PS/Int. AR/60-33
November 18th, 1960.

SOME THOUGHTS ON THE STACKING EFFICIENCY OF THE STORAGE RING.

1. General Remarks.

One of the main purposes of the electron storage ring is to make experiments in which one stacks beam with an R.F. programme that gives a reasonably high computed stacking efficiency, and measures the stacking efficiency achieved in practice.[x)]

Apart from complete computations of the whole stacking process with definite specified R.F. programmes, it is desirable to have some rough quantitative information about the influence of the various parameters of the programme on the stacking efficiency. This information is needed to determine in a general way what types of amplitude programme and frequency programme the R.F. system is likely to be called upon to produce, and what types of programme are worth computing in full. It is a bad thing to make the R.F. system unnecessarily flexible, as this almost certainly increases the difficulty of obtaining close tolerances and low noise.

For these rough quantitative estimates it is probably valid to treat the whole stacking process as consisting of a number of separate processes, each of which has a certain degenerative effect (depending on the parameters used for the process) on the longitudinal phase-plane density. The overall stacking efficiency is then the product of these separate phase-space efficiencies: -

$$\eta = \eta_1 \cdot \eta_2 \cdot \eta_3 \cdot \cdots\cdots \eta_i \cdots\cdots \tag{1}$$

Many of these processes are such that they must be carried out slowly enough to be nearly adiabatic if one wishes to obtain an η_i near to unity. Since the time available to make a stack is limited by the gas-scattering, it will be necessary to make compromises between speed and efficiency. If, after making such compromises, the time taken to stack is too long, one is forced to use a higher harmonic number.

x) We define stacking efficiency as the ratio of the phase-space density in the stack to that in the injected beam.

Some of the processes involved in stacking are sufficiently simple that one can calculate approximate expressions for the corresponding η_i , others are so complicated that analytic considerations give only some rough guidance on what parameter values are most worth putting into digital computation. But even in the latter case it will be a great advantage if the factorisation(1) is valid and one can determine the individual η_i separately, for the information obtained in such a way gives a more useful picture of what goes on in a stacking machine, and a better basis for making speed-efficiency compromises, than would be obtained from computations of the overall η of various complete programmes. It is also likely to be more economical of computing time, as it cuts down the necessity for investigating very many combinations of parameter values.

2. Trapping.

In PS/Int. AR/60-8, Swenson considers initial capture into stationary buckets of length 2π R.F.-radians and width equal to the energy spread of the injected beam. This method has three very attractive features: -

(a) The phase-space density in the bucket is the same as that in the injected beam *): -

$$\eta_1 = 1 \tag{2}$$

(b) The time taken for the "trapping process" is zero: -

$$t_1 = 0 \tag{3}$$

(c) It requires no computation.

On the other hand the fraction of particles trapped is only $2/\pi$ *): -

$$f_1 = 0.637 \tag{4}$$

It may be argued that a more sophisticated trapping process, such as the use of stationary slowly growing buckets, would be capable of yielding a higher f_1 with

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*) These remarks are only exact if the injected energy spectrum is rectangular between its limits: if it has a maximum in the middle then f_1 is higher and the mean phase-space density in the bucket is a favourably weighted average of that in the injected beam.

η_1 still nearly unity and t_1 small compared with the total time of stacking one pulse; but it seems to me that we ought first to study stacking with the simplest possible trapping process. As a separate problem it may be interesting to study high-f trapping processes, both theoretically and using the storage ring, and later to combine high-f trapping with stacking.

The possibility of making the injected energy spread larger, say by a factor two, than that of the trapping buckets has also been discussted. This makes f_1 even lower, but retains the merits (a), (b), (c) above and relaxes some tolerances. It also enables a larger current to be injected in the face of the longitudinal space-charge instability phenomenon.

For the moment, therefore, we shall regard the method of capture into stationary buckets as adopted.

3. Change-over to Accelerating Buckets.

To convert the full trapping-buckets to ones that accelerate and become no smaller one must increase the R.F. amplitude V and raise $|\dot{f}|$ from zero. These can be done: -

(a) In that order, separately

(b) In that order but with substantial overlap

(c) Simultaneously.

In any case we shall want to do things reasonably adiabatically, which means that the fractional change of shape of a particle trajectory in the $\Delta\phi$, ΔE plane per cycle of synchrotron oscillations should not be large.

In PS/Int. AR/60-8, Swenson considers case (a) and compares several different choices of γ_{Tr} and of ϕ_s, all with the accelerating bucket arranged to be twice the area of the initial trapping bucket. If we hold to this factor two the phase-space efficiency η_2 of this stage is automatically 0.5, provided we do not again change the bucket area before entering the stack, and assuming that a negligible fraction of the particles is spilt out and lost in this stage: then the interesting questions are whether the speeds proposed in AR/60-8 are unnecessarily slow, sufficiently slow, or insufficiently slow to ensure small spill-out; and whether methods (b) or (c) would be better than (a).

One may note that in the table on page 8 of PS/Int. AR/60-8, this change-over process takes anything from 21 o/o to 97 o/o of the total time: unless we go to values of ϕ_s less than 30° it is always big enough to be worth reducing.

Other questions of interest are whether we have time enough to make this change-over so adiabatic that a factor of less than 2 between the accelerating bucket and the trapping bucket areas would be sufficient, or alternatively whether the accelerating bucket can profitably be reduced before entering the stack. Either of these could give η_2 greater than 0.5 with some sacrifice either in time or increased spill-out.

