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On the integrability of differential equations by quadratures according to Maximovič

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Dedicated to W.N. Everitt on the occasion of his 80th birthday.

Abstract

We show that a first-order ordinary differential equation can be integrated by quadratures in the sense of Maximovič only if it arises from the linear equation by a diffeomorphic transformation of the dependent variable. In the appendix this result is applied to show that the linear second-order equation can be integrated by quadratures in a restricted sense only if it has constant coefficients. A brief outline of Maximovič's life is also included.

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0. Introduction

It is a well-known fundamental fact of differential equation theory that there is no general method or algorithm to solve a given ordinary differential equation; in fact there are only very few differential equations whose solutions can be given explicitly in terms of elementary functions and processes. Several attempts at a systematic treatment of the question which equations can be integrated in finite terms have been undertaken, beginning with Liouville's theory (cf. [7]); later Lie's study of transformation groups was motivated by a classification of (partial) differential equations and integration methods for their solution (see [5] for an account). His ideas have subsequently given rise to the voluminous theory of Picard–Vessiot extensions in differential algebra (cf. [6]).

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In his 1885 treatise [4], Maximovič has developed an independent approach. Motivated by the well-known explicit solution formula for the linear first-order equation

$$y' + py = q, \tag{1}$$

$$y(x) = \exp\left(-\int_{x_0}^{x} p\right)\left(y(x_0) + \int_{x_0}^{x} q(t)\exp\left(\int_{x_0}^{t} p\right)\,\mathrm{d}t\right), \tag{2}$$

which contains the two *quadratures* $s = \int p$ and $S = \int q\mathrm{e}^{s}$, he considers the question under which circumstances a first-order ordinary differential equation can be *integrated by quadratures*, i.e. its general solution can be expressed as an (elementary) function of finitely many arbitrarily nested quadratures, their integration constants serving as parameters for the solution manifold. (In (2) and the following, the variable of integration is not shown in integrals unless necessary.) Note that in the above example the same solution formula is valid for all (reasonable) coefficient functions p and q: similarly, Maximovič studies symbolic differential equations containing a number of undetermined coefficient functions, and seeks a solution formula which holds independently of the special form these coefficients take.

In the first part of [4] he claims to show that a symbolic first-order ordinary differential equation can be integrated by quadratures in this sense if and only if it arises from the linear first-order equation (1) by means of a transformation of the unknown variable y. In the second part he proceeds to find criteria for a given equation to have this property, and concludes, among other things, that the linear second-order equation (which is intimately connected with the non-linear first-order Riccati equation) cannot be integrated by quadratures in general.

Despite Maximovič's professed intention 'to lay the foundation of an entirely new theory' of an importance 'comparable to that of the theory of general algebraic equations' [4, p. 1], his work does not seem to have found a wider audience. An apparent outward reason for this is its practical inaccessibility as a monograph printed at the Imperial University at Kazan', unavailable even at the larger American and Western European libraries; thus Ritt in his classical treatise [7, p.77], regrets that as he 'has not been able to secure Maximovich's paper or any account of it except those given in an abstract in the *Jahrbuch* and in one in the Paris *Comptes Rendus*, he is unable to make a definite statement in regard to it.' When mentioning Maximovič's result in his annotations to Lie's Collected Works [2, p.686], Engel also refers to Vasil'ev's review in the *Jahrbuch* [8] only. Moreover, Maximovič's statements are often obscure and open to interpretation. Nevertheless, his work seems to contain some original, useful and justifiable ideas which deserve to be brought to light.

It is the purpose of the present paper to give a precise form to Maximovič's definition of integrability by quadratures and to prove that first-order equations which can be integrated in this sense are essentially linear. Although [4] has served as a source of inspiration, no attempt is made to reconstruct Maximovič's thought. Specifically, our exposition fundamentally differs from his in that we consider, instead of a symbolic ordinary differential equation, i.e. a class of ordinary differential equations of similar structure, a single differential equation without any hypotheses regarding its structure or coefficients. This clearly does not restrict the generality of our main result as compared to the corresponding theorem of Maximovič. We assume however (as does Maximovič implicitly) that all initial value problems have unique solutions defined on a fixed interval.

The paper is organized as follows. In Section 1 we state the main result (Corollary 1.7) that, up to diffeomorphic transformations of the unknown variable, the linear first-order equation is the only equation which can be integrated by quadratures in the sense of Definition 1.3. This result is a consequence of a normal-form theorem (Theorem 1.6) for integrals by quadratures which represent the general solution of a first-order ordinary differential equation: the number of quadratures can always be reduced to at most two, which moreover enter the integral in a very specific way. Sections 2 and 3 are devoted to the proof of Theorem 1.6. We remark that although the functions occurring in the quadrature expression must be assumed to be elementary in some sense in order to avoid a tautology (cf. Remark 3 to Definition 1.3), the proof of Theorem 1.6 does not make use of the fact or nature of this elementary property, only certain mild regularity properties are assumed. In an appendix, we apply Corollary 1.7 to show that the linear second-order equation can be integrated by quadratures in a somewhat restricted sense (which excludes, in particular, the well-known examples by Bernoulli, cf. [7, VI Section 3]) only if it has constant coefficients. We conclude with a brief account of Maximovič's life, based on the information given in the preface of [4] and in the obituary notice [9].

1. Effectively one-parametric integrals by quadratures

In this section we give a definition of integrability by quadratures modelled roughly on [4], and state (Theorem 1.6) that if such an integral is essentially one-parametric, e.g. if it represents the general solution of a sufficiently well-behaved first-order ordinary differential equation, then it can be reduced to a simple normal form which is very similar to the integral of the linear equation (2). From this our main result (Corollary 1.7) follows.

Throughout the paper, we fix an interval $I \subset \mathbb{R}$, and a point $x_0 \in I$.

Definition 1.1. We define a *system of quadratures* recursively as follows:

(i) If $\varphi_1 : I \to \mathbb{R}$ is a locally integrable function, then, with

$$s_1(x) := \int_{x_0}^{x} \varphi_1(t)\,\mathrm{d}t \quad (x \in I),$$

(s_1) is a system of quadratures.

(ii) If $(s_1, \ldots, s_n)$ is a system of quadratures, $n \in \mathbb{N}$, and $\varphi_{n+1} : I \times \mathbb{R}^n \to \mathbb{R}$ is locally integrable in the first and has continuous partial derivatives in the other variables, then setting for $x \in I, c_1, \ldots, c_n \in \mathbb{R}$,

$$s_{n+1}(x, c_1, \ldots, c_n) := \int_{x_0}^{x} \varphi_{n+1}(t, s_1(t) + c_1, \ldots, s_n(t, c_1, \ldots, c_{n-1}) + c_n)\,\mathrm{d}t,$$

$(s_1, \ldots, s_n, s_{n+1})$ is a system of quadratures. We call φ_j the *integrand* of s_j.

Remark. Thus a system of quadratures is a collection of finite quadrature expressions which is *ordered* in that a quadrature can occur, along with its integration constant, only in the integrands of quadratures with higher index; and *complete* in that all quadratures occurring in integrands are included in the system. Note that any collection of quadratures which is complete in the latter sense can be re-arranged to form a system of quadratures, since φ_n is permitted to be a constant function of one or more of its arguments.

