ON A CLASS OF ÉTALE ANALYTIC SHEAVES

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"Aber das ist eine andere Geschichte
und soll ein andermal erzählt werden"
Michael Ende - Die Unendliche Geschichte

1. Introduction

The bias of this work has been slowly but steadily shifting since its inception. Its immediate predecessor is the preprint [Ra2], which was itself based on the author's Ph.D. thesis [Ra1]. This new, extensively revised version has benefited considerably from the remarks of a referee who has weeded out of [Ra2] an embarassingly high number of naiveties, silly mistakes, misprints, and even proposed some nice improvements which I have adopted in what is now chapter 10. I present my wholehearted thanks to this person, who is unknown to me.

According to N.Katz (see [Ka4], Introduction), it was B.Dwork the first person to understand that classical differential equations with *irregular* singularities had deep meaning in arithmetic algebraic geometry (against the "prevailing dogma" which held that only equations with regular singular points should have meaning). Since then, the irregular differential equations have been gradually reappropriated into the mainstream of geometry. Initially only some specific areas were affected, such as p-adic analysis and positive characteristic geometry, but the trend is now spreading even to the domain of complex analysis, as witnessed *e.g.* by the recent book [Mal], which reports on ideas of Deligne, Malgrange *et al.* towards establishing an irregular Riemann-Hilbert correspondence.

The aim of this paper is to explore a p-adic version of the theory developed in [Mal]. In truth, in our work the differential equations remain on the background, while the emphasis is on the "dual world" of étale local systems naturally attached to them. In this we are guided by a well known heuristic, which translates many concepts arising from the study of differential equations, into dual topological notions (see *e.g.* the table at the end of [Ka1]). In particular, it is well understood that the notion of irregular singular point should be related to the appearance of *wild ramification* on a local system. Now, in our framework, all the varieties are defined over some p-adic field k of *characteristic zero*. But for such varieties, the étale topology is very close to the classical complex analytic topology, in particular, all ramification is tame: in other words, the algebraic étale topology in characteristic zero is too coarse to describe the monodromy of irregular differential equations.

We remedy this problem by replacing the algebraic étale topology with the much finer *analytic étale topology* recently introduced by Berkovich. In this sense, the upgrade from algebraic to analytic étale topology is analogous to the introduction of the space $\tilde{E}$ of Deligne, which plays a major role in chapter XI of [Mal].

In technical terms, what we need to do is to consider our algebraic varieties as special analytic spaces, and then work systematically inside the framework developed by Berkovich. We should stress here, that our main object of interest remains the category of algebraic schemes (over a fixed local field) and algebraic morphisms: the analytic spaces are always intended as auxiliary tools to define the finer topology and perform certain crucial constructions.

Once we have our candidate topology, we need to describe the class of analytic étale local systems we are interested in. In this paper, we limit ourselves to the study of local systems on smooth curves (notice that also the book [Mal] is mainly concerned with the one-dimensional case), and we will deal with general varieties in a planned sequel to this article.

A priori one may see no reasons why one should not consider the category of *all* such locally constant sheaves of finite rank. However it turns out that, if the curve is not compact (and this is really the only non-trivial case), certain bounds on the ramification of the sheaf around the points at infinity must be imposed in order to obtain a reasonable theory.

In order to conveniently express this condition, we introduce a notion of analytic local fundamental group: the finite rank representations of this group classify the admissible ramification behaviours of our class of sheaves. Chapter 3 and 4 are devoted to this construction, and we refer to the introductory remarks of chapter 3 for more details. This local fundamental group should really be thought of as a topological incarnation of the local differential Galois group of [Ka1]. In particular, the upper numbering filtration defined in *loc.cit.* has a very satisfactory counterpart: that is, we have a canonical higher ramification filtration on our local fundamental group, which behaves pretty much the way it is expected of these gadgets. In terms of this filtration we define also a notion of analytic Swan conductor, which is one of the main characters in our story.

However, at present we cannot yet claim that we completely understand the local theory of sheaves on a punctured curve: there are still a few important questions which need to be clarified, the main, according to our opinion, being conjecture 1 in section 3.1. On the other hand, we emphasize that none of the results in this paper depend on any conjecture: everything is proved unconditionally. But lest the reader should fear of being dragged on some wild Swan chase, let us highlight few firm points already established: first, the definition of the Swan conductor itself, is given in section 4.3, together with the usual paraphernalia of representations, their slopes and so on. Second, we can prove (theorem 9) a version of the Arf-Hasse theorem: the Swan conductor of a representation of finite rank is always an integer. Third, we construct (section 4.2) a functor of *meromorphic vanishing cycles* for analytic étale sheaves, for a basis of dimension one (*i.e.* essentially for a family of varieties over an open disc). This functor takes values in the category of sheaves with an action of the local fundamental group.

In view of its ties with the local differential Galois group, and since the latter group classifies connections with poles of finite order, the label "meromorphic fundamental group" which we bestow on our construction, seems appropriate enough. Hence we derive a notion of *meromorphically ramified* local system on an open curve, and the class of such sheaves is the chief object of study in this paper.

Our main tool for the investigation of the meromorphically ramified sheaves is the Fourier transform. The construction of the Fourier transform for analytic étale sheaves of Λ-modules (where Λ is some "big" torsion ring) is accomplished in chapter 6: it is really what one expects: we take the (essentially unique) rank one local system $\mathcal{L}_\psi$ on the affine line which has Swan conductor equal to one at infinity, then, for any vector bundle $\mathsf{E} \to S$ with dual $\mathsf{E}' \to S$, we have the dual pairing $\langle,\rangle : \mathsf{E} \times_S \mathsf{E}' \to S$, and the Fourier transform on E is the anti-involution

$$\mathcal{F}_\psi : \mathbf{D}(\mathsf{E}, \Lambda) \to \mathbf{D}(\mathsf{E}', \Lambda)$$

with "kernel" given by $\langle,\rangle^*\mathcal{L}_\psi$. We actually give a somewhat more general construction of the kernel, using Lubin-Tate theory: all these alternative kernels become isomorphic on the completion of the algebraic closure of our base field, but the extra generality could be useful for future arithmetic applications.

Our first application of the Fourier transform is contained in section 8.4: there we prove (see theorem 17) that the cohomology of any meromorphically ramified local system on a curve, has finite rank. We also show by a counterexample, that finiteness does not hold if the ramification is worse than meromorphic.

Wherever there is a Swan conductor, one expects also a formula of the Grothendieck-Ogg-Shafarevich type. As a second application, we verify such a formula for some very special sheaf on the affine line (see corollary 8 in section 8.4). This result is of course very modest, but is significant nevertheless. In a future paper I will show how to derive the full conjecture 2 of section 8.4 from these very special cases (and from the principle of the stationary phase). The proof of corollary 8 relies on (beside the Fourier transform) a deformation argument, based on the Kummer-Artin-Schreier-Witt theory recently developed by Suwa and Sekiguchi. For more details, we refer the reader to the beginning of chapter 7.

In chapter 9 we prove our principle of the stationary phase, and we sketch a study of the local Fourier transform by the usual global to local method. The knowledgeable reader will recognize the influence of Katz's paper [Ka2] on our presentation (except that our poor style cannot match Katz's elegant exposition). In particular our theorem 19 is formally identical to theorem 3, pag.114 in *loc.cit.*

Our last application of the Fourier transform is of arithmetic nature: the inspiration comes from the classical work [We] of Weil. In that paper, a special role is played by certain quadratic characters of a locally compact topological field F. Let $\psi : F \to \mathbb{C}^\times$ be a fixed additive character of F, V a finite dimensional F-vector space and $q : V \to F$ a non-degenerate quadratic form. Weil defines a Fourier transform $f \mapsto \hat{f}$ from the space of distributions on V to the space of distributions on the dual V'. Next he proves the following formula (see [We],chapt.I,n.14]:

$$\widehat{\psi \circ q}(\xi) = \gamma(q) \cdot |q|^{-1/2}(\psi \circ q')(\xi) \qquad (\xi \in V^*)$$

where $\gamma(q)$ is a complex number of absolute value equal to one, $|q|$ is a volume factor and $q^t : V' \to k$ is the transpose of q (see *loc.cit.*).
Of the two factors, the most interesting one is, by far, $\gamma(q)$. In [We], the properties of γ as a function of the quadratic form q are studied at length. The main result is that the assignment

$$q \mapsto \gamma(q)$$

descends to a group homomorphism from the Witt group $W(F)$ of the given base field F to the group of complex roots of unity.

In case F is a finite field, a simple application of the sheaves-to-functions dictionary of [SGA4$\frac{1}{2}$] allows us to recover the value of $\gamma(q)$ by cohomological means. In fact, in this case it boils down to a finite (Gauss) sum, and one has the formula:

$$\gamma(q) = \mathrm{Tr}(Fr, H_c^{\dim V}(V \times_F F^a, q^*\mathcal{L}_\psi)(\dim V/2)) \tag{1}$$

where $\mathcal{L}_\psi$ is the Lang torsor associated to the character ψ (which acts as a kernel for the ℓ-adic Fourier transform in the finite field case), F^a is the algebraic closure of F and $\mathrm{Tr}(Fr, M)$ denotes the trace of the action of the Frobenius generator $Fr \in Gal(F^a/F)$ on a Galois module M.

The cohomology group appearing in (1) has an obvious analogue in our theory (after all, $q^*\mathcal{L}_\psi$ is a meromorphically ramified sheaf), except that for the time being, we can only deal with torsion coefficient sheaves. But this limitation cannot stop us from considering an inverse system of kernels $\{\mathcal{L}_{\psi_n}\}$ (see chapter 10 for the notation) and then define

$$\Gamma(q) = \varprojlim_{n} H_c^{\dim V}(V \times_k \widehat{k}^a, q^*\mathcal{L}_{\psi_n})(\dim V/2) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

In chapter 10 we show that $\Gamma(q)$ descends to a homomorphism from the Witt group of k to the group of isomorphism classes of one-dimensional ℓ-adic Galois representations of (a certain extension of) k. Furthermore, many formal properties of Weil's γ-invariant have adequate counterpart for Γ. The precise relationship between Γ and Weil's invariant is not completely clear yet; nevertheless, we hope that this example may offer a glimpse of the kind of applications which we foresee for our theory.

To conclude, we want to mention two important differences between this version of the paper and the previous one [Ra2]. First of all, in [Ra2] an ℓ-adic formalism was proposed for analytic étale cohomology: I recognize now that this issue presents non-trivial aspects, and the fierce criticism of a referee advised a tactical retreat from that front. Nevertheless, I am confident that an ℓ-adic theory of *meromorphically ramified constructible sheaves* will eventually appear, and I plan to come back to this subject in a sequel to this paper.

Second, in [Ra2] an incomplete proof was given of theorem 10, which states that the Fourier transform commutes with Verdier duality. The "proof" amounted to a reproduction of a sketch of the unpublished argument of Verdier. Another referee pointed out gaps in this approach which cannot be easily filled. Therefore in this new version we have given a different proof, closer in style to the method of [Ka-La].

2. Preliminaries

2.1. Lubin-Tate theory. We recall here some well known facts from Lubin-Tate theory. The paper [LT] is the original source, but a complete account can be found in Lang's book [La].

Let k be a one-dimensional local field with valuation $|\cdot|$; denote by k° and resp. π the ring of integers of k and a uniformizing parameter in k°. Let q be the cardinality of the residue field $\widetilde{k} = k^\circ/\mathfrak{m}$, where as usual $\mathfrak{m} = (\pi)$ is the maximal ideal. Set $p = \operatorname{char} \widetilde{k} > 0$. Let also k^a be the algebraic closure of k and $\widehat{k}^a$ its completion, with the unique valuation $|\cdot|$ that extends the valuation of k.

Following Lubin-Tate [LT], we let $\mathfrak{F}_\pi$ be the set of power series $f \in k^\circ[[X]]$ such that

$$f(X) \cong \pi X \text{ mod degree } 2$$
$$f(X) \cong X^q \text{ mod } \pi.$$

The simplest example is just the polynomial $f(X) = \pi X + X^q$. Recall that a formal group F is a power series $F(X,Y) = \sum_{ij} a_{ij} X^i Y^j$ with coefficients $a_{ij} \in k$, satisfying the identities $F(F(X,Y),Z) = F(X,F(Y,Z))$, $F(X,Y) = F(Y,X)$ and $F(X,0) = 0$. A homomorphism of the formal group F into the formal group F' is a power series $f(X) \in k[[X]]$ such that $f(F(X,Y)) = F'(f(X), f(Y))$. In particular an endomorphism of F is a homomorphism of F into itself. We say that a formal group is defined over k° if its coefficients a_{ij} are in k°.

The following theorem summarizes the main features of the Lubin-Tate construction:

Theorem 1. *a) For each $f \in \mathfrak{F}_\pi$ there exists a unique formal group F_f, defined over k° such that f is a (formal) endomorphism of F_f. Moreover, for any two power series $f, g \in \mathfrak{F}_\pi$ and every $a \in k^\circ$ there is a unique $[a]_{f,g} \in k^\circ[[X]]$ such that $[a]_{f,g} \in Hom(F_f, F_g)$ and $[a]_{f,g} \cong aX$ mod degree 2.*

b) The map $a \mapsto [a]_{f,g}$ gives a group homomorphism $k^\circ \to Hom(F_f, F_g)$ satisfying the composition rule

$$[a]_{g,h} \circ [a]_{f,g} = [ab]_{f,h}.$$

In particular, if $f = g$, this map is a ring homomorphism $k^\circ \to End(F_f)$.

Proof. This is theorem 1.2, chapt. 8 of [La]. □

We will write $[a]_f$ in place of $[a]_{f,f}$; in particular notice that $[\pi]_f = f$.

Given $f \in \mathfrak{F}_\pi$, the associated formal group F_f converges, as a power series, for all pairs (x, y) of elements of $\widehat{k}^a$ such that $|x|, |y| < 1$. We introduce the notation $\Delta(a, \rho)$ for the set $\{x \in \widehat{k}^a, \ |x - a| < \rho\}$. Here $a \in \widehat{k}^a$ and ρ is a real number. Then it is clear that F induces a group structure on $\Delta(0, 1)$. Any $a \in k^\circ$ induces an endomorphism $[a]_f$ of this group.

Definition 1. *For any positive integer n we let $G_n \subset k^a$ be the kernel of the iterated power $[\pi]_f^n$. Also we define $G_\infty = \cup_{n>0} G_n$.*

We collect here some well known results about G_n:

Theorem 2. *1) The action of k° on $\Delta(0, 1)$ induces an isomorphism of k°-modules between G_n and the additive group $k^\circ/\mathfrak{m}^n$.*

2) The field $k(G_n)$ is a totally ramified abelian extension k with Galois group isomorphic to $(k^\circ/\mathfrak{m}^n)^\times$.

Proof. See theorem 2.1, chapt. 8 of [La]. □

We specialize now to characteristic zero, that is $\mathrm{char}(k) = 0$. In this case it is known (see [La], section 8.6) that for any formal group F over k, there exists a formal isomorphism

$$\lambda : F \to \mathbb{G}_a$$

where $\mathbb{G}_a$ is the usual additive formal group over k, that is $\mathbb{G}_a(X, Y) = X + Y$. The isomorphism λ is called the logarithm of F, and it is uniquely determined by F and by the condition $d\lambda(0)/dX = 1$.

Lemma 1. *Let F be a Lubin-Tate formal group, i.e. $F = F_f$ for some $f \in \mathfrak{F}_\pi$. Then the logarithm $\lambda = \lambda_F$ can be written in the form:*

$$\lambda(X) = \sum_i g_i(X) \frac{X^{q^i}}{\pi^i}$$

with $g_i(X) \in k^\circ[[X]]$.

Proof. This is lemma 6.3, chapt. 8 of [La]. □

It follows easily from the lemma that λ converges over $\Delta(0, 1)$, therefore it induces a group homomorphism

$$\lambda : \Delta(0, 1) \longrightarrow \mathbb{G}_a(\widehat{k}^a).$$

The following theorem measures the extent to which λ fails to be an isomorphism of groups:

Theorem 3. *Let $e_F(Z)$ be the power series (with coefficient in k) which is the inverse of $\boldsymbol{\lambda}_F(X)$. Then $e_F(Z)$ converges on the disc $\Delta(0, |\pi|^{1/(q-1)})$ and induces the inverse homomorphism to $\boldsymbol{\lambda}_F$ on the subgroups*

$$\Delta(0, |\pi|^{1/(q-1)}) \underset{\boldsymbol{\lambda}_F}{\overset{e_F}{\rightleftarrows}} \mathbb{G}_a(0, |\pi|^{1/(q-1)}).$$

(the group on the right coincides set-theoretically with the group on the left, and we use the notation $\mathbb{G}_a$ to emphasize that it is endowed with additive group structure).

Proof. See lemma 6.4, chapt. 8 of [La]. □

Remark: *a*) It can be shown that $\boldsymbol{\lambda}$ is a homomorphism of k°-modules, i.e. for all $a \in k^\circ$ there is an equality of power series:

$$a \cdot \boldsymbol{\lambda} = \boldsymbol{\lambda} \circ [a]_f.$$

b) Using theorem 3 and (*a*) it is not hard to show that the kernel of $\boldsymbol{\lambda}$ is the subgroup G_∞.

In what follows we will reserve the symbol ρ_1 for the constant $|\pi|^{1/(q-1)}$.

2.2. Complements of étale cohomology. Berkovich has defined an étale topology on his analytic varieties, and has studied the corresponding cohomology. In the work [B1], which is the reference for all the definitions which are implicit in this paper, he establishes the usual properties for his cohomology, like proper and smooth base change and Poincaré duality. In [B2] and [B3] he introduces two constructions of vanishing cycles.

In this chapter k will denote an arbitrary complete valuation field.

We denote by $\text{Ét}(X)$ the category of étale analytic varieties over X and for any ring Λ, we let $\mathbf{S}(X, \Lambda)$ be the category of sheaves of Λ-modules on $\text{Ét}(X)$.

In his paper, Berkovich considers mainly finite rings of coefficients, of the form $\Lambda = \mathbf{Z}/n\mathbf{Z}$. For our purposes, these are not quite enough, since we have to consider characters of an infinite divisible group G_∞ into $\Lambda^\times$.

Our first task is to extend the main results to more general torsion rings Λ. Instead of trying to reprove all the statements that we need beginning from scratch, we take a shortcut: we will show that in order to compute the effect of a cohomological functor on a sheaf F of Λ-modules, it suffices to regard F as a sheaf of abelian groups and compute the cohomological functor inside the category of sheaves of abelian groups. This will allow us to quickly derive our results from the theorems of Berkovich.

To start with, let Λ be any torsion ring and let $\mathbb{D}(X, \Lambda)$ (resp.($\mathbb{D}^+(X, \Lambda)$) be the derived category of complexes (resp. of complexes vanishing in large negative degrees) of sheaves $K^\cdot$ of Λ-modules and similarly define $\mathbb{D}^-(X, \Lambda)$; denote by Ψ_X the forgetful functor from $\mathbb{D}(X, \Lambda)$ to $\mathbb{D}(X, \mathbf{Z})$.

Let $f : X \to Y$ be a map of analytic spaces over k. First of all there is a direct image functor $Rf_* : \mathbb{D}^+(X, \Lambda) \to \mathbb{D}^+(Y, \Lambda)$.

Proposition 1. *The functor Rf_* commutes with the forgetful functor, i.e.*

$$Rf_* \circ \Psi_X = \Psi_Y \circ Rf_*.$$

Proof. For any sheaf F we will construct a resolution $I^\cdot$ by sheaves which are both injective as sheaves of Λ-modules and flabby as sheaves of abelian groups. One checks as in the algebraic case that flabby resolutions are f_*-acyclic : to do this one can look at [Mi] chapt. III sections 1,2,3 and convince oneself that all the arguments work without change in the present situation.

Then $I^{\cdot}$ computes at the same time Rf_* in the categories $D(Y, \Lambda)$ and $\mathbf{D}(X, \mathbf{Z})$, and the proposition follows.

For each $x \in X$, choose a geometric point x' localized at x, *i.e.* an imbedding of the residue field $\mathcal{H}(x)$ of x in the completion of its algebraic closure. We form the locally ringed space $X' = \cup_{x \in X} x'$ that we endow with the discrete topology. This space is an inductive limit of analytic spaces and therefore carries a natural étale site X'_{et}. Let $\pi : X'_{et} \to X_{et}$ be the obvious map.

The sheaf $\pi^* F$ is the direct product over the stalks $F_{x'} = x'^* F$ at the points $x' \in X'$. For every $x' \in X'$ choose an imbedding into an injective Λ-module $F_{x'} \hookrightarrow I_{x'}$: we see $I_{x'}$ as an injective sheaf of Λ-modules over the point x'. The product $I^0 = \Pi_{x' \in X'} I_{x'}$ is an injective sheaf of Λ-modules on X' and clearly F imbeds into $\pi_* I$. Since π_* preserves injective sheaves, we have constructed the first step of an injective resolution of Λ-modules; if we iterate this construction we obtain a full Godement resolution $I^{\cdot}$ for F. On the other hand, I is also flabby as a sheaves of abelian groups (since every sheaf on X' is flabby) and π_* preserves flabby sheaves, therefore $I^{\cdot}$ is also a flabby resolution, as wanted. $\square$

Next we turn to cohomology with support. For the notation we follow section 5.1 of [B1], to which we refer the reader for all the relevant definitions.

Recall (see *loc.cit*) that a ϕ-family of supports Φ defines a left exact functor $\phi_\Phi : \mathbf{S}(Y, \Lambda) \to \mathbf{S}(X, \Lambda)$ as follows. If $F \in \mathbf{S}(Y, \Lambda)$ and $f : U \to X$ is etale, then

$$(\phi_\Phi F)(U) = \{s \in F(U_\phi) | \mathrm{Supp}(s) \in \Phi(f)\}.$$

For example, if Φ is the family of all closed subsets, then $\phi_\Phi = \phi_*$. If the map $\phi : X \to Y$ is separated then the family of all ϕ-proper subsets of X is a paracompactifying ϕ-family, and we get a left exact functor which is denoted by $\phi_!$.

We can derive the functor ϕ_Φ in the two categories $\mathbf{D}^+(X, \mathbf{Z})$ and $\mathbf{D}^+(X, \Lambda)$, and in this way we obtain two functors that we denote both by $R\phi_\Phi$. The following proposition shows that in the cases of interest no ambiguity arises from this choice of notation.

Proposition 2. *Suppose that the family Φ is paracompactifying. Then the two functors defined above coincide, i.e.*

$$R\phi_\Phi \circ \Psi_X = \Psi_Y \circ R\phi_\Phi.$$

Proof. The proof of proposition 1 produces for any sheaf of Λ-modules a resolution that is injective in the category of sheaves of Λ-modules and flabby in the category of sheaves of abelian groups.

To prove the theorem, it suffices to show that this resolution is acyclic for the functor ϕ_Φ defined on the category $\mathbf{S}(X, \mathbf{Z})$, thus the proposition follows from lemma 2 below. $\square$

Lemma 2. *Suppose that the family Φ is paracompactifying. Let F be a flabby sheaf of abelian groups. Then $R^n\phi_\Phi(F) = 0$ for all $n > 0$.*

Proof. It is shown in [B1], proposition 5.2.1, that $R^n\phi_\Phi(F)$ is the sheaf associated with the presheaf $(U \to X) \mapsto H^n_{\Phi(f)}(U_\phi, F)$. Therefore it suffices to show that under the stated hypothesis, $H^n_{\Phi(f)}(U_\phi, F) = 0$ for all étale morphisms $U \to X$ and all $n > 0$. Since the restriction to U of a flabby sheaf of abelian groups on X, is a flabby sheaf, we have only to prove this for $U = X$.

