or there are infinitely many formal solutions at the origin with vertical tangent, all of them linear branches.*

(b2) *If* $\lambda/\mu \notin \mathbb{N}$ *then there is one unique linear branch formal solution at the origin with vertical tangent* $F(x, y) = x + \cdots$.

(2) *Let* $(0, 0)$ *be a* logarithmic *critical point. Then, the formal differential system* $\dot{x} = \lambda x + y + \cdots$, $\dot{y} = \lambda y + \cdots$, *where* $\lambda \neq 0$ *has a unique formal solution at the origin, which is a linear branch with horizontal tangent* $F(x, y) = y + \cdots$.

(3) *Let* $(0, 0)$ *be an* elementary degenerate *critical point. Then, the formal differential system* $\dot{x} = x + \cdots$, $\dot{y} = \cdots$, *has exactly two formal solutions at the origin* $F_i(x, y) = 0$ *with* $i = 1, 2$. *They are linear branches with horizontal and vertical tangent respectively, i.e.,* $F_1(x, y) = x + \cdots$, $F_2(x, y) = y + \cdots$.

(4) *Let* $(0, 0)$ *be a* nilpotent *critical point. Then, the formal differential system* $\dot{x} = y + \cdots$, $\dot{y} = \cdots$, *can have either one, two or infinitely many formal solutions at the origin. If there are just two, they are linear branches.*

Let us consider an irreducible algebraic curve $f(x, y) = 0$ with $f \in \mathbb{C}[x, y]$ such that $f(x_0, y_0) = 0$. We translate the point (x_0, y_0) to the origin. In particular, $f \in \mathbb{C}[[x, y]]$ with $f(0, 0) = 0$, hence f is not a unit element in $\mathbb{C}[[x, y]]$ and in this ring it is possible that f be a reducible element. By using the Newton–Puiseux algorithm (see [2]) one can see that there are ℓ irreducible elements $\phi_i(x, y) \in \mathbb{C}[[x, y]]$, with $i = 1, \ldots, \ell$ such that f factorizes as

$$f(x, y) = x^r U(x, y) \prod_{i=1}^{\ell} \phi_i(x, y), \tag{11}$$

being $r \in \mathbb{N} \cup \{0\}$ and $U \in \mathbb{C}[[x, y]]$ a unit element. Later on, in [3], it was proved that the above decomposition (11) is square free, that is, there is no repeated element ϕ_i and neither do we have $r \geqslant 2$.

Let the origin $(0, 0)$ be a singular point of system (1) and let $f = 0$ be an irreducible invariant algebraic curve of that system such that $f(0, 0) = 0$. The curve $f(x, y) = \sum_{i=s}^{n} f_i(x, y) = 0$ with f_i real homogeneous polynomials of degree i and $s \geqslant 1$, defines a finite number of branches at the origin corresponding to its irreducible non-unit factors in $\mathbb{C}[[x, y]]$. As f_s is homogeneous, it can be factorized over $\mathbb{C}[x, y]$ as $f_s(x, y) = \prod_{i=1}^{s} L_i(x, y)$ where $L_i(x, y) = a_i x + b_i y$ are called the *tangents* of the curve $f = 0$ at the origin and $a_i, b_i \in \mathbb{C}$.

Finally, it is easy to see that each of the irreducible elements appearing in the above formal decomposition (11) of f is a formal solution of (1). Moreover, the tangents at the origin of these branches are given by $f_s = 0$ as defined above.

Let $(x_0, y_0) \in \mathbb{C}^2$ be a critical point with eigenvalues $\lambda, \mu \in \mathbb{C}$. Denoting by $v_\lambda, v_\mu \in \mathbb{C}^2$ the corresponding eigenvectors, we will call $L_\lambda(x, y)$ and $L_\mu(x, y)$ the non-null homogeneous polynomials of degree one belonging to $\mathbb{C}[x, y]$ such that $\nabla L_\lambda \perp v_\lambda$ and $\nabla L_\mu \perp v_\mu$, respectively. Here $\nabla := (\partial/\partial x, \partial/\partial y)$ is the gradient operator and $\perp$ means orthogonality with respect to the standard Euclidean scalar product in $\mathbb{C}^2$. Taking into account all this background, in [3] the following results are proved, which describe the tangents and the value of the cofactor at some generic class of critical points.