Rather rough estimates can be made on the questions by working in the linear approximation. If we use θ to represent the azimuthal position of the particle measured forward round the machine, in units of R.F. radians, from the phase-stationary particle; and E to represent the particle energy referred to that of the phase-stationary particle, the linearised phase-oscillation equations become: -

$$\begin{aligned} \dot{\theta} &= -aE \\ \dot{E} &= b\,\theta \end{aligned} \qquad (5)$$

In general a and b are both time-dependent coefficients, but in our storage ring a is nearly constant. The coefficient b is proportional to $V \cos \phi_s$.

The instantaneous phase-oscillation frequency is $\sqrt{ab}$, and we first change the time scale to one in which this is constant and equal to one: i.e., we measure time in radians of the phase oscillations. Then

$$\begin{aligned} \theta' &= -\sqrt{\frac{a}{b}}\, E \\ E' &= \sqrt{\frac{b}{a}}\, \theta \end{aligned} \qquad (6)$$

where the primes indicate differentiation with respect to the new time.

If we change variables to

$$\begin{aligned} x &= \sqrt[4]{\frac{b}{a}}\, \theta \\ y &= \sqrt[4]{\frac{a}{b}}\, E \end{aligned} \qquad (7)$$

we obtain

$$x' = -y + \varepsilon x \qquad (8)$$

$$y' = x - \varepsilon y$$

where ε is $\frac{1}{4} (\log \frac{b}{a})'$. (9)

In the case of constant b/a, where ε is zero, the trajectories are circles in the x,y plane and $\pi(x^2 + y^2)$ is invariant on a trajectory and equal to the area within it.

With a finite but constant ε, a little algebra shows that

$$\pi\,(x^2 - 2\,\varepsilon\, x\, y + y^2)\,(1 - \varepsilon^2)^{-1/2} \qquad (10)$$

is the trajectory invariant equal to the enclosed area, and that the trajectories are ellipses with principal axes at 45°, 135°, and axis ratio of

$$\left(\frac{1 + |\varepsilon|}{1 - |\varepsilon|}\right)^{1/2}. \qquad (11)$$

It follows that if we start with $\varepsilon = 0$ and a stationary distribution consisting of an occupied circle in the x,y plane, and make an abrupt change to a constant ε, the occupied region now begins to sweep out (in the course of the phase oscillations) an area which is increased by a factor: -

$$\left(\frac{1 + |\varepsilon|}{1 - |\varepsilon|}\right)^{1/2} \qquad (12)$$

In practice a process of increasing b at constant ε will have an end, as well as the beginning that we have just considered. When this is taken into account one sees that it does not matter whether one has ε non-zero for a small number of synchrotron oscillations or a large number, but it does matter whether the number of quarter synchrotron oscillations is near to an even integer or an odd one.

We can treat the beginning and the end of the process as independent or incoherent, so that their η's are multiplicative, if we are prepared to make one of the following assumptions: -

(a) The number of phase oscillations between the beginning and the end is large, and they are sufficiently non-linear to result in a range of phase-oscillation frequencies being present. Then the region of phase space that is swept out early in the process must, before the end, be regarded as occupied as a result of a filamentation process.

(b) The number of phase oscillations is large enough that one cannot expect to fix it or know it with an accuracy better than $\pm 1/4$, so we treat it as indeterminate and add the effects of the beginning and the end in the worst possible phase relationship ≡).

On either of these bases the η for the whole process is

$$\frac{1 - |\varepsilon|}{1 + |\varepsilon|} \tag{13}$$

In PS/Int. AR/60-8 the voltage-raising process was considered with constant ϕ_s and

$$(\log V)^{\bullet} = 2\,(m-1)/\tau_p \tag{14}$$

where V is the R.F. voltage, τ_p is the period of the synchrotron oscillations, and $(m - 1)$ was made 0.5.

Postponing for the moment the question whether in fact we shall want to raise voltage at constant ϕ_s, let us see what this gives. We find: -

$$(\log \frac{b}{a})' = + \frac{1}{2\pi} \tag{15}$$

so

$$|\varepsilon| = \frac{1}{8\pi} = 0.0398 \tag{16}$$

≡) If the assumption (a) is not the case, and we have altogether to deal with a large number of steps in ε, this "worst possible" computation may be unreasonably pessimistic. It is then of interest, instead of adding up the log η_i values of all the steps, to calculate a sort of expectation-value using the process of adding by squares.

and

$$\eta_V = \frac{1 - |\varepsilon|}{1 + \varepsilon} = 0.92 \qquad (17)$$

This is sufficiently close to unity that it would be reasonable, if desired, to make this process somewhat faster and less efficient, but not faster by an order of magnitude. Alternatively it would be reasonable to increase the bucket area by less than a factor of 2.

It is of considerable interest that the phase-space blow-up in this process does not occur during the ε = constant, V rising, time; but instantaneously at the steps in ε. In principle (and in the linear approximation that we are using) the whole blow-up could be eliminated by making each ε-change either in two steps separated by one quarter cycle of synchrotron oscillations, or spread suitably over one half cycle; and a very substantial reduction in the blow-up can be expected if the ε-change is spread in any reasonably smooth way over one or two or so cycles. We can call this a second-order smoothing, for it basically amounts to devoting some of the available time to keeping the second (logarithmic) derivative of the coefficient b reasonably low, instead of devoting all the time to keeping the first derivative as low as possible. Some further discussion of it is in Appendix A.