Consider the morphism of sites $\pi : X_{et} \to |X|$, where $|X|$ is the space X with its underlying analytic topology. The morphism π induces a spectral sequence

$$H^p_\Phi(|X|, R^q\pi_* F) \Rightarrow H^{p+q}_\Phi(X, F).$$

We will prove that $R^q\pi_*F = 0$ for all $q > 0$. Assuming this for the moment, we show how to conclude. It follows from the vanishing that $H^p_\Phi(|X|, \pi_*F) = H^p_\Phi(X, F)$. Since F is flabby by hypothesis, we obtain from [B1], corollary 4.2.5, that π_*F is flabby in the analytic topology. Then π_*F is Γ_Φ-acyclic, by lemma 3.7.1 from [Gro] and the lemma is proved.

To see that $R^q\pi_*F = 0$, we can look at the stalks of this sheaf. For any point $x \in X$, let G_x be the Galois group of the algebraic closure of the residue field $\mathcal{H}(x)$. According to [B1], proposition 4.2.4, we have $(R^q\pi_*F)_x \simeq H^q(G_x, F_x)$, $q \geq 0$. Since F is flabby, it follows from [B1], corollary 4.2.5 that F_x is an acyclic G_x-module, as wanted. □

As a corollary, we get a proper base change theorem for sheaves of Λ-modules.

Theorem 4. *Assume that char($\widetilde{k}$) is invertible in Λ. Let $\phi : Y \to X$ be a separated morphism of k-analytic spaces, and let $f : X' \to X$ be a morphism of analytic spaces over k, which gives rise to a cartesian diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & Y \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\ X' & \xrightarrow{f} & X \end{array}$$

Then for any complex $K^\cdot \in \mathbf{D}^+(Y, \Lambda)$ there is a canonical isomorphism in $\mathbf{D}^+(X', \Lambda)$

$$f^*(R\phi_! K^\cdot) \simeq R\phi'_!(f'^* K^\cdot).$$

Proof. The usual devissage reduces to the case where $K^\cdot$ is concentrated in degree 0. Then the theorem follows from proposition 2 and theorem 7.7.1 of [B1]. □

Let $\mathbf{D}^b(X, \Lambda)$ be the subcategory of $\mathbf{D}^+(X, \Lambda)$ consisting of cohomologically bounded complexes. Let $\phi : Y \to X$ be as in theorem 5 and suppose that the fibres of ϕ have bounded dimension. Then, by corollary 5.3.8 of [B1] and proposition 2 we deduce that $R\phi_!$ takes $\mathbf{D}^b(X, \Lambda)$ to $\mathbf{D}^b(Y, \Lambda)$ and extends to a functor $R\phi_! : \mathbf{D}^-(X, \Lambda) \to \mathbf{D}^-(Y, \Lambda)$.

The following projection formula is proved as in [B1], theorem 5.3.9.

Theorem 5. *Suppose that $F^\cdot \in \mathbf{D}^-(X, \Lambda)$ and $G^\cdot \in D^-(Y, \Lambda)$ or that $F^\cdot \in \mathbf{D}^b(X, \Lambda)$ has finite Tor-dimension and $G^\cdot \in \mathbf{D}(Y, \Lambda)$. Then there is a canonical isomorphism*

$$F^\cdot \overset{\mathbf{L}}{\otimes} R\phi_!(G^\cdot) \simeq R\phi_!(\phi^*(F^\cdot) \overset{\mathbf{L}}{\otimes} G^\cdot). \tag{2}$$

□

Remark: we point out that the isomorphism of the theorem is functorial in both $F^\cdot$ and $G^\cdot$. Explicitly, let $f : F^\cdot \to F'^\cdot$ and $g : G^\cdot \to G'^\cdot$ be maps of complexes; then the isomorphism (2) induces the following commutative diagram

$$\begin{array}{ccc} F^\cdot \overset{\mathbf{L}}{\otimes} R\phi_!(G^\cdot) & \xrightarrow{\sim} & R\phi_!(\phi^*(F^\cdot) \overset{\mathbf{L}}{\otimes} G^\cdot) \\ \downarrow{\scriptstyle f \overset{\mathbf{L}}{\otimes} R\phi_!(g)} & & \downarrow{\scriptstyle R\phi_!(\phi^*(f) \overset{\mathbf{L}}{\otimes} g)} \\ F^\cdot \overset{\mathbf{L}}{\otimes} R\phi_!(G^\cdot) & \xrightarrow{\sim} & R\phi_!(\phi^*(F^\cdot) \overset{\mathbf{L}}{\otimes} G^\cdot). \end{array}$$

Finally we explain briefly how to deal with Poincaré duality for sheaves of Λ-modules.

Let $\Lambda' \to \Lambda$ be a ring homomorphism and let F (resp. G) be a sheaf of Λ-modules (resp. of Λ'-modules) on the analytic space X. Then F becomes a sheaf of Λ'-modules by restriction of scalars, and we can form the tensor product $F \otimes_{\Lambda'} G$. The sheaf of Λ'-modules $F \otimes_{\Lambda'} G$ carries also a canonical structure of sheaf of Λ-modules. To describe this structure, recall that a sheaf of Λ-modules S is by definition a Λ-module object in the category of sheaves of abelian groups; in other words, the structure of S is determined by a collection of endomorphisms $\lambda^*_S : S \to S$ for all $\lambda \in \Lambda$, such that $\lambda^*_S \circ \lambda'^*_S = (\lambda\lambda')^*_S$ and $1^*_S = \mathrm{id}_S$. Then the structure of $F \otimes_{\Lambda'} G$ is given by the rule: $\lambda^*_{F\otimes_{\Lambda'}G} = \lambda^*_F \otimes_{\Lambda'} \mathrm{id}_G$.

Proposition 3. *One can assign to every separated flat quasifinite morphism $\phi : Y \to X$ and every sheaf of Λ-modules on X a trace mapping*

$$Tr_\phi : \phi_! \phi^*(F) \to F.$$

These mappings are functorial on F and are compatible with base change and with composition. If ϕ is finite of constant rank d, then composition with the adjunction map

$$F \to \phi_*\phi^*(F) = \phi_!\phi^*(F) \xrightarrow{Tr_\phi} F$$

gives the multiplication by d. These properties determine uniquely the trace mappings.

Proof. In theorem 5.4.1 from [B1] the mappings are constructed in the category of sheaves of abelian groups, but the construction shows that Tr_ϕ commutes with the homomorphisms λ^*, $\lambda \in \Lambda$, i.e. it preserves the structure of Λ-module. □

Let X be an analytic variety over k. Denote by μ_n the sheaf of roots of unity of order n. We write μ_n^d for the d-th tensor power of μ_n with itself. Then we define the sheaves $\Lambda(d)_X = \mu_n^d \otimes_{\mathbf{Z}/n} \Lambda_X$. By the argument above, $\Lambda(d)_X$ is a sheaf of Λ-modules.

Let n be an integer prime to $\mathrm{char}(\widetilde{k})$. Here we specialize further and assume that $n\Lambda = 0$.

Proposition 4. *Suppose that k is algebraically closed. Then one can assign to every smooth connected k-analytic curve X a trace mapping isomorphism*

$$Tr_X : H^2_c(X, \Lambda(1)_X) \to \Lambda.$$

Proof. Theorem 6.2.1 of [B1] constructs trace mappings

$$\mathrm{Tr}_X : H^2_c(X, \mu_n) \to \mathbf{Z}/n\mathbf{Z}$$

with corresponding properties. By theorem 5, these mappings induce isomorphisms of abelian groups

$$\mathrm{Tr}_X : H^2_c(X, \Lambda(1)_X) \simeq H^2_c(X, \mu_n) \otimes_{\mathbf{Z}/n\mathbf{Z}} \Lambda \to \mathbf{Z}/n\mathbf{Z} \otimes_{\mathbf{Z}/n\mathbf{Z}} \Lambda \simeq \Lambda.$$

But the remark after the proof of theorem 5 implies that this isomorphism preserves the Λ-module structure. □

Using proposition 4 we can now establish the usual formalism of trace maps, just by following [B1] and making the obvious modifications. We leave the details as an exercise for the referee.

Let $G^\cdot, G'^\cdot \in \mathbb{D}^b(Y, \Lambda)$; a general nonsense argument provides us with a canonical morphism in $\mathbb{D}^+(X, \Lambda)$

$$R\phi_*(\mathcal{H}om(G^\cdot, G'^\cdot)) \longrightarrow \mathcal{H}om(R\phi_! G^\cdot, R\phi_! G'^\cdot).$$

Applying this morphism to complexes of the form $\phi^* F^{\cdot}(d)[2d]$ and using the trace mapping $R\phi_!(\phi^* F^{\cdot}(d)[2d]) \to F^{\cdot}$ we obtain for any $G^{\cdot} \in \mathbf{D}^b(Y, \Lambda)$ and $F^{\cdot} \in \mathbf{D}^b(X, \Lambda)$ a duality morphism

$$R\phi_*(\mathcal{H}om(G^{\cdot}, \phi^* F^{\cdot}(d)[2d])) \longrightarrow \mathcal{H}om(R\phi_! G^{\cdot}, F^{\cdot}).$$

Theorem 6. *The duality morphism is an isomorphism.*

Proof. The proof is given in [B1], theorem 7.3.1, with $\Lambda = \mathbf{Z}/n\mathbf{Z}$. With the help of the remarks above, the reader can verify that the same proof goes through with no change for a general ring Λ such that $n\Lambda = 0$. □

3. The analytic fundamental group of an affine curve

The aim of this section is the construction of a canonical descending filtration on the analytic fundamental group of $\mathbb{G}_m$. Some remarks about the parallel positive characteristic situation may motivate our idea. Let $\mathbf{F}$ be a field of positive characteristic, and let 0 be the origin of $\mathbf{A}^1_{\mathbf{F}}$. The henselization of $\mathbf{A}^1_{\mathbf{F}}$ at the point 0 is by definition $S = \mathrm{Spec}\mathcal{O}^h_{\mathbf{A}^1,0}$, where $\mathcal{O}^h_{\mathbf{A}^1,0}$ is the henselization of the local ring of germs of regular functions around 0. Then S has a generic point η_0, and for any choice of a geometric point $\overline{\eta}_0$ localized at η_0, we have the local fundamental group at 0 of $\mathbf{A}^1_{\mathbf{F}} - \{0\}$ which is the étale fundamental group $\pi_1(\eta_0, \overline{\eta}_0)$, in other words, the Galois group of the separable closure of the field of fractions of $\mathcal{O}^h_{\mathbf{A}^1,0}$. It is well known that the inertia subgroup I of $\pi_1(\eta_0, \overline{\eta}_0)$ has a distinguished subgroup P, called the wild inertia subgroup. The quotient I/P is the tame inertia, which is isomorphic to the product $\Pi_\ell \mathbf{Z}_\ell$ ranging over all primes $\ell \neq p$. Moreover, I is endowed with a canonical descending filtration, $I^{(r)}$, indexed by the positive real numbers $r \geq 0$, such that $I = I^{(0)}$ and $P = \overline{\bigcup_{r>0} I^{(r)}}$. The Swan conductor of an ℓ-adic representation of $\pi_1(\eta_0, \overline{\eta}_0)$ is given in terms of this higher ramification filtration.

In the analytic setting, the replacement for η_0 is a certain huge and rather unwyieldy pro-analytic space, from which it does not seem to be feasible to extract any detailed information. To our rescue comes Gabber-Katz's theorem on the local-to-global extension of local ℓ-adic representations. Briefly put, this states that any finite rank ℓ-adic representation of $\pi_1(\eta_0, \overline{\eta}_0)$ extends functorially to a smooth ℓ-adic sheaf on $\mathbb{G}_m$, which has at most tame ramification at infinity.

The foundations of the theory of the fundamental group of analytic spaces have been established recently by de Jong [deJ], and we are therefore encouraged to proceed in the following way. We will begin a close study of $\pi_1^{an}(\mathbb{G}_m, \overline{x})$, and find a certain canonical filtration on it. Then we will try to define a notion of local fundamental group, basically by decreeing that the local-to-global extension theorem should hold, and we will seek to convince the reader that this approach gives rise to a reasonable theory. In particular, the higher ramification filtration will be exported from the global to the local fundamental group.

3.1. The asymptotic Kummer exact sequence. Let X be any good analytic space over the complete non-archimedean field k of characteristic zero and residue characteristic p. We introduce the sheaves μ_{p^∞} and $\mathcal{U}^1_X$ on the étale site of X, by setting

$$\begin{aligned} \mu_{p^\infty}(V) &= \{f \in \mathcal{O}_V(V) \quad | \quad f^{p^n} = 1 \quad (n >> 0)\} \\ \mathcal{U}^1_X(V) &= \{f \in \mathcal{O}_V(V) \quad | \quad |1 - f|_{\sup} < 1\} \end{aligned}$$

for any étale morphism $V \to X$; the usual multiplication of functions defines an abelian sheaf structure on $\mathcal{U}^1_X$.

Lemma 3. *(Asymptotic Kummer exact sequence) There exists a short exact sequence of étale sheaves*

$$\begin{array}{c} 0 \longrightarrow \mu_{p^\infty} \longrightarrow \mathcal{U}^1 \xrightarrow{\lambda} \mathcal{O}_X \longrightarrow 0. \\ f \longmapsto \log(f) \end{array} \tag{3}$$

Proof. We only have to prove the surjectivity of λ, and for this we can check on the stalks. Let $p \in X$ be any point, and $f \in \mathcal{O}_{X,p}$. Choose some pointed étale morphism $(V, q) \to (X, p)$ where f extends to an element $f \in \mathcal{O}_V(V)$. Take a compact neighborhood W of q in V so that f is bounded on W, and we can find an integer N such that $|p^N f|_{\mathrm{sup},W} < \rho_1 = p^{1/(1-p)}$. Then $g = \exp(p^N f)$ is defined and belongs to $\mathcal{O}_W(W)$; moreover, g vanishes nowhere on W. Hence g defines an analytic map $W \to \mathbb{G}_m$. Define W' as the fibre product in the following square diagram

$$\begin{tikzcd} W' \arrow[r, "\phi"] \arrow[d] & W \arrow[d, "g"] \\ \mathbb{G}_m \arrow[r, "f \mapsto f^{p^N}"] & \mathbb{G}_m . \end{tikzcd}$$

Then W' is étale over W and $h = g^{1/p^N}$ is defined as an element of $\mathcal{O}_{W'}(W')$. One sees easily that $\lambda(h) = \phi^*(f)$ and the claim follows. □

Suppose now in addition, that X is the analytification of a connected algebraic scheme $\mathcal{X}$. Then $H^0(X, \mathcal{U}^1)$ is the group U^1 of elements $x \in k^\circ$ which are congruent to 1 modulo $\mathfrak{m}$. Taking the cohomology of the exact sequence (3) we obtain

$$0 \longrightarrow H^0(X, \mathcal{O}_X)/\lambda(U^1) \longrightarrow H^1(X, \mu_{p^\infty}) \longrightarrow H^1(X, \mathcal{U}^1) \longrightarrow H^1(X, \mathcal{O}_X). \tag{4}$$

Now we make the further assumption that the field k be algebraically closed. This hypotesis will be lifted only towards the end of section 4.1. Under this hypothesis, we have $\lambda(U^1) = k$. Furthermore, we point out that, if the variety X is proper, then the leftmost term in (4) vanishes. For this reason, and for others which will shortly be apparent, the case when X is proper is scarcely interesting.

Lemma 4. *Suppose $X = \mathcal{X}^{an}$ for some k-algebraic scheme $\mathcal{X}$. Then the inclusion of sheaves $\mu_{p^\infty} \hookrightarrow \mathcal{U}^1_X$ induces a natural imbedding*

$$\lim_{n \to \infty} H^1(X, \mu_{p^n}) \hookrightarrow H^1(X, \mathcal{U}^1_X).$$

Proof. It suffices to consider the usual Kummer exact sequence

$$0 \longrightarrow \mu_{p^n} \longrightarrow \mathcal{U}^1_X \longrightarrow \mathcal{U}^1_X \longrightarrow 0$$

and observe that the induced sequence

$$0 \longrightarrow \mu_{p^n} \longrightarrow H^0(X, \mathcal{U}^1_X) \longrightarrow H^0(X, \mathcal{U}^1_X) \longrightarrow 0$$

is exact. □

We notice that, due to the well-known compactness properties of the algebraic étale topology, the group $\lim_{n\to\infty} H^1(X, \mu_{p^n})$ can be suggestively rewritten as $H^1(\mathcal{X}, \mu_{p^\infty})$. What seems to be happening is that the analytic and algebraic contributions to the (abelianized) fundamental groups are distribuited onto respectively $H^0(X, \mathcal{O}_X)$ and $H^1(X, \mathcal{U}^1_X)$. Accordingly, I do not expect any exotic coverings coming from the cohomology of $\mathcal{U}^1$, but I do not know how to compute it completely. Therefore I will just conjecture this problem away:

Conjecture 1. *The imbedding of lemma* 4 *induces an exact sequence*

$$0 \longrightarrow H^1(\mathcal{X}, \mu_{p^\infty}) \longrightarrow H^1(X, \mathcal{U}^1_X) \longrightarrow H^1(X, \mathcal{O}_X)$$

whenever X is the analytification of a k-algebraic scheme $\mathcal{X}$.

We take the time out to make some side remarks on the cohomology of $\mathcal{U}^1$. These will not have any bearings on the continuation, so the hurried reader is invited to skip them.

The question of the structure of $H^1(X, \mathcal{U}^1)$ is meaningful and not trivial even in the proper case. Suppose now that X is the analytification of a proper scheme. Then, as it is well known, the group $H^1(X, \mathcal{O}^*_X)$ is invested of geometric meaning, by means of its identification with the set of k-rational points of the Picard scheme $\mathrm{Pic}(X)$. The natural imbedding $\mathcal{U}^1_X \hookrightarrow \mathcal{O}^*_X$ yields a morphism $H^1(X, \mathcal{U}^1_X) \to H^1(X, \mathcal{O}^*_X)$, so we may ask whether $H^1(X, \mathcal{U}^1_X)$ has geometric meaning as well. I propose the following conjectural picture. First of all, let us introduce the sheaves $\mathcal{U}^{\rho_1}_X$, $\mathcal{O}^{\rho_1}_X$ defined by

$$\begin{array}{lcl} \mathcal{U}^{\rho_1}_X(V) = \{f \in \mathcal{U}^1_X(V) & | & |1-f|_{\sup} < \rho_1\} \\ \mathcal{O}^{\rho_1}_X(V) = \{f \in \mathcal{O}_X(V) & | & |f|_{\sup} < \rho_1\} \end{array}$$

for any étale map $V \to X$. The restriction of λ induces an isomorphism $\mathcal{U}^{\rho_1}_X \xrightarrow{\sim} \mathcal{O}^{\rho_1}_X$. The situation is summarized by the following diagram

$$H^1(X, \mathcal{O}^*_X) \xleftarrow{j_1} H^1(X, \mathcal{U}^1_X) \xleftarrow{j_2} H^1(X, \mathcal{U}^{\rho_1}_X) \xrightarrow{H^1(\lambda)} H^1(X, \mathcal{O}^{\rho_1}_X) \xrightarrow{j_3} H^1(X, \mathcal{O}_X).$$

We recall that $H^1(X, \mathcal{O}_X)$ is canonically identified with the tangent space $T_0\mathrm{Pic}(X)$ of $\mathrm{Pic}(X)$ at the point $0 \in \mathrm{Pic}(X)$. Hence the following conjectures arise naturally:

1) the map j_3 is injective and identifies $H^1(X, \mathcal{O}^{\rho_1}_X)$ with an open neighborhood (with the topology inherited from k) of the origin in $T_0\mathrm{Pic}(X)$;

2) the composition $j_1 \circ j_2 \circ H^1(\lambda)^{-1}$ corresponds, via the identification in (1) and the standard identification $H^1(X, \mathcal{O}^*_X) = \mathrm{Pic}(X)$, to the classical exponential map for (p-adic) Lie groups.

Then we expect that also the map j_1 be the imbedding of an open disc around $0 \in \mathrm{Pic}(X)$. This should be in fact the smallest open disc which contains all the p-power torsion elements in $\mathrm{Pic}(X)$.

Back to business: in any case, we notice that the composition of the morphisms $\mu_{p^n} \to \mathcal{U}^1_X \to \mathcal{O}_X$ is the zero map. Thus, if we let $\mathcal{H}(X)$ be the preimage of $H^1(\mathcal{X}, \mu_{p^\infty})$ inside $H^1(X, \mu_{p^\infty})$, we get an exact sequence

$$0 \longrightarrow H^0(X, \mathcal{O}_X)/k \longrightarrow \mathcal{H}(X) \longrightarrow H^1(\mathcal{X}, \mu_{p^\infty}) \longrightarrow 0$$

with a canonical splitting, coming from the map of sheaves $\mu_{p^n} \hookrightarrow \mu_{p^\infty}$.

After these generalities, let us specialize to the case $X = \mathbf{A}^{1,an}$. Let x be a cooordinate on $\mathbf{A}^1$; we get $\mathcal{H}(\mathbf{A}^{1,an}) = k\langle x\rangle/k$, where $k\langle x\rangle$ denotes the group of entire power series over $\mathbf{A}^{1,an}$. This group should be dual to a certain quotient of the yet-to-be local fundamental group around infinity. According to our philosophy, there should be a canonical filtration on this group, and that in turns should leave a print on $\mathcal{H}(\mathbf{A}^{1,an})$. As a first step, we are going to define a canonical filtration on $\mathcal{H}(A^{1,an})$.

Definition 2. *Let $k[x]/k \hookrightarrow \mathcal{H}(\mathbf{A}^{1,an})$ be the imbedding of the group of polynomials in x with vanishing constant term, induced by our choice of a coordinate x over $\mathbf{A}^{1,an}$. For any integer $n \geq 0$, we define $F_n(\mathbf{A}^{1,an}) \subset \mathcal{H}(\mathbf{A}^{1,an})$ as the subgroup of polynomials of degree $\leq n$. $\{F_n\}_{n\geq 0}$ is the meromorphic ramification filtration in cohomology.*

The union $\bigcup_{n\geq 0} F_n(\mathbf{A}^{1,an})$ is called the meromorphic ramification in the cohomology of $\mathbf{A}^{1,an}$ and is denoted by $\mathcal{H}(\mathbf{A}^{1,an})_{mer.\infty}$.

We need to show that the meromorphic filtration is canonical, *i.e.* that it does not depend on the choice of the coordinate x. This is taken care of by the following lemma.

Lemma 5. *Let x be some coordinate function on $\mathbf{A}^{1,an}$. Any other coordinate is of the form $ax + b$ for some $a, b \in k$.*

For the proof we require the following version of Weierstrass preparation theorem, whose proof can be found for instance in Lang's book [La], theorem 2.2, chap.5.