Theorem 8. *Let* $f(x, y) = 0$ *with* $f \in \mathbb{C}[x, y]$ *be an irreducible invariant algebraic curve with associated cofactor* $k(x, y)$ *of a real polynomial differential system. Let* $(x_0, y_0) \in \mathbb{C}^2$

be a non-degenerate or elementary degenerate critical point of the system with different associated eigenvalues λ *and* μ *verifying* $f(x_0, y_0) = 0$. *Then, the equation of the tangents of the curve* $f = 0$ *at* (x_0, y_0) *is* $f_s(x, y) = L_\lambda^r(x, y)L_\mu^{s-r}(x, y)$ *with* $s, r \in \mathbb{N}$, $r \leqslant s$. *Moreover,* $k(x_0, y_0) = r\mu + (s - r)\lambda$.

Lemma 9. *Let* $f(x, y) = 0$ *with* $f \in \mathbb{R}[x, y]$ *be an irreducible invariant algebraic curve in* $\mathbb{R}[x, y]$ *with associated cofactor* $k(x, y)$ *of a real polynomial differential system. Let* $(x_0, y_0) \in \mathbb{R}^2$ *be a real critical point of the system with complex eigenvalues* $\lambda = a + \mathrm{i}b$ *and* $\mu = a - \mathrm{i}b$, *where* $b \neq 0$ *and verifying* $f(x_0, y_0) = 0$. *Then, the equation of the tangents of the curve* $f = 0$ *at* (x_0, y_0) *is* $f_2(x, y) = L_\lambda(x, y)L_\mu(x, y)$. *Moreover,* $k(x_0, y_0) = \mu + \lambda$ *and no other invariant algebraic curve* $\tilde{f}(x, y) = 0$ *irreducible in* $\mathbb{R}[x, y]$ *with* $\tilde{f}(x_0, y_0) = 0$ *can exist.*

Remark 2. The above two results and Remark 1 give the possible values of the cofactor k of an invariant algebraic curve $f = 0$ of system (1) at a non-degenerate or degenerate elementary critical point $(x_0, y_0) \in \mathbb{C}^2$ whose ratio of eigenvalues does not equal one. Of course, we can extend system (1) to $\mathbb{C}P^2$. Hence, if $p_0 = (X_0 : Y_0 : Z_0)$ is a singular point of the projective equation (7), we can take local coordinates at this point and Theorem 8 and Remark 1 can be applied. We remark that, for an infinite critical point $p_0 = (X_0 : Y_0 : 0)$ we will obtain by the above procedure conditions on the degree n of the curve $f = 0$ because the coefficients of the cofactor also depend on n.

Notation. We will write $k(p) = \operatorname{div} \mathcal{X}(p)$ in case that $k(p) = \lambda + \mu$.

3. Preliminary results

We will study the algebraic limit cycles of system (1) under the next assumption:

Hypothesis A. Let us suppose that system (1) has two non-nested algebraic limit cycles γ_i with $i = 1, 2$. We will assume, moreover, that system (1) has two different irreducible real invariant algebraic curves $f_i(x, y) = 0$ with $i = 1, 2$, such that $\gamma_i \subset \{(x, y) \in \mathbb{R}^2\colon\ f_i(x, y) = 0\}$.

Since system (1) is a quadratic system, a consequence of Hypothesis A is the existence of two different critical points of non-degenerate focus type p_i with $i = 1, 2$, such that $p_i \subset \operatorname{Int}(\gamma_i)$, the bounded component of the complement of γ_i, see [8].

Lemma 10. *Under Hypothesis* A *the following hold*:

(i) $f_i(p_1)f_i(p_2) = 0$ *for* $i = 1, 2$.
(ii) $f_1^2(p_j) + f_2^2(p_j) \neq 0$ *for* $j = 1, 2$.

Proof. Let $k_i(x, y)$ be the cofactor of the invariant algebraic curve $f_i(x, y) = 0$. Let us assume the contrary of statement (i), that is, suppose that $f_i(p_1)f_i(p_2) \neq 0$ for some $i \in \{1, 2\}$. Then, from (2) it follows $k_i(p_j) = 0$ for $j = 1, 2$ and, since $\deg f_i > 1$, by Theorem 6 all the algebraic limit cycles of the quadratic system must be contained in either curve $f_1(x, y) = 0$ or $f_2(x, y) = 0$. Of course, this is in contradiction with Hypothesis A and so either p_1 or p_2 must belong to the zero level set of f_i for $i = 1, 2$ proving thus statement (i).