Definition 1.2. A system $(s_1,\dots,s_n)$ of quadratures is called *independent* if there exist $c_1,\dots,c_{n-1}\in\mathbb{R}$ such that the functions $\varphi_1,\varphi_2(\cdot,c_1),\dots,\varphi_n(\cdot,c_1,\dots,c_{n-1}):I\to\mathbb{R}$ are linearly independent (φ_j being the integrand of s_j, $j\in\{1,\dots,n\}$).

In the following, we denote by ∂_n the partial derivative with respect to the n-th variable.

Definition 1.3. A function $F:\mathbb{R}^n\to\mathbb{R}$, $n\in\mathbb{N}$, is called *admissible* if it is four times continuously differentiable, and $\partial_n F$ has no zeros.

Let $\Theta: I\times\mathbb{R}\to\mathbb{R}$ be continuously differentiable in the second variable such that $\partial_2\Theta$ has no zeros, $(s_1,\dots,s_n)$ an independent system of quadratures, and $F:\mathbb{R}^n\to\mathbb{R}$ an admissible function, $n\in\mathbb{N}$. Then we call the family of functions

$$f(x,c_1,\dots,c_n)=\Theta(x,F(s_1(x)+c_1,\dots,s_n(x,c_1,\dots,c_{n-1})+c_n)),\quad (x\in I,(c_1,\dots,c_n)\in\mathbb{R}^n),$$

an *integral by quadratures*.

Remarks. 1. The name 'integral' reflects that we are primarily interested in families $f(\cdot,c_1,\dots,c_n)$ which represent the general solution of an ordinary differential equation.

2. Maximovič claims that the above structure of an integral by quadratures replaces, without loss of generality, the more general expression

$$f(x,c_1,\dots,c_n)=F(x,s_1(x)+c_1,s_2(x,c_1)+c_2,\dots,s_n(x,c_1,\dots,c_{n-1})+c_n)$$

if f represents the general solution of a first-order ordinary differential equation [4, Section II].

3. Note that according to the above definition, *every* one-parameter family of functions (with sufficiently regular dependence on the parameter) can be represented by an integral by quadratures: given $f(\cdot,c): I\to\mathbb{R}$, $c\in\mathbb{R}$, we take any locally integrable function φ and set $s(x):=\int_{x_0}^x\varphi$, $\Theta(x,c):=f(x,c-s(x))$, $F(c):=c$ $(x\in I,c\in\mathbb{R})$; then trivially

$$f(x,c)=\Theta(x,F(s(x)+c))\quad (x\in I,c\in\mathbb{R}).$$

In order to give some meaning to the concept of 'integrability by quadratures', it is therefore necessary to add the assumption that Θ and/or F be elementary in some sense. We emphasize, however, that neither the fact nor the nature of this elementary property are made use of in the considerations of this paper, and we therefore do not refer to it specifically in our definition.

4. The assumptions that the system of quadratures be independent, and the conditions on Θ and F are of a technical nature, necessary in the arguments of Section 3.

Definition 1.4 (cf. [3, Section II, Définition I]). Let $n,m\in\mathbb{N}$. Two families $f(\cdot,\mathbf{c}),\mathbf{c}\in\mathbb{R}^n$, and $g(\cdot,\mathbf{d}),\mathbf{d}\in\mathbb{R}^m$, of functions $:I\to\mathbb{R}$ are called *equivalent* if for each $\mathbf{c}\in\mathbb{R}^n$ there is a $\mathbf{d}\in\mathbb{R}^m$ such that $f(\cdot,\mathbf{c})\equiv g(\cdot,\mathbf{d})$, and vice versa.

Now consider a first-order ordinary differential equation which, for each real initial value at x_0, has a unique solution defined at least on I. We say that such an equation can be *integrated by quadratures* if its general solution on $I\times\mathbb{R}$ is equivalent to an integral by quadratures. As the general solution is a one-parameter family of functions parametrized by their value at x_0, the integral, though containing n free integration constants, has only one effective parameter, which means that the integration constants are not distinct, but compensate each other. This observation is central to Maximovič's work.

Definition 1.5. A family of functions $f(\cdot,\mathbf{c}) : I \to \mathbb{R}$, $\mathbf{c} \in \mathbb{R}^n$, is called *effectively one-parametric* if it is equivalent to a one-parameter family $\hat{f}y(\cdot,d) : I \to \mathbb{R}$, $d \in \mathbb{R}$, with $\hat{f}(x_0,d) = d$ $(d \in \mathbb{R})$.

Example. Let $p,q : \mathbb{R} \to \mathbb{R}$ be linearly independent, locally integrable functions; then with $\varphi_1(x) := p(x)$, $\varphi_2(x,c_1) := q(x)e^{c_1}$ $(x,c_1 \in \mathbb{R})$, (s_1,s_2) is an independent system of quadratures. If $\Theta : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable in the second variable, $\partial_2\Theta \neq 0$, and $\Theta(0,\cdot)$ is surjective, then

$$f(x,c_1,c_2) := \Theta\left(x, \exp\left(-\left(\int_0^x p\right) - c_1\right)\left(\int_0^x q(t)\exp\left(\left(\int_0^t p\right) + c_1\right)\,dt + c_2\right)\right) \quad (x,c_1,c_2 \in \mathbb{R})$$

is an effectively one-parametric integral by quadratures (with $F(c_1,c_2) = e^{-c_1}c_2$).

Indeed, it is easy to see that the parameter c_1 can be eliminated, and $f(\cdot,c_1,c_2)$ is equivalent to $\hat{f}(\cdot,d) := f(\cdot,0,\Theta^{-1}(0,d))$ (where $\Theta^{-1}(0,\cdot)$ is the inverse of $\Theta(0,\cdot)$).

Our main result shows that the structure of $\hat{f}$ in this simple example is in fact universal; specifically, we are going to prove

Theorem 1.6 (Normal form of effectively one-parametric integrals by quadratures). *If f is an effectively one-parametric integral by quadratures, then there are locally integrable functions $p,q : I \to \mathbb{R}$ and a function $\hat{\Theta} : I \times \mathbb{R} \to \mathbb{R}$ which is continuously differentiable in the second variable, such that f is equivalent to the family*

$$\hat{f}(x,C) := \hat{\Theta}\left(x, \exp\left(-\int_{x_0}^x p\right)\left(\int_{x_0}^x q(t)\exp\left(\int_{x_0}^t p\right)\,dt + C\right)\right)$$

$(x \in I, C \in \mathbb{R})$.

Remark. Moreover, if f is given in the form

$$f(x,c_1,\ldots,c_n) = \Theta(x, F(s_1(x)+c_1,\ldots,s_n(x,c_1,\ldots,c_{n-1})+c_n)),$$

then—as will be apparent from the proof of Theorem 1.6—$\hat{\Theta}(x,c) = \Theta(x,F(0,\ldots,0,c))$ $(x \in I, c \in \mathbb{R})$. The final p and q arise from F and the integrands of $s_1,\ldots,s_n$ by the basic arithmetical operations, exponentiation, differentiation and integration. Thus the process leading to the normal form can be traced explicitly in principle, and in particular it can be inferred that $\hat{\Theta}$ will be elementary if Θ and F are.

Corollary 1.7. *A first-order ordinary differential equation*

$$0 = \mathscr{F}(x,y,y')$$

which, for each real initial value at x_0, has a unique solution which is defined at least on I, can be integrated by quadratures only if it arises from the linear first-order equation by a diffeomorphic transformation of the unknown variable.