Proposition 5. *Let $\mathfrak{o}$ be any complete local ring, and suppose that $f \in \mathfrak{o}[[x]]$ is a power series whose coefficients are not all in the maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$. Say that $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $a_0, ..., a_{n-1} \in \mathfrak{m}$, $a_n \notin m$. Then f factors uniquely as a product*

$$f(x) = (x^n + b_1 x^{n-1} + ... + b_n) g(x)$$

where $b_1, ..., b_n \in \mathfrak{m}$ and g is a unit in $\mathfrak{o}[[x]]$. $\square$

Proof. (of lemma 5) Let $f(x) = \sum_{j=0}^{\infty} a_j x^j$ be the power series expression for some other coordinate function. We want to show that $a_j = 0$ for $j > 1$. Suppose this is not the case. The supremum of the values $|a_j|$ $(j = 0, 1, ...)$ is attained for some index $j_{\max}$. In order to apply proposition 5 we would like that $j_{\max} > 1$. We can easily "jolt" the coefficients, by replacing $f(x)$ with $f(cx)$ where $c \in k$ has large norm. We can also arrange that $|a_{j_{\max}}| = 1$, by replacing $f(cx)$ with $\tilde{f}(x) = a_{j_{\max}}^{-1} f(cx)$. Then proposition 5 gives us a factorization $\tilde{f}(x) = P(x)g(x)$ where $P(x)$ is monic of degree $j_{\max}$ with coefficients in $\mathfrak{m}$. It follows that the roots of $P(x)$ have norm strictly less than 1. Let ρ be the maximum of these norms. In the closed disc of radius ρ centered at 0, the above formal factorization decomposes the analytic function expressed by $\tilde{f}$ as the product of P and another analytic function g. In particular, $\tilde{f}$ is not an injective map on the disc, hence it cannnot be a coordinate function. $\square$

The next example is $\mathcal{X} = \mathbb{G}_m$ and $X = \mathbb{G}_m^{an}$. Repeating the same considerations we get a canonically split short exact sequence

$$0 \longrightarrow k\langle x, x^{-1}\rangle/k \longrightarrow \mathcal{H}(\mathbb{G}_m^{an}) \longrightarrow H^1(\mathbb{G}_m, \mu_{p^\infty}) \longrightarrow 0.$$

Here we have chosen some coordinate x on $\mathbf{A}^1$ and have denoted by $k\langle x, x^{-1}\rangle$ the group of power series in x and x^{-1} which are entire over all of $\mathbb{G}_m$. Loosely speaking, this rather large group describes all the characters of the analytic fundamental group of $\mathbb{G}_m$, which are wildly ramified at either 0 or infinity.

Let $\overline{x}$ be some geometric point of $\mathbb{G}_m^{an}$. De Jong [deJ] has defined the analytic fundamental group $\pi_1^{an}(\mathbb{G}_m^{an}, \overline{x})$. In keeping with our philosophy, there should be an imbedding of the local fundamental group into $\pi_1^{an}(\mathbb{G}_m^{an}, \overline{x})$, which should be canonical up to conjugate action and such that the image should inject into a quotient group $\pi_1^{mod.0}(\mathbb{G}_m^{an}, \overline{x})$ of elements which are "tamely ramified" at the point 0. This translates as follows.

First of all, the tamely ramification comes from the term $H^1(\mathbb{G}_m, \mu_{p^\infty}) \simeq \mathbf{Q}_p/\mathbf{Z}_p$. We isolate a subgroup $\mathcal{H}(\mathbb{G}_m^{an})^{mod.0} = k(\langle x\rangle/k) \oplus (\mathbf{Q}_p/\mathbf{Z}_p) \subset \mathcal{H}(\mathbb{G}_m^{an})$. Inside $\mathcal{H}(\mathbb{G}_m^{an})^{mod.0}$ we further select the subgroup of characters with meromorphic ramification at infinity:

$$\mathcal{H}(\mathbb{G}_m^{an})^{mod.0}_{mer.\infty} = (k[x]/k) \oplus (\mathbf{Q}_p/\mathbf{Z}_p) \subset \mathcal{H}(\mathbb{G}_m^{an}).$$

The proof that these subgroups are canonically determined is the same as for lemma 5, so we omit the details.

It is obvious that the image of the canonical map $i : \mathcal{H}(\mathbf{A}^{1,an})_{mer.\infty} \to \mathcal{H}(\mathbb{G}_m^{an})$ lands into $\mathcal{H}(\mathbb{G}_m^{an})^{mod.0}_{mer.\infty}$.

Definition 3. *The meromorphic ramification filtration on $\mathcal{H}(\mathbb{G}_m^{an})$ is the increasing sequence of subgroups $F_n(\mathbb{G}_m^{an})$, $n = 0, 1, 2 \ldots$, defined as $i(F_n(\mathbf{A}^{1,an})) + (\mathbf{Q}_p/\mathbf{Z}_p)$.*

3.2. prime-to-p torsion cohomology. In this section we want to determine the group

$$H^1(\mathbb{G}_m^{an}, \varinjlim_{(p,N)=1} \mu_N)$$

where N ranges over all positive integers prime with p. As predictable, the answer is the same for both algebraic and étale cohomology. Nevertheless, this result does not seem to descend directly from Berkovich's general comparison theorems in [B1]. The following result will not be used in the sequel, and it is included only for the sake of completeness.

The proof uses the so-called Mittag-Leffler technique. This material should be pretty standard, but since Berkovich takes pains to prove a very special case (see [B1], lemma 6.3.2) of proposition 6 below, we do not feel too ashamed for including some extra details.

Definition 4. *Let $A = \{A_n, \phi_{n,m} : A_m \to A_n\}$ be a projective system of abelian groups. One says that A satisfies the Mittag-Leffler condition if for any $n \in \mathbf{N}$ the decreasing sequence $\{\phi_{n,m}(A_m)\}$ of subgroups of A_n is stationary.*

Proposition 6. *Let $\{X_n\}_{n\in\mathbf{N}}$ be an increasing family of subsets of the analytic space X satisfying $X = \bigcup_n X_n$ and $X_n \subset Int(X_{n+1})$ for all n. Let F be an abelian étale sheaf on X and assume that for a given j, the projective system $\{H^{j-1}(X_n, F)\}_n$ satisfies the Mittag-Leffler condition. Then the canonical map*

$$H^j(X, F) \to \varprojlim_n H^j(X_n, F)$$

is bijective.

Proof. A proof for the topological category is given in [Ka-Sh], prop. 2.7.1. To handle the analytic étale case requires hardly any changes. □

Proposition 7. *There exists a canonical isomorphism*

$$H^1(\mathbb{G}_m^{an}, \varinjlim_{(p,N)=1} \mu_N) \simeq \varinjlim_{(p,N)=1} \mathbf{Z}/N\mathbf{Z}.$$

Proof. Write $\mathbb{G}_m^{an} = \bigcup_{\epsilon>0} A(\epsilon, \epsilon^{-1})$ where $A(\epsilon,\epsilon^{-1})$ is a closed annulus of inner radius ϵ and outer radius ϵ^{-1}. We apply proposition 6 with $j = 1$ and $F = \varinjlim_{(p,N)=1} \mu_N$ (to be picky, choose a sequence of positive real numbers $\epsilon_n \to 0$). We need to compute $\varprojlim_{\epsilon} H^1(A(\epsilon,\epsilon^{-1}), F)$. As $A(\epsilon,\epsilon^{-1})$ is compact, we have (see [B1], prop. 5.2.9)

$$H^1(A(\epsilon,\epsilon^{-1}), F) = \varinjlim_{(p,N)=1} H^1(A(\epsilon,\epsilon^{-1}), \mu_N).$$

On the other hand, it follows easily from [B1], theorem 6.3.5 that

$$H^1(A(\epsilon,\epsilon^{-1}), \mu_N) \simeq \mathbf{Z}/N\mathbf{Z}$$

and the claim ensues. □

4. The local fundamental group

4.1. Meromorphic ramification filtration. In this section we complete the job started in the previous one, in that we define the local fundamental group of $\mathbf{A}^{1,an}$ at infinity, or better, what should be thought as a certain canonical quotient of it. Furthermore, we will exploit the meromorphic ramification filtration in cohomology to define a canonical descending filtration on it.

To justify our procedure, we have to unveil another (not so) secret source of inspiration. This is the theory of the differential Galois group in characteristic zero, and our model is Katz's paper [Ka1]. For the convenience of the reader, we give a quick digest of some aspects of it, in so far as they are relevant to our situation (k algebraically closed of characteristic zero).

The theory comes in both global and local flavors. For the global theory, one is given a smooth, connected and separated k-scheme and a category $DE(X)$ is introduced whose objects are all the pairs (M, ∇) consisting of a locally free $\mathcal{O}_X$-module of finite rank M, together with an integrable connection ∇ on M. Morphisms are the horizontal $\mathcal{O}_X$-linear maps. With the obvious notions of tensor product and internal Hom, $DE(X)$ is a "neutral Tannakian category over k". Any rational point $x \in X(k)$ defines a k-valued fibre functor.

If ω is any such fibre functor, we denote by $\pi_1^{diff}(X,\omega)$ the affine k-groupscheme $\mathrm{Aut}(\omega)$: this is the differential fundamental group of X/k with base point ω. Let $\underline{\mathrm{Rep}}(\pi_1^{diff}(X,\omega))$ be the category of finite dimensional k-representations of $\pi_1^{diff}(X,\omega)$. By some general theorem on neutral Tannakian categories, the functor ω defines an equivalence of tensor categories

$$DE(X) \xrightarrow{\sim} \underline{\mathrm{Rep}}(\pi_1^{diff}(X,\omega)).$$

One also knows that any two k-valued fibre functors are (non-canonically) isomorphic.

For the local theory, let K be a complete discrete valuation field with residue field k. After choosing a uniformizing parameter t, we can identify $K = k((t))$. We denote by $\mathcal{D}$ the ring of all t-adically continuous k-linear differential operators of K to itself. If θ is any non-zero derivation in $\mathcal{D}$, its powers $1, \theta, \theta^2 \ldots$ form a K-basis as left K-module.

We denote by $DE(K/k)$ the category of those left $\mathcal{D}$-modules whose underlying K-vector space is finite dimensional. In terms of a choice of θ, an object of $DE(K/k)$ is a pair $(M, \nabla(\theta))$ of a finite dimensional K-vector space M and a k-linear map $\nabla(\theta) : M \to M$ satisfying the usual Leibnitz rule

$$\nabla(\theta)(fm) = \theta(f)m + f\nabla(\theta)(m)$$

for all $f \in K$, $m \in M$. $DE(K/k)$ has natural internal Hom and tensor products, which make it into a rigid abelian tensor category.

It is convenient to choose the derivation $\theta = t\frac{d}{dt}$ in $\mathcal{D}$. Then any one-dimensional object V in $DE(K/k)$ is of the form $\mathcal{D}/\mathcal{D}(\theta - f)$ for some $f \in K$ and the isomorphism class of V is the image of f in

$$K/t\frac{d}{dt}\log(K^\times) = k((t))/\mathbf{Z} + tk[[t]] \simeq k[t^{-1}]/\mathbf{Z}. \tag{5}$$

One knows that, if V and W are two non-isomorphic one dimensional objects in $DE(K/k)$, then

$$\mathrm{Hom}_{\mathcal{D}}(V, W) = 0 = \mathrm{Ext}^1_{\mathcal{D}}(V, W)$$

and for $V = \mathcal{D}(\theta - f)$ we have $\mathrm{Hom}_{\mathcal{D}}(V, V) = k$ while $\mathrm{Ext}^1_{\mathcal{D}}(V, V)$ a one-dimensional k-space with basis the class of $\mathcal{D}/\mathcal{D}(\theta - f)^2$. In particular, notice that the trivial object has non-trivial extensions by itself. An iterated extension of the trivial object is called *R.S.-unipotent.* This notion applies to the global theory as well.

The crucial result of the local theory is Levelt's theorem in [Le], which states that, given any non-zero object V in $DE(K/k)$, there exists a finite extension L of K, which can be written in the form $K(t^{1/N})$ for some integer N, and such that the inverse image V_L of V from $DE(K/k)$ to $DE(L/k)$ is an iterated extension of one-dimensional objects of $DE(L/k)$.

The global and local theories are tied together thanks to the following observation.

Let $DE(\mathbb{G}_m)_{RS.0}$ be the subcategory of $DE(\mathbb{G}_m)$ consisting of all objects which are regular singular at zero. In terms of the choice of a coordinate x on $\mathbb{G}_m$, the rank-one objects L of $DE(\mathbb{G}_m)_{RS.0}$ are of the form

$$\left(k[x, x^{-1}], x\frac{d}{dx} + P(x)\right)$$

where $P(x) \in k[x]$ and the group of isomorphism classes of such L is the additive group $k[x]/\mathbf{Z}$ via the map $L \mapsto P(x) \bmod \mathbf{Z}$. Comparing with (5), we see that the inverse image functor

$$DE(\mathbb{G}_m)_{RS.0} \to DE(K/k)$$

induces an equivalence between the full subcategories of rank-one objects.

This prompts us to make the following

Definition 5. *We say that an object V of $DE(\mathbb{G}_m)$ is special if there exists a positive integer N such that the inverse image of V by the morphism*

$$\mathbb{G}_m \xrightarrow{x \mapsto x^N} \mathbb{G}_m$$

is a finite direct sum of objects of the form $L \otimes U$, where
L is of rank one, regular singular at zero
U is R.S.-unipotent.

Then Levelt's result implies

Theorem 7. *(see* [Ka1]*, 2.4.10) The inverse image functor*

$$DE(\mathbb{G}_m) \to DE(K/k)$$

when restricted to the full subcategory of $DE(\mathbb{G}_m)$ consisting of the special objects, induces an equivalence of categories. □

The quasi-inverse functor is called the *canonical extension.*

The ℓ-adic analogue of theorem 7 is Gabber-Katz's theorem on the local-to-global extension of representations of fundamental groups (see [Ka3], theorem 1.4.1). The notion of slopes and Swan conductor familiar from local class field theory have also appropriate counterparts in the differential equations setting. These are obtained as follows. First, consider a one-dimensional object $L \in DE(K/k)$, and let $L = \mathcal{D}/\mathcal{D}(t\frac{d}{dt} - f)$ for some $f \in K$; then the *slope* λ of L is defined to be the integer

$$\max(0, -\mathrm{ord}_t(f)).$$

Next, if V is an arbitrary object in $DE(K/k)$, find an extension $E = k((t^{1/N})$ such that V_E is a succcessive extension of one-dimensional objects $L_1, ..., L_n$. To each of them we associate its slope $\lambda_1, ..., \lambda_n$ (computed with respect to the uniformizing parameter of E). Then the set of slopes (with multiplicity) of V is defined to be the collection of the numbers $\lambda_1/N, ..., \lambda_n/N$. One can show that the slopes are intrinsic invariants of V, independent of all the choices made.

In place of the Swan conductor, we have the *irregularity index* of V which is defined as the sum with multiplicities of its slopes.

Theorem 7 allows to construct plenty of k-valued fibre functors on $DE(K/k)$, namely choose any point $a \in k^\times = \mathbb{G}_m(k)$, and associate to it the functor ω_a obtained by composing the canonical extension functor with the functor "fibre at a".

Hence the local differential Galois group can now be defined as $I_{diff} = \mathrm{Aut}(\omega_a)$. It is endowed we a canonical upper numbering filtration, given as follows. For any real number $x \geq 0$, denote by $DE^{(\leq x)}(K/k)$ the full subcategory of $DE(K/k)$ of objects all of whose slopes are $\leq x$. Dual to the inclusions

$$DE^{(\leq x)}(K/k) \subset DE(K/k)$$

we have homomorphisms of corresponding groups

$$I_{diff} \to \mathrm{Aut}(\omega_a | DE^{(\leq x)}(K/k)).$$

Their kernels are closed normal subgroups of I_{diff}, denoted $I^{(x)}_{diff}$. The usual properties of the filtration have satisfactory differential analogues; see [Ka1] for further details.

To see how this may be relevant to our discussion, we go back to formula (5). One way to interpret (5) is as describing the group of k-valued characters of I_{diff}. By the observation above, this is also the group of characters of a certain quotient of $\pi_1^{diff}(\mathbb{G}_m, \omega)$.

On the other hand we have the standard formula

$$\mathrm{Hom}(\pi_1^{an}(\mathbb{G}_m^{an}, \overline{x}), \mu_{p^\infty}) \simeq H^1(\mathbb{G}_m^{an}, \mu_{p^\infty}).$$

Dual to the imbedding $\mathcal{H}^{mod.0}_{mer.\infty} \subset H^1(\mathbb{G}_m^{an}, \mu_{p^\infty})$ we have a quotient map $\pi_1^{an}(\mathbb{G}_m^{an}, \overline{x}) \to \widetilde{I}$.

We are therefore led to compare the two formulas

$$\begin{aligned} \mathrm{Hom}(I_{diff}, k) &= (k[t]/k) \oplus (k/\mathbf{Z}) \\ \mathrm{Hom}(\widetilde{I}, \mu_{p^\infty}) &= (k[t]/k) \oplus (\mathbf{Q}_p/\mathbf{Z}_p). \end{aligned} \tag{6}$$

In the first one we have isolated the summand $k/\mathbf{Z}$, to stress the similarity between differential and analytic settings. This summand is responsible for the differential equations of rank one with regular singularities at the origin. In other words, the complement $k[x]/k$ classifies the differential equations of rank one defined over all of $\mathbf{A}^1$. The upshot is that

(7) *the rank one differential equations on the affine line correspond exactly to the analytic étale sheaves with meromorphic ramification.*

The situation is less mysterious than what it may seem. To understand what is going on, we consider our standard μ_{p^∞}-torsor $\mathcal{L}$, given as the sheaf of local sections of the logarithm

$$\log : \Delta(0,1) \to \mathbf{A}^1.$$

An easy computation shows that the class in $H^1(\mathbf{A}^1, \mu_{p^\infty})$ of $\mathcal{L}$ is the element $t \in k\langle t\rangle$. The same class in $\mathrm{Hom}(I_{diff}, k)$ represents the differential equation

$$\frac{d}{dt} f = f$$

whose sheaf of horizontal sections is given by the scalar multiples of the exponential function.

We may ask why bound ourselves to the meromorphically ramified étale sheaves, since we could as well try to extend the theory of the differential Galois group, to comprehend also the p-adic differential equations with essential singularities.

I will come back later on more extensively on this matter. One of the main reasons is that I have come to believe that the essentially ramified sheaves are in a sense "too wild". For instance, I can show that if a sheaf has only meromorphic singularities, then its cohomology has finite rank, while for an essentially ramified sheaf this never happens.

We should comment briefly on the discrepancies appearing on the "regular singularities" components of the character formulas (6). Let $a \in k$ be any element. The corresponding differential equation of rank one is

$$\frac{d}{dt} f = \frac{a}{t} f.$$

When a is a rational number, the sheaf of solutions is already defined on the algebraic étale topology, and in fact it is a Kummer sheaf with finite monodromy. This accounts for the appearance of the term $\mathbf{Q}_p/\mathbf{Z}_p$ in the second formula in (6). On the other hand, when $a \notin \mathbf{Q}$, the corresponding horizontal sections converge only on small discs, and there is no analytic étale covering of $\mathbb{G}_m^{an}$ over which these sections can be prolonged. Yet the corresponding differential module has its independent life, which explains the term $k/\mathbf{Z}$ on the differential side of (6).

We recall next a few notions from [deJ] section 2. For any analytic space X, let $\underline{\mathrm{Cov}}(X)$ be the category of analytic étale covering spaces of X. Every geometric point $\overline{x}$ of X defines a set-valued fibre functor $F_{\overline{x}} : \underline{\mathrm{Cov}}(X) \to \underline{\mathrm{Set}}$, which assigns to a covering $Y \to X$ its fibre $Y_{\overline{x}}$ over $\overline{x}$. Then $\pi_1^{an}(X, \overline{x})$ is the group of automorphisms of $F_{\overline{x}}$. For a topological group G, denote by $G - \underline{\mathrm{Set}}$ the category of discrete sets with a continuous G-action. The fundamental group of X has a natural pro-discrete topology, and $F_{\overline{x}}$ can be refined to a functor from $\underline{\mathrm{Cov}}(X)$ to the category $\pi_1^{an}(X, \overline{x}) - \underline{\mathrm{Set}}$.

After these preliminaries we can proceed with the construction of the canonical filtration on $\pi_1^{an}(\mathbb{G}_m^{an}, \overline{x})$. The idea is to mimick definition 5, and in this way give ourselves a "good" quotient to work on.

For convenience, let $\overline{x}$ be a geometric point localized at the point $1 \in \mathbb{G}_m^{an}$ and let $\tau : \pi_1^{an}(\mathbb{G}_m^{an}, \overline{x}) \to \pi_1^{alg}(\mathbb{G}_m^{an}, \overline{x})$ be the canonical map (see [deJ], section 2). We denote by P the kernel of τ. It is the intersection of the images of all the maps

$$\phi_{N*} : \pi_1^{an}(\mathbb{G}_m^{an}, \overline{x}) \to \pi_1^{an}(\mathbb{G}_m^{an}, \overline{x})$$

induced by the endomorphisms $\phi_N : x \mapsto x^N$ of $\mathbb{G}_m^{an}$. From (4) we obtain a natural map:

$$\varinjlim_N H^0(\mathbb{G}_m^{an}, \mathcal{O})/k \to \mathrm{Hom}(\varprojlim_N \mathrm{Im}\phi_{N*}, \mu_{p^\infty}) = \mathrm{Hom}(P, \mu_{p^\infty}). \tag{8}$$

Definition 6. *We denote by $\mathcal{A}$ the subgroup of the direct limit group in (8), consisting of all polynomials in the rational powers $x^{1/N}$ ($N \in \mathbf{N}$) with vanishing constant term. By the usual argument we see that this subgroup is canonically determined, in particular is independent of the choice of the coordinate x.*

The essential ramification subgroup of $\pi_1^{an}(\mathbb{G}_m^{an}, \overline{x})$ is the normal subgroup of P

$$I_{ess} = \bigcap_{f \in \mathcal{A}} Ker(f : P \to \mu_{p^\infty}).$$

Notice that $P_{mer} = P/I_{ess}$ is an abelian group.

Lemma 6. *I_{ess} is a normal subgroup of $\pi_1^{an}(\mathbb{G}_m^{an}, \overline{x})$.*

Proof. Let $\gamma \in \pi_1^{an}(\mathbb{G}_m^{an}, \overline{x})$ and $f \in \mathcal{A}$. Then $f : P \to \mu_{p^\infty}$ extends to a character $\overline{f}(x^{1/N})$: $\mathrm{Im}\phi_{N*} \to \mu_{p^\infty}$ for some N. The conjugate $\gamma(\mathrm{Ker} f)\gamma^{-1}$ depends only on the class

$$\overline{\gamma} \in \pi_1^{an}(\mathbb{G}_m^{an}, \overline{x})/\mathrm{Im}\phi_{N*} \simeq \mathbf{Z}/N\mathbf{Z}.$$

Clearly $\mathbf{Z}/N\mathbf{Z}$ acts as the group of deck transformations of the covering $\phi_N : \mathbb{G}_m^{an} \to \mathbb{G}_m^{an}$, *i.e.*, $\overline{\gamma}$ corresponds to a morphism

$$\begin{aligned} \overline{\gamma} : \mathbb{G}_m^{an} &\longrightarrow \mathbb{G}_m^{an} \\ x &\longmapsto \zeta x \end{aligned}$$

where ζ is an N-th root of 1. Unwinding the definitions one checks easily that $\overline{\gamma}(\mathrm{Ker}\overline{f}(x^{1/N}))\overline{\gamma}^{-1} = \mathrm{Ker}\overline{f}(\zeta x^{1/N})$ and from this the claim follows. □

Definition 7. *The quotient $\pi_1^{an}(\mathbb{G}_m^{an}, \overline{x})/I_{ess}$ is called the meromorphic fundamental group of $\mathbb{G}_m^{an}$ and is denoted by $\pi_1^{mer}(\mathbb{G}_m^{an}, \overline{x})$. We impose a topology on $\pi_1^{mer}(\mathbb{G}_m^{an}, \overline{x})$ by declaring that the intersection of finitely many subgroups*

$$Ker(\overline{f}(x^{1/N}) : Im\phi_{N*} \to \mu_{p^\infty}) \qquad\qquad (f \in \mathcal{A})$$

introduced in the proof of the lemma, form a cofinal system of open neighborhoods of the identity element. (By [Bou], *chapter III.2 the topology is well defined and unique).*

Remark: I tend to think that the topology of $\pi_1^{mer}(\mathbb{G}_m^{an}, \overline{x})$ is just the quotient topology induced by the projection

$$\pi_1^{an}(\mathbb{G}_m^{an}, \overline{x}) \to \pi_1^{mer}(\mathbb{G}_m^{an}, \overline{x}) \tag{9}$$

but I do not know how to prove (or disprove) this statement. Is it perhaps buried in the generalities of [SGA1]? In any case, the map (9) is continuous, and this suffices to prove the following

Theorem 8. *The fibre functor $F_{\overline{x}}$ restricts to an equivalence between a full subcategory $\underline{Cov}^{mer}(\mathbb{G}_m^{an})$ of $\underline{Cov}(\mathbb{G}_m^{an})$ and $\pi_1^{mer}(\mathbb{G}_m^{an}, \overline{x}) - \underline{Set}$. Moreover, for each object of $Y \in \underline{Cov}^{mer}(\mathbb{G}_m^{an})$ there exists an integer N such that the fibre product Y_N in the diagram*

$$\begin{tikzcd} Y_N \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathbb{G}_m^{an} \arrow[r, "\phi_N"] & \mathbb{G}_m^{an} \end{tikzcd}$$

extends to an abelian Galois covering of $\mathbf{A}^{1,an}$.