In order to prove statement (ii) we suppose the contrary, i.e., $f_1(p_j) = f_2(p_j) = 0$ for some $j \in \{1, 2\}$. Since p_j is a non-degenerate focus, its associated eigenvalues λ and μ are complex numbers $\alpha \pm i\beta$ verifying $\lambda/\mu \notin \mathbb{Q}^+$. We can translate the focus p_j to the origin and make a linear change of coordinates in order to bring system (1) to the form $\dot{x} = \lambda x + \cdots$, $\dot{y} = \mu y + \cdots$. After, applying statement (1)(i) of Theorem 7, we conclude that there are exactly two formal solutions at the origin $F_i(x, y) = 0$ with $i = 1, 2$. More concretely $F_1(x, y) = T_1(x, y) + \cdots$, $F_2(x, y) = \bar{F}_1(x, y) = \bar{T}_1(x, y) + \cdots$, being T_1 the tangent of F_1 at the origin and where the over bar denotes complex conjugation operation. Finally, since $f_i(x, y) = 0$ are real invariant algebraic curves, we conclude that $f_1 = f_2 = F_1\bar{F}_1$ in contradiction with Hypothesis A. □

Lemma 11. *Under Hypothesis* A*, either*

$$k_i(p_j) = \begin{cases} \operatorname{div} \mathcal{X}(p_j), & i = j, \\ 0, & i \neq j, \end{cases} \quad \textit{for } i, j = 1, 2,$$

or

$$k_i(p_j) = \begin{cases} 0, & i = j, \\ \operatorname{div} \mathcal{X}(p_j), & i \neq j, \end{cases} \quad \textit{for } i, j = 1, 2.$$

Proof. From Lemma 10 it follows that one focus belongs to a curve and the other one belongs to the other curve of Hypothesis A. Taking into account Remark 1 it follows that the cofactor is zero over at least one of the foci. On the other hand, if any cofactor vanishes at more that one foci, from (6) we get a contradiction with Hypothesis A. In short, any cofactor is zero exactly at one focus. The value of the cofactor at the other focus is given by Lemma 9. □

Anyway, respect to the configuration of the real or complex finite critical points of system (1), the next possibilities are presented. Two foci p_1 and p_2 exist always and:

(a) There are not more finite critical points;
(b) There is exactly one more finite critical point p_3 which has multiplicity one;
(c) The rest of finite critical points p_3 and p_4 are real. Here it is possible $p_3 = p_4$;
(d) The rest of finite critical points p_3 and p_4 have complex conjugate coordinates.

We will see that the first two former cases (a) and (b) are in contradiction with Hypothesis A. First we present this preliminary result.

Lemma 12. *Let us assume that quadratic system* (1) *has a common factor in their highest order terms, i.e.,* $P_2 = \Lambda L_1$ *and* $Q_2 = \Lambda L_2$ *where* Λ*,* L_1 *and* L_2 *are linear polynomials. Then system* (1) *does not satisfy Hypothesis* A.

Proof. By linear change of variables we consider the case $\Lambda = x$ without lost of generality. Then the point $q_1 = (0 : 1 : 0) \in \mathbb{C}P^2$ is an singular point of (1) at infinity.

Assume the contrary of the thesis, i.e., Hypothesis A is verified. Let $F_i(X, Y, Z) = 0$ and $K_i(X, Y, Z)$ be the projectivizations of the invariant algebraic curves $f_i(x, y) = 0$ and its associated cofactors for $i = 1, 2$, respectively.