Remarks. 1. As a consequence, a first-order differential equation which can be integrated by quadratures is of the type

$$\partial_1 \Phi(x, y) + \partial_2 \Phi(x, y) y' = q(x) - p(x)\Phi(x, y),$$

with $p, q : I \to \mathbb{R}$ locally integrable and $\Phi : I \times \mathbb{R} \to \mathbb{R}$ continuously differentiable, $\partial_2 \Phi \neq 0$. (Φ is the inverse of $\hat{\Theta}$ with respect to the second variable.)

If $p \equiv 0$, $\partial_1 \Phi \equiv 0$, then the resulting equation $\Phi'(y)\, y' = q(x)$ has separated variables; note that its integral by quadratures,

$$f(x, c) = \Phi^{-1}\left(\int_{x_0}^{x} q(t)\,\mathrm{d}t + c\right),$$

contains only one quadrature. The second quadrature in the usual solution formula for differential equations of separated type, which arises in the process of passing from Φ' to Φ^{-1}, does not introduce an independent integration constant and does not figure explicitly in Maximovič's form of the integral.

2. By the Picard–Lindelöf Theorem, differential equations of the type

$$y' = g(y, x)$$

satisfy the requirement of having a unique solution on I for all initial data if g is uniformly Lipschitz in the first variable. Our hypothesis excludes cases with movable singularities, such as the Riccati equation. However, the Riccati equation $y' = y^2 + Q$ can easily be transformed into the Prüfer equation $\vartheta' = \cos^2 \vartheta + Q \sin^2 \vartheta$, which is uniformly Lipschitz in ϑ.

It seems likely that the statement and proof of Theorem 1.6 can be made local and that this restriction can be removed in this way; yet we prefer to study the global general solution in order to avoid notational complications which would obscure the underlying ideas.

2. Independent systems of quadratures

In this section we prove the following property of independent systems of quadratures which will play a central role in the reduction process of Section 3.

Lemma 2.1. *Let $(s_1, \ldots, s_n)$ be an independent system of quadratures. If $g : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function such that*

$$g(s_1(x) + c_1, \ldots, s_n(x, c_1, \ldots, c_{n-1}) + c_n) = g(c_1, \ldots, c_n) \tag{3}$$

$(x \in I, (c_1, \ldots, c_n) \in \mathbb{R}^n)$, *then g is constant.*

Remark. This property could also be used as a definition of independence for the purposes of Section 3. However, our definition emphasizes the local character of independence, and proves more convenient in the reduction process.

It may seem surprising that the global property stated in the above assertion follows from the very local condition of linear independence of the integrands at *one* point; as will be apparent from the proof, the decisive point is the hierarchical order of the quadratures which ensures that the jth component of the curve

$$t \mapsto (s_1(t) + c_1, \dots, s_n(t, c_1, \dots, c_{n-1}) + c_n) \quad (t \in I)$$

—along which g is constant by hypothesis—depends only on $c_1, \dots, c_{j-1}$, but not on c_j.

Proof. Let $\gamma_1, \dots, \gamma_n \in \mathbb{R}$ and $\xi \in I$. In Eq. (3) set

$$c_1 := \gamma_1 - s_1(\xi),$$

$$c_2 := \gamma_2 - s_2(\xi, c_1),$$

$$c_3 := \gamma_3 - s_3(\xi, c_1, c_2), \dots,$$

$$c_n := \gamma_n - s_n(\xi, c_1, \dots, c_{n-1}).$$

Then, differentiating with respect to x at $x = \xi$, we obtain

$$0 = \sum_{j=1}^{n} \partial_j g(\gamma_1, \dots, \gamma_n) \varphi_j(\xi, \gamma_1, \dots, \gamma_{j-1}), \quad (\xi \in I, \gamma_1, \dots, \gamma_n \in \mathbb{R}). \tag{4}$$

The independence of the quadratures means that there is a point $(\hat{c}_1, \dots, \hat{c}_{n-1}) \in \mathbb{R}^{n-1}$ such that the n functions

$$\varphi_1, \varphi_2(\cdot, \hat{c}_1), \dots, \varphi_n(\cdot, \hat{c}_1, \dots, \hat{c}_{n-1})$$

are linearly independent. It is easy to prove by induction over k that then there exist n points $x_1, \dots, x_n \in I$ such that the matrix $(\varphi_j(x_i, \hat{c}_1, \dots, \hat{c}_{j-1}))_{i,j \in \{1,\dots,k\}}$ has rank k, for $k \in \{1, \dots, n\}$.

Fix $\hat{c}_n \in \mathbb{R}$ arbitrary, and set $\hat{g} := g(\hat{c}_1, \dots, \hat{c}_n)$. We shall prove by induction over $k \in \{0, \dots, n\}$ that

$$g(\hat{c}_1, \dots, \hat{c}_{n-k}, c_{n-k+1}, \dots, c_n) = \hat{g}, \quad (c_{n-k+1}, \dots, c_n \in \mathbb{R}).$$

In the case $k = 0$ this is clear from the definition of $\hat{g}$.

Now assume we know the assertion is true for some $k \in \{0, \dots, n-1\}$; we want to show that it also holds for $k + 1$.

As the vectors $(\varphi_1(x_i), \varphi_2(x_i, \hat{c}_1), \dots, \varphi_{n-k}(x_i, \hat{c}_1, \dots, \hat{c}_{n-k-1}))_{i \in \{1,\dots,n-k\}}$ are linearly independent, there are constants $\alpha_1, \dots, \alpha_{n-k} \in \mathbb{R}$ such that

$$\sum_{i=1}^{n-k} \alpha_i \varphi_j(x_i, \hat{c}_1, \dots, \hat{c}_{j-1}) = \delta_{j,n-k} \quad (j \in \{1, \dots, n-k\}).$$

Multiplying (4) by α_i and summing over $i \in \{1, \dots, n-k\}$, and setting for $j \in \{n-k+1, \dots, n\}$

$$\psi_j(c_{n-k}, \dots, c_{j-1}) := \sum_{i=1}^{n-k} \alpha_i \varphi_j(x_i, \hat{c}_1, \dots, \hat{c}_{n-k-1}, c_{n-k}, \dots, c_{j-1}),$$

we obtain the linear first-order partial differential equation

$$0 = \partial_{n-k} g(\hat{c}_1, \ldots, \hat{c}_{n-k-1}, c_{n-k}, \ldots, c_n) + \sum_{j=n-k+1}^{n} \partial_j g(\hat{c}_1, \ldots, \hat{c}_{n-k-1}, c_{n-k}, \ldots, c_n) \psi_j(c_{n-k}, \ldots, c_{j-1})$$

with initial data $\hat{g}$ on the (non-characteristic) surface $\{(\hat{c}_1, \ldots, \hat{c}_{n-k})\} \times \mathbb{R}^k$. In order to solve this equation we consider the characteristic initial value problems (cf. [1, Section I.5])

$$\gamma'_{n-k}(t) = 1, \quad \gamma_{n-k}(0) = \hat{c}_{n-k},$$
$$\gamma'_j(t) = \psi_j(\gamma_{n-k}(t), \ldots, \gamma_{j-1}(t)), \quad \gamma_j(0) = \hat{\gamma}_j \quad (j \in \{n-k+1, \ldots, n\})$$

with $\hat{\gamma}_{n-k+1}, \ldots, \hat{\gamma}_n \in \mathbb{R}$ arbitrary.