Proof. For the first statement, we only need to show that every $\pi_1^{mer}(\mathbb{G}_m^{an}, \bar{x})$-set which consists of a single orbit is in the image of $F_{\bar{x}}$. Since the map (9) is continuous and surjective, this is a consequence of [deJ], theorem 2.10. The second statement follows easily from the definition of the topology of $\pi_1^{mer}(\mathbb{G}_m^{an}, \bar{x})$. □

We regard the second part of theorem 8 as an analogue of Levelt's main theorem. Of course, in our case the result is built into the construction.

We also notice that τ descends to a natural map on $\pi_1^{mer}(\mathbb{G}_m^{an}, \bar{x})$ and we have a short exact sequence

$$0 \to P_{mer} \to \pi_1^{mer}(\mathbb{G}_m^{an}, \bar{x}) \to \pi_1^{alg}(\mathbb{G}_m^{an}, \bar{x}) \to 0. \tag{10}$$

Next, our filtration is defined on the meromorphic fundamental group:

Definition 8. *We set $I^{(0)} = \pi_1^{mer}(\mathbb{G}_m^{an}, \bar{x})$ and for any positive real number r we let $I^{(r)}$ be the subgroup of P_{mer} consisting of the elements σ such that*

$$f(\sigma) = 0$$

for all $f \in \mathcal{A}$ which have degree less than or equal to r. It is clear that $r > r' \Rightarrow I^{(r)} \subset I^{(r')}$, hence $\{I^{(r)}\}_{r \in \mathbb{R}}$ defines a descending filtration on the meromorphic fundamental group, which we call the meromorphic ramification filtration.

Finally we go local. Let ∞ be the "point at infinity" on the projective line $\mathbb{P}^1$. Choose some local coordinate z on $\mathbb{P}^1$, centered at ∞ (*i.e.* $z(\infty) = 0$).

Definition 9. *The pro-analytic space η_∞ is the projective system $\{\Delta(\infty, r)^*\}_{0<r<1}$ of pointed open discs*

$$\Delta(\infty, r)^* = \{p \in \mathbb{P}^1 \quad | \quad 0 < |z(p)| < r\}$$

where $\phi_{r,s} : \Delta(\infty, r) \to \Delta(\infty, s)$ $(r < s)$ is the natural imbedding.

Recall (see [B2], section 2) that an étale space over η_∞ is just an object of the direct limit category $\text{Ét}(\eta_\infty) = \varinjlim_{0<r<1} \text{Ét}(\Delta(\infty, r)^*)$ (where $\text{Ét}(X)$ denotes the category of étale morphisms $Y \to X$ and the transition maps are given by the pull-back functors). The natural restriction map defines a functor

$$\begin{array}{ccc} \text{Ét}(\mathbb{G}_m^{an}) & \xrightarrow{\mathfrak{R}} & \text{Ét}(\eta_\infty). \\ Y & \longmapsto & Y_{\eta_\infty} \end{array}$$

Definition 10. *The category $\underline{Cov}^{mer}(\eta_\infty)$ is the full subcategory of $\acute{E}t(\eta_\infty)$ consisting of all objects of the form Y_{η_∞} for some object $Y \in \underline{Cov}^{mer}(\mathbb{G}_m^{an})$.*

Proposition 8. *The restriction map induces an equivalence of categories*

$$\mathfrak{R} : \underline{Cov}^{mer}(\mathbb{G}_m^{an}) \xrightarrow{\sim} \underline{Cov}^{mer}(\eta_\infty).$$

Proof. Suppose Y_1, Y_2 are two connected étale coverings in $\underline{\mathrm{Cov}}^{mer}(\mathbb{G}_m^{an})$ such that there exists an isomorphism $\psi : Y_{1\eta} \simeq Y_{2\eta}$. Take an integer N as in theorem 8, so that both $Y_{1,N}$ and $Y_{2,N}$ extend to abelian Galois coverings of $\mathbf{A}^{1,an}$. Then the isomorphism ψ induces an isomorphism $\psi_N : Y_{1,N\eta} \simeq Y_{2,N\eta}$ between abelian Galois coverings of η_∞. In order to show that ψ_N comes from a global isomorphism, it suffices to check that the canonical map

$$\mathcal{H}(\mathbf{A}^{1,an})_{mer.\infty} \to H^1(\eta_\infty, \mu_{p^\infty})$$

is injective. In turns, this is equivalent to showing that for all $r > 0$ the map

$$k[z^{-1}]/k \to H^0(\Delta^*(\infty, r), \mathcal{O})/\log(H^0(\Delta^*(\infty, r), \mathcal{U}^1))$$

is injective, which can be checked explicitly.

This shows that $Y_{1,N}$ and $Y_{2,N}$ are isomorphic. Let G_N be the group of deck automorphisms of the covering $\phi_N : \mathbb{G}_m \to \mathbb{G}_m$. For $i = 1, 2$, the descent data from $Y_{i,N}$ to Y_i is given by a set of isomorphisms

$$\sigma^* Y_{i,N} \to Y_{i,N} \qquad (\sigma \in G_N)$$

with the usual cocycle conditions. Similarly, the descent data for the isomorphism ψ is given by a set of square diagrams

$$\begin{array}{ccc} Y_{1,N\eta} & \xrightarrow{\psi_N} & Y_{2,N\eta} \\ \uparrow & & \uparrow \\ \sigma^* Y_{1,N\eta} & \xrightarrow{\sigma^*\psi_N} & \sigma^* Y_{2,N\eta} \end{array}$$

Since all the spaces involved are étale over $\mathbb{G}_m$, all the maps in these diagrams are determined by the image of any chosen point in some disc $\Delta^*(\infty, r)$. Therefore the diagrams extend uniquely to descent data over all of $\mathbb{G}_m$, which shows that Y_1 and Y_2 are isomorphic. □

In analogy with the differential case, we can call $\mathfrak{R}^{-1}$ the *canonical extension* functor.

Definition 11. *Let $F_{\overline{x}}$ be the fibre functor over $\underline{Cov}(\mathbb{G}_m^{an})$ defined by a geometric point $\overline{x}$. Then $G_{\overline{x}} = F_{\overline{x}} \circ \mathfrak{R}^{-1}$ is a fibre functor for $\underline{Cov}^{mer}(\eta_\infty)$. The local meromorphic fundamental group $\pi_1^{mer}(\eta_\infty)$ is the automorphism group of $G_{\overline{x}}$. It is well defined up to an isomorphism which depends only on the choice of $\overline{x}$.*

Corollary 1. *Each choice of a geometric point in $\mathbb{G}_m$ determines an isomorphism*

$$\pi_1^{mer}(\mathbb{G}_m^{an}, \overline{x}) \simeq \pi_1^{mer}(\eta_\infty).$$

The meromorphic filtration on $\pi_1^{mer}(\mathbb{G}_m^{an})$ carries over to a canonical filtration on the local meromorphic fundamental group, which is in particular independent of the choice of base point. □

To ease notation sometime we will write π instead of $\pi_1^{mer}(\eta_\infty)$. The short exact sequence (10) has a local counterpart

$$0 \to P \to \pi_1^{mer}(\eta_\infty) \to \pi_1^{alg}(\eta_\infty) \to 0$$

where the rightmost term is the algebraic local fundamental group, which is canonically isomorphic to $\widehat{\mathbf{Z}}(1)$.

We give here a sample of the first few elementary properties of the meromorphic ramification filtration. First of all, since the map

$$\phi_N^* : \mathcal{A} \longrightarrow \mathcal{A}$$

is an isomorphism (you can always take an N-th root of $t^{1/M}$) one checks easily that for all real numbers $r > 0$ there is an induced isomorphism

$$\phi_{N*} : I^{(r)} \xrightarrow{\sim} I^{(r/N)}.$$

Next, we would like to explicit the connection between the meromorphic filtrations on the fundamental group and on the cohomology of $\mathbb{G}_m^{an}$, as given in definition 3. The compatibility between the two is expressed by the following

Proposition 9. *Let $f \in \mathcal{H}(\mathbb{G}_m^{an})$ and let n be the smallest integer such that $f \in F_n(\mathbb{G}_m^{an})$. Then n is also the smallest real number such that $I^{(n)} \subset Ker(f : P_{mer} \to \mu_{p^\infty})$*

Proof. First of all, it is obvious that $I^{(n)} \subset \mathrm{Ker}(f : P_{mer} \to \mu_{p^\infty})$, so that the infimum over the set of real numbers with this property is smaller than or equal to n. Suppose that this infimum r is strictly smaller than n. For any $g \in \mathcal{A}$ of degree less than r, set $C_g = f(\mathrm{Ker} g) \subset \mu_{p^\infty}$. By hypothesis:

$$\bigcap_g C_g = 0.$$

Since all the proper subgroups of μ_{p^∞} are finite and nested into each other, this means that for some g we have already $C_g = 0$. Take N an integer large enough so that both $f_N = \phi_N^*(f)$ and $g_N = \phi_N^*(g)$ extend to homomorphisms $\pi_1(\mathbf{A}^{1,an}, \overline{x}) \to \mu_{p^\infty}$.

By construction we have $\mathrm{Ker} g_N \subset \mathrm{Ker} f_N$ and therefore we can find an endomorphism ω of μ_{p^∞} which makes the following diagram commute:

$$\begin{tikzcd} \pi_1(\mathbf{A}^{1,an}, \overline{x}) \arrow[r, "f_N"] \arrow[d, "g_N"'] & \mu_{p^\infty} \\ \mu_{p^\infty} \arrow[ur, "\omega"'] & \end{tikzcd}$$

We have $\mathrm{End}(\mu_{p^\infty}) \simeq \mathbf{Z}_p$, the isomorphism being given by

$$\gamma \mapsto (\zeta \mapsto \zeta^\gamma) \qquad (\gamma \in \mathbf{Z}_p, \zeta \in \mu_{p^\infty}).$$

Suppose that $\omega = (-)^\gamma$ and consider the ladder diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mu_{p^\infty} \arrow[r] \arrow[d, "(-)^\gamma"] & \mathcal{U}^1 \arrow[r, "\lambda"] \arrow[d, "(-)^\gamma"] & \mathcal{O}_X \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & \mu_{p^\infty} \arrow[r] & \mathcal{U}^1 \arrow[r, "\lambda"] & \mathcal{O}_X \arrow[r] & 0. \end{tikzcd} \tag{11}$$

From the long exact ladder for the cohomology of (11) we derive that

$$f_N = \omega_* g_N = \gamma \cdot g_N.$$

But this is a contradiction, since the degree of g_N is strictly smaller than the degree of f_N. The claim follows. □

By inspecting the proof of lemma 6 it easy to see that the topological group $\pi_1^{mer}(\eta_\infty)$ has a cofinal system of open normal subgroups. Let $\mathcal{I}$ be the partially ordered set of all these open normal subgroups (with order given by inclusion). Since the intersection of any two normal subgroups is again a normal subgroup, we see that $\mathcal{I}$ is a small cofiltered category in a natural way. We identify $\mathcal{I}$ with a subcategory of $\pi_1^{mer} - \underline{\mathrm{Set}}$ by sending the normal subgroup S to quotient set π_1^{mer}/S with π_1^{mer}-action given by translation. Select a right inverse functor $\mathfrak{F}$ of the functor $F_{\overline{x}}$ of theorem 8:

$$\mathfrak{F} : \mathcal{I} \to \underline{\mathrm{Cov}}^{mer}(\mathbb{G}_m).$$

Then the composition

$$\mathfrak{R} \circ \mathfrak{F} : \mathcal{I} \to \underline{\mathrm{Cov}}^{mer}(\eta_\infty)$$

defines a pro-analytic space over η_∞, which we denote by $\mathcal{G}^{mer}$.

Let Λ be any ring. Recall that, for any analytic space $\mathbf{X} = \varprojlim_{i \in I} X_i$, the category $\mathbf{S}(\mathbf{X}, \Lambda)$ of sheaves of Λ-modules on $\mathbf{X}$ is defined as $\varinjlim_{i \in I} \mathbf{S}(X_i, \Lambda)$, where the maps in the direct limit are the pull-back functors.

If G_i is a group of automorphisms of X_i, we let $\mathbf{S}_{G_i}(X_i, \Lambda)$ be the category of sheaves F on X_i with a G_i-action, *i.e.*, the datum for all $g \in G_i$ of a map $\rho_g : g^*F \to F$ satisfying the usual associativity condition. If it is given a compatible system of groups $\{G_i\}$, *i.e.* such that for $i > j$ there is a map $\phi_{ij} : G_i \to G_j$ with the property that every diagram of the kind

$$\begin{array}{ccc} X_i & \xrightarrow{g} & X_i \\ \downarrow & & \downarrow \\ X_j & \xrightarrow{\phi_{ij}(g)} & X_j \end{array}$$

commutes, then we can let $G = \varprojlim_{i \in I} G_i$ and define the category $\mathbf{S}_G(\mathbf{X}, \Lambda) = \varinjlim_{i \in I} \mathbf{S}_{G_i}(X_i, \Lambda)$.

All this applies in particular to the pro-analytic space $\mathcal{G}^{mer}$. Let F be any sheaf of Λ-modules on η_∞ and $X \in \mathcal{G}^{mer}$ any object. Then the pull-back of F to X is an object in $\mathbf{S}_\eta(\mathcal{G}^{mer}, \Lambda)$, well defined independently of the choice of X. This construction defines a functor

$$\lambda : \mathbf{S}(\eta_\infty, \Lambda) \to \mathbf{S}_\pi(\mathcal{G}^{mer}, \Lambda).$$

Proposition 10. *The functor λ is an equivalence of categories.*

Proof. Let $F \in \mathbf{S}_\pi(\mathcal{G}^{mer}, \Lambda)$; by definition, there is a meromorphic étale covering X of some pointed disc $\Delta^*(\infty, r)$ on which F lives. Set $Y = \mathfrak{R}^{-1}(X)$ and let $j : X \to Y$ be the imbedding of X into Y. Clearly $\phi : Y \to \mathbb{G}_m$ is a connected Galois covering, and it suffices to show that $G = j_*F$ can be descended to a unique sheaf on $\mathbb{G}_m$.

Choose any point $p \in Y$ and a coordinate x on $\mathbb{G}_m$ centered at 0. For any positive real number r, let $\Delta^*(0, r)$ be the pointed open disc in $\mathbb{G}_m$ with radius r and center 0. Since Y is tamely ramified at 0, the connected component Y_r of $\phi^{-1}(\Delta^*(0, r))$ containing p is an étale covering of $\Delta^*(0, r)$ of finite degree. Let $\pi^{(r)}$ be the subgroup of π which stabilizes Y_r. Then $G_r = G_{|Y_r}$ is a sheaf with $\pi^{(r)}$ action, and it suffices to show that for any r we can descend G_r to a unique sheaf on $\Delta^*(0, r)$, since $\bigcup_{r>0} Y_r = Y$ and $\bigcup_{r>0} \pi^{(r)} = \pi$. But the map $Y_r \to \Delta^*(0, r)$ is finite étale and descent theory for such morphisms is a standard result. $\square$

Lastly, we mention that all the constructions above have also "arithmetic" variants: suppose that X_k is an analytic variety over the complete but not necessarily algebraically closed base field k. Let $\widehat{k}^a$ be the completion of an algebraic closure of k and let $X_{\widehat{k}^a}$ be the base change of X_k to $\widehat{k}^a$. Then there is a short exact sequence (see [deJ], proposition 2.13)

$$0 \longrightarrow \pi_1(X_{\widehat{k}^a}, \overline{x}) \longrightarrow \pi_1(X_k, \overline{x}) \longrightarrow \mathrm{Gal}(k^a/k) \longrightarrow 0. \tag{12}$$

Lemma 7. *Suppose as above that k is a general complete field. The subgroup $I_{ess} \subset \pi_1(\mathbb{G}_{m,\widehat{k^a}}, \overline{x})$ is normal inside $\pi_1(\mathbb{G}_{m,k}, \overline{x})$.*

Proof. Let $f \in \mathcal{A}$ be an element and take $\sigma \in \mathrm{Gal}(k^a/k)$. As in lemma 6 we see f as a homomorphism $f : P \to \mu_{p^\infty}$. Unwinding the definitions one checks easily that

$$\sigma(\mathrm{Ker} f)\sigma^{-1} = \ker f^\sigma$$

where $f \mapsto f^\sigma$ denotes the natural action of $\mathrm{Gal}(k^a/k)$ on the group $H^0(\mathbb{G}_{m,\widehat{k^a}}, \mathcal{O})$. The claim follows. □

The definitions 2 and 6 have to be slightly modified: the exact sequence (4) yields a natural imbedding of the group $k[x]/\lambda(U^1)$ inside the group $\mathrm{Hom}(\pi_1(\mathbb{G}_{m,k}, \overline{x}), \mu_{p^\infty})$. For every $N \in \mathbf{N}$ and every finite field extension E of k we have a map $\phi_{N,E} : \mathbb{G}_{m,E} \to \mathbb{G}_{m,k}$ and the analogue of formula (8) leads us to replace $\mathcal{A}$ with the group $\mathcal{A}_k$ consisting of all polynomials in the rational powers of x and with coefficients in k^a. Then each element of $f \in \mathcal{A}_k$ determines a character $f : \mathrm{Im}\phi_{N,E*} \to \mu_{p^\infty}$ for some N, E.

Definition 12. *The group $\pi_1^{mer}(\mathbb{G}_{m,k}, \overline{x})$ is the quotient $\pi_1(\mathbb{G}_{m,k}, \overline{x})/I_{ess}$. We define a topology on $\pi_1^{mer}(\mathbb{G}_{m,k}, \overline{x})$ by declaring that the intersection of finitely many subgroups*

$$Ker(\overline{f}(x^{1/N}) : Im\phi_{N,E*} \to \mu_{p^\infty}) \qquad (f \in \mathcal{A}_k)$$

form a cofinal system of open neighborhoods of the identity element.

We obtain from (12) the exact sequence

$$0 \longrightarrow \pi_1^{mer}(\mathbb{G}_{m,\widehat{k^a}}, \overline{x}) \longrightarrow \pi_1^{mer}(\mathbb{G}_{m,k}, \overline{x}) \longrightarrow \mathrm{Gal}(k^a/k) \longrightarrow 0.$$

Remark: from the definition we see in particular that the topology of $\pi_1^{mer}(\mathbb{G}_{m,\widehat{k^a}}, \overline{x})$ is strictly finer than the topology which is induced by its imbedding in $\pi_1^{mer}(\mathbb{G}_{m,k}, \overline{x})$.

With this topology, an argument like in theorem 8 shows that the category $\pi_1^{mer}(\mathbb{G}_{m,k}, \overline{x}) - \underline{\mathrm{Set}}$ is equivalent to a certain category $\underline{\mathrm{Cov}}^{mer}(\mathbb{G}_{m,k})$ of étale coverings of $\mathbb{G}_{m,k}$.

Next, we define in the obvious way the pro-analytic space $\eta_{\infty,k}$ and a restriction functor $\mathfrak{R} : \underline{\mathrm{Cov}}^{mer}(\mathbb{G}_{m,k}) \to \mathrm{\acute{E}t}(\eta_{\infty,k})$ which is an equivalence onto its essential image $\underline{\mathrm{Cov}}^{mer}(\eta_{\infty,k})$. Then the group $\pi_1^{mer}(\eta_{\infty,k})$ is given as in definition 11. In particular we derive a short exact sequence

$$0 \longrightarrow \pi_1^{mer}(\eta_{\infty,\widehat{k^a}}) \longrightarrow \pi_1^{mer}(\eta_{\infty,k}) \longrightarrow \mathrm{Gal}(k^a/k) \longrightarrow 0. \tag{13}$$

Similarly we can construct the pro-analytic space $\mathcal{G}_k^{mer}$ in this more general setting, and it is easily seen that the analogue of proposition 10 still holds.

4.2. Meromorphic vanishing cycles. Let now C be a smooth curve defined over the field k, and s some k-rational point on C. Fix also a $\widehat{k}^a$-geometric point $\overline{s}$, localized at s. To this data we associate the germ of k-analytic space (C, s). It follows from [B1] theorem 3.4.1, that we can find an isomorphism of k-germs $\phi : (C, s) \to (\mathbb{P}^1_k, \infty)$.

Recall from section 4 of [B2] that any k-germ (X, x) determines a pro-k-analytic space $X(x)$ and the assignment $(X, x) \to X(x)$ induces a functor from k-germs to pro-k-analytic spaces. There is an obvious map of pro-k-analytic spaces $\eta_{\infty,k} \to \mathbb{P}^1_k(\infty)$, induced by the natural imbedding and hence we can define the pro-k-analytic space $\mathcal{G}^{mer}_s$ as the fibre product in the following fibre diagram:

$$\begin{array}{ccc} \mathcal{G}^{mer}_s & \longrightarrow & \mathcal{G}^{mer}_k \\ \big\downarrow & & \big\downarrow \\ C(s) & \longrightarrow & \mathbf{P}^1_k(\infty). \end{array}$$

The pro-k-analytic space $\mathcal{G}^{mer}_s$ is determined up to isomorphism: in fact, suppose $\phi_1, \phi_2 : (C, s) \to (\mathbb{P}^1_k, \infty)$ are two isomorphisms as above. Then we can write $\phi_1 = \phi_2\psi$ for an automorphism $\psi : (\mathbf{P}^1_k, \infty) \to (\mathbb{P}^1_k, \infty)$ and this implies easily the claim. We also define $\eta_s = C(s) \times_{\mathbf{P}^1_k(\infty)} \eta_{\infty,k}$ and then we have a category $\underline{\mathrm{Cov}}^{mer}(\eta_s)$ consisting of all the fibre products $\mathcal{C} \times_{\eta_\infty} \eta_s$, with $\mathcal{C} \in \underline{\mathrm{Cov}}^{mer}(\eta_\infty)$. By composing the fibre product functor $\underline{\mathrm{Cov}}^{mer}(\eta_\infty) \to \underline{\mathrm{Cov}}^{mer}(\eta_s)$ with the fibre functor $G_{\overline{x}}$ of definition 11, we get a fibre functor for $\underline{\mathrm{Cov}}^{mer}(\eta_s)$, whose group of automorphisms we denote $\pi_1^{mer}(\eta_s)$ and sometime just π, to ease notation. It is isomorphic to $\pi_1^{mer}(\eta_\infty)$, but the isomorphism depends on the choice of the map ϕ. A topology can be defined on this group, so that the mentioned isomorphism becomes a homeomorphism. In particular, we can see the pro-k-analytic space $\mathcal{G}^{mer}_s$ as a functor

$$\begin{array}{ccc} \mathcal{I}_s & \longrightarrow & \mathrm{Cov}^{mer}(\eta_s) \\ T & \longmapsto & \mathcal{G}_T \end{array}$$

where $\mathcal{I}_s$ is the small cofiltered category of open normal subgroups of $\pi_1^{mer}(\eta_s)$.