To complete the change-over to accelerating buckets we must raise $|\dot{f}|$ from zero, so changing the stable phase from zero to some value suitable for acceleration. The first thing to be considered is the choice (a), (b) or (c) of page 3. One may note that raising the voltage is an ε-positive process, while a change-over to an accelerating ϕ_s at constant voltage is an ε-negative one, so placing these two processes end to end involves three steps in ε, the middle one being largest. Once the voltage has been raised at constant frequency to the point where the bucket shape (in linear approx.) is the same as that of the required accelerating bucket, it is a retrograde step to go on increasing V without starting to increase $|\dot{f}|$.

In principle one possibility is to increase V first to the point mentioned. then jump V, ϕ_s, $\dot{f}$ to the values adopted for acceleration. If they are jumped together this is an $\varepsilon = 0$ process and can be done as fast as is practically convenient. The objections to this method are that simultaneity is important (within a fraction of a quarter-cycle synchrotron oscillations) and that the R.F. phase ought to be jumped too, implying a delta-function in $\dot{f}$. If we rule out this discontinuous method the following seems to be a reasonable way of devising a programme from trapping bucket to accelerating bucket: -

(a) Choose some reasonably low value of ε , say 0.1 or 0.05.

(b) Decide on the ratio of areas between the accelerating bucket and the trapping bucket. In PS/Int. AR/60-8 a ratio of 2 was used, but the results of page 6 and 7 suggest that a ratio of 1.5 would catch nearly as many particles, at higher mean density. Probably both these values would be worth computing.

(c) Raise voltage at constant f and ϕ_s and chosen ε until the bucket area has increased by the chosen factor.

(d) Now raise $|\dot{f}|$, and raise V faster than before in order to maintain the same ε . Relate $\dot{f}$ and V in such a way that the bucket area is approximately constant. V , $\dot{f}$ and ϕ_s will reach the values chosen for acceleration simultaneously, and then one stops changing them.

(e) Apply some second-order smoothing to the programme constructed by the above procedure.

The main purpose of (c) is to get most of the particles out of the grossly non-linear region as soon as possible. But see also the remarks at the end of Appendix A.

Before we can go through the above putting in numbers, there is one more parameter of the acceleration process to be decided, for the accelerating bucket area fixed in (b) can be realised either by a high voltage and rate of acceleration near peak R.F., or a low voltage and rate, well away from peak. This choice will be discussed in later sections.

4. Noise during Acceleration.

The basic theory of R.F.-programme noise is in CERN 60-38. For a given amount of noise the r.m.s. increase in phase-oscillation amplitude is proportional to the square root of the time taken to accelerate. On the other hand the relative importance, in diluting the effective phase-space density, of a given noise-induced phase-amplitude, is evidently inversely as the phase-spread of the unperturbed bunches.

At constant bucket area the R.F. voltage is proportional to α^{-2}, the rate of acceleration to $\alpha^{-2}\dot{\Gamma}$, and the length of a bucket can conveniently be taken as proportional to the reciprocal of its width, and therefore to

$$\alpha\ (\cos \phi_s + (\phi_s - \pi/2)\sin \phi_s)^{-1/2}$$

Thus the relative importance of a given amount of noise can be taken as proportional to

$$\alpha\ \dot{\Gamma}^{-1/2}\ \alpha^{-1}\ (\cos \phi_s + (\phi_s - \pi/2)\sin \phi_s)^{1/2} \qquad (18)$$

Some values of this are tabulated below. One sees that, if noise is a serious problem, there is every incentive to accelerate near the peak field, where acceleration rates are high and buckets are, relatively, long and narrow.

Table I.

$\dot{\Gamma} = \sin \phi_s$	ϕ_s	(18)
0	0	∞
0.1	$5^\circ\ 44'$	2.9
0.3	$17^\circ\ 28'$	1.4
0.5	30°	0.83
0.7071	45°	0.46
0.8660	60°	0.23
0.9	$64^\circ\ 10'$	0.18

5. Stacking.

The process of adding another pulse to an existing stack can be expected to disturb it, and to increase its energy spread by more than the amount corresponding to the buckets brought up. For this process, or for separately-considered parts of this process, one will therefore in general fiend a phase-space efficiency $\eta(n)$ which is a function of n, the number of pulses stacked.

We shall assume that the non-uniformity in azimuthal density distribution, which will exist in the stack at the moment when the n'th pulse is deposited, will effectively have disappeared by the time the next is being deposited. On this assumption, its energy spectrum is the only thing that the stack remembers from one pulse to the next, and we are allowed, while calculating the energy spectrum after the (n + 1)st pulse from that after the n'th, to treat the latter as azimuth-independant.

The validity of this assumption depends on the parameters used, and has to be checked for each case considered. As a concrete example, suppose we stack 50 pulses per second at an R.F. frequency of 25 MHz. Any group of particles with a revolution-frequency spread of 2 parts per million will become spread azimuthally over one R.F. cycle between one pulse and the next. Comparing this with, for example, a revolution-frequency spread of 300 parts per million - about that of one adiabatically deposited typical pulse - it is clear that the azimuthal structure does practically disappear x).

(1) Non-adiabatic turn-off.