This system of $k+1$ ordinary differential equations fully decouples; the equations can be solved one after the other by simple integration:

$$\gamma_{n-k}(t) = \hat{c}_{n-k} + t,$$
$$\gamma_{n-k+1}(t) = \hat{\gamma}_{n-k+1} + \int_0^t \psi_{n-k+1}(\gamma_{n-k}(s))\, \mathrm{d}s,$$
$$\ldots, \gamma_n(t) = \hat{\gamma}_n + \int_0^t \psi_n(\gamma_{n-k}(s), \ldots, \gamma_{n-1}(s))\, \mathrm{d}s.$$

g is constant along the curves $t \mapsto (\hat{c}_1, \ldots, \hat{c}_{n-k-1}, \gamma_{n-k}(t), \ldots, \gamma_n(t))$, $t \in \mathbb{R}$; thus

$$g(\hat{c}_1, \ldots, \hat{c}_{n-k-1}, \hat{c}_{n-k} + t, \gamma_{n-k+1}(t), \ldots, \gamma_n(t)) = g(\hat{c}_1, \ldots, \hat{c}_{n-k-1}, \hat{c}_{n-k}, \hat{\gamma}_{n-k+1}, \ldots, \hat{\gamma}_n).$$

Now given $c_{n-k}, \ldots, c_n \in \mathbb{R}$, we set

$$t := c_{n-k} - \hat{c}_{n-k},$$
$$\hat{\gamma}_{n-k+1} := c_{n-k+1} - \int_0^t \psi_{n-k+1}(\gamma_{n-k}(s))\, \mathrm{d}s,$$
$$\ldots, \hat{\gamma}_n := c_n - \int_0^t \psi_n(\gamma_{n-k}(s), \ldots, \gamma_{n-1}(s))\, \mathrm{d}s,$$

and find by induction hypothesis

$$g(\hat{c}_1, \ldots, \hat{c}_{n-k-1}, c_{n-k}, c_{n-k+1}, \ldots, c_n) = g(\hat{c}_1, \ldots, \hat{c}_{n-k-1}, \hat{c}_{n-k}, \hat{\gamma}_{n-k+1}, \ldots, \hat{\gamma}_n) = \hat{g}. \qquad \square$$

3. The reduction procedure

In this section we prove Theorem 1.6. As a starting point we show that the fact that the integral by quadratures is essentially one-parametric can be expressed by a simple relationship between its partial derivatives with respect to the free integration constants (Proposition 3.1). Then we provide two auxiliary propositions which capture two fundamental processes which are repeatedly applied in the reduction procedure: the explicit solution of a very simply structured first-order linear partial differential equation (Lemma 3.2); and the inclusion of an arbitrary function of a number of quadratures in the integrand of a quadrature with higher index (Lemma 3.3). Then we proceed to the heart of the matter in Propositions 3.4 and 3.5, which demonstrate how the number of quadratures in the integral can be reduced iteratively by combining two quadratures into one with the help of Proposition 3.1. After the proof of Proposition 3.5 we outline the reduction algorithm, and finish the proof of Theorem 1.6.

Proposition 3.1. *If $f(\cdot,\mathbf{c}) : I \to \mathbb{R}$, $\mathbf{c}\in\mathbb{R}^n$, is an effectively one-parametric family continuously differentiable with respect to* $\mathbf{c}$, *then the following* Fundamental Equality (*for the pair c_i, c_j*), *$i,j\in\{1,\dots,n\}$, holds*:

$$\partial_{1+i} f(x,\mathbf{c})\partial_{1+j} f(x_0,\mathbf{c}) = \partial_{1+j} f(x,\mathbf{c})\partial_{1+i} f(x_0,\mathbf{c}) \quad (x\in I, \mathbf{c}\in\mathbb{R}^n).$$

Proof. By the equivalence of f and a one-parameter family $\hat{f}$ parametrized by its values at x_0, there is a function $d : \mathbb{R}^n \to \mathbb{R}$ such that $f(\cdot,\mathbf{c}) = \hat{f}(\cdot, d(\mathbf{c}))$ $\quad(\mathbf{c}\in\mathbb{R}^n)$; in particular $f(x_0,\mathbf{c}) = \hat{f}(x_0, d(\mathbf{c})) = d(\mathbf{c})$ $\quad(\mathbf{c}\in\mathbb{R}^n)$. Differentiating the identity $f(x,c_1,\dots,c_n) = \hat{f}(x, f(x_0,c_1,\dots,c_n))$ with respect to c_j, we find

$$\partial_{1+j} f(x,c_1,\dots,c_n) = \partial_2 \hat{f}(x, f(x_0,c_1,\dots,c_n))\,\partial_{1+j} f(x_0,c_1,\dots,c_n)$$

$(x\in I, c_1,\dots,c_n\in\mathbb{R})$, from which the Fundamental Equality follows upon elimination of $\partial_2\hat{f}$. □

Lemma 3.2. *Let $H : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, and $a, b : \mathbb{R}^{n-1} \to \mathbb{R}$ continuous such that*

$$0 = \partial_{n-1} H(\tilde{x}, x_{n-1}, x_n) - (a(\tilde{x}, x_{n-1})x_n + b(\tilde{x}, x_{n-1}))\partial_n H(\tilde{x}, x_{n-1}, x_n).$$

Then, with $G(\tilde{x}, x_n) := H(\tilde{x}, 0, x_n)$,

$$H(\tilde{x}, x_{n-1}, x_n) = G\left(\tilde{x}, \exp\left(\int_0^{x_{n-1}} a(\tilde{x},t)\,\mathrm{d}t\right) x_n + \int_0^{x_{n-1}} b(\tilde{x},t)\exp\left(\int_0^t a(\tilde{x},s)\,\mathrm{d}s\right)\,\mathrm{d}t\right)$$

$(\tilde{x}\in\mathbb{R}^{n-2}, x_{n-1}, x_n\in\mathbb{R})$.

Proof. We apply the standard procedure to solve a (quasi-) linear partial differential equation of first order, cf. [1, Section I.5]. For fixed $\tilde{x}\in\mathbb{R}^{n-2}$, we solve the characteristic initial value problems

$$\xi_{n-1}'(t) = 1,\ \ \xi_{n-1}(0) = 0, \quad \xi_n'(t) = -a(\tilde{x}, \xi_{n-1}(t))\,\xi_n(t) - b(\tilde{x}, \xi_{n-1}(t)),\ \ \xi_n(0) = \hat{\xi}_n \in\mathbb{R}.$$

The solution is given by formula (2):

$$\xi_{n-1}(t) = t, \quad \xi_n(t) = \exp\left(-\int_0^t a(\tilde{x},s)\,\mathrm{d}s\right)\left(\hat{\xi}_n - \int_0^t b(\tilde{x},s)\exp\left(\int_0^t a(\tilde{x},r)\,\mathrm{d}r\right)\,\mathrm{d}s\right)$$