For a k-analytic space X, we denote by $X - \mathcal{A}n$ the category of X-analytic spaces, defined in the obvious way; then if $\mathbf{Z} = \varprojlim_{i \in I} Z_i$ is a pro-k-analytic space, a $\mathbf{Z}$-analytic space $\mathbf{X}$ is by definition an object of the direct limit category $\varinjlim_{i \in I^o} Z_i - \mathcal{A}n$, where the maps in the direct system are induced by the fiber products $X_j = X \times_{Z_i} Z_j \to X$ (for $X \in Z_i - \mathcal{A}n$ and $j > i$).

We remark that the category of $\mathbf{Z}$-analytic spaces admits fibre products and cofiltered projective limits. For any ring Λ, the category $\mathbf{S}(\mathbf{X}, \Lambda)$ of sheaves of Λ-modules on $\mathbf{X}$ is by definition $\mathbf{S}(\varprojlim_{j \geq i} X_i, \Lambda)$. More generally, if G is a topological group, we let $\mathbf{S}_G(\mathbf{X}, \Lambda)$ be the category of the sheaves of Λ-modules on $\mathbf{X}$ endowed with a continuous G-action. If all the maps $Z_i \to Z_j$ are étale, then the category $\mathbf{S}_G(\mathbf{X}, \Lambda)$ has enough injectives and every injective object of $\mathbf{S}_G(\mathbf{X}, \Lambda)$ is injective also in $\mathbf{S}(\mathbf{X}, \Lambda)$.

In particular, we obtain the category $C(s) - \mathcal{A}n$ of $C(s)$-analytic spaces; if $\mathbf{X}$ is an object of this category, the special fibre $\mathbf{X}_s$ is defined and it is a k-analytic space.

For a $C(s)$-analytic space $\mathbf{X}$, we let $\mathbf{X}_{\overline{s}} = \mathbf{X}_s \times_s \overline{s}$, $\mathbf{X}_{\eta_s} = \mathbf{X} \times_{C(s)} \eta_s$, $\mathbf{X}_{\overline{\eta}_s} = \mathbf{X} \times_{C(s)} \mathcal{G}^{mer}_s$ and for any object $T \in \mathcal{I}_s$ we set $\mathbf{X}_T = \mathbf{X} \times_{C(s)} \mathcal{G}_T$. Similarly, if $F \in \mathbf{S}(\mathbf{X}, \Lambda)$, we denote by F_{η_s} (resp. F_T) the restriction of F to $\mathbf{X}_{\eta_s}$ (resp. to $\mathbf{X}_T$) and for any morphism $\phi : \mathbf{Y} \to \mathbf{X}$ of

$C(s)$-anaytic spaces we write $\phi_{\eta_s} : \mathbf{Y}_{\eta_s} \to \mathbf{X}_{\eta_s}$ (resp. $\phi_{\bar{s}} : \mathbf{Y}_{\bar{s}} \to \mathbf{X}_{\eta_s}$) for the map induced by the base change $\eta_s \to C(s)$ (resp. $\bar{s} \to C(s)$).

Moreover, let $\omega : \pi_1^{mer}(\eta_s) \to \mathrm{Gal}(k^a/k)$ be the map as in (13); then $\omega(T)$ is a subgroup of finite index in $\mathrm{Gal}(k^a/k)$, corresponding to a finite extension k_T of k. The morphism $\mathcal{G}_T \to C(s)$ factors as a composition

$$\mathcal{G}_T \to C(s) \times_k k_T \to C(s).$$

If we let $\mathbf{X}_{k_T} = \mathbf{X} \times_k k_T$, we obtain a diagram

$$\begin{tikzcd} X_T \arrow[r, "j_T"] \arrow[d] & X_{k_T} \arrow[d] & X_{\bar{s}} \arrow[l, "i_T"'] \arrow[d] \\ \mathcal{G}_T \arrow[r] & C(s) \times_k k_T & \bar{s} \arrow[l] \end{tikzcd}$$

For $F \in \mathbf{S}(\mathbf{X}_{\eta_s}, \Lambda)$, we define a left exact functor

$$\Psi_{\eta_s}^{mer} : \mathbf{S}(\mathbf{X}_{\eta_s}, \Lambda) \to \mathbf{S}_\pi(\mathbf{X}_{\bar{s}}, \Lambda) : \quad F \mapsto \varinjlim_{T \in \mathcal{I}_s} i_T^* j_{T*}(F_T).$$

By deriving $\Psi_{\eta_s}^{mer}$ we obtain a functor $R\Psi_{\eta_s}^{mer}$ which we call the *functor of meromorphic vanishing cycles.*

The functor of meromorphic vanishing cycles enjoys analogues of most of the properties which Berkovich proves for his functor $R\Psi_\eta$. We give hereafter a sample of such results. The proofs are minor variations of those for the corresponding statements in Berkovich's paper, therefore we omit the details.

Let Λ be any torsion ring in which the residue characteristic of k is invertible and for any $\mathbf{X}$ as above, let $\mathbb{D}_G(\mathbf{X}, \Lambda)$ be the derived category of $\mathbf{S}_G(\mathbf{X}, \Lambda)$ (and similarly for $\mathbb{D}_G^+$, $\mathbb{D}_G^-$).

Proposition 11. *Let $\phi : Y \to X$ be a smooth morphism of $C(s)$-analytic spaces. Then for any sheaf F of Λ-modules and any $q \geq 0$, there is a canonical isomorphism*

$$\phi_{\bar{s}}^*(R^q\Psi_{\eta_s}^{mer} F) \simeq R^q\Psi_{\eta_s}^{mer}(\phi_{\eta_s}^* F).$$

□

Proposition 12. *Let $\phi : Y \to X$ be a compact morphism of $C(s)$-analytic spaces. Then for any $F^\cdot \in \mathbb{D}^+(Y_\eta, \Lambda)$ there is an isomorphism in $\mathbb{D}_\pi^+(X_{\bar{s}}, \Lambda)$*

$$R\Psi_{\eta_s}^{mer}(R\phi_{\eta_s*} F^\cdot) \simeq R\phi_{\bar{s}*}(R\Psi_{\eta_s}^{mer} F^\cdot).$$

□

Corollary 2. *Let X be a $C(s)$-analytic space compact over $C(s)$. Then for any sheaf F of Λ-modules on X_{η_s} there is a spectral sequence*

$$E_2^{p,q} = H^p(X_{\bar{s}}, R^q\Psi_{\eta_s}^{mer}(F)) \Longrightarrow H^{p+q}(X_{\bar{\eta}_s}, F).$$

□

For technical reasons, we will need the following slight generalization of the meromorphic vanishing cycle. Suppose that the field k is the completion of an algebraic extension of a complete subfield k_0, and moreover, that there is a k_0-germ (C_0, s_0) such that $(C, s) = (C_0, s_0) \times_{k_0} k$.

Then define $\mathcal{I}_{s,k_0}$ as the subset of $\mathcal{I}_s$ consisting of all elements of the form $S \cap \pi_1^{mer}(\eta_{s,k})$ for some $S \in \mathcal{I}_{s_0}$. We obtain a left exact functor

$$\Psi^{mer}_{\eta_s,k_0} : \mathbf{S}(\mathbf{X}_{\eta_s}, \Lambda) \to \mathbf{S}_\pi(\mathbf{X}_{\bar{s}}, \Lambda) : \quad F \mapsto \varinjlim_{T \in \mathcal{I}_{s,k_0}} i_T^* j_{T*}(F_T)$$

and hence its derived functor $R\Psi^{mer}_{\eta_s,k_0}$ as above.

Notice that the action of π on $R^q\Psi^{mer}_{\eta_s,k_0}$ is continuous with respect to the coarser topology induced by the imbedding $\pi \hookrightarrow \pi_1^{mer}(\eta_{s_0})$. Notice moreover, that propositions 11 and 12 have obvious variants for $R\Psi^{mer}_{\eta_s,k_0}$.

The next proposition relates the more general functor $R\Psi^{mer}_{\eta_s,k_0}$ to our functor of vanishing cycles. Let

$$\mathfrak{f} : \mathbf{S}_{\pi_1^{mer}(\eta_{s_0})}(\mathbf{X}_{\bar{s}}) \to \mathbf{S}_\pi(\mathbf{X}_{\bar{s}})$$

be the natural forgetful functor (which forgets part of the group action).

Proposition 13. *With the notation above, suppose that $\boldsymbol{X}_0$ is a $C_0(s_0)$-analytic space and set $\boldsymbol{X} = \boldsymbol{X}_0 \times_{k_0} k$. Let F_0 be a sheaf of Λ-modules on $\boldsymbol{X}_{0,\eta_{s_0}}$ and denote by F the inverse image of F_0 on $\boldsymbol{X}_{\eta_s}$. Then for any integer q there is a canonical isomorphism*

$$\mathfrak{f}(R^q\Psi^{mer}_{\eta_{s_0}}(F)) \simeq R^q\Psi^{mer}_{\eta_s,k_0}(F).$$

Proof. Let $T = S \cap \pi_1^{mer}(\eta_{s,k}) \in \mathcal{I}_{s,k_0}$. For $q = 0$ it suffices to remark the isomorphism

$$i_T^* j_{T*}(F_T) \simeq i_S^* j_{S*}(F_{0,S}).$$

The case $q > 0$ follows from this, by observing that all injective sheaves are acyclic for both functors $\Psi^{mer}_{\eta_{s_0}}$ and $\Psi^{mer}_{\eta_s,k_0}$. $\square$

To end this section, we want to derive a workable formula for the stacks of the vanishing cycles functor. With the notation above, let t be any point in $\mathbf{X}_{\bar{s}}$ and for each $T \in \mathcal{I}_s$, denote by t_T the image of t in $\mathbf{X}_s \times k_T$. Let us write $\eta_s = \varprojlim_\alpha Z_\alpha$ for a certain family $\{Z_\alpha\}$. Then there exists a C-analytic space X such that $\mathbf{X} = \varprojlim_\alpha X \times_{(C,s)} Z_\alpha$. If F is a sheaf of Λ-modules defined on $\mathbf{X}_{\eta_s}$, then by definition we can find α such that F has a representative F_α on $X_\alpha = X \times_{(C,s)} Z_\alpha$. Let $F_{\alpha,T}$ be the restriction of F_α to $X_\alpha \times_C \mathcal{G}_T$ and finally, denote by j_T the morphism $X_\alpha \times_C \mathcal{G}_T \to X_\alpha \times_k k_T$ which represents j_T. We have

$$\begin{aligned} R^q\Psi^{mer}_{\eta_s}(F)_t &\simeq (\varinjlim_{T \in \mathcal{I}_s} R^q(i_T^* j_{T*} F_T))_t \\ &\simeq \varinjlim_{T \in \mathcal{I}_s} (i_T^* R^q j_{T*} F_T)_t \\ &\simeq \varinjlim_{T \in \mathcal{I}_s} (R^q j_{T*} F_T)_{t_T} \end{aligned}$$

Next a standard argument yields

$$(R^q j_{T*} F_T)_{t_T} \simeq \varinjlim_{U_T} H^q(\mathrm{j}_T^{-1} U_T, F_{\alpha,T})$$

where U_T ranges over all the étale neighborhoods of t_T in $X_\alpha \times_k k_T$. Putting everything together we get

$$R^q\Psi^{mer}_{\eta_*}(F)_t \simeq \varinjlim_{T\in\mathcal{I}} \varinjlim_{U_T} H^q(\mathrm{j}_T^{-1}U_T, F_{\alpha,T}). \tag{14}$$

4.3. Swan conductor. In this section we establish some basic facts about the linear representations of the local fundamental group. The representations we have in mind are continuous group homomorphisms

$$\rho : \pi_1^{mer}(\eta_\infty) \to GL(n, A)$$

where A is a ring of a certain kind. Rather than axiomatizing the properties of A which we need, we give a list of typical rings with which we will deal in our applications. Hence in the sequel A will be either

1) a finite extension $\mathbb{E}_\lambda$ of the completion $\mathbb{B}_\ell$ of the field $\mathbb{Q}_\ell(\mu_{p^\infty})$ (taken with its natural topology), or

2) the ring of integers $\mathbb{O}$ of $\mathbb{E}_\lambda$, which is a complete discrete valuation ring whose residue field we call $\mathbf{F}$ and whose maximal ideal is $\mathfrak{m}$ (with the $\mathfrak{m}$-adic topology), or

3) any of the artinian rings $\mathbb{O}_n$ defined as $\mathbb{O}/\mathfrak{m}^n$ (with the discrete topology).

Lemma 8. *Suppose that A is of type (1) or (2). Let V be a finite rank free A-module with a continuous P_{mer}-action $\rho : P_{mer} \to GL(V)$. Then the image of ρ consists of semisimple elements.*

Proof. Let r be the rank of V over A and let G be the Zariski closure of the image of P_{mer} in $GL(r, E_\lambda)$. Since P_{mer} is commutative, up to replacing $\mathbf{E}_\lambda$ with some finite extension, G decomposes as a direct product TU, where T is a torus and U is a unipotent group. It suffices to show that $U = 0$. The group U is isomorphic to an affine space $\mathbf{A}^m_{E_\lambda}$. Let L be a lattice inside U (*i.e.* a compact $\mathbb{O}$-submodule of maximal rank). Then clearly $\bigcap_n \ell^n L = 0$ and $\rho^{-1}(Id_V) = \bigcap_n \ell^n \rho^{-1}(L)$. But since multiplication by ℓ is an automorphism in P_{mer}, we derive $\rho^{-1}(L) = \mathrm{Ker}\rho$. Let $u \in U$ be any element. Then for some big enough $n \in \mathbf{N}$ we have $\ell^n u \in L$, and the claim follows easily. □

Proposition 14. *Let A and $\rho : P_{mer} \to GL(V)$ be as in lemma 8. Then P_{mer} acts on V through a discrete quotient.*

Proof. First of all, the lemma implies that, up to replacing $\mathbb{E}_\lambda$ with some finite extension, V decomposes as a direct sum of one-dimensional P_{mer}-modules and hence we can assume that V itself is one-dimensional. Then the P_{mer}-action is given by a continuous character $\chi : P_{mer} \to \mathbf{E}_\lambda^\times$. If we compose with the valuation map $\mathbf{E}_\lambda^\times \to \mathbf{Z}$, we obtain a continuous map $P_{mer} \to \mathbf{Z}$. But all the discrete quotients of P_{mer} are torsion groups, hence this map is trivial, and χ lands into $\mathbb{O}^\times$. The group $\mathbb{O}^\times$ is homeomorphic to a direct sum $\mathbf{F}^\times \oplus \mathbb{O}$, by means of the identification of $\mathbf{F}^\times$ with the group of roots of unit inside $\mathbb{O}$, and the logarithm map $\mathbb{O} \simeq 1 + \ell\mathbb{O}$. It suffices to show that the induced map $\pi : P_{mer} \to \mathbb{O}$, obtained by projecting on the second factor, is the trivial group homomorphism. For this, we write: $\pi^{-1}(0) = \pi^{-1}(\bigcap_n \ell^n \mathbb{O}) = \bigcap_n \ell^n \pi^{-1}(\mathbb{O})$. Since multiplication by ℓ is an automorphism in P_{mer}, the claim follows. □

To take care of torsion rings A of type (3) we need some preparation. Let G be some group and $\rho : G \to GL(V)$ a representation of G on some finite rank free A-module V. For any character $\chi \in \mathrm{Hom}(G, \mu_{p^\infty})$ we let V_χ the maximal submodule of V on which G acts as χ *i.e.*

$$\rho(g)v = \chi(g)v \qquad (g \in G, \quad v \in V_\chi).$$

Notice that this definition makes sense since any A of type (3) contains the multiplicative subgroup μ_{p^∞}.

Proposition 15. *Assume that A is of type (3). Let G be a finite commutative p-group and $\rho : G \to GL(V)$ a representation of G as above. Then there is a canonical decomposition*

$$V \simeq \bigoplus_{\chi \in Hom(G,\mu_{p^\infty})} V_\chi.$$

Proof. Let g be some element in G. Let p^n be the exponent of G and choose a primitive root of unity $\zeta \in \mu_{p^\infty}$ of order p^n. First of all we remark that all elements of the form $\zeta^i - \zeta^j$ ($i \not\equiv j \bmod p^n$) are invertible in A. This follows easily from [Wa] proposition 2.1. For $1 \leq j \leq p^n$ we define

$$C_j = \prod_{i \neq j}(\zeta^j - \zeta^i).$$

Clearly we have

$$\prod_{1 \leq i \leq p^n} (\rho(g) - \zeta^i) = 0 \tag{15}$$

as an element of $\mathrm{End}_A(V)$. Define the element $\pi_j \in \mathrm{End}_A(V)$ by setting

$$\pi_j = C_j^{-1} \prod_{j \neq i}(\rho(g) - \zeta^i).$$

From (15) it follows that the image of π_j lands into the submodule $V_{g,\zeta^j} = \mathrm{Ker}(\rho(g) - \zeta^j)$.

Lemma 9. *The morphism*

$$\oplus_{1 \leq j \leq p^n} \pi_j : V \to \bigoplus_{1 \leq j \leq p^n} V_{g,\zeta^j}$$

is injective.

Proof of the lemma: For any subset $S \subset \{1, 2, ..., p^n\}$ define more generally

$$\pi_S = \prod_{i \notin S}(\rho(g) - \zeta^i).$$

For any such S and any two distinct elements i, j in the complement of S we show that

$$\mathrm{Ker}\pi_{S \cup \{i\}} \cap \mathrm{Ker}\pi_{S \cup \{j\}} = \mathrm{Ker}\pi_S. \tag{16}$$

The lemma will follow easily from (16) and a simple induction argument.

Let $v \in \mathrm{Ker}\pi_{S \cup \{i\}} \cap \mathrm{Ker}\pi_{S \cup \{j\}}$ and set $w = \pi_S(v)$. Then we have

$$(\rho(g) - \zeta^i)w = (\rho(g) - \zeta^j)w = 0$$

which implies $(\zeta^i - \zeta^j)w = 0$. Since $(\zeta^i - \zeta^j)$ is invertible, this yields $w = 0$ and proves (16). □

Next we show that the composition

$$\begin{array}{ccccc} \bigoplus_j V_{g,\zeta^j} & \longrightarrow & V & \xrightarrow{\oplus_j \pi_j} & \bigoplus_j V_{g,\zeta^j} \\ (v_1, ..., v_{p^n}) & \longmapsto & \sum_j v_j & & \end{array}$$

is the identity map. This is a direct calculation:

$$\begin{aligned}\pi_j(\textstyle\sum_k v_k) &= C_j^{-1}\prod_{j\neq i}(\rho(g)-\zeta^i)(\textstyle\sum_k v_k)\\ &= \textstyle\sum_k C_j^{-1}\prod_{j\neq i}(\rho(g)-\zeta^i)v_k\\ &= C_j^{-1}\prod_{j\neq i}(\rho(g)-\zeta^i)v_j\\ &= C_j^{-1}\prod_{j\neq i}(\zeta^j-\zeta^i)v_j\\ &= v_j.\end{aligned}$$

Together with lemma 9 this shows that V is isomorphic to the direct sum of G-stable A-modules $\bigoplus_j V_{g,\zeta^j}$.

Let $g_1, ..., g_m$ be a set of generators of G. To conclude the proof, it suffices to remark that, for any character $\chi \in \mathrm{Hom}(G, \mu_{p^\infty})$,

$$V_\chi = V_{g_1,\chi(g_1)} \cap \ldots \cap V_{g_m,\chi(g_m)}$$

and that this intersection of A-modules is a direct summand of V. □

Corollary 3. *Assume again that A is a ring of type (3). Let $\rho : P_{mer} \to GL(V)$ be a representation of P_{mer} into a finite rank free A-module V. Then there is a direct sum decomposition*

$$V \simeq \bigoplus_{\chi\in Hom(P_{mer},\mu_{p^\infty})} V_\chi.$$

Proof. The proof is a typical "filter" argument: since V has the discrete topology, ρ factors through a discrete quotient $\widetilde{P}$ of P_{mer}. Then $\widetilde{P}$ is a commutative p-power torsion group, and hence it is the direct limit of the filtered family $\mathcal{F}$ of its finite subgroups.

We argue by induction on the rank r of V. Thanks to proposition 15 we can choose for each subgroup $S \in \mathcal{F}$ a character $\chi_S : S \to \mu_{p^\infty}$ and a non-zero G-stable direct summand V_S in V such that

1) $\rho_{|V_S} = \chi_S$;

2)$V_T \subset V_S$ and χ_T restricts to χ_S on S for any $S, T \in \mathcal{F}$ such that $S \subset T$.

Then, since the rank r is finite, the submodule

$$V' = \varprojlim_{S\in\mathcal{F}^\circ} V_S$$

is non-zero and it is clearly a direct summand in V. On V' the action ρ is given by the character $\varprojlim_{S\in\mathcal{F}^\circ} \chi_S$ and the complement of V' has rank strictly less than r, which shows the claim. □

Suppose A is a ring of any of the types above. Let M be a one-dimensional A-representation of P_{mer}. Since all the discrete quotients of P_{mer} are p-power torsion groups, the proposition implies that P_{mer} acts on M through a continuous character $\chi : P_{mer} \to \mu_{p^\infty}$.

This character corresponds to an element f_χ of the algebra $\mathcal{A}$ introduced in section 3.

Definition 13. *The degree of the element f_χ is called the slope of the P_{mer}-module M and it is denoted by $\lambda(M)$. In particular, the slope of a simple P_{mer}-module is always a rational number.*

Finally, let V an arbitrary P_{mer}-module free of finite A-rank. If A is of type (3), corollary 3 shows that V decomposes as direct sum of P_{mer}-stable rank one A-submodules.

The same holds for a ring of type (1) or (2), at least after replacing A by a A' still of the same type and contained into a finite extension $\mathbf{E}'_\lambda$ of $\mathbf{E}_\lambda$.

Remark: It is easy to verify that the direct sum of the simple components of $V_{A'} = V \otimes_A A'$ which have the same fixed slope λ is a submodule $V_{A',\lambda}$ of $V_{A'}$ which is already defined over A, *i.e.* there is a submodule V_λ of V such that $V_{A',\lambda} = V_\lambda \otimes_A A'$.

Hence we denote by $\Lambda(V)$ the set of the slopes of the simple rank one components of $V_{A'}$; clearly $\Lambda(V)$ is a finite subset of $\mathbf{Q}$, whose elements are called the slopes of V. Gathering the simple components of $V_{A'}$ which have same slope, and using the remark above we obtain a canonical decomposition of V as direct sum

$$V = \bigoplus_{\lambda \in \Lambda(V)} V_\lambda$$

where each V_λ is purely of slope λ.

Definition 14. *The Swan conductor $sw(V)$ of a P_{mer}-module V, is the rational number*

$$sw(V) = \Sigma_{\lambda \in \Lambda(V)} \lambda \cdot rk_A V_\lambda.$$

The next result is our version of the Hasse-Arf theorem.

Theorem 9. *Let V be a finite rank free A-module with an action of $\pi_1^{mer}(\eta_\infty)$. Then $sw(V)$ is a positive integer.*

Proof. For an element $f(x^{1/N}) \in \mathcal{A}$ let us denote by M_f the one-dimensional A-module on which P_{mer} acts through the character f. Then, at least after replacing A by a finite extension, the P_{mer}-module V has a decomposition of the kind

$$V \simeq \bigoplus_{f \in S} M_f^{n_f}$$

for some finite set S of elements of $\mathcal{A}$. Let $\gamma \in \pi_1^{alg}(\eta_\infty)$ be any element. We can define a new action of P_{mer} on V, by setting

$$(p, v) \mapsto \gamma p \gamma^{-1}(v) \qquad (p \in P_{mer}, \quad \gamma \in \pi^{alg}(\eta_\infty)).$$

Let V^γ be the module V with the new P_{mer}-action. Since $\gamma P_{mer} \gamma^{-1} = P_{mer}$ as subgroups of $\pi_1^{mer}(\eta_\infty)$, it follows that $V^\gamma \simeq V$. On the other hand, we can write

$$M_f^\gamma = M_{f^\gamma}$$

where $f^\gamma \in \mathcal{A}$ denotes an element of the form $f(\zeta x^{1/N})$ for some $\zeta \in \mu_N$. Hence we see that the set S must be stable under the substitution $f \mapsto f^\gamma$ for any γ as above. Suppose that N has been chosen minimal among the integers such that we can write f as a polynomial in $x^{1/N}$. Then it is easy to see that the orbit $\{f^\gamma \mid \gamma \in \pi_1^{alg}(\eta_\infty)\}$ consists of exactly N elements. On the other hand, $\lambda(M_f) = \lambda(M_{f^\gamma})$ is a rational number of the form n/N ($n \in \mathbf{N}$). The claim follows directly from these facts. □

5. The Lubin-Tate torsor

In this chapter we introduce and study the sheaf that plays the role covered by the Lang torsor in positive characteristic. I believe the name "Lubin-Tate torsor" is appropriate enough for this object. We return to the setup of chapter 2: here k is a one-dimensional local field of zero characteristic, i.e. a p-adic field.