We shall attempt to get some idea whether it is of interest to make a slow turn-off of R.F. voltage when the buckets have reached stacking energy. Let us suppose that buckets have been raised to and into the stack, and have been converted into stationary buckets of the same area, all without disturbing the stack: then we are in a position to make a direct comparison between turning off the R.F. slowly enough to be adiabatic, and turning it off instantaneously. We shall assume that the buckets are completely and uniformly full before turn-off.

Stationary buckets of length 2π and total energy width ΔE have area in ϕ, E space of $4\,\Delta E$. Their average width is $\frac{2}{\pi}\,\Delta E$ and a region of this width would be occupied if they were turned off adiabatically.

With instantaneous turn-off the occupied width would be ΔE, so the η for stacking a single pulse is then $\frac{2}{\pi} = 0.637$.

For simplicity let us assume that subsequent pulses are always deposited in the middle of the stack. After n pulses let the stack have total energy-width

x) The stack does, of course, exhibit an azimuthal variation of both density and mean energy, whenever the R.F. is on. But this structure runs round with the R.F. wave, not with the stack revolution frequency, and it does not invalidate this estimate of what happens to the azimuthal structure belonging to the previous pulse, even if successive R.F. pulses are phase-coherent.

2 D(n). When the next pulse has been brought to rest in the middle of such a stack the trajectories in ϕ, E space are given by the invariants.

$$1 - \cos\phi + 2\left(\frac{2}{\Delta E}\right)^2 \delta E^2 = \text{const.} = 2/k^2 \tag{19}$$

Calling this constant $2/k^2$ simplifies some of the later expressions. In this equation ΔE is still the total energy spread of the buckets and δE is the particle energy referred to the mean. Hence

$$\frac{\delta E}{\Delta E} = \pm \frac{1}{2\,k} \sqrt{1 - \frac{k^2}{2}(1 - \cos\phi)} \tag{20}$$

Assuming the $(n + 1)$st pulse is brought up to the middle adiabatically with respect to the stack, the stack will then be bounded by two trajectories with a k-value such that the area between them is the old area $2\pi \cdot 2\,D(n)$ plus the added area $4\Delta E$, so we obtain a k determined by

$$4\pi\,\frac{D(n)}{\Delta E} + 4 = 2\int_0^{2\pi} \frac{1}{2\,k}\quad 1 - \frac{k^2}{2}(1 - \cos\phi)\;d\phi \tag{21}$$

Using the complete elliptic integral E, which is defined by

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 z}\quad dz$$

we get

$$\pi\,\frac{D(n)}{\Delta E} + 1 = \frac{1}{k}\,E(k) \tag{22}$$

Published tables of E against k can be used to construct a table of k against $\frac{1}{k}\,E(k)$, so that we are in a position to read off k as soon as the left hand side is known.

Substituted into (19) or (20), this k gives us the shape of the stack boundaries when the $(n + 1)$st set of buckets is in the middle of it, having been brought there adiabatically. The energy extrema of this stack are then obtained by substituting $\cos\phi = 1$ into (20): -

$$\left(\frac{\delta E}{\Delta E}\right)_{\text{extr.}} = \pm\frac{1}{2\,k} \tag{23}$$

The R.F. is now turned off instantaneously, and, for purposes of considering what happens when the subsequent pulses are stacked, we shall have to regard the whole area of phase-space between these energies as being occupied. Then

$$\frac{D\ (n+1)}{\Delta E} = \frac{1}{2\ k} \tag{24}$$

Successive use of (22) and the tables (to get the next k) and (24) to get the next D enable one to tabulate the stack width against the number of pulses brought up.

It is clear from the nature of this calculation that the result would be very little different if the buckets (of given area) were in fact non-uniformly filled: the stack would then have a tendency to be striated, but its width would be little different*).

The convenient quantities to know are in fact: -

$$\frac{\pi\ D\ (n)}{\Delta E} \tag{25}$$

which is the ratio of the stack width after n pulses, to what it would be <u>per pulse</u> for adiabatic turn-off; and n divided by this expression, which is the $\eta(n)$ associated with the instantaneous turn-off of n pulses: -

$$\eta_{inst.t.o.}(n) = \frac{n}{\pi}\ \frac{\Delta E}{D\ (n)} \tag{26}$$

Results of such calculations are shown in Table II. It is seen that $\eta(n)$ rises from 0.637 , fairly rapidly at first, later converging rather slowly on unity.

It should be remarked that, for reasonably large n , these figures are in a sense rather pessimistic estimates, as they relate to the absolute extremes of the stack width, and a particle must be on the worst possible phase on every successive pulse in order to reach such an energy deviation. It is a rather thin tail whose

*) Of course, if one knew that the buckets had a substantially higher density near their centres, it would become of interest to make them smaller at some stage before they reach the stack.

and we have calculated.

In the 4th column is shown

$$\frac{\pi}{\Delta E} (D(n) - D(n - 1)) - 1 \qquad (26)$$

This is the part of the increase in (25) on the nth pulse that is atributable to the non-adiabatic turn-off. It is of interest because it retains some validity when the circumstances are not entirely as we have assumed: for example, if the stack width reaches a value equivalent to 16.75 adiabatically deposited pulses (which it may do as a result of 15 deposited in the way considered, or as a result of some smaller number deposited in a more disturbing manner), then the next pulse adds to the width, one unit by virtue of its area, together with 0.03 units if, and only if, the R.F. is turned off instantaneously, together with any other non-adiabatic effect not yet considered.