$(t\in\mathbb{R})$. Now setting for $x_{n-1}, x_n\in\mathbb{R}$

$$\hat{\xi}_n := \exp\left(\int_0^{x_{n-1}} a(\tilde{x},t)\,\mathrm{d}t\right) x_n + \int_0^{x_{n-1}} b(\tilde{x},t)\exp\left(\int_0^t a(\tilde{x},s)\,\mathrm{d}s\right)\,\mathrm{d}t$$

and observing that

$$0=\frac{\mathrm{d}}{\mathrm{d}t}H(\tilde{x},\xi_{n-1}(t),\xi_n(t))\quad (t\in\mathbb{R}),$$

the assertion follows. □

Lemma 3.3. *Let* $(s_1,\dots,s_n)$ *be an independent system of quadratures,* $n\geqslant 2$, *and* $B:\mathbb{R}^{n-1}\to\mathbb{R}$ *twice continuously differentiable. Then there is a quadrature* $\hat{s}_n$ *such that* $(s_1,\dots,s_{n-1},\hat{s}_n)$ *is an independent system of quadratures, and*

$$\begin{aligned}&s_n(x,c_1,\dots,c_{n-1})+B(s_1(x)+c_1,\dots,s_{n-1}(x,c_1,\dots,c_{n-2})+c_{n-1})\\&\quad=\hat{s}_n(x,c_1,\dots,c_{n-1})+B(c_1,\dots,c_{n-1})\end{aligned}$$

$(x\in I,c_1,\dots,c_{n-1}\in\mathbb{R})$.

Proof. Let $\varphi_1,\dots,\varphi_n$ denote the integrands of $s_1,\dots,s_n$. Set

$$\hat{\varphi}_n(x,c_1,\dots,c_{n-1}):=\varphi_n(x,c_1,\dots,c_{n-1})+\sum_{j=1}^{n-1}\partial_jB(c_1,\dots,c_{n-1})\varphi_j(x,c_1,\dots,c_{j-1}),$$

$$\hat{s}_n(x,c_1,\dots,c_{n-1}):=\int_{x_0}^{x}\hat{\varphi}_n(t,s_1(t)+c_1,\dots,s_{n-1}(t,c_1,\dots,c_{n-2})+c_{n-1})\,\mathrm{d}t$$

$(x\in I,c_1,\dots,c_n\in\mathbb{R})$. □

In the following the last (highest-order) quadrature plays a special role and is therefore distinguished in the notation.

Proposition 3.4. *Let* f *be an effectively one-parametric integral by quadratures of the form*

$$f(x,c_1,\dots,c_n,C):=\Theta(x,F(s_1(x)+c_1,\dots,s_n(x,c_1,\dots,c_{n-1})+c_n,S(x,c_1,\dots,c_n)+C))$$

$(x\in I,c_1,\dots,c_n,C\in\mathbb{R})$, $n\in\mathbb{N}$.

Set $G(c_1,\dots,c_{n-1},C):=F(c_1,\dots,c_{n-1},0,C)$ $(c_1,\dots,c_{n-1},C\in\mathbb{R})$.

Then one of the following statements is true:

1. *There is a quadrature* $\hat{S}$ *such that* $(s_1,\dots,s_{n-1},\hat{S})$ *is an independent system of quadratures, and the integral by quadratures*

$$\begin{aligned}&\hat{f}(x,c_1,\dots,c_{n-1},\hat{C})\\&\quad:=\Theta(x,G(s_1(x)+c_1,\dots,s_{n-1}(x,c_1,\dots,c_{n-2})+c_{n-1},\hat{S}(x,c_1,\dots,c_{n-1})+\hat{C}))\end{aligned}$$

$(x\in I,c_1,\dots,c_{n-1},\hat{C}\in\mathbb{R})$, *is equivalent to* f.

2. *There are quadratures* $s, \hat{S}$ *such that* $(s_1, \dots, s_{n-1}, s, \hat{S})$ *is an independent system of quadratures, and the integral by quadratures*

$$\hat{f}(x, c_1, \dots, c_{n-1}, c, \hat{C}) := \Theta(x, G(s_1(x) + c_1, \dots, s_{n-1}(x, c_1, \dots, c_{n-2}) + c_{n-1}, \mathrm{e}^{s(x, c_1, \dots, c_{n-1}) + c}(\hat{S}(x, c_1, \dots, c_{n-1}, c) + \hat{C})))$$

$(x \in I, c_1, \dots, c_{n-1}, c, \hat{C} \in \mathbb{R})$, *is equivalent to* f.

Proof. Writing out the Fundamental Equality for f and the pair c_n, C in terms of the above expression, and dividing by factors of $\partial_2 \Theta$, $\partial_{n+1} F \neq 0$, we find, for all $x \in I, c_1, \dots, c_n, C \in \mathbb{R}$:

$$\frac{\partial_n F}{\partial_{n+1} F}(c_1, \dots, c_n, C) = \frac{\partial_n F}{\partial_{n+1} F}(s_1(x) + c_1, \dots, S(x, c_1, \dots, c_n) + C) + \partial_{1+n} S(x, c_1, \dots, c_n).$$

Differentiation of this identity with respect to C yields

$$\partial_{n+1} \frac{\partial_n F}{\partial_{n+1} F}(c_1, \dots, c_n, C) = \partial_{n+1} \frac{\partial_n F}{\partial_{n+1} F}(s_1(x) + c_1, \dots, S(x, c_1, \dots, c_n) + C).$$

By the independence of the quadratures and Lemma 2.1 it follows that there is a constant $\alpha \in \mathbb{R}$ such that

$$\partial_{n+1} \frac{\partial_n F}{\partial_{n+1} F}(c_1, \dots, c_n, C) = \alpha, \quad (c_1, \dots, c_n, C \in \mathbb{R});$$

and consequently

$$\partial_n F(c_1, \dots, c_n, C) = (\alpha C + \beta(c_1, \dots, c_n)) \partial_{n+1} F(c_1, \dots, c_n, C) \quad (c_1, \dots, c_n, C \in \mathbb{R})$$

with some twice continuously differentiable function $\beta : \mathbb{R}^n \to \mathbb{R}$. Hence by Lemma 3.2

$$F(c_1, \dots, c_n, C) = G\left(c_1, \dots, c_{n-1}, \mathrm{e}^{\alpha c_n}\left(C + \mathrm{e}^{-\alpha c_n} \int_0^{c_n} \beta(c_1, \dots, c_{n-1}, t) \mathrm{e}^{\alpha t} \, \mathrm{d}t\right)\right)$$

$(c_1, \dots, c_n, C \in \mathbb{R})$.

Now there are two different cases to consider:

Case 1: $\alpha = 0$. Then

$$F(c_1, \dots, c_n, C) = G(c_1, \dots, c_{n-1}, C + B(c_1, \dots, c_n)),$$

where

$$B(c_1, \dots, c_n) := \int_0^{c_n} \beta(c_1, \dots, c_{n-1}, t) \, \mathrm{d}t \quad (c_1, \dots, c_n \in \mathbb{R}).$$

By Lemma 3.3 there is a quadrature S' (here and in corresponding cases below the prime is used merely for distinction and does not signify a derivative) such that $(s_1, \dots, s_n, S')$ is an independent system of quadratures and

$$f(x, c_1, \dots, c_n, C) = \Theta(x, G(s_1(x) + c_1, \dots, s_{n-1}(x, c_1, \dots, c_{n-2}) + c_{n-1}, S'(x, c_1, \dots, c_n) + C + B(c_1, \dots, c_n)))$$

$(x \in I, c_1, \dots, c_n, C \in \mathbb{R})$.