We take local coordinates in a neighborhood of the singular point q_1 and denote by $\tilde{F}_i(X,1,Z)=0$ and $\tilde{K}_i(X,1,Z)$ the transformed invariant curves and cofactors in such coordinates respectively, see the preliminaries. Since $\tilde{K}_i(X,1,Z)=K_i(X,1,Z)-\deg f_i M(X,1,Z)$ by definition and $M(q_1)=0$ it follows

$$\tilde{K}_i(q_1)=K_i(q_1). \tag{12}$$

Additionally, it is easy to see that the linear part of the system in local coordinates at q_1 is given by

$$\begin{pmatrix} L_1(0,1) & P_1(0,1) \\ 0 & 0 \end{pmatrix}.$$

This means that q_1 has at least one associated eigenvalue different from zero. If both eigenvalues vanish then $\tilde{K}_i(q_1)=0$ for $i=1,2$. Otherwise, if exactly one eigenvalue is zero then, from statement (3) of Seidenberg's Theorem 7, it follows that there are two formal solutions trough q_1. Since one of them is the line at infinity $Z=0$, it is clear that $\tilde{F}_1(q_1)\neq 0$ or $\tilde{F}_2(q_1)\neq 0$. This implies that $\tilde{K}_1(q_1)=0$ or $\tilde{K}_2(q_1)=0$, respectively. Hence, taking into account (12) we get $K_1(q_1)=0$ (re-indexing if necessary).

We know that the affine cofactor $k_1(x,y)$ vanishes also at one of the two foci by Lemma 11. Hence $K_1(X,Y,Z)$ vanishes at such focus, too. Therefore, we are under hypothesis of Theorem 6 and we get a contradiction with Hypothesis A. □

Proposition 13. *Let us assume that quadratic system* (1) *has two real finite different critical points p_1 and p_2 of non-degenerate focus type. If either there are not more finite critical points or there is exactly one more finite critical point p_3 with multiplicity one then system* (1) *does not satisfy Hypothesis* A.

Proof. We consider the homogeneous polynomials $L(X,Y,Z)=Z^2P(X/Z,Y/Z)$ and $M(X,Y,Z)=Z^2Q(X/Z,Y/Z)$. We denote $I_p(L,M)$ the intersection index of $L=0$ and $M=0$ at the point $p\in\mathbb{C}P^2$, see a formal definition in [10]. From Bézout Theorem it follows $\sum_p I_p(L,M)=4$. Since p_1 and p_2 are non-degenerate foci, its associated eigenvalues are different from zero and then p_1 and p_2 have multiplicity one as common roots of $P(x,y)$ and $Q(x,y)$. Hence $I_{p_i}(L,M)=1$ for $i=1,2$. We split the study of each situation described in the proposition.

- If there are not more finite critical points of system (1) then there are points $q_j\in\{Z=0\}\cap\{L=0\}\cap\{M=0\}$ such that $\sum_{q_j} I_{q_j}(L,M)=2$. Therefore $Q_2(x,y)=\alpha P_2(x,y)$ with $\alpha\in\mathbb{R}$ and from Lemma 12 system (1) does not satisfy Hypothesis A.
- If there is exactly one more finite critical point p_3 of system (1) with multiplicity one then $\sum_{i=1}^{3} I_{p_i}(L,M)=3$. So there is exactly one point $q\in\{Z=0\}\cap\{L=0\}\cap\{M=0\}$ such that $I_q(L,M)=1$. Therefore P_2 and Q_2 have exactly one real common divisor of degree 1. Hence, applying Lemma 12, system (1) does not verifies Hypothesis A. □

The next two propositions explore the possibilities of the above cases (c) and (d). In such study we shall consider the real straight line $L(x,y):=pk_1(x,y)+qk_2(x,y)-\operatorname{div}\mathcal{X}(x,y)=0$, with $p,q\in\mathbb{R}$. The main idea in what follows consists on to look for three finite critical points of system (1) such that L vanishes at them. Of course such critical points are not in any straight line because in this case $P(x, y)$ and $Q(x, y)$ are not coprime. So the only possibility is $L(x, y) \equiv 0$ and therefore, applying Darboux's integrability theory we conclude that $f_1^p f_2^q$ is an inverse integrating factor of the system.

Proposition 14. *Let us assume that quadratic system* (1) *verifies Hypothesis* A *and, moreover, the other finite critical points p_3 and p_4 are real. Then $f_1(x, y)f_2(x, y)$ is an inverse integrating factor of the system.*

Proof. We will start with two different cases which are either $p_3 \neq p_4$ or $p_3 = p_4$.