It is known that, if the R.F. programme is the same on every pulse, each set of accelerated bunches passes through most of the stack, displacing it downwards, and is deposited near the top of the stack (MURA 477). Thus the calculations that we have just done relate to a non-repetitive R.F. programme in which turn-off is earlier on successive pulses. The equivalent calculations have been done for the repetitive case and results shown in Table III. Here D^+ is the width between stacking energy and the top of the stack, D^- between stacking energy and the bottom, $D^+ + D^- = D$ is the total width.

We see that the repetitive R.F. programme is a little worse, from this point of view, than depositing in the middle of the stack. In neither case is there a strong argument in favour of an adiabatic R.F. turn-off, provided one is proposing to stack of the order of 15 or more pulses.

Table II.

Stack widths and Phase-density Efficiencies due to Instantaneous R.F. Turn-off of Stationary Buckets at Stack Centre.

n	$\frac{\pi D(n)}{\Delta E}$	$\eta(n)$	expr. (26)
1	1.57	0.637	0.57
2	2.81	0.713	0.23
3	3.97	0.756	0.16
4.	5.09	0.786	0.12
5	6.19	0.808	0.10
6	7.28	0.825	0.08
7	8.35	0.838	0.07
8	9.42	0.850	0.07
9	10.47	0.859	0.06
10	11.53	0.867	0.05
11	12.58	0.874	0.05
12	13.63	0.881	0.05
13	14.67	0.886	0.04
14	15.71	0.891	0.04
15	16.75	0.896	0.04
16	17.78	0.900	0.03
30	32.12	0.934	0.02

Table III.

Stack Widths and Phase-density Efficiencies due to Instantaneous R.F.

Turn-off of Stationary Buckets with Repetitive R.F. Programme

n	$\frac{\pi D^{-}(n)}{\Delta E}$	$\frac{\pi D^{+}(n)}{\Delta E}$	$\frac{\pi D(n)}{\Delta E}$	$\eta(n)$	expr. (26)
1	0.78	0.78	1.57	0.637	0.57
2	1.87	0.97	2.84	0.704	0.27
3	2.92	1.12	4.05	0.741	0.21
4	3.96	1.26	5.22	0.766	0.17
5	4.99	1.38	6.37	0.784	0.15
6	6.02	1.49	7.51	0.799	0.14
7	7.04	1.59	8.63	0.811	0.12
8	8.06	1.68	9.75	0.821	0.11
9	9.08	1.78	10.86	0.829	0.11
10	10.09	1.86	11.96	0.836	0.10
11	11.11	1.94	13.05	0.843	0.10
12	12.12	2.02	14.14	0.848	0.09
13	13.13	2.10	15.23	0.854	0.09
14	14.15	2.17	16.32	0.858	0.08
15	15.16	2.24	17.40	0.862	0.08
16	16.17	2.31	18.48	0.866	0.08
30	30.26	3.16	33.42	0.898	0.05

The calculations summarized in Tables II and III are for the case in which the abolished buckets are stationary: it would be useful to have equivalent results for moving buckets, especially those of the shapes that could reasonably be used for acceleration. There is a difficulty here, which is not just a matter of more elaborate calculations, but comes from the fact that one cannot arrive at stacking energy with buckets moving at a finite rate without disturbing the stack non-adiabatically, and makes it difficult to separate into two independent blow-up factors the effects of the instantaneous R.F. turn-off and of the immediately preceding rapid bucket arrival.

For a given bucket area, moving buckets imply more R.F. voltage than stationary ones (the buckets used for acceleration in PS/Int. AR/60-8 have ϵ^{-2} = 9 to 284 times as much voltage as is required for the same area of stationary bucket: for other possible programmes suggested by Swenson, the factor is 2 to 3), so some estimate, even if rather crude, of the effect of turn-off of this R.F. would be desirable.

In the calculation of Tables II and III the non-adiabatic part of the stack-width increase between one n and the next is due to the energy oscillations, caused by the R.F., of particles at the top and bottom of the stack. The energy oscillations of particles separated by $\overline{\delta E}$ from the bucket energy ($\overline{\delta E}$ being an average over the oscillations) have an amplitude that depends on $\overline{\delta E}$ and is approximately proportional to the R.F. voltage and independent of whether the buckets are moving or stationary, provided $\overline{\delta E}$ is large compared to the bucket width.

On this approximation one can make the necessary modifications to the procedure used in calculating Table III to obtain corresponding results for the case where the R.F. voltage is, for example, four times the value appropriate to stationary buckets of the given area [x]. Since the approximation is not good for small n we have not tabulated the result, but quote only

$$\eta\ (16) \approx 0.73$$

$$\eta\ (30) \approx 0.81$$

for this case.

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[x] So α is 0.5 and $\Gamma \sim 0.33$.

Bearing in mind that these are calculations of the extreme limits of the stack, it still seems that the question of adiabatic or non-adiabatic turn-off may not be of much consequence if the R.F. voltage for acceleration is only a few times that corresponding to the stationary bucket case. If we add to this the fact that a semi-adiabatic turn-off should not be too difficult to arrange, it seems that the effect of the moving bucket passing into and through the stack, rather than that of the turn-off, is likely to determine how high a rate of acceleration can be used.