Note that the quadrature s_n does no longer occur explicitly as an argument of G, but only in the integrand of S'; we now show that it is redundant and can be eliminated.

Clearly $\hat{C} := C + B(c_1,\dots,c_n)$ takes the role of a new integration constant for S'. Thus f is equivalent to the integral by quadratures

$$\begin{aligned} &f'(x,c_1,\dots,c_n,\hat{C}) \\ &\quad = \Theta(x, G(s_1(x)+c_1,\dots,s_{n-1}(x,c_1,\dots,c_{n-2})+c_{n-1}, S'(x,c_1,\dots,c_n)+\hat{C})) \end{aligned}$$

$(x \in I, c_1,\dots,c_n,\hat{C} \in \mathbb{R})$.

Therefore—since equivalence is transitive—f' is effectively one-parametric, and by Proposition 3.1 the Fundamental Equality holds for the pair $c_n,\hat{C}$; its left-hand part vanishes as the initial value $f'(x_0,c_1,\dots,c_n,\hat{C})$ is independent of c_n:

$$\begin{aligned} 0 &= \partial_n G(s_1(x)+c_1,\dots,s_{n-1}(x,c_1,\dots,c_{n-2})+c_{n-1}, S'(x,c_1,\dots,c_n)+\hat{C}) \\ &\quad \times \partial_{1+n} S'(x,c_1,\dots,c_n)\partial_n G(c_1,\dots,c_{n-1},\hat{C}); \end{aligned}$$

since $\partial_n G \neq 0$, it follows that $\partial_{1+n} S' = 0$. Differentiating with respect to x (and denoting the integrand of S' by Φ') we find

$$0 = \partial_{1+n}\Phi'(x, s_1(x)+c_1,\dots,s_n(x,c_1,\dots,c_{n-1})+c_n) \quad (x \in I, c_1,\dots,c_n \in \mathbb{R}).$$

This means that for fixed $x,c_1,\dots,c_{n-1}$ the integrand has the same value for all real values of c_n; therefore setting $\hat{\Phi}(x,c_1,\dots,c_{n-1}) := \Phi'(x,c_1,\dots,c_{n-1},0)$,

$$\hat{S}(x,c_1,\dots,c_{n-1}) := \int_{x_0}^{x} \hat{\Phi}(t, s_1(t)+c_1,\dots,s_{n-1}(t,c_1,\dots,c_{n-2})+c_{n-1})\,\mathrm{d}t$$

has the required properties.

Case 2: $\alpha \neq 0$. Denoting by φ_n and Φ the integrands of s_n and S, resp., define new quadratures s, S' by giving their integrands

$$\begin{aligned} \varphi(x,c_1,\dots,c_{n-1}) &:= \alpha\,\varphi_n(x,c_1,\dots,c_{n-1}), \\ \Phi'(x,c_1,\dots,c_{n-1},c) &:= \Phi(x,c_1,\dots,c_{n-1},c/\alpha), \\ s(x,c_1,\dots,c_{n-1}) &:= \int_{x_0}^{x} \varphi(t, s_1(t)+c_1,\dots,s_{n-1}(t,c_1,\dots,c_{n-2})+c_{n-1}))\,\mathrm{d}t, \\ S'(x,c_1,\dots,c_{n-1},c) &:= \int_{x_0}^{x} \Phi'(t, s_1(t)+c_1,\dots,s_{n-1}(t,c_1,\dots,c_{n-2})+c_{n-1}, \\ &\qquad s(t,c_1,\dots,c_{n-1})+c)\,\mathrm{d}t. \end{aligned}$$

Then $(s_1,\dots,s_{n-1},s,S')$ is an independent system of quadratures, and $S(x,c_1,\dots,c_n) = S'(x,c_1,\dots,c_{n-1},\alpha c_n)$. Furthermore, set

$$B(c_1,\dots,c_{n-1},c) := \mathrm{e}^{-c}\int_{x_0}^{c/\alpha} \beta(c_1,\dots,c_{n-1},t)\mathrm{e}^{\alpha t}\,\mathrm{d}t \quad (c_1,\dots,c_n,c \in \mathbb{R}),$$

then by Lemma 3.3 we find a quadrature $\hat{S}$ such that $(s_1,\dots,s_{n-1},s,\hat{S})$ is an independent system of quadratures, and

$$f(x,c_1,\dots,c_n,C)=\Theta(x,G(s_1(x)+c_1,\dots,s_{n-1}(x,c_1,\dots,c_{n-2})+c_{n-1},\\ e^{s(x,c_1,\dots,c_{n-1})+\alpha c_n}(\hat{S}(x,c_1,\dots,c_{n-1},\alpha c_n)+C+B(c_1,\dots,c_{n-1},\alpha c_n)))).$$

Hence, with $c:=\alpha c_n$ and $\hat{C}=C+B(c_1,\dots,c_{n-1},\alpha c_n)$, f is equivalent to the integral $\hat{f}$. $\square$

Proposition 3.5. *Let f be an effectively one-parametric integral by quadratures of the form*

$$f(x,c_1,\dots,c_n,c,C):=\Theta(x,F(s_1(x)+c_1,\dots,s_n(x,c_1,\dots,c_{n-1})+c_n,\\ e^{s(x,c_1,\dots,c_n)+c}(S(x,c_1,\dots,c_n,c)+C)))$$

$(x\in I,c_1,\dots,c_n,c,C\in\mathbb{R})$, $n\in\mathbb{N}$.

Set $G(c_1,\dots,c_{n-1},D):=F(c_1,\dots,c_{n-1},0,D)$ $(c_1,\dots,c_{n-1},D\in\mathbb{R})$.

Then there are quadratures $\hat{s}$, $\hat{S}$ such that $(s_1,\dots,s_{n-1},\hat{s},\hat{S})$ is an independent system of quadratures and the integral by quadratures

$$\hat{f}(x,c_1,\dots,c_{n-1},\hat{c},\hat{C}):=\Theta(x,G(s_1(x)+c_1,\dots,s_{n-1}(x,c_1,\dots,c_{n-2})+c_{n-1},\\ e^{\hat{s}(x,c_1,\dots,c_{n-1})+\hat{c}}(\hat{S}(x,c_1,\dots,c_{n-1},\hat{c})+\hat{C})))$$

$(x\in I,c_1,\dots,c_{n-1},\hat{c},\hat{C}\in\mathbb{R})$ *is equivalent to* f.