Let F be a fixed Lubin-Tate group and let $\Delta(0,\rho)$ be the open disc of the affine line centered at the origin and of radius ρ. Then $\Delta(0,\rho)$ is an analytic variety and we can regard F as an analytic map $\Delta(0,1)\times\Delta(0,1)\to\Delta(0,1)$; the functional identities for F say that $\Delta(0,1)$ becomes a commutative analytic group with addition given by the power series F. Similarly, the logarithm $\boldsymbol{\lambda}_F$ defines a morphism of analytic groups $\boldsymbol{\lambda}_F:\Delta(0,1)\to\mathbf{A}^1_k$.

5.1. Construction of the torsor.

Lemma 10. *The logarithm $\boldsymbol{\lambda}_F:\Delta(0,1)\to\mathbf{A}^1_k$ is an étale covering of $\mathbf{A}^1_k$.*

Proof. Let $\mathbf{A}^1_k=\cup_{r>0}D_r$ be the covering of the affine line by closed discs of radius r centered at the origin. Denote by E_r the connected component of $\boldsymbol{\lambda}^{-1}(D_r)$ containing 0.

From remark (a) following theorem 3 we get an equality of formal power series: $\boldsymbol{\lambda}\circ[\pi]^n_f=\pi^n\cdot\boldsymbol{\lambda}$. By analytic continuation, this formal identity gives rise to a commutative diagram of analytic maps:

$$\begin{tikzcd}
\Delta(0,1) \arrow[r, "{[\pi]^n_f}"] \arrow[d, "\lambda"'] & \Delta(0,1) \arrow[d, "\lambda"'] & \Delta(0,\rho_1) \arrow[l] \arrow[d, "\lambda"'] \\
\mathbf{A}^1 \arrow[r, "\pi^n"] & \mathbf{A}^1 & \mathbb{G}_a(\rho_1). \arrow[l]
\end{tikzcd}$$

We remark that, for sufficiently large n_r, E_r is the connected component of the inverse image of $e_F(\pi^n_r D_r)$ by $[\pi^n_r]$. Looking at the diagram above, we see that the restriction of $\boldsymbol{\lambda}$ to E_r is a finite map, hence E_r is an affinoid domain in $\Delta(0,1)$ for all r and $\Delta(0,1)=\cup_{r>0}E_r$. Note that for $r<s$, E_s is a closed neighborhood of E_r. It follows easily that $\boldsymbol{\lambda}$ is étale and surjective if and only if the induced maps $E_r\to D_r$ are étale and surjective for all r.

Given $r>0$, choose an integer n_r large enough such that $[\pi]^{n_r}_f(E_r)\subset\Delta(0,\rho_1)$. By theorem 3, the power series e_F converges on $\Delta(0,\rho_1)$. This means that e_F defines a morphism on the quasiaffinoid space $\Delta(0,\rho_1)$, and therefore the restriction of $\boldsymbol{\lambda}$ to $\Delta(0,\rho_1)$ is an isomorphism of quasiffinoid spaces. It follows that $\boldsymbol{\lambda}:E_r\to D_r$ is an étale covering if and only if $[\pi]^{n_r}_f:E_r\to\pi^{n_r}\cdot D_r$ is an étale covering. Let $g\in\mathfrak{F}_\pi$ be any other power series; the homomorphism $[1]_{f,g}:\Delta(0,1)\to\Delta(0,1)$ of quasiaffinoid spaces has an inverse $[1]_{g,f}$ and therefore it is an isomorphism. From theorem 1.(b) we see that $[1]_{f,g}\circ[\pi]_f\circ[1]_{g,f}=[\pi]_g$. Therefore it suffices to prove that for some $g\in\mathfrak{F}$ the morphism $g=[\pi]_g:\Delta(0,1)\to\Delta(0,1)$ is finite and étale. Then we select $g(Z)=\pi Z+Z^q$. Now consider the map of schemes $\mathbf{A}^1_k\to\mathbf{A}^1_k$ defined by the polynomial $g(Z)$: this map ramifies over a finite set of points $x_1,\dots,x_m\in\mathbf{A}^1_k(\overline{k})=\overline{k}$, and using the jacobian criterion one checks easily that $|x_i|\geq 1$ for all i. On the complement of $x_1,\dots,x_n$, g restricts to an étale covering $U\to\mathbf{A}^1_k-\{x_1,\dots,x_n\}$. By proposition 3.3.11 of [B1], it follows that the map $g^{an}:U^{an}\to{\mathbf{A}^1_k}^{an}-\{x_1,\dots,x_n\}$ is also an étale covering. But clearly $[\pi]_g$ is obtained from g^{an} by base change to $\Delta(0,1)\subset{\mathbf{A}^1_k}^{an}$, and the lemma follows from corollary 3.3.8 of [B1]. $\square$

Remark: the proof of the lemma shows in particular that the restriction of the analytic covering $\boldsymbol{\lambda} : \Delta(0,1) \to \mathbf{A}^1_k$ to any bounded disc $\Delta(0,\rho) \hookrightarrow \mathbf{A}^1_k$ factors as a trivial (split) covering followed by an algebraic covering of finite degree.

For any positive integer n, let $k_n = k(G_n)$, $k_\infty = \cup_{n>0} k_n$ and $\widehat{k}_\infty$ the completion of k_∞.

It has already been remarked that $G_\infty = \mathrm{Ker}(\boldsymbol{\lambda} : \Delta(0,1) \to \mathbf{A}^1_{\widehat{k}^a})$. In particular, this kernel is contained in k_∞.

As usual we obtain a sheaf of sets (in the rigid etale topology) over $\mathbf{A}^1_k$ by taking the étale local sections of the morphism $\boldsymbol{\lambda}$; let us denote by ϕ this sheaf.

For any given complete field extension E of k, there is a base change map $p : \mathbf{A}^1_E \to \mathbf{A}^1_k$ and we can form the pull back $\phi_E = p^*\phi$. For our purposes, the really useful sheaf is $\phi_{\widehat{k}_\infty}$; for brevity we will denote it simply by ϕ_∞.

Definition 15. *The sheaf ϕ_∞ acquires a translation action of the discrete group G_∞, which as usual makes it into a G_∞-torsor. We call ϕ_∞ the Lubin-Tate torsor.*

Let Λ be some torsion ring in which the residue characteristic of k is invertible, and $\psi : G_\infty \to \Lambda^\times$ be a character of G_∞. We can form the associated sheaf

$$\mathcal{L}_\psi = \phi_\infty \times_\psi \Lambda$$

which is a rank one local system of Λ-modules on $\mathbb{G}_a$.

A note about notation: for a map $f : X \to \mathbb{G}_a$ sometime we will write $\mathcal{L}(f)$ in place of $f^*\mathcal{L}$. Also, if F is a complete extension of k_∞, the base change map $\pi : \mathbb{G}_{a,F} \to \mathbb{G}$ gives us a new sheaf $\mathcal{L}_F = \pi^*\mathcal{L}$. If it is clear from the context which base field we have in mind, we will omit the subscript F. Given a linear coordinate t on $\mathbb{G}_a$, sometime we will write $\mathbb{G}_a(\rho,t)$ for the analytic group obtained by restricting the addition law of $\mathbb{G}_a$ to the disc $\Delta(0,\rho) = \{x \in \mathbb{G}_a,\ |t(x)| < \rho\}$.

We list here some elementary properties of $\mathcal{L}_\psi$, that follow from the general yoga of torsors. Let $m : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ be the addition map, and $\mathrm{pr}_1, \mathrm{pr}_2 : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ the projection maps on the first and second factor. Then $\mathcal{L}_\psi$ comes with:

LT1) a rigidification at the origin:

$$\mathcal{L}_{\psi,\{0\}} \simeq \Lambda_{\psi,\{0\}}$$

LT2) a trivialization:

$$m^*\mathcal{L}_\psi \otimes \mathrm{pr}_1^*\mathcal{L}_\psi^{-1} \otimes \mathrm{pr}_2^*\mathcal{L}_\psi^{-1} \simeq \Lambda_{,\mathbb{G}_a\times\mathbb{G}_a}$$

compatible with the rigidification at the origin $\{0,0\}$ induced by **LT1**.

LT3) In particular:

$$\mathcal{L}_{\psi^{-1}} \simeq \mathcal{L}_\psi^{-1}.$$

We will denote by $\rho(\psi,t)$ the supremum of all real numbers ρ such that $\mathcal{L}_\psi$ trivializes on $\mathbb{G}_a(\rho,t)$. If t happens to be the same parameter which we chose to give the power series expansion for the morphism $\boldsymbol{\lambda}$, we get $\rho(\psi,t) \geq \rho_1$ and equality holds if and only if ψ is injective. Moreover $\rho(\psi,t) = \infty$ if and only if ψ is trivial.

Before moving on, we should remark that the difference between one choice or another of the underlying Lubin-Tate group, is purely arithmetic. By this we mean the following: suppose that F, F' are two Lubin-Tate groups, and G_∞, G'_∞ the respective torsion groups. Take two characters ψ, ψ' of G_∞ and respectively G'_∞. Then over the completion of $k(G_\infty)$ (resp. of $k(G'_\infty)$) we obtain the Lubin-Tate torsor $\mathcal{L}_\psi$ (resp. $\mathcal{L}_{\psi'}$). We can pull-back both of them to the common overfield $\widehat{k}^a$, and there we have

Proposition 16. *For $a \in \widehat{k}^a$, let $\mu_a : \mathbb{A}^1_{\widehat{k}^a} \to \mathbb{A}^1_{\widehat{k}^a}$ be the morphism $x \mapsto ax$. Then there exists $a \in \widehat{k}^a$ such that, with the above notation*

$$\mathcal{L}_{\psi,\widehat{k}^a} \simeq \mu_a^* \mathcal{L}_{\psi',\widehat{k}^a}.$$

Proof. I am grateful to G. Faltings for furnishing the following explanation. It suffices to compare a general Lubin-Tate torsor F with the classical $\mathbb{G}_m$. To distinguish the two analytic groups, call Δ_F (resp. $\Delta_{\mathbb{G}_m}$) the analytic space $\Delta(0,1)$ endowed with the group law F (resp. the multiplicative group law). The torsion of $\mathbb{G}_m$ is of course μ_{p^∞}. To prove the claim it suffices to show that the group homomorphism $\psi : G_\infty \to \mu_{p^\infty}$ is induced by a morphism of analytic groups $\widetilde{\psi} : \Delta_F \to \Delta_{\mathbb{G}_m}$, because in that case we can find out the right $a \in \widehat{k}^a$ by noticing that $\lambda_{\mathbb{G}_m} \circ \widetilde{\psi}\lambda_F^{-1}$ is an endoomorphism of $\mathbb{G}_a$, hence of the form μ_a for a certain a.. Now, the map ψ induces a map on the Tate groups $\widehat{\psi} : T(F) \to T(\mathbb{G}_m)$, or what is the same, an element of $T(G)^* \simeq T(G^t)$ (here G^t is the Cartier dual group of G). This is the same as giving a compatible system of group scheme homomorphisms

$$\widehat{\psi}_n : F[n] \to \mu_{p^n} \qquad (n > 0)$$

defined over $\mathcal{O}_{k^a}$. In turns, this is a map of p-divisible group schemes $F[p^\infty] \to \mu_{p^\infty}$ which determines the needed morphism $\widetilde{\psi} : \Delta_F \to \Delta_{\mathbb{G}_m}$ over $\widehat{\mathcal{O}}_{k^a}$. □

The proof of the following proposition is taken from [SGA4$\frac{1}{2}$], Sommes trig. We reproduce it here to stay on the safe side.

Proposition 17. *Let $\psi : G_\infty \to \Lambda^\times$ be a non-trivial character. Then:*

$$H_c^*(\mathbb{G}_a(\rho,t)_{\widehat{k}_\infty}, \mathcal{L}_\psi) = 0$$

for all $\rho > \rho(\psi,t)$.

Proof. Let Δ_ρ be the connected component of $\lambda^{-1}(\mathbb{G}_a(\rho,t))$ containing 0. For a $\widehat{k}_\infty$-rational point x of Δ_ρ, let τ_x be the translation $\tau_x(g) = g[+_f]x$ on Δ_ρ, where $[+_f]$ is Lubin-Tate group law. Also, let τ'_y be the translation by $y \in \mathbb{G}_a$, with respect to usual addition law on $\mathbb{G}_a$. The formula $\lambda \circ \tau_x = \tau'_{\lambda(x)} \circ \lambda$ states that the pair $(\tau_x, \tau'_{\lambda(x)})$ is an automorphism of the diagram $\Delta_\rho \to \mathbb{G}_a(\rho,t)$.

Let $\psi(x)$ be the induced automorphism of $(\mathbb{G}_a(\rho,t), \mathcal{L}_\psi)$. For $x \in G_\infty$ this automorphism gives the identity on $\mathbb{G}_a(\rho,t)$, and multiplication by $\psi(x)^{-1}$ on $\mathcal{L}_\psi$.

Let $\psi_H(x)$ be the automorphism of $H_c^*(\mathbb{G}_a(\rho,t)_{\widehat{k}_\infty}, \mathcal{L}_\psi)$ induced by $\psi(x)$. Then $\psi_H(x)$ is multiplication by $\psi(x)^{-1}$. On the other hand, the following "homotopy" lemma shows that $\psi_H(x) = \psi_H(0)$. Since by hypothesis $\rho > \rho(\psi,t)$, we can find $x \in G(\rho,t) \cap G_\infty$ such that $1 - \psi(x)^{-1}$ is invertible; but we have seen that multiplication by $(1 - \psi(x)^{-1}) \neq 0$ is the zero map, therefore the claim follows.

Lemma 11 ("Homotopy" lemma). *Let X and Y be two rigid analytic varieties over a complete valued field F, with Y connected. Let $\mathcal{G}$ be a sheaf on X and (ψ, ϵ) a family of endomorphisms of $(X, \mathcal{G})$ parametrized by Y, i.e.:*

$$\begin{aligned} \psi : Y \times X &\longrightarrow Y \times X && \text{is a } Y\text{-morphism and} \\ \epsilon : \psi^* pr_2^* \mathcal{G} &\longrightarrow pr_2^* \mathcal{G} && \text{a morphism of sheaves.} \end{aligned}$$

Assume ψ is proper. For $y \in Y(F)$, let $\psi_H(y)^*$ the endomorphism of $H_c^*(X, \mathcal{G})$ induced by $\psi_y : X \longrightarrow X$ and $\epsilon_y : \psi_y^* \mathcal{G} \longrightarrow \mathcal{G}$. Then $\psi_H(y)^*$ is independent of y.

Proof. In fact, $R^p\mathrm{pr}_{1!}\mathrm{pr}_2^*\mathcal{G}$ is the constant sheaf on Y with stalk $H_c^p(X,\mathcal{G})$, and $\psi_H(y)^*$ is the fiber at y of the endomorphism :

$$R^p\mathrm{pr}_{1!}\mathrm{pr}_2^*\mathcal{G} \xrightarrow{\psi^*} R^p\mathrm{pr}_{1!}\psi^*\mathrm{pr}_2^*\mathcal{G} \xrightarrow{\epsilon} R^p\mathrm{pr}_{1!}\mathrm{pr}_2^*\mathcal{G}.$$

□

To apply the homotopy lemma to the present situation, we take $\psi : \Delta_\rho \times \mathbb{G}_a(\rho,t) \longrightarrow \Delta_\rho \times \mathbb{G}_a(\rho,t)$ defined by $\psi(x,y) = (x, y+\lambda(x))$. □

5.2. The character induced by Galois action. We conclude this chapter with some observations about the Galois action on $\mathcal{L}_\psi$. Let $\overline{\phi}$ be the pull back of ϕ_∞ to $\mathbf{A}^1_{\widehat{k^a}}$; by transport of structure we get a natural action of $\mathrm{Gal}(k^a/k_\infty)$ on $\overline{\phi}$, covering the action on $\mathbf{A}^1_{\widehat{k^a}}$. This action is inherited by $\mathcal{L}_{\psi,\widehat{k^a}}$. In particular, if p is a k_∞-rational point of $\mathbf{A}^1_{\widehat{k^a}}$, then the stalk $\mathcal{L}_{\psi,p}$ becomes a representation of $\mathrm{Gal}(k^a/k_\infty)$ of rank one. For any $n \leq \infty$, let k_n^{ab} denote the maximal abelian extension of k_n. It is clear that the action on $\mathcal{L}_{\psi,p}$ factors through $\mathrm{Gal}(k_\infty^{ab}/k_\infty)$. I do not know the complete structure of $\mathrm{Gal}(k_\infty^{ab}/k_\infty)$; in particular I don't know whether there is a canonical generator that takes the place of the Frobenius element as in the finite field case. Instead we make the following:

Definition 16. *Let k^{ur} be the maximal unramified extension of k. Clearly $k^{ur} \subset k_\infty^{ab}$ and $k^{ur} \cap k_\infty = k$. We say that an element $\sigma \in Gal(k_\infty^{ab}/k_\infty)$ is a Frobenius element if the image of σ in $Gal(k^{ur}/k)$ is the canonical Frobenius generator.*

Our aim is to give an explicit formula for the trace $\mathrm{Tr}(\sigma, \mathcal{L}_{\psi,p})$ of the endomorphism induced by the Frobenius element σ on the stalk of $\mathcal{L}_\psi$ at the point p. We start with two elementary lemmas:

Lemma 12. *The map $p \mapsto Tr(\sigma, \mathcal{L}_{\psi,p})$ is a continuous group homomorphism $Tr_\sigma : k_\infty \to \Lambda^\times$.*

Proof. It follows easily from **LT1** and **LT2** that the map Tr_σ is a group homomorphism. Moreover, it follows from lemma 3 that the restriction of ϕ_∞ to $\Delta(0,\rho_1)$ is the trivial G_∞-torsor; therefore the restriction of $\mathcal{L}_\psi$ to the same disc is a trivial line bundle, and we conclude that the kernel of Tr_σ contains this entire disc, i.e. the map is continuous. □

Lemma 13. $k_\infty^{ab} = \cup_{n<\infty} k_n^{ab}$.

Proof. It is clear that $k_n^{ab} \subset k_\infty^{ab}$. On the other hand, let $x \in k_\infty^{ab}$ and let $x_1, ..., x_m$ be the orbit of x for the action of the full Galois group $\mathrm{Gal}(k^a/k)$; take n big enough such that $[k_n(x_1,...,x_m) : k_n] = [k_\infty(x_1,...,x_m) : k_\infty]$. Then there is a natural isomorphism $\mathrm{Gal}(k_n(x_1,...,x_m)/k_n) \simeq \mathrm{Gal}(k_\infty(x_1,...,x_m)/k_\infty)$, and this last group is abelian, being a quotient of $\mathrm{Gal}(k_\infty^{ab}/k_\infty)$. □

It follows from the lemma that the choice of a Frobenius element σ in $\mathrm{Gal}(k_\infty^{ab}/k_\infty)$ is equivalent to the choice of a sequence $\sigma_0, \sigma_1, ...$ of liftings of Frobenius $\sigma_n \in \mathrm{Gal}(k_n^{ab}/k_n)$ such that the restriction of σ_{n+1} to k_n^{ab} acts as σ_n. Let $\beta_n \in k_n$ such that the Artin symbol $(\beta_n, k_n^{ab}/k_n)$ acts on k_n^{ab} as σ_n. Then by local class field theory, it follows $\mathrm{Nm}_{k_{n+1}/k_n}(\beta_{n+1}) = \beta_n$. Also, by Lubin-Tate theory it follows $\beta_0 = \pi$.

Conversely, the choice of a compatible system of elements $\beta_n \in k_n$ as before is equivalent to the choice of a Frobenius element σ.

For the next result we need some notation. First of all we select for each positive integer n:

1) a generator v_n of G_n as an k^o-module, such that $[\pi^{m-n}]_f(v_m) = v_n$;

2) an element $\beta_n \in k_n$ such that the sequence of these elements satisfies the compatibility condition above, and corresponds to the choice of a Frobenius element σ_β;

3) a power series $b_n(z) = z \cdot r_n(z)$, where $r(z) \in k^\circ[[z]]$ satisfies $r(0) \neq 0$ and such that $b_n(v_n) = \beta_n$.

Finally, let T_n be the trace map from k_n to k.

Theorem 10. *Let p be a point in $\mathbf{A}^1_{\widehat{k^a}}(k_\infty) = k_\infty$, and choose an integer n such that:*

(a) $|\pi^n p| < \rho_1$;

(b) $[k(p) : k] \leq n$.

Let m be any integer $\geq 2n+1$. Then, with reference to the notation above:

$$Tr(\sigma_\beta, \mathcal{L}_{\psi,p}) = \psi\left(\left[\frac{1}{\pi^{m-n}} T_m\left(\frac{p}{\lambda'(v_m)} \frac{db_m}{dz}\Big|_{z=v_m}\right)\right]_f (v_n)\right).$$

Proof. First of all, notice that the group $\mathrm{Gal}(k^a/k_\infty)$ acts also on $\Delta(0,1)_{\widehat{k^a}}$ in such a way that the logarithm becomes an equivariant morphism. Let $q \in \lambda^{-1}(p)$. Let $\tilde{\sigma}$ be any lifting of σ_β to $\mathrm{Gal}(\overline{k}/k_\infty)$; then essentially by definition we have:

$$\mathrm{Tr}(\sigma_\beta, \mathcal{L}_{\psi,p}) = \psi(\tilde{\sigma}(q)[-_f]q) \tag{17}$$

(where $[-]_f$ denotes subtraction in the formal group). Obviously this formula is independent of the choices involved. Take n such that (a) is satisfied; by inspecting the proof of lemma 10 and the remark that follows it, we obtain:

$$\lambda^{-1}(p) = [\pi^n]_f^{-1}(e(\pi^n p))[+_f]G_\infty.$$

In particular we can take $q \in [\pi^n]_f^{-1}(e(\pi^n p))$ in equation (17). We recall now the definition of the generalized Kummer pairing, introduced by Fröhlich in [Fr]: let $F(k_n)$ be the subgroup of $\Delta(0,1)(k_\infty)$ consisting of the elements rational over k_n; then there is a bilinear map:

$$(\ ,\ )_n^F : F(k_n) \times k_n^\times \longrightarrow G_n$$

defined as follows. If $\beta \in k_n^\times$, let τ_β be the element of the $\mathrm{Gal}(k_n^{ab}/k_n)$ which is attached to β by the Artin symbol. If $\alpha \in F(k_n)$, choose γ in $\Delta(0,1)(\overline{k})$ such that $[\pi^n]_f(\gamma) = \alpha$. Then $(\alpha, \beta)_n^F = \tau_\beta(\gamma)[-_f]\gamma$. Clearly, if we take n such that both (a) and (b) are satisfied, the right side in formula (17) translates as $\psi((e(\pi^n p), \beta_n)_n^F)$.