We have considered the repetitive R.F. programme in which one stacks at the theoretical stack top, and the non-repetitive one in which one stacks in the middle. There is another non-repetitive programme that is of interest: that in which one stacks at the theoretical stack bottom. In respect of the effect of R.F. turn-off, this is just the same as stacking at the theoretical top, so Table III (with the columns 2 and 3 interchanged) is applicable, and so is our estimate of $\eta(16)$ and $\eta(30)$ for $\Gamma = 0.33$. If the effects (considered in the next section) of passing into and through the stack with the bucket should turn out to be rather bad, one could largely eliminate them by stacking at the bottom; the instantaneous turn-off would be the main disturbing influence on the stack, and these estimates would become of more consequence.

(2) Effect of moving buckets on the stack.

If we consider the case of a repetitive R.F. programme, it seems certain that after a moderate number [x] of pulses have been stacked, the upper limit of the stack will be a little above stacking energy and the lower limit will be fairly well below stacking energy, both of these limits being effectively straight lines $E = \text{const.}$ in the absence of R.F.

When the next pulse is brought up, we wish to know how these two lines are deformed, in particular what is the energy of the lowest particle on the lower line and of the highest particle on the upper one.

The precise effect of passing through this lower line with buckets that come from $-\infty$ and go to $+\infty$ has already been computed (by MURA, CERN Symp. 1959, p. 58). It is reasonable to assume that our slow increase of R.F. volts at injection frequency will not look very different (from the point of view of the stack) from their movement of buckets up from $-\infty$, and that their continuing on to $+\infty$ will

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[x] Say 5 or 10.

look (from the point of view of the bottom of the stack) more adiabatic than the case where the bucket is destroyed quickly when it gets about to the top of the stack. The results of Vogt-Nilsen (CERN 58-9, Figures 2 to 7) show that the energy spread of particles that have been passed by a bucket divides rather clearly into two parts; (a) associated with variations of relative phase of particle and bucket, remaining when the bucket recedes to $+\infty$; and (b), energy oscillations that damp to zero as the bucket recedes.

The relevant MURA results are given in Table 1 on page 61 of the reference. If we take $I_{max}/\langle I \rangle_{av}$ we obtain a figure which corresponds to the case where, for different Γ, the voltage has been arranged to give the same bucket area, and which gives the maximum downward displacement of the stack bottom in units of one adiabatically deposited pulse.

Table V.

Γ	$I_{max}/\langle I \rangle_{av}$	$\eta(\infty)$
0.5	2.37	0.42
0.3	1.59	0.64
0.1	1.17	0.88
0.0	1.00	1

The case $\Gamma = 0$ is for buckets whose form is that of stationary buckets, and which consequently move infinitesimally slowly and only disturb the stack in an adiabatic way.

It seems plausible that when the stack is sufficiently wide, say many times the total energy width of a buckets, these figures will be a reasonably accurate measure of how much its bottom moves down on each pulse, and that the contribution to width increase from the top of the stack will become relatively unimportant; we therefore have in the third column put $\langle I \rangle_{av}/I_{max}$ and called it $\eta(\infty)$, The phase-density efficiency of this process in the limit of a large number of pulses.

It also seems likely that as n increases this $\eta(\infty)$ will be approached from below: for the effects of the energy oscillations, and spread at the top of the stack, are relatively more important for small n .

We could also consider a repetitive R.F. programme in which the last bit of the programme is devoted to slowing-down to rest and turning-off the bucket, more or less adiabatically, these being done within the energy range occupied by the upper part of the final stack. In this case $\eta(1)$ would be unity, but the phenomena at the bottom of the stack as it approaches final size would be very little different, and the third column of Table V can be taken as a reasonable estimate of an $\eta(\infty)$ that will now be approached from above.

Taking Table V at its face value, one could conclude as follows: if we aim at an overall stacking efficiency of around 50 o/o, these values of $\eta(\infty)$ indicate that we are not likely to be interested in Γ for acceleration greater than about 0.4 ; but if one wants to verify that one can come close to the theoretical stacking efficiency when this is nearly unity, a Γ around 0.1 is of more interest.

A semi-adiabatic slow-down of acceleration and reduction of R.F. voltage takes about the same time as the converse processes (just after trapping) that we have already considered: and the values of Table V seem to force one into the range of Γ values where this time is relatively small. One should therefore not be too worried by the fact that the Table V results may be (especially for small n) on the safe side only for the semi-adiabatic deposition case.

For large *) values of n , there is one respect in which Table V can be regarded as unduly pessimistic: it is based on the MURA figures for the largest downward displacement of any particle, so as n increases we approach more and more the situation of having calculated the extreme limit of a very thin tail, consisting of particles that have been most unfortunate in their phase on every pulse.

It is worth having a look at the opposite approach, ignoring extreme cases and estimating, for example, the root-mean-square energy spread of a stack.

In MURA 477, Reilly gives the average displacement and r.m.s. scatter due to the passage and turn-off of buckets, for various values of initial energy (referred to the bucket turn-off point). His results are all expressed in units of "expected

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*) An in this connection "large" may only mean >10 or so.

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mean displacement for a beam passed by a bucket": this is the same quantity that we have been using under the name of "width of one adiabatically deposited pulse". He treats the case $\Gamma = 0.55$: unfortunately similar data for other Γ does not seem to be available.