Proof. Writing out the Fundamental Equality for f and the pair c_n,C in terms of Θ and F, and dividing by factors of $\partial_2\Theta$, $\partial_{n+1}F\neq 0$, we find

$$\frac{\partial_n F}{\partial_{n+1}F}(c_1,\dots,c_n,e^c C)e^{-c}$$
$$=\frac{\partial_n F}{\partial_{n+1}F}(s_1(x)+c_1,\dots,s_n(x,\dots)+c_n,e^{s(x,\dots)+c}(S(x,\dots)+C))e^{-s(x,c_1,\dots,c_n)-c}$$
$$+(S(x,c_1,\dots,c_n,c)+C)\partial_{1+n}s(x,c_1,\dots,c_n)+\partial_{1+n}S(x,c_1,\dots,c_n,c).$$

Differentiating this equality twice with respect to C we obtain

$$\partial_{n+1}\partial_{n+1}\frac{\partial_n F}{\partial_{n+1}F}(c_1,\dots,c_n,e^c C)e^c$$
$$=\partial_{n+1}\partial_{n+1}\frac{\partial_n F}{\partial_{n+1}F}(s_1(x)+c_1,\dots,s_n(x,\dots)+c_n,e^{s(x,\dots)+c}(S(x,\dots)+C))e^{s(x,c_1,\dots,c_n)+c}.$$

Since the quadratures are independent, this implies by Lemma 2.1 that there is a constant $\gamma\in\mathbb{R}$ such that

$$\partial_{n+1}\partial_{n+1}\frac{\partial_n F}{\partial_{n+1}F}(c_1,\dots,c_n,D)e^c=\gamma\quad(c_1,\dots,c_n,c,D\in\mathbb{R}).$$

As c is arbitrary and the first factor on the left-hand side does not depend on c, γ must be 0. Thus there are twice continuously differentiable functions $\alpha, \beta : \mathbb{R}^n \to \mathbb{R}$ such that

$$\partial_n F(c_1,\ldots,c_n,D) = (\alpha(c_1,\ldots,c_n)D + \beta(c_1,\ldots,c_n))\partial_{n+1}F(c_1,\ldots,c_n,D)$$

$(c_1,\ldots,c_n,D \in \mathbb{R})$.

Hence by Lemma 3.2

$$F(c_1,\ldots,c_n,D) = G(c_1,\ldots,c_{n-1},\mathrm{e}^{A(c_1,\ldots,c_n)}D + B'(c_1,\ldots,c_n)), \quad (c_1,\ldots,c_n,D \in \mathbb{R}),$$

where

$$A(c_1,\ldots,c_n) := \int_0^{c_n} \alpha(c_1,\ldots,c_{n-1},t)\,\mathrm{d}t$$

and

$$B'(c_1,\ldots,c_n) := \int_0^{c_n} \beta(c_1,\ldots,c_{n-1},t)\mathrm{e}^{A(c_1,\ldots,c_{n-1},t)}\,\mathrm{d}t$$

$(c_1,\ldots,c_n \in \mathbb{R})$.

By Lemma 3.3 there is a quadrature s' such that

$$s(x,c_1,\ldots,c_n) + A(s_1(x)+c_1,\ldots,s_n(x,c_1,\ldots,c_{n-1})+c_n) = s'(x,c_1,\ldots,c_n) + A(c_1,\ldots,c_n).$$

Denoting by Φ the integrand of S and setting

$$\Phi'(x,c_1,\ldots,c_n,c') := \Phi(x,c_1,\ldots,c_n,c' - A(c_1,\ldots,c_n)),$$

we have the new quadrature

$$S'(x,c_1,\ldots,c_n,c') := \int_{x_0}^{x} \Phi'(t,s_1(t)+c_1,\ldots,s_n(t,c_1,\ldots,c_{n-1})+c_n,s'(t,c_1,\ldots,c_n)+c')\,\mathrm{d}t$$

$(x \in I, c_1,\ldots,c_n,c' \in \mathbb{R})$; then $(s_1,\ldots,s_n,s',S')$ is an independent system of quadratures, and

$$\begin{aligned} f(x,c_1,\ldots,c_n,c,C) = \Theta(x, G(&s_1(x)+c_1,\ldots,s_{n-1}(x,c_1,\ldots,c_{n-2})+c_{n-1},\\ &\mathrm{e}^{s'(x,c_1,\ldots,c_n)+c+A(c_1,\ldots,c_n)}(S'(x,c_1,\ldots,c_n,c+A(c_1,\ldots,c_n))+C)\\ &+B'(s_1(x)+c_1,\ldots,s_n(x,c_1,\ldots,c_{n-1})+c_n))). \end{aligned}$$

With $B(c_1,\ldots,c_n,c') := \mathrm{e}^{-c'}B'(c_1,\ldots,c_n)$, there is by Lemma 3.3 a quadrature S'' such that $(s_1, \ldots,s_{n-1},s',S'')$ is an independent system of quadratures, and

$$\begin{aligned} f(x,c_1,\ldots,c_n,c,C) = \Theta(x, G(&s_1(x)+c_1,\ldots,s_{n-1}(x,c_1,\ldots,c_{n-2})+c_{n-1},\\ &\mathrm{e}^{s'(x,c_1,\ldots,c_n)+c+A(c_1,\ldots,c_n)}(S''(x,c_1,\ldots,c_n,c+A(c_1,\ldots,c_n))+C\\ &+B(c_1,\ldots,c_n,c+A(c_1,\ldots,c_n))))), \end{aligned}$$

which is clearly equivalent to

$$\begin{aligned} f'(x,c_1,\ldots,c_n,\hat{c},\hat{C}) = \Theta(x, G(&s_1(x)+c_1,\ldots,s_{n-1}(x,c_1,\ldots,c_{n-2})+c_{n-1},\\ &\mathrm{e}^{s'(x,c_1,\ldots,c_n)+\hat{c}}(S''(x,c_1,\ldots,c_n,\hat{c})+\hat{C}))). \end{aligned}$$

Note that the quadrature s_n occurs only in the integrands of s' and S'', but not as an explicit argument of G: we now show that it is redundant and can be eliminated.

From the Fundamental Equality for f' and the pair $c_n, \hat{C}$ (note that the initial value $f'(x_0, c_1, \ldots, c_n, \hat{c}, \hat{C})$ does not depend on c_n) we find after division by factors $\partial_2\Theta$, $\partial_n G \neq 0$,

$$0 = \partial_{1+n} s'(x, c_1, \ldots, c_n)(S''(x, c_1, \ldots, c_n, \hat{c}) + \hat{C}) + \partial_{1+n} S''(x, c_1, \ldots, c_n, \hat{c}).$$

Making use of the fact that this holds for all $\hat{C} \in \mathbb{R}$ and all $x \in I$, we conclude $0 = \partial_{1+n}\varphi'(x, c_1, \ldots, c_n)$, and $0 = \partial_{1+n}\Phi''(x, c_1, \ldots, c_n, \hat{c})$.

Therefore, setting $\hat{\varphi}(x, c_1, \ldots, c_{n-1}) := \varphi'(x, c_1, \ldots, c_{n-1}, 0)$ and $\hat{\Phi}(x, c_1, \ldots, c_{n-1}, \hat{c}) := \Phi''(x, c_1, \ldots, c_{n-1}, 0, \hat{c})$, we obtain quadratures $\hat{s}$, $\hat{S}$ with the required properties. □

Starting with an arbitrary effectively one-parametric integral in finite form, we can apply Propositions 3.4 and 3.5 in the following iterative algorithm to eliminate all except at most two of the quadratures:

Step 1: **if** there is only one quadrature, **stop**.
else apply Proposition 3.4;
if this results in case 1, **repeat** step 1.
else proceed with
Step 2: **if** there are only two quadratures, **stop.**
else apply Proposition 3.5; **repeat** step 2.