Then the formula of the theorem follows immediately from theorem 1 of [Wi]. □

5.3. Semilinear Galois action. Since the sheaf ϕ is already defined over k, it is natural to expect the full Galois group $\mathrm{Gal}(k^a/k)$ to act on $\mathcal{L}_\psi$. In this section we show that this is indeed the case, at least when the Lubin-Tate formal group under consideration is the classical multiplicative group $\mathbb{G}_m$. The action thus obtained will not be linear, but rather semilinear in a precise sense. In this way, our theory acquires a "p-adic flavour" which is unusual in an ℓ-adic setting. Of course there should be a parallel p-adic Fourier transform over p-adic fields, where the full meaning of this semilinear action is revealed. We feel that the present limitation to ℓ-adic (or ℓ-torsion) coefficients is only due to our current incomplete understanding, and it will be eventually removed.

As announced, in this section we restrict to the Lubin-Tate group $\mathbb{G}_m$. Take a prime ℓ whose residue class generates $\mathbf{Z}/p^2$; by Dirichlet theorem on primes in arithmetic progressions, there are plenty of such ℓ. With this choice, the Galois group of $\mathbb{Q}_\ell(\mu_{p^\infty})$ over $\mathbb{Q}_\ell$ is easily seen to be isomorphic to $\mathbf{Z}_p^\times$. Let $\mathbf{O}$ be the ring of integers of $\mathbf{B}_\ell$.

The group G_∞ attached to $\mathbb{G}_m$ is just μ_{p^∞} and any character $\psi : \mu_{p^\infty} \to \mathbb{O}_n^\times$ lifts to a character $\widetilde{\psi} : \mu_{p^\infty} \to \mathbb{Q}_\ell(\mu_{p^\infty})^\times$; conversely, we can deal with $\widetilde{\psi}$ and then obtain ψ by projecting onto $\mathbb{O}_n^\times$. Clearly we can assign $\widetilde{\psi}$ by identifying the two copies of μ_{p^∞}, one in $\mathbb{Q}_\ell(\mu_{p^\infty})$ and the other in $\mathbb{Q}_p(\mu_{p^\infty})$. Such identification also induces a unique isomorphism χ between $\mathrm{Gal}(\mathbb{Q}_\ell(\mu_{p^\infty})/\mathbb{Q}_\ell)$ and $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$, given explicitly by the rule

$$\sigma(\widetilde{\psi}(g)) = \widetilde{\psi}(\chi(\sigma)g)$$

for all $\sigma \in \mathrm{Gal}(\mathbb{Q}_\ell(\mu_{p^\infty})/\mathbb{Q}_\ell$ and all $g \in \mu_{p^\infty}$. Another way of seeing this is as saying that $\widetilde{\psi}$ becomes $\mathcal{G}$-equivariant, if we endow $\mathbb{B}_\ell$ with the $\mathcal{G}$-action extended by continuity from $\mathbb{Q}_\ell(\mu_{p^\infty})$

$$(\sigma, x) \mapsto \chi(\sigma)x$$

for $\sigma \in \mathcal{G}$, $x \in \mathbb{B}_\ell$. Having equivariance for $\widetilde{\psi}$ is exactly the condition needed to transfer the $\mathcal{G}$-action from ϕ_∞ to the associated locally constant sheaf $\mathcal{L}_\psi$. The $\mathcal{G}$-action on $\mathcal{L}_\psi$ is not linear, but has the following semilinearity property:

$$\sigma(bs) = (\chi(\sigma)b) \cdot \sigma(s)$$

for any local section s of $\mathcal{L}_\psi$ and all $\sigma \in \mathcal{G}$, $b \in \mathbb{B}_\ell$.

Next, let K be any algebraic extension of k_∞, and $\widehat{K}$ its completion. There is a natural surjection $\pi : \mathrm{Gal}(K/k) \to \mathcal{G}$ and the $\mathcal{G}$-action on $\mathcal{L}_{\psi,\widehat{k}_\infty}$ lifts in a natural way to an action of $\mathrm{Gal}(K/k)$ on $\mathcal{L}_{\psi,\widehat{K}}$, which satisfies again the same semilinearity condition above (after replacing χ by its composition with π). For a detailed proof the reader is referred e.g. to proposition 1.4 of [B2].

Remark: in the algebraic setting, one usually introduces the topos $\mathbf{S}_X$ of sheaves of sets on the scheme X, and then, for any given ring Λ, assigns to $\mathbf{S}_X$ a structure of Λ-ringed topos. by selecting the ring object Λ_X defined by the *constant* sheaf on X with stalks isomorphic to Λ. As the above construction illustrates, in the étale analytic setting, the choice of the constant $\mathbb{O}_n$-sheaf is not the most natural: one should rather take the geometrically constant sheaf $\mathbb{O}_{n,X}$, twisted by the semilinear $\mathrm{Gal}(k^a/k)$-action defined in this section.

6. Fourier Transform

We are now ready to define the Fourier transform. With the set-up of the previous chapters, we only have to mimic the construction of the Deligne-Fourier transform. The proofs of most of the main properties reduce to routine verifications, carried out by applying projection formulas, proper base change theorem and Poincaré duality, exactly as in Laumon's paper.

6.1. Definition and main properties. We consider complexes of sheaves of $\mathbb{O}_n$-modules, where $\mathbb{O}_n$ is one of the rings of type (3) according to the list in section 4.3. Let $\mathcal{L}_\psi$ be the locally constant Lubin-Tate $\mathbb{O}_n$-sheaf of rank 1 associated to the Lubin-Tate group F defined over the field k, and the character $\psi : G_\infty \to \mathbb{O}_n^\times$. In this chapter and the following one, the base field is a complete extension E of k_∞.

Let S be an analytic variety over E and $\pi : \mathsf{E} \to S$ an analytic vector bundle (defined in the obvious way) of constant rank $r \geq 1$. We denote by $\pi' : \mathsf{E}' \to S$ the vector bundle dual to $\mathsf{E} \to S$, by $\langle , \rangle : \mathsf{E} \times_S \mathsf{E}' \to \mathbb{G}_{a,E}$ the canonical dual pairing and by $\mathrm{pr} : \mathsf{E} \times_S \mathsf{E}' \to \mathsf{E}$, $\mathrm{pr}' : \mathsf{E} \times_S \mathsf{E}' \to \mathsf{E}'$ the two canonical projections.

Definition 17. *The Fourier transform for* $\mathrm{E} \to S$, *associated to the character* ψ, *is the triangulated functor*

$$\mathcal{F}_\psi : \mathbb{D}^b(\mathrm{E}, \mathbb{O}_n) \longrightarrow \mathbb{D}^b(\mathrm{E}', \mathbb{O}_n)$$

defined by

$$\mathcal{F}_\psi(K^\cdot) = Rpr'_!(\mathcal{L}_\psi(\langle , \rangle) \otimes pr^* K^\cdot))[r].$$

We will usually drop the subscript ψ, unless we have to deal with more than one character at the same time. For later use we also introduce a special notation for a closely related functor: the operator $\mathcal{F}_{\psi,*}$ is given by the following formula:

$$\mathcal{F}_{\psi,*} = R\mathrm{pr}'_*(\mathcal{L}_\psi(\langle , \rangle) \otimes \mathrm{pr}^* K^\cdot))[r].$$

Next we would like to show that $\mathcal{F}$ shares some interesting properties with the Fourier transform defined over finite fields.

To start with, we state and establish involutivity: denote by $\pi'' : \mathrm{E}'' \to S$ the double dual vector bundle of E. The previous construction applies to E' and its dual E'' to give a Fourier transform $\mathcal{F}'$ (and the related functor $\mathcal{F}'_*$). We consider the composition:

$$\mathbb{D}(\mathrm{E}, \mathbb{O}_n) \xrightarrow{\mathcal{F}} \mathbb{D}^b(\mathrm{E}', \mathbb{O}_n) \xrightarrow{\mathcal{F}'} \mathbb{D}^b(\mathrm{E}'', \mathbb{O}_n).$$

Denote by $a : \mathrm{E} \xrightarrow{\sim} \mathrm{E}''$ the S-isomorphism defined by $a(v) = -\langle v, \cdot \rangle$. Also, let $\sigma : S \hookrightarrow \mathrm{E}$, $\sigma' : S \hookrightarrow \mathrm{E}'$, $\sigma'' : S \hookrightarrow \mathrm{E}''$ the zero sections of π, π', π'' respectively. We denote by $s : \mathrm{E} \times_S \mathrm{E} \to \mathrm{E}$ (resp. by $s' : \mathrm{E}' \times_S \mathrm{E}' \to \mathrm{E}'$) the addition law in the vector bundle $\mathrm{E} \to S$ (resp. in $\mathrm{E}' \to S$) and by $[-1] : \mathrm{E} \to \mathrm{E}$ the inverse map for this addition law.

Theorem 11. *There is a functorial isomorphism:*

$$\mathcal{F}' \circ \mathcal{F}(K^\cdot) \simeq a_*(K^\cdot)(-r)$$

for $K^\cdot \in \mathbb{D}^b(V, \mathbb{B}_\ell)$ *(The brackets denoting Tate twist, as usual).*

Proof. (Cp. [Lau2], theorem (1.2.2.1)). We fix some notation: let $\alpha : \mathrm{E} \times_S \mathrm{E}' \times_S \mathrm{E}'' \longrightarrow \mathrm{E}' \times_S \mathrm{E}''$ be defined as $\alpha(e, e', e'') = (e', e'' - a(e))$ and $\beta : \mathrm{E} \times \mathrm{E}'' \longrightarrow \mathrm{E}''$ as $\beta(e, e'') = e'' - a(e)$.

Consider the commutative diagram:

$$\begin{tikzcd}
& & \mathrm{E} \times_S \mathrm{E}'' \arrow[rrr, "\beta"] \arrow[dddll, bend right, "\mathrm{pr}"'] \arrow[dddrr, bend left, "\mathrm{pr}''"] & & & \mathrm{E}'' \\
& & \mathrm{E} \times_S \mathrm{E}' \times \mathrm{E}'' \arrow[u, "\mathrm{pr}_{13}"'] \arrow[dl, "\mathrm{pr}_{12}"'] \arrow[dr, "\mathrm{pr}_{23}"] \arrow[rrr, "\alpha"] & & & \mathrm{E}' \times \mathrm{E}'' \arrow[u, "\mathrm{pr}''"'] \\
& \mathrm{E} \times_S \mathrm{E}' \arrow[dl, "\mathrm{pr}"'] \arrow[dr, "\mathrm{pr}'"] & & \mathrm{E}' \times_S \mathrm{E}'' \arrow[dl, "\mathrm{pr}'"'] \arrow[dr, "\mathrm{pr}''"] & & \\
\mathrm{E} & & \mathrm{E}' & & \mathrm{E}'' &
\end{tikzcd}$$

where the two squares are fiber diagrams.

It follows easily from property **LT2** that

$$\mathrm{pr}_{12}^{*}\mathcal{L}(\langle,\rangle)\otimes \mathrm{pr}_{23}^{*}\mathcal{L}(\langle,\rangle) = \alpha^{*}\mathcal{L}(\langle,\rangle). \tag{18}$$

Then we have:

$$\begin{aligned}
\mathcal{F}'\circ\mathcal{F}(K^{\cdot}) &\simeq \mathcal{F}'(R\mathrm{pr}'_{!}(\mathcal{L}(\langle,\rangle)\otimes \mathrm{pr}^{*}K^{\cdot})[r]) && \\
&\simeq R\mathrm{pr}''_{!}(\mathcal{L}(\langle,\rangle)\otimes \mathrm{pr}'^{*}(R\mathrm{pr}'_{!}(\mathcal{L}(\langle,\rangle)\otimes \mathrm{pr}^{*}K^{\cdot})))[2r] && \\
&\simeq R\mathrm{pr}''_{!}(\mathcal{L}(\langle,\rangle)\otimes R\mathrm{pr}_{23!}\mathrm{pr}_{12}^{*}(\mathcal{L}(\langle,\rangle)\otimes \mathrm{pr}^{*}K^{\cdot}))[2r] && \text{(proper base change)}\\
&\simeq R\mathrm{pr}''_{!}R\mathrm{pr}_{23!}(\mathrm{pr}_{23}^{*}\mathcal{L}(\langle,\rangle)\otimes \mathrm{pr}_{12}^{*}\mathcal{L}(\langle,\rangle)\otimes \mathrm{pr}_{12}^{*}\mathrm{pr}^{*}K^{\cdot})[2r] && \text{(proj.formula)}\\
&\simeq R\mathrm{pr}''_{!}R\mathrm{pr}_{23!}(\alpha^{*}\mathcal{L}(\langle,\rangle)\otimes \mathrm{pr}_{12}^{*}\mathrm{pr}^{*}K^{\cdot})[2r] && \text{(by formula (18))}\\
&\simeq R\mathrm{pr}''_{!}R\mathrm{pr}_{13!}(\alpha^{*}\mathcal{L}(\langle,\rangle)\otimes \mathrm{pr}_{13}^{*}\mathrm{pr}^{*}K^{\cdot})[2r] && \text{(functoriality)}\\
&\simeq R\mathrm{pr}''_{!}(\mathrm{pr}^{*}K^{\cdot}\otimes R\mathrm{pr}_{13!}\alpha^{*}\mathcal{L}(\langle,\rangle))[2r] && \text{(projection formula)}\\
&\simeq R\mathrm{pr}''_{!}(\mathrm{pr}^{*}K^{\cdot}\otimes \beta^{*}R\mathrm{pr}''_{!}\mathcal{L}(\langle,\rangle))[2r]. && \text{(proper base change)}
\end{aligned}$$

To end the proof we apply to $\pi' : \mathbf{E}' \to S$ and $L = \mathbb{O}_n$ the lemma 14 below. □

Lemma 14. *For any $L^{\cdot} \in \mathbf{D}^b(S, \mathbb{O}_n)$ we have:*

$$\mathcal{F}(\pi^{*}L^{\cdot}[r]) \simeq \sigma'_{*}L^{\cdot}(-r).$$

Proof. By the projection formula:

$$\mathcal{F}(\pi^{*}L^{\cdot}[r]) = L^{\cdot}\otimes R\mathrm{pr}'_{!}\mathcal{L}(\langle,\rangle)[2r].$$

On the other hand, using proper base change, property **LT1** and proposition 17, we get:

$$\begin{aligned}
&\sigma'^{*}R\mathrm{pr}'_{!}\mathcal{L}(\langle,\rangle) = R\pi_{!}\mathbb{O}_n = \mathbb{O}_{n,S}(-r)[-2r]\\
&R\mathrm{pr}'_{!}\mathcal{L}(\langle,\rangle)_{|\mathbf{E}'-\sigma'(S)} = 0.
\end{aligned}$$

□

Corollary 4. *$\mathcal{F}$ is an equivalence of triangulated categories of $\mathbf{D}^b(\mathbf{E}, \mathbb{O}_n)$ onto $\mathbf{D}^b(\mathbf{E}', \mathbb{O}_n)$, with inverse $a^{*}\mathcal{F}'(-)(r)$.* □

In the case of the Fourier transform over a finite field, it is known moreover that $\mathcal{F}$ preserves the t-structure coming from middle perversity. As explained in [Lau2], this boils down to the equality of functors $\mathcal{F}_{\psi} = \mathcal{F}_{\psi,*}$. Even in absence of a theory of perverse sheaves for analytic varieties, we can still prove the corresponding statement:

Theorem 12. *The canonical map of "forget support" induces an isomorphism of functors:*

$$\phi : \mathcal{F}_{\psi}(-) \xrightarrow{\sim} \mathcal{F}_{\psi,*}(-).$$

Proof. Fix as usual a coordinate t on $\mathbf{A}^1_E$. First of all, an argument like at the beginning of the proof of [Ka-La] Theorème 2.4.1 reduces us to the case $r = 1$. Moreover, the assertion is obviously local on S, hence we can suppose that there exists a fiberwise linear isomorphism $x : \mathbf{E} \xrightarrow{\sim} \mathbf{A}^1_S$. Then also $\mathbf{E}'$ is trivialized by a coordinate $y : \mathbf{E}' \to \mathbf{A}^1_S$ such that $t(\langle e, e'\rangle) = x(e)y(e')$ for all local sections e, e'. Next we can find a unique $\overline{\mathbf{E}} \supset \mathbf{E}$ such that x extends to a (unique) isomorphism $\overline{x} : \overline{\mathbf{E}} \to \mathbf{P}^1_S$,

Let $j : \mathbf{E}\times_S \mathbf{E}' \hookrightarrow \overline{\mathbf{E}}\times_S \mathbf{E}'$ be the natural imbedding. Clearly it suffices to show that for all points of the type $(\infty, p) \in \overline{\mathbf{E}}\times_S \mathbf{E}'$

$$Rj_{*}(\mathcal{L}_{\psi}(\langle,\rangle)\otimes \mathrm{pr}^{*}K)_{(\infty,p)} = 0.$$

We consider the map $\tau : \mathsf{E}' \times_S \overline{\mathsf{E}} \times_S \mathsf{E}' \to \overline{\mathsf{E}} \times_S \mathsf{E}'$ defined as $(e_1', e, e_2') \mapsto (e, s'(e_1', e_2'))$. We form the fibre product diagram

$$\begin{tikzcd}
\mathsf{E}' \times_S \mathsf{E} \times_S \mathsf{E}' \arrow[r, "j^\circ"] \arrow[d, "\tau^\circ"'] & \mathsf{E}' \times_S \overline{\mathsf{E}} \times_S \mathsf{E}' \arrow[d, "\tau"] \\
\mathsf{E} \times_S \mathsf{E}' \arrow[r, "j"] & \overline{\mathsf{E}} \times_S \mathsf{E}'
\end{tikzcd}$$

and by smooth base change

$$\tau^* Rj_*(\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}^* K) \simeq Rj^\circ_* \tau^{\circ *}(\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}^* K).$$

In particular

$$Rj_*(\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}^* K)_{(\infty,p)} \simeq Rj^\circ_* \tau^{\circ *}(\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}^* K)_{(p,\infty,0)}.$$

Let $\mathcal{C}_S$ be the partially ordered set of all the étale neighborhoods of $(y(p), \infty, 0)$ in $\mathbf{A}^1_S \times_S \mathbf{P}^1_S \times \mathbf{A}^1_S$. We introduce the family $\mathcal{C}^\delta_S$ consisting of all the varieties of the form $W \times_E B$ such that

1) B is an open disc in $\mathbf{A}^1_E$, centered at zero, *i.e.* $B = \{a \in \mathbf{A}^1_E |\ |t(a)| < r_B\}$, and $W \xrightarrow{\phi} \mathbf{A}^1_S \times_S \mathbf{P}^1_S$ is an étale neighborhood of $(p, \infty) \in \mathbf{P}^1_S \times_S \mathbf{A}^1_S$;

2) the image $\phi(W)$ is contained in an open subset of the form $N(p) \times_E C$, with C an open disc in $\mathbf{P}^1_E$ of radius r_C around ∞ *i.e.* $C = \{a \in \mathbf{P}^1_E |\ |t(a)| > r_C^{-1}\}$ and $N(p)$ some fixed open neighborhood of $y(p)$ in $\mathbf{A}^1_S$;

3) the ratio r_B/r_C is equal to the constant δ.

Lemma 15. *For any real number $\delta > 0$ the family $\mathcal{C}^\delta_S$ is cofinal in $\mathcal{C}_S$.*

Proof. Let $\sigma : U \to \mathbf{A}^1_S \times_S \mathbf{P}^1_S \times_S \mathbf{A}^1_S$ be any étale open neighborhood of $(p, \infty, 0)$ and $q \in U$ a chosen lifting of $(p, \infty, 0)$. We have an induced map of germs

$$(U, q) \to (\mathbf{A}^1_S \times_S \mathbf{P}^1_S \times_S \mathbf{A}^1_S, (p, \infty, 0)).$$

Notice that the residue fields of the points $(p, \infty) \in \mathbf{A}^1_S \times_S \mathbf{P}^1_S$ and $(p, \infty, 0) \in \mathbf{A}^1_S \times_S \mathbf{P}^1_S \times_S \mathbf{A}^1_S$ are naturally isomorphic. Therefore, it follows from theorem 3.4.1 of [B1] that the germ (U, q) is isomorphic to a product of germs $(W', q') \times_E (\mathbf{A}^1_E, 0)$, where $\phi : (W', q') \to (\mathbf{A}^1_S \times_S \mathbf{P}^1_S, (p, \infty))$ is a morphism of germs with an étale representative. Concretely this means that there exists an open subset $V \subset W' \times_E \mathbf{A}^1_E$ with an open imbedding $V \hookrightarrow U$ which make the following diagram commute

$$\begin{tikzcd}
W' \times_E \mathbf{A}^1_E \arrow[d, "\phi \times 1_{\mathbf{A}^1}"'] & V \arrow[l] \arrow[r] & U \arrow[d, "\sigma"] \\
(\mathbf{A}^1_S \times_S \mathbf{P}^1_S) \times_E \mathbf{A}^1_E \arrow[rr, equal] & & \mathbf{A}^1_S \times_S \mathbf{P}^1_S \times_S \mathbf{A}^1_S.
\end{tikzcd}$$

Then proposition 3.7.8 of [B1] says that we can find inside V a subset of the form $W'' \times B'$ which fulfills condition (1) above. Conditions (2) and (3) are easy to fix, by taking open subsets $B \subset B'$ and $W \subset W''$. □

Fix a real number δ strictly greater than $\rho(\psi, t)$. Let $W \times_E B \in \mathcal{C}^\delta_S$ be any neighborhood as above and set $B_S = y^{-1}(B \times_E S)$, $C^\circ = C \cap \mathbf{A}^1_E$, $C^\circ_S = x^{-1}(C^\circ \times_E S)$, $W^\circ = W \times_{\mathbf{A}^1_S \times_S \mathbf{P}^1_S} (\mathsf{E}' \times_S \mathsf{E})$. Furthermore, we obtain obvious projection maps $\alpha : W^\circ \to C^\circ_S$ and $\beta : C^\circ_S \to C^\circ$.

In view of the lemma, the theorem will follow if we show that

$$H^i(W^\circ \times_S B_S, \tau^{\circ *}\mathcal{L}(\langle,\rangle) \otimes \tau^{\circ *}\mathrm{pr}^* K) = 0. \tag{19}$$

We remark the commutative diagram

$$(20)\qquad \begin{tikzcd} \mathsf{E}' \times_S \mathsf{E} \times_S \mathsf{E}' \arrow[r, "\mathrm{pr}_{12}"] \arrow[d, "\tau^\circ"'] & \mathsf{E}' \times_S \mathsf{E} \arrow[d, "\mathrm{pr}_2"] \\ \mathsf{E} \times_S \mathsf{E}' \arrow[r, "\mathrm{pr}"] & \mathsf{E}. \end{tikzcd}$$

Moreover, let $\mu : \mathsf{E}' \times_S \mathsf{E} \to \mathbf{A}^1_E$ be the map $(e', e) \mapsto \langle e, e' \rangle$; an easy application of the Yoga of torsors yields

$$(21)\qquad \tau^{\circ *}\mathcal{L}(\langle , \rangle) \simeq \mathrm{pr}_{23}^*\mathcal{L}(\langle , \rangle) \otimes \mathrm{pr}_{12}^*\mathcal{L}(\mu).$$

We apply the Leray spectral sequence for the morphism $\mathrm{pr}_{12} : W^\circ \times_E B \to W^\circ$.