If $p_3 \neq p_4$ then each one have multiplicity one. Since p_1 and p_2 are foci of the quadratic system, using Theorem 3, we can suppose that p_3 is a topological saddle. Hence the quotient of the eigenvalues associated to p_3 is negative. So following Seidenberg's results and more concretely statement (1)(i) of Theorem 7, there are exactly two formal solutions (linear branch) with different tangent at p_3.

If $f_1(p_3) \neq 0$ or $f_2(p_3) \neq 0$, then $k_1(p_3) = 0$ or $k_2(p_3) = 0$. Then applying Theorem 6 and Lemma 11 we have that all the limit cycles are contained in $f_1 = 0$ or $f_2 = 0$, respectively. This is a contradiction with Hypothesis A. Therefore, the only possibility consists in that the invariant algebraic curve $f_1 = 0$ contains exactly one branch at p_3 and $f_2 = 0$ the other one.

Hence, translating the critical point p_3 to the origin, and making a linear change of coordinates we will continue assuming $f_1(x, y) = x + \cdots$, $f_2(x, y) = y + \cdots$ and the system becomes $\dot{x} = \lambda x + \cdots$, $\dot{y} = \mu y + \cdots$, where λ and μ are the eigenvalues associated to p_3. Now, equating the same powers of x and y in both members of the equations $\mathcal{X} f_i = k_i f_i$ for $i = 1, 2$, we have that $k_1(p_3) = \lambda$ and $k_2(p_3) = \mu$. Since $\operatorname{div} \mathcal{X}(p_3) = \lambda + \mu$ we have in short $k_1(p_3) + k_2(p_3) - \operatorname{div} \mathcal{X}(p_3) = 0$. As we also knew that $k_1(p_i) + k_2(p_i) - \operatorname{div} \mathcal{X}(p_i) = 0$ for $i = 1, 2$, this implies $k_1(x, y) + k_2(x, y) \equiv \operatorname{div} \mathcal{X}(x, y)$ because k_1, k_2 and $\operatorname{div} \mathcal{X}$ are polynomials of degree at most one. Finally, by Darboux's integrability theory we conclude that $f_1(x, y)f_2(x, y)$ is an inverse integrating factor of system (1).

In the second option, i.e., when $p_3 = p_4$, we have that p_3 is a critical point of system (1) with multiplicity two and therefore either p_3 is a nilpotent singular point or exactly one of the eigenvalues associated to p_3 is null. Now we put p_3 at the origin and in the first case the quadratic system can be written after a linear change of coordinates as $\dot{x} = y + \cdots$, $\dot{y} = \cdots$. From (2) at lower degree it follows $k_i(p_3) = 0$ for $i = 1, 2$. Taking into account Lemma 11 and Theorem 6 we get that $f_1 = 0$ and $f_2 = 0$ contain each one all the limit cycles. This is a contradiction because $f_1 \neq f_2$ and are irreducible.

We can assume that exactly one eigenvalue associated to p_3 (now at the origin) is equal zero. Then we can write the system as $\dot{x} = \lambda x + \cdots$, $\dot{y} = \cdots$. By statement (3) of Seidenberg's Theorem 7, it follows that the above system has exactly two formal solutions at the origin $F_i(x, y) = 0$ with $i = 1, 2$ of the form $F_1(x, y) = x + \cdots$ and $F_2(x, y) = y + \cdots$. The following possibilities appear: either $f_i(p_3) \neq 0$ for some $i \in \{1, 2\}$ and so $k_i(p_3) = 0$ for such i or $f_1(p_3) = 0$ and $f_2(p_3) = 0$. The first case leads to a contradiction with Hypothesis A because we have two critical points (p_3 and one focus) in the straight line $k_1(x, y) = 0$ and we can apply Theorem 6. In the second option, when $f_i(p_3) = 0$ for $i = 1, 2$, it follows that $f_1 = 0$ contains exactly one branch and $f_2 = 0$ the other one. Moreover, from Theorem 8 we have either $k_1(p_3) = 0$ or $k_2(p_3) = 0$. Again, using Theorem 6 we get a contradiction with Hypothesis A. □

Proposition 15. *Let us assume that quadratic system* (1) *verifies Hypothesis* A *and moreover the other finite critical points* p_3 *and* p_4 *are not real. Then* $f_1(x, y) f_2(x, y)$ *is an inverse integrating factor of the system.*

Proof. Of course, since system (1) is real, if $p_3 = (x_3, y_3)$ and $p_4 = (x_4, y_4)$ are not real then its coordinates are complex conjugates, i.e., $x_4 = \bar{x}_3$ and $y_4 = \bar{y}_3$. This will be denoted by $p_4 = \bar{p}_3$. Moreover, the eigenvalues associated to each point verify the same property. So if λ and μ are the eigenvalues associated to p_3 then $\bar{\lambda}$ and $\bar{\mu}$ are the eigenvalues associated to p_4.