Depending on whether this algorithm terminates in step 1 or 2 (which it must since the number of quadratures is finite at the beginning and decreases by 1 at each step, except in one single step in which Proposition 3.4 winds up in case 2), one of the following situations is reached:

1. The original integral is equivalent to the integral by quadratures

$$\hat{f}(x, C) = \Theta(x, G(S(x) + C)) \quad (x \in I, C \in \mathbb{R}).$$

This clearly is of the form stated in Theorem 1.6, with $\hat{\Theta}(x, c) := \Theta(x, G(c))$ $(x \in I, c \in \mathbb{R})$, $p := 0$, and $q := \Phi$ (the integrand of S).

2. The original integral is equivalent to the integral by quadratures

$$\hat{f}(x, c, C) = \Theta(x, G(e^{s(x)+c}(S(x, c) + C))) \quad (x \in I, c, C \in \mathbb{R}).$$

The Fundamental Equality for $\hat{f}$ and the pair c, C yields after cancellation of factors $\partial_2\Theta$, $\partial_1 G \neq 0$

$$0 = S(x, c) + \partial_2 S(x, c) \quad (x \in I, c \in \mathbb{R}).$$

Differentiating with respect to x and using that c is arbitrary, we find

$$0 = \Phi(x, c) + \partial_2\Phi(x, c) \quad (x \in I, c \in \mathbb{R}).$$

Thus there is a locally integrable function $q : I \to \mathbb{R}$ such that $\Phi(x, c) = q(x)e^{-c}$. Eliminating the redundant integration constant of the quadrature s, we arrive at the desired expression.

This completes the proof of Theorem 1.6.

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Appendix. Integration by quadratures of the linear second-order equation

As is well known, the homogeneous linear second-order equation

$$u'' + Qu = 0, \tag{A.1}$$

$Q : I \to \mathbb{R}$ locally integrable, is equivalent to the first-order system

$$\vartheta' = \cos^2 \vartheta + Q \sin^2 \vartheta, \tag{A.2}$$

$$(\log \varrho)' = (Q - 1) \sin \vartheta \cos \vartheta, \tag{A.3}$$

by the Prüfer transformation

$$\begin{pmatrix} u \\ u' \end{pmatrix} = \varrho \begin{pmatrix} \sin \vartheta \\ \cos \vartheta \end{pmatrix}.$$

Clearly, the solution of (A.3) can be obtained from that of (A.2) by a simple integration; therefore it seems reasonable to say that (A.1) can be *integrated by quadratures* if and only if (A.2) can be integrated by quadratures in the sense specified in Section 1.

We call the family

$$f(x, c_1, \ldots, c_n) := F(s_1(x) + c_1, \ldots, s_n(x, c_1, \ldots, c_{n-1}) + c_n)$$

($x \in I, c_1, \ldots, c_n \in \mathbb{R}$), with $(s_1, \ldots, s_n)$ an independent system of quadratures and $F : \mathbb{R}^n \to \mathbb{R}$ admissible, an *integral by quadratures in the restricted sense* (excluding any explicit dependence of the integral on the independent variable x). This restriction narrows the concept of integration by quadratures considerably; however, it is interesting to observe that then the following consequence can be drawn from Theorem 1.6 without any assumptions of elementarity of the functions involved:

Theorem A.1. *If the homogeneous linear second-order equation* (A.1) *can be integrated by quadratures in the restricted sense*, *then* Q *is constant*.

Proof. The right-hand side of (A.2) is uniformly Lipschitz with respect to ϑ; we can therefore apply Theorem 1.6 (with $\Theta(x, c) = c$) to learn that (A.2) can be integrated by quadratures in the restricted sense only if there are a (four times continuously differentiable) diffeomorphism $\phi : \mathbb{R} \to \mathbb{R}$, and locally integrable functions p, q such that

$$(\phi \circ \vartheta)'(x) + \phi \circ \vartheta(x) p(x) = q(x), \quad (x \in I)$$

whenever $\vartheta : I \to \mathbb{R}$ is a solution of (A.2). Applying the chain rule and noting that through every point $(x, y) \in I \times \mathbb{R}$ there passes some solution of (A.2), we conclude that

$$\phi'(y)(\cos^2 y + Q(x) \sin^2 y) + \phi(y) p(x) = q(x), \quad (x \in I, y \in \mathbb{R}).$$

Differentiating this identity with respect to y, dividing by $\phi'(y) \neq 0$, and differentiating again with respect to y in order to eliminate p and q, we find

$$\left(\frac{\phi''}{\phi'}\right)'(y)(\cos^2 y + Q(x)\sin^2 y) + \frac{\phi''}{\phi'}(y)((\cos^2)'(y) + Q(x)(\sin^2)'(y))$$
$$+(\cos^2)''(y) + Q(x)(\sin^2)''(y) = 0 \quad (x \in I, y \in \mathbb{R}). \tag{A.4}$$

If Q is not constant, there are $x_1, x_2 \in I$ such that $Q(x_1) \neq Q(x_2)$. Taking the difference of (A.4) at $x = x_1$ and $x = x_2$, it follows that

$$\left(\frac{\phi''}{\phi'}\right)'(y)\sin^2 y + \frac{\phi''}{\phi'}(y)(\sin^2)'(y) + (\sin^2)''(y) = 0 \quad (y \in \mathbb{R}). \tag{A.5}$$

When we multiply this equation by $1 - Q(x)$ and add the result to (A.4), we obtain

$$\left(\frac{\phi''}{\phi'}\right)' = 0,$$

which by (A.5) implies that $(\sin^2)' = 2\sin\cos$ and $(\sin^2)'' = 2(\cos^2 - \sin^2)$ are linearly dependent, which is not the case. □

Biographical Note

The mathematical abilities of Vladimir Pavlovič Maximovič early attracted Čebyšev's attention. After leaving the 1st class at the physical-mathematical faculty of St. Petersburg University in 1867, he completed his mathematical education as a student of Bertrand at the Paris École Polytechnique, where he obtained in 1879 the doctoral degree for his thesis 'Nouvelle méthode pour intégrer les equations simultanées aux differentielles totales'. His 1880 paper [3] was motivated by his hope to find, based on the principle of the confluence, or mutual compensation, of integration constants, a proof for the impossibility of integrating the general linear second-order equation by quadratures. These investigations, of which he had, already on 1st July, 1880, deposited a preliminary version 'Mémoire sur les équations différentielles générales du premier ordre qui s'intègrent au moyen d'un nombre fini de quadratures. Démonstration de l'impossibilité d'une telle intégration de l'équation linéaire du second ordre' at the Paris Academy in a sealed envelope under the motto *nihil optimum nisi mathesis, et non est mortale quod opto*, are published in his 1885 habilitation thesis [4] at Kazan', where he was subsequently *Privatdozent*, and later *Dozent* at the chair for pure mathematics at the physical-mathematical faculty. There he published six more papers on the integration of differential equations, on interpolation, function expansions and the roots of algebraic equations. After he had moved to Kiev, Maximovič's interest turned to probability theory, and to the construction of a computing machine. His last published work, and his only one on probability theory, was a talk (Kiev, 1888) on the application of probabilistic laws to school statistics, using empirical data from the admission exams of the École Polytechnique. Early in the following year, signs of severe mental illness began to manifest themselves, which led to the premature death in his 40th year, in St. Petersburg on 17 October, 1889, of 'the talented Russian mathematician whose name in the history of science will be inseparably linked to the important question of the integration of differential equations by quadratures' [9, p.55 seq.].

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