His results can approximately be summarised (with a bit of averaging and interpolation) into the following statements: -

(a) The r.m.s. scatter of any particles passed by a bucket, including those initially at the turn-off energy, is about 1.2 units.

(b) The mean downward displacement of particles initially at energy zero is 1.41. Of particles initially at -1.41 it is 1.21, for particles at -2.62 and below it is 1.00

If we permit ourselves to calculate the additional displacement and scatter of each pulse due to a later one in the approximation that all its particles are at their mean position, we have the following situation. After n pulses, the i'th pulse has been passed by n - i pulses; it will be found at a mean position of

$$- 1.41 - 1.21 - 1 - 1 \ldots\ldots \text{ to } (n - i) \text{ terms} \tag{29}$$

and will have a mean square spread about this mean of

$$\overline{\delta E_1^2} + (1.2)^2 (n - i) \tag{30}$$

where $\overline{\delta E_1^2}$ is the mean square energy spread of a single pulse after its R.F. has been turned off.

After n pulses [*], the whole stack has a mean position of

$$\overline{\delta E(n)} = -\left(\frac{n}{2} + 0.12 - \frac{0.83}{n}\right) \tag{31}$$

and a mean square deviation from "zero" of

$$\overline{\delta E^2(n)} = \overline{\delta E_1^2} + \tfrac{1}{3} n^2 + 0.84\, n - 0.789 + 13.71/n \tag{32}$$

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*) $n \geqslant 2$

The mean square deviation from the mean is therefore

$$\overline{\delta E^2(n)} - (\overline{\delta E(n)})^2 = \overline{\delta E_1^2} + \frac{1}{12} n^2 + 0.72\, n + 0.027$$

$$+ 13.91/n - 0.69/n^2 \qquad (33)$$

The half-width of the full bucket is, in the units that we are using, 1.41, and we shall take $\overline{\delta E_1^2}$ as 0.5 (one quarter the square of the half-width is the exyct value for a uniformly filled ellipse. It is clear from (33) that the value used for $\overline{\delta E_1^2}$ is not of much consequence if n is say 10 or more.)

In Table VI we give some values of this expression, together with its square root, and in the 4th column we have

$$\frac{n}{\sqrt{12}} \Big/ \left(\overline{\delta E^2}(n) - (\overline{\delta E(n)})^2\right)^{1/2} \qquad (34)$$

which is some sort of r.m.s. measure of stacking efficiency, as $n/\sqrt{12}$ is the r.m.s. width of a rectangular spectrum of total width n, i.e. of a stack consisting of n perfectly adiabatically deposited pulses.

Table VI

n	m.s.	r.m.s.	$\eta(n)$ r.m.s.	g(n)
5	8.96	2.99	0.48	0.48
10	17.44	4.18	0.69	0.69
15	31.00	5.57	0.78	0.75
20	48.95	7.00	0.83	0.80
30	97.59	9.88	0.88	0.84
50	245.1	15.65	0.92	0.88
∞			1.00	1.00

The r.m.s. width alone gives rather little information about the shape of a distribution, so it is difficult to decide how low one would be prepared to have $\eta(n)$ r.m.s., but it can be shown that one may derive from it another quantity $g(n)$ with a rather more direct significance, if one is prepared to assume that the distribution has only one maximum and that this is no higher than the density one would get with adiabatic stacking. The meaning of $g(n)$ is as follows: there must be at least a fraction $g(n)$ of the particles within an energy width equal to the theoretical adiabatic width. Or, alternatively expressed, at least a fraction $g(n)$ of the particles have an average phase-space-density, compared with the adiabatic case, of $g(n)$. This quantity is in the fifth column of Table VI. If we compare these values with Table V, and believe both, one must conclude that calculations of the extreme stack width are almost useless as a way of obtaining an estimate of stacking efficiency, presumably because of the long-thin-tail phenomenon. Possibly Table V retains a little usefulness in giving a general idea of how η depends on Γ' : one can argue that if Γ were 0.28 instead of 0.55, the η values of Table VI would probably be about twice as close to one.

The weak point in the calculation of Table VI is the assumption that all particles in the stack receive their downward displacements each time another pulse is raised. In fact the scattering will produce an upward tail on the stack, and particles in this will not be reached by subsequent pulses. This could be remedied by adiabatic turn-off of the R.F. in the upper part of the stack. Since part of the scatter in Reilly's figures is due to the instantaneous turn-off, it is possible to claim that one would do better than the η and g values of Table VI if one turned off adiabatically, but one would probably do worse than these values if one turned off instantaneously. It is not practicable to estimate how much the difference would be on the data available.

One can conclude that for n of the order of 20 it is possible to do reasonably well, say $g \approx 0.8$, even at a high Γ like 0.55, but it may be necessary to turn off adiabatically. The range of Γ values most likely to be of interest to us is, say, 0.2 to 0.3.

Calculation of the stack spread after n pulses by this type of method; that is to say, by first calculating the displacement and scatter for one R.F. pulse as a function of the initial particle energy within the stack, and then using this data statistically to find the overall situation after n pulses, is quite amenable to elaboration so that one obtains the final spectrum rather than just its mean and r.m.s. spread. Its justification rests on the randomisation of the phases between one pulse and the next, which we have already discussed.