Let us suppose that p_3 (and therefore p_4) is not a resonant node. This means that $\lambda/\mu \notin \mathbb{Q}^+$. In this case we may simply repeat verbatim the first paragraph in the proof of Proposition 14 when we apply statement (1)(i) of Theorem 7 to conclude a contradiction with Hypothesis A.

We continue supposing that p_3 and $p_4 = \bar{p}_3$ are resonant nodes. Hence the ratio of the eigenvalues λ and μ associated to p_3 is a positive rational number and are related by means of $\mu = \kappa\lambda$ with $\kappa \in \mathbb{Q}^+$. Of course the eigenvalues $\bar{\lambda}$ and $\bar{\mu}$ associated to p_4 verify $\bar{\mu} = \kappa\bar{\lambda}$. Moreover, $\operatorname{div} \mathcal{X}(p_3) = (\kappa + 1)\lambda$ and $\operatorname{div} \mathcal{X}(p_4) = (\kappa + 1)\bar{\lambda}$.

If $f_i(p_3) = 0$ with $i = 1, 2$ then, applying Theorem 8 we have that $k_i(p_3) = r_i\mu + (s_i - r_i)\lambda$ for $i = 1, 2$ where $s_i, r_i \in \mathbb{N}$ and $r_i \leqslant s_i$. Clearly this implies $k_i(p_3) = \alpha_i\lambda$ where $\alpha_i := r_i\kappa + s_i - r_i \in \mathbb{Q}^+$. Furthermore since $k_i \in \mathbb{R}[x, y]$ and $p_4 = \bar{p}_3$ then $k_i(p_4) = \alpha_i\bar{\lambda}$ for $i = 1, 2$.

Now let us consider the real straight line $L(x, y) := pk_1(x, y) + qk_2(x, y) - \operatorname{div} \mathcal{X}(x, y) = 0$, with $p, q \in \mathbb{R}$. We have

$$L(p_3) = \big[p\alpha_1 + q\alpha_2 - (\kappa + 1)\big]\lambda, \tag{13}$$

where $\lambda \neq 0$. We recall here that, since $p_4 = \bar{p}_3$ and $L \in \mathbb{R}[x, y]$, if $L(p_3) = 0$ then $L(p_4) = 0$.

If we are in the first case of Lemma 11, then $k_i(p_i) = \operatorname{div} \mathcal{X}(p_i)$ and $k_i(p_j) = 0$ for $i \neq j$ and $i, j \in \{1, 2\}$. This implies

$$L(p_1) = (p - 1) \operatorname{div} \mathcal{X}(p_1), \qquad L(p_2) = (q - 1) \operatorname{div} \mathcal{X}(p_2). \tag{14}$$

First of all we claim that none of the foci p_1 and p_2 can be weak foci because in this case $\operatorname{div} \mathcal{X}(p_i) = 0$ for some $i \in \{1, 2\}$ and so either $k_1(p_i) = 0$ for $i = 1, 2$ or $k_2(p_i) = 0$ for $i = 1, 2$ in contradiction with Hypothesis A by Theorem 6.

So we continue the proof assuming $\operatorname{div} \mathcal{X}(p_i) \neq 0$ for $i = 1, 2$. If we impose $L(p_1) = 0$ then $p = 1$ from the first equation (14). Moreover, from (13) we can take $q = (\kappa + 1 - \alpha_1)/\alpha_2$ so that $L(p_3) = L(p_4) = 0$. Hence $L(p_i) = 0$ for $i = 1, 3, 4$ and therefore $L(x, y) \equiv 0$. But now, from the second equation of (13) we have that, in fact, $q = 1$. So, quadratic system (1) admits the polynomial inverse integrating factor $f_1(x, y) f_2(x, y)$.

If the second case of Lemma 11 is verified then the proof is similar. □

Proposition 16. *Under Hypothesis* A, *the curves* $f_1 = 0$ *and* $f_2 = 0$ *are the unique invariant algebraic curves of system* (1).

Proof. We suppose that another invariant algebraic curve $f_3 = 0$ irreducible in $\mathbb{R}[x, y]$ exists with $\mathcal{X} f_3 = k_3 f_3$ for some polynomial k_3. Assuming Hypothesis A, f_3 must have degree greater than one because it is well known that a quadratic system with an invariant straight line has at most one limit cycle, see [7] or [18].

As we have proved in Lemma 10, the foci p_i, $i=1,2$, are contained in the curves $f_i=0$, $i=1,2$ (each focus in one curve). Then, from Lemma 9, $f_3(p_i)\neq 0$ and so $k_3(p_i)=0$, $i=1,2$. Now, applying Theorem 6, it follows that all the limit cycles of system (1) must be contained in $f_3=0$, against Hypothesis A. □

4. Proof of the main result

We will see that Hypothesis A cannot be satisfied for system (1). Assuming the contrary, i.e., Hypothesis A is fulfilled, we have shown that system (1) has the polynomial inverse integrating factor $V=f_1f_2$. Hence it must have a Darboux first integral H, see Corollary 5. Since $f_1=0$ and $f_2=0$ are real curves and, from Proposition 16, they are the unique invariant algebraic curves of system (1) it follows that

$$H=f_1^{\lambda_1}f_2^{\lambda_2}\left[\exp\left(\frac{h_1}{f_1^{n_1}}\right)\right]^{\mu_1}\left[\exp\left(\frac{h_2}{f_2^{n_2}}\right)\right]^{\mu_2},$$

for some $\lambda_i,\mu_i\in\mathbb{C}$, $n_i\in\mathbb{N}\setminus\{0\}$, $h_i\in\mathbb{C}[x,y]$, where h_i and f_i are coprime polynomials for $i=1,2$.

Following the ideas of [4], we compute

$$\log H=\lambda_1\log f_1+\lambda_2\log f_2+\mu_1\frac{h_1}{f_1^{n_1}}+\mu_2\frac{h_2}{f_2^{n_2}}$$

which is also a first integral for system (1) whose partial derivatives are rational functions.

The inverse integrating factor $\hat{V}$ related to the first integral $\log H$ is given by

$$\hat{V}=-\frac{P}{\frac{\partial}{\partial y}\log H}=\frac{Q}{\frac{\partial}{\partial x}\log H}.$$

It must be verified $\hat{V}=V$ (modulus a multiplicative constant). Otherwise, $\hat{H}=\frac{\hat{V}}{V}$ is a rational first integral and excludes the existence of limit cycles. In other words

$$f_1f_2\frac{\partial}{\partial x}\log H=Q, \tag{15}$$

must be verified. Moreover, it can be checked that $\frac{\partial}{\partial x}\log H=\frac{\Phi}{\Lambda}$, where $\Lambda=f_1^{n_1+2}f_2^{n_2+2}$ and

$$\begin{aligned}\Phi&=\lambda_1f_1^{n_1+1}f_2^{n_2+2}\frac{\partial f_1}{\partial x}+\lambda_2f_1^{n_1+2}f_2^{n_2+1}\frac{\partial f_2}{\partial x}\\&\quad+\mu_1f_1f_2^{n_2+2}\left(f_1\frac{\partial h_1}{\partial x}-n_1h_1\frac{\partial f_1}{\partial x}\right)+\mu_2f_1^{n_1+2}f_2\left(f_2\frac{\partial h_2}{\partial x}-n_2h_2\frac{\partial f_2}{\partial x}\right).\end{aligned}$$

Relation (15) becomes $\Phi=Qf_1^{n_1+1}f_2^{n_2+1}$, from where $f_1^{n_1+1}f_2^{n_2+1}$ divides Φ. Therefore, f_1 must divide $-n_1h_1\frac{\partial f_1}{\partial x}$ and then $h_1=\Omega f_1$ for certain polynomial $\Omega\in\mathbb{R}[x,y]$. Thus, h_1 and f_1 are not coprime, which is a contradiction. □

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