It is worth pointing out that this type of method is very economical in computation time, especially for the case where the R.F. is to be turned off instantaneously. One takes a number of particles spread over all initial phases, to represent a line spectrum, runs the R.F. up to a certain frequency, and punches out the energy spectrum in some suitable form. To obtain the spectrum for the next interesting value of ΔE (difference between initial particle energy and turn-off energy) one merely continues with the R.F. programme on the same particles for a while. So the quantity of particle dynamics that needs to be computed amounts only to one sweep-through R.F. cycle acting on enough particles to represent all phases. The spectra obtained in this way can then be fed into statistical combining programmes for several n values, including, if desired, cases where one deposits in the middle or at the bottom of the stack, or with jitter in the turn-off frequency, etc.

6. Conclusions.

Even values of Γ as high as about 0.5 may be capable of giving reasonably high stacing efficiencies for a fiarly large number of pulses, but may require slow turn-off of the R.F. at stacking energy. Taking advantage of the merits of second-order smooting it should be possible to increase V and $|\dot{f}|$ after trapping sufficiently adiabatically and still in a time that is not a large fraction of the total. The same is true of the slow turn-off at stacking energy if this is needed.

For relatively small numbers of pulses, high stacking efficiency will require lower values of Γ, but it is difficult to say how much lower. In this regime the increase of V and $|\dot{f}|$ costs so little time that one need not look for the fastest way of doing it sufficiently adiabatically; the use of second-order smoothing is then mainly to eliminate noise and other unwanted irregularities in the programme.

If Γ is taken low enough, good stacking efficiency can be obtained without slow turn-off of the R.F., but in the low Γ region this slow turn-off costs relatively so little time that it is probably worth having.

The noise problem, if it is serious att all, is more serious if Γ is low.

Taken together with the calculations of Swenson, it seems that it will be possible to make a stack reasonably efficiently in somewhat under one second, even with harmonic numbers as low as two or three.

APPENDIX A

Second-order Smoothing.

We are interested in the equations (8) when ε is not being changed in discontinuous steps, but continuously with time; for example linearly in the "time" of the equations. If one changes to the variables $x + y$ and $x - y$ one obtains equations the same in form as (5), adn can then repeat the analysis from (5) to (13) - (and so on to any order). It is possibly more informative to suppose that, having devised a programme for changing V or $|\dot{f}|$ or both, and estimated the η that it will contribute, we then smooth it a little mathematically (and perhaps also in physical practice) by passing it through an integrating time constant, represented by the operator: -

$$\frac{1}{1 + j\omega\tau} \tag{40}$$

τ being the timeconstant in question.

In the linear approximation and for small perturbations, changes of trajectory-shape blow-up the phase-oscillations entirely by virtue of their Fourier components of frequency around twice the phase oscillation frequency.

The reduction of these components by our smoothing operator or circuit can be obtained directly from (40). For example, consider a smoothing timeconstant equal to one half a cycle (π radians) of phase oscillations; (40) becomes, at twice the phase oscillation frequency

$$\frac{1}{1 + 2\pi j}$$

whose modulus is $1/6.4$

It may therefore be quite profitable to design a programme with relatively high values of ε, and then smooth it in this way.

A smoothing timeconstant τ used in this way does not, of course, cost an extra time of the order of τ each time it does its job of smoothing a step in ε. Because of the resulting overlapping of processes, it rather costs an extra time of the order of τ on the whole R.F. programme.

A further advantage is to be gained if the programme-generating equipment is made to generate first the unsmoothed programme and then smooth it with a physical integrating-timeconstant circuit, for such a circuit also gives an attenuation of any noise or other unwanted irregularities in the programme *). The dominant effect of noise is by way of changes in the value of ϕ_s, and here the Fourier components around the phase oscillation frequency are the relevant ones, so our example attenuates them only by: -

$$\left| \frac{1}{1 + \pi j} \right| = 1/3.3$$

but this is still a useful factor.

A problem arises from the fact that the phase-oscillation frequency increases, and a timeconstant that is big enough to be useful at the beginning will be unnecessarily big later. Mathematically one can consider programmes in which each step in ε is smoothed with the timeconstant appropriate to it, but we hardly want to include a programmed timeconstant in the programming equipment. But up to now we have not made use of the fact that we propose to trap with a bucket width equal or less than that of the injected energy spread; this will result in the initial buckets being surrounded by an area occupied at nearly the full density, and the first step in ε, when the buckets begin to enlarge, can well be neglected and need not be appreciably smoothed.

As the voltage is increased at constant frequency one can in fact expect many of these particles surrounding the trapped area to be sucked into the increasing

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*) Only noise, etc., originating before the smoothing circuit can be smoothed in this way. Any noise originating in the devices that take the signal called "programme" and modulate the R.F. with it cannot, of course, be so dealt with. In principle this noise too could be smoothed by having a sufficient Q in the cavity or somewhere in the modulated part of the R.F. system. But Q-values of the order of $\omega_{rf}/\omega_{\phi}$ would be needed!

stable area, and values of f_1 appreciably higher than:

$$0.637 \Big/ \frac{\text{injected spread}}{\text{initial bucket width}}$$

together with values of η_2 appreciably higher than: -

$$\frac{\text{Initial bucket area}}{\text{Accelerated bucket area}}$$

can be hoped for in practice. In section II and III we disregarded these extra particles. Although they may make a useful increase in overall efficiency, they probably have rather little effect on the types of voltage-raising programme that it is desirable to use.

H.G. Hereward.

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