Set $M = \mathrm{pr}_2^* K \otimes \mathcal{L}(\mu)$; then, in virtue of (20) and (21) it suffices to show that

$$R\mathrm{pr}_{12*}(\mathrm{pr}_{23}^*\mathcal{L}(\langle , \rangle) \otimes \mathrm{pr}_{12}^* M)_w = 0$$

for all points $w \in W^\circ$. We consider the commutative diagram

$$\begin{tikzcd} W^\circ \times_S B_S \arrow[r, "\alpha \times 1_B"] \arrow[d] & C_S^\delta \times_S B_S \arrow[r, "\beta \times 1_B"] \arrow[d] & C^\circ \times_E B \arrow[d, "m"] \\ \mathsf{E}' \times_S \mathsf{E} \times_S \mathsf{E}' \arrow[r, "\mathrm{pr}_{23}"] & \mathsf{E} \times_S \mathsf{E}' \arrow[r, "\langle , \rangle"] & \mathbb{G}_a \end{tikzcd}$$

where $m(a, b) = ab$. Set $u = \beta \circ \alpha(w)$, take a small open neighborhood $U \subset C'^\circ$ around u, and let $\Delta_r = m(U \times_E B)$. One checks easily that, if U has been chosen small enough, then Δ_r is some open disc of finite radius r, centered at the origin. Denote by E the connected component of $\lambda^{-1}(\Delta_r) \subset \Delta(0, 1)$ which contains $0 \in \Delta(0, 1)$. We form the fibre diagram

$$\begin{tikzcd} E \arrow[d, "\lambda"'] & V_1 \arrow[l, "m'"'] \arrow[d, "f"'] & V_2 \arrow[l] \arrow[d, "g"'] \\ \Delta_r & U \times_E B \arrow[l, "m"'] & ((\beta \circ \alpha)^{-1} U) \times_E B. \arrow[l, "(\beta\circ\alpha) \times 1_B"'] \end{tikzcd}$$

By construction, the sheaf $m^*\mathcal{L}$ trivializes on the étale covering of finite degree $f : V_1 \to U \times_E B$. It follows that $m^*\mathcal{L}_{|U \times_E B}$ is a direct summand in $f_*\mathbb{O}_n$ and hence we obtain an imbedding

$$(22)\quad R^q\mathrm{pr}_{12*}(\mathrm{pr}_{23}^*\mathcal{L}(\langle , \rangle) \otimes \mathrm{pr}_{12}^* M)_{|(\beta\circ\alpha)^{-1}(U)} \hookrightarrow R^q(\mathrm{pr}_{12} \circ g)_*(\mathrm{pr}_{12} \circ g)^* M_{|(\beta\circ\alpha)^{-1}(U)} \quad (q \geq 0).$$

Notice also that for all $y \in U$, the geometric fibre $(\mathrm{pr}_1 \circ f)^{-1}(y)$ is a finite union of open discs. In order to apply this observation, we need the following lemma, which is a minor variation of [B1] Corollary 7.4.2, and whose proof we leave therefore as an exercise for the referee.

Lemma 16. *Let $\phi : X \to Y$ be a separated smooth morphism of pure dimension d, and suppose that the geometric fibres of ϕ are non-empty and have trivial cohomology groups H_c^q with coefficients in $\mathbb{O}_n$ for $q < 2d$. Then for all $F \in S(X, \mathbb{O}_n)$ we have $R^q\phi_*\phi^* F = 0$ for $q > 0$.* □

Next, since we have taken $W \times_E B \in C_S^\delta$ and $\delta > \rho(\psi, t)$, we see that the sheaf $m^*\mathcal{L}_{|U \times_E B}$ is never trivial on any of the geometric fibres $\{y\} \times_E B$ $(y \in U)$. From this, together with (22) and lemma 16 (applied to $\mathrm{pr}_{12} \circ g$) we derive easily that

$$R\mathrm{pr}_{12*}(\mathrm{pr}_{23}^*\mathcal{L}(\langle , \rangle) \otimes \mathrm{pr}_{12}^* M)_{|(\beta\circ\alpha)^{-1}U} = 0.$$

This proves (19) and the claim of the theorem. □

Remark: it is well known that theorem 12 formally implies that the Fourier transform commutes with Verdier duality. A Verdier duality theory for étale analytic sheaves has been established by Berkovich in [B5].

We list hereafter a few of the other main formal properties of the Fourier transform. The proofs have the same flavour as the previous proof of involutivity, and proceed exactly as in Laumon's paper, therefore we limit ourself to give the statements and refer the reader to the corresponding results in [Lau2].

Theorem 13. *(Cp.* [Lau2], *theorem (1.2.2.4)) Let* $\mathsf{E}_1 \to \mathsf{E}_2$ *a morphism of vector bundles over* S *of constant ranks* r_1 *and* r_2 *respectively, and let* $f' : \mathsf{E}_2' \to \mathsf{E}_1'$ *be the transpose of* f*. Then there is a canonical isomorphism*

$$\mathcal{F}_2(Rf_! K_1^{\cdot}) \simeq f'^* \mathcal{F}_1(K_1^{\cdot})[r_2 - r_1]$$

for all $K_1^{\cdot} \in \mathbb{D}^b(\mathsf{E}_1, \mathbb{O}_n)$. □

Corollary 5. *There is a canonical isomorphism*

$$R\pi_!' \mathcal{F}(K^{\cdot}) \simeq \sigma^* K^{\cdot}(-r)[-r]$$

for all $K^{\cdot} \in \mathbb{D}^b(\mathsf{E}_1, \mathbb{O}_n)$. □

Definition 18. *The convolution product on* $\mathsf{E} \to S$ *is the operation*

$$* : \mathbb{D}^b(\mathsf{E}, \mathbb{O}_n) \times \mathbb{D}^b(\mathsf{E}, \mathbb{O}_n) \to \mathbb{D}^b(\mathsf{E}, \mathbb{O}_n)$$

defined as

$$K_1^{\cdot} * K_2^{\cdot} = Rs_!(K_1^{\cdot} \overset{L}{\boxtimes} K_2^{\cdot}).$$

Proposition 18. *(Cp.* [Lau2], *proposition (1.2.2.7)) There is a canonical isomorphism*

$$\mathcal{F}(K_1^{\cdot} * K_2^{\cdot}) \simeq \mathcal{F}(K_1^{\cdot}) \overset{L}{\otimes} \mathcal{F}(K_2^{\cdot})[-r]$$

for all $K_1^{\cdot}, K_2^{\cdot} \in \mathbb{D}^b(\mathsf{E}, \mathbb{O}_n)$. □

Proposition 19. *(Cp.* [Lau2], *proposition (1.2.2.8)). There is a canonical "Plancherel" isomorphism*

$$R\pi_!'(\mathcal{F}(K_1^{\cdot}) \overset{L}{\otimes} \mathcal{F}(K_2^{\cdot})) \simeq R\pi_!(K_1^{\cdot} \overset{L}{\otimes} [-1]^* K_2^{\cdot})(-r)$$

for all $K_1^{\cdot}, K_2^{\cdot} \in \mathbb{D}^b(\mathsf{E}, \mathbb{O}_n)$. □

Proposition 20. *(Cp.* [Lau2], *proposition (1.2.3.5)). Let* $S_1 \xrightarrow{f} S$ *be a morphism of E-analytic varieties. Let* $\mathsf{E}_1 \xrightarrow{\pi_1} S_1$ *and* $\mathsf{E}_1' \xrightarrow{\pi_1'} S_1$ *the vector bundles over* S_1 *obtained by base change from* $\mathsf{E} \xrightarrow{\pi} S$ *and* $\mathsf{E}' \xrightarrow{\pi'} S$*. Denote by* $f_{\mathsf{E}} : \mathsf{E}_1 \to \mathsf{E}$ *and* $f_{\mathsf{E}'} : \mathsf{E}_1' \to \mathsf{E}'$ *the canonical projections. Then there exists a canonical isomorphism*

$$\mathcal{F}(Rf_{\mathsf{E}!} K^{\cdot}) \simeq Rf_{\mathsf{E}'!} \mathcal{F}_1(K^{\cdot})$$

for all $K^{\cdot} \in \mathbb{D}^b(\mathsf{E}_1, \mathbb{O}_n)$ *(we have denoted by* $\mathcal{F}_1$ *the Fourier transform for the vector bundle* $\mathsf{E}_1 \to S_1$*).* □

6.2. Computation of some Fourier transforms. The following examples of calculation of Fourier trasforms are taken from [Lau2], with the exception of proposition 24, which has no analogue in positive characteristic.

Proposition 21. *Let* $\mathsf{F} \overset{i}{\hookrightarrow} \mathsf{E}$ *be a vector sub-bundle over* S *of constant rank* s. *Denote by* $\mathsf{F}^{\perp} \overset{i^{\perp}}{\hookrightarrow} \mathsf{E}'$ *the orthogonal of* F *in* E'. *Then there is a canonical isomorphism*

$$\mathcal{F}(i_*\mathbb{O}_{n,\mathsf{F}}[s]) \simeq i^{\perp}_*\mathbb{O}_{n,\mathsf{F}^{\perp}}(-s)[r-s].$$

□

Proposition 22. *Let* $e \in \mathsf{E}(S)$ *(i.e. a section of* $\mathsf{E} \xrightarrow{\pi} S$*). Denote by* $\tau_e : \mathsf{E} \to \mathsf{E}$ *the translation by* e. *Then there is a canonical isomorphism*

$$\mathcal{F}(\tau_{e*}K^{\cdot}) \simeq \mathcal{F}(K^{\cdot}) \otimes \mathcal{L}(\langle e, \ \rangle)$$

for all $K^{\cdot} \in \mathbb{D}^b(V, \mathbb{O}_n)$. □

Proposition 23. *Let* $\alpha : \mathsf{E} \xrightarrow{\sim} \mathsf{E}'$ *be a symmetric isomorphism. Denote by* $q : \mathsf{E} \to \mathbb{G}_a$ *and* $q' : \mathsf{E}' \to \mathbb{G}_a$ *the quadratic forms associated to* α *(i.e.* $q(e) = \langle e, \alpha(e)\rangle$ *and* $q'(e') = \langle \alpha^{-1}(e'), e'\rangle$*). Let* $[2] : \mathsf{E}' \to \mathsf{E}'$ *be multiplication by* 2 *on the vector bundle* E'. *Then there is a canonical isomorphism*

$$[2]^*\mathcal{F}(\mathcal{L}(q)) \simeq \mathcal{L}(-q') \otimes \pi'^* R\pi_!\mathcal{L}(q)[r].$$

□

For the next result, we suppose $\mathsf{E} \xrightarrow{\pi} S$ has rank one for simplicity. Let $B \xrightarrow{\beta} S$ be a sphere bundle inside E, *i.e.* a fibre bundle over S with an open imbedding $j : B \hookrightarrow \mathsf{E}$ which is a morphism of S-varieties, and such that over each point $s \in S$, the restriction $j_s : \beta^{-1}(s) \hookrightarrow \pi^{-1}(s)$ is the imbedding of an open ball of finite radius centered at $\sigma(s) \in \pi^{-1}(s)$.

We also fix some linear coordinate t on $\mathbb{G}_a$ and let $D \xrightarrow{\beta'} S$ be the dual bundle of $B \to S$, *i.e.* the fibre bundle over S with a closed S-imbedding $i : D \to \mathsf{E}'$, defined by the equation

$$|t(\langle e, e'\rangle)| < \rho(\psi, t) \qquad (e \in B, e' \in D).$$

In other words, the restriction $i_s : \beta'^*(s) \to \pi'^*(s)$ is the imbedding of a closed disc centered at $\sigma'(s)$.

Proposition 24. *i)* $\mathcal{F}(i_*\mathbb{O}_{n,D}) = j_!\mathbb{O}_{n,B}[1]$,
ii) $\mathcal{F}(j_!\mathbb{O}_{n,B}) = i_*\mathbb{O}_{n,D}(-1)[-1]$.

Proof. By theorem 11 we see that (i) and (ii) are equivalent. We will prove (ii). By proper base change we can assume that S is a point; then B is an open disc $\mathbb{G}_a(\alpha, t)$ and $D = D_\beta$ is a closed disc of radius $\beta = \rho(\psi, t)/\alpha$. Set $T = \mathbb{G}_a(\alpha, t) \times D_\beta$. Note that the condition $\alpha\beta = \rho(\psi, t)$ implies that $\mathcal{L}(\langle , \rangle)$ trivializes on T. It follows that the restriction of $\mathcal{F}(j_!\mathbb{O}_{n,\mathbb{G}_a(\alpha,t)})$ to D_β coincides with $\mathbb{O}_n[-1]$. Therefore it suffices to show that $\mathcal{F}(j_!\mathbb{O}_{n,\mathbb{G}_a(\alpha,t)})$ vanishes outside D_β. To this purpose we can check on the stalks, and then the claim follows from proposition 17. □

7. Kummer-Artin-Schreier-Witt theory

This chapter is a prelude to the following one: we review and complement the theory of the deformation from Artin-Schreier-Witt to Kummer, which has been developed by Sekiguchi and Suwa in [Se-Su].

To start with, let n be any positive integer. Let μ_{p^n} be the group of p^n-order roots of unit in the algebraic closure $\mathbf{Q}^a$ of the field $\mathbf{Q}$ of rational numbers; we fix once for all a generator ζ_n of μ_{p^n} and denote by K the field of fractions of the ring $A = \mathbf{Z}_{(p)}[\mu_{p^n}]$ with maximal ideal $\mathfrak{m}$. Recall that if $B = \mathbf{Z}[p^{-1}]$, then the sequence of sheaves of groups on the étale site on Spec$B$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_{p^n} & \longrightarrow & \mathbb{G}_{m,B} & \xrightarrow{\theta} & \mathbb{G}_{m,B} & \longrightarrow & 1 \\ & & & & t & \longmapsto & t^{p^n} & & \end{array}$$

is exact and is called a Kummer exact sequence.

On the other hand, let $\mathbb{F}$ be a field of positive characteristic p, and let $W_{n,\mathbb{F}}$ be the Witt group scheme over $\mathbb{F}$ of dimension n. Then we have the Artin-Schreier-Witt exact sequence of groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}/p^n & \longrightarrow & W_{n,\mathbb{F}} & \xrightarrow{\mathfrak{f}} & W_{n,\mathbb{F}} & \longrightarrow & 0 \\ & & & & x & \longmapsto & x^p - x & & \end{array}$$

on the étale site on Spec$\mathbb{F}$. Here the p-th power map is given by the canonical ring structure on $W_{n,\mathbb{F}}$.

The purpose of the Kummer-Artin-Schreier-Witt theory (in short KASW theory) is to assemble the two above short exact sequences into a single diagram of group schemes on the étale site over Spec$A$. To this purpose, we must first replace the Kummer exact sequence by another, essentially equivalent one. In detail, define maps $\pi, \Sigma_n : \mathbb{G}^n_{m,B} \to \mathbb{G}_{m,B}$ by $\pi(u_1, ..., u_n) = u_n$ and $\Sigma_n(u_1, ..., u_n) = u_1 u_2^p \cdot ... \cdot u_n^{p^{n-1}}$. Also, let $\Theta : \mathbb{G}^n_{m,B} \to \mathbb{G}^n_{m,B}$ be the morphism $(u_1, ..., u_n) \mapsto (u_1^p, u_1^{-1}u_2^p, ..., u_{n-1}^{-1}u_n^p)$. All these maps are group scheme homomorphisms, and the kernel of Θ is the subgroup of all points of the form $a_\zeta = (\zeta^{p^{n-1}}, \zeta^{p^{n-2}}, ..., \zeta)$, for ζ ranging over the elements of μ_{p^n}. Clearly the assignment $\zeta \mapsto a_\zeta$ defines an isomorphism onto KerΘ, and hence we have a commutative diagram with exact rows

$$\begin{tikzcd} 1 \arrow[r] & \mu_{p^n} \arrow[r] \arrow[d, equal] & \mathbb{G}^n_{m,B} \arrow[r, "\Theta"] \arrow[d, "\pi"] & \mathbb{G}^n_{m,B} \arrow[r] \arrow[d, "\Sigma_n"] & 1 \\ 1 \arrow[r] & \mu_{p^n} \arrow[r] & \mathbb{G}_{m,B} \arrow[r, "\theta"] & \mathbb{G}_{m,B} \arrow[r] & 1. \end{tikzcd}$$

We can now state the fundamental

Theorem 14. *(See* [Se-Su], *Assertion 1) For every integer $n > 0$ there exists a smooth group scheme $\mathcal{W}_n$ over SpecA, containing the constant group scheme $(\mathbf{Z}/p^n)_A$, such that*

(a): the exact sequence

$$0 \longrightarrow (\mathbf{Z}/p^n)_A \longrightarrow \mathcal{W}_n \xrightarrow{\chi} \mathcal{W}_n/(\mathbf{Z}/p^n)A \longrightarrow 0$$

has the Artin-Schreier-Witt exact sequence as the special fibre, and

(b): is isomorphic to the Kummer exact sequence

$$1 \longrightarrow \mu_{p^n} \longrightarrow \mathbb{G}^n_{m,B} \xrightarrow{\Theta} \mathbb{G}^n_{m,B} \longrightarrow 1$$

on the generic fibre. □

Following [Se-Su] we denote by $\mathcal{V}_n$ the quotient group scheme $\mathcal{W}_n/(\mathbf{Z}/p^n)_A$. The groups $\mathcal{W}_n$ and $\mathcal{V}_n$ abound with structure, most of which will be useful for our purposes. We proceed to describe some of this structure.

Recall that, for every $n \geq 2$, the truncation map from $W_{n,\mathbb{F}}$ to $W_{n-1,\mathbb{F}}$ induces a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{Z}/p \arrow[r] \arrow[d, hook] & \mathbf{Z}/p^n \arrow[r] \arrow[d, hook] & \mathbf{Z}/p^{n-1} \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathbb{G}_{a,\mathbb{F}} \arrow[r] & W_{n,\mathbb{F}} \arrow[r] & W_{n-1,\mathbb{F}} \arrow[r] & 0. \end{tikzcd} \tag{23}$$

The counterpart of this diagram is the following

Theorem 15. *(See [Se-Su], Assertion 1) For each $n \geq 2$ there exists a commutative diagram with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & (\mathbf{Z}/p)_A \arrow[r] \arrow[d, hook] & (\mathbf{Z}/p^n)_A \arrow[r] \arrow[d, hook] & (\mathbf{Z}/p^{n-1})_A \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathcal{W}_{1,A} \arrow[r] & \mathcal{W}_{n,A} \arrow[r] & \mathcal{W}_{n-1,A} \arrow[r] & 0. \end{tikzcd} \tag{24}$$

which gives a deformation of the exact sequence (23) *to a diagram of exact sequences of multiplicative groups*

$$\begin{tikzcd} 1 \arrow[r] & \mu_p \arrow[r] \arrow[d, hook] & \mu_{p^n} \arrow[r] \arrow[d, hook] & \mu_{p^{n-1}} \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & \mathbb{G}_m \arrow[r] & (\mathbb{G}_m)^n \arrow[r] & (\mathbb{G}_m)^{n-1} \arrow[r] & 1. \end{tikzcd}$$

In the latter diagram, the imbedding $\mathbb{G}_m \hookrightarrow (\mathbb{G}_m)^n$ is the map $u \mapsto (1, ..., 1, u)$ and the epimorphism $(\mathbb{G}_m)^n \to (\mathbb{G}_m)^{n-1}$ is given by $(u_1, ..., u_n) \mapsto (u_1, ..., u_{n-1})$. □

The underlying schemes of $\mathcal{W}_n$ and $\mathcal{V}_n$ can be described somewhat more explicitly. Recall that, as a scheme, $W_{n,\mathbb{F}}$ is isomorphic to $\mathbf{A}^n_{\mathbb{F}}$. The sequence of truncation maps $W_{n,\mathbb{F}} \to W_{n-1,\mathbb{F}} \to ... \to W_{1,\mathbb{F}} \simeq \mathbf{A}^1_{\mathbb{F}}$ corresponds to a chain of linear projections $\mathbf{A}^n_{\mathbb{F}} \to \mathbf{A}^{n-1}_{\mathbb{F}} \to ... \to \mathbf{A}^1_{\mathbb{F}}$. Moreover, all the subquotients $W_{j+1,\mathbb{F}}/W_{j,\mathbb{F}}$ are canonically isomorphic to $\mathbb{G}_{a,\mathbb{F}}$.

We can similarly consider the sequence of epimorphisms $\mathcal{W}_{n,A} \to \mathcal{W}_{n-1,A} \to ... \to \mathcal{W}_{1,A}$ derived from diagram (24). The subquotients are isomorphic to the group scheme $\mathcal{W}_{1,A}$. The underlying scheme of $\mathcal{W}_{1,A}$ can be given as follows. For every $j < n$ set $\zeta_j = \zeta_n^{p^{n-j}}$ and let $\lambda = 1 - \zeta_1$; then $\mathcal{W}_{1,A} \simeq \mathrm{Spec}A[X, 1/(1+\lambda X)]$. Notice in particular that $\mathcal{W}_{1,A}$ is already defined on the smaller ring $A_1 = \mathbf{Z}[\mu_p]$. It is also true that the group law of $\mathcal{W}_{1,A}$ is defined on A_1 as well. Moreover, the group scheme $\mathcal{W}_{1,A_1}$ is independent of the choice of λ. A similar discussion holds for the group scheme $\mathcal{V}_{n,A}$ and the chain of epimorphisms $\mathcal{V}_{n,A} \to \mathcal{V}_{n-1,A} \to ... \to \mathcal{V}_{1,A}$ obtained by taking the cokernels of the vertical arrows in diagram (24). Let $\iota : \mathrm{Spec}A/\lambda \to \mathrm{Spec}A$ be the canonical inclusion. With this notation, one has

Proposition 25. *(See [Se-Su], Theorem 3.3) For each j $(1 \leq j \leq n-1)$ there exists a polynomial $F_j \in \mathbf{Z}[\zeta_{j+1}, X_1, ..., X_j]$ inducing a group homomorphism*

$$\overline{F}_j : \mathcal{W}_{j+1,A} \to \iota_* \mathbb{G}_{m,A/\lambda}$$

and each $\mathcal{W}_{j,A}$ is given by

$$\mathcal{W}_{j,A} \simeq SpecA[X_1, ..., X_j, \frac{1}{1+\lambda X_1}, \frac{1}{F_1(X_1)+\lambda X_2}, ..., \frac{1}{F_{j-1}(X_1,...,X_{j-1})+\lambda X_j}].$$

Moreover, the group law on $\mathcal{W}_{j,A}$ is determined by requiring that the morphism $\alpha^j_{\mathcal{W}} : \mathcal{W}_{j,A} \to (\mathbb{G}_{m,A})^j$ given by

$$(X_1, ..., X_j) \mapsto (1+\lambda X_1, F_1(X_1)+\lambda X_2, ..., F_{j-1}(X_1,...,X_{j-1})+\lambda X_j)$$

be a group homomorphism.

Similarly, for each j there are polynomials $G_j \in \mathbf{Z}[\zeta_{j+1}, X_1, ..., X_j]$ such that

$$\mathcal{V}_{n,A} \simeq SpecA[X_1, ..., X_n, \frac{1}{1+\lambda^p X_1}, \frac{1}{F_1(X_1)+\lambda^p X_2}, ..., \frac{1}{F_{n-1}(X_1,...,X_{j-1})+\lambda^p X_n}].$$

and such that the above statements remain valid after replacing $\mathcal{W}_{j,A}$ by $\mathcal{V}_{j,A}$, F_j by G_j, λ by λ^p and $\alpha^j_{\mathcal{W}}$ by $\alpha^j_{\mathcal{V}}$. □

Remarks. 1): Proposition 25 allows to complete the statement of theorem 15 and of part (b) of theorem 14. In fact, the epimorhisms $\mathcal{W}_{n,A} \to \mathcal{W}_{n-1,A}$ can be expressed in terms of the coordinates introduced in proposition 25, as the projections $(X_1, ..., X_n) \mapsto (X_1, ..., X_{n-1})$ and similarly for the maps $\mathcal{V}_{n,A} \to \mathcal{V}_{n,A}$. Moreover, the isomorphism on the generic fibre in theorem 14(b) extends to a commutative diagram with exact rows (in the fttp topology)

$$\begin{tikzcd}
0 \arrow[r] & (\mathbf{Z}/p^n)_A \arrow[r] \arrow[d] & \mathcal{W}_{n,A} \arrow[r, "\chi"] \arrow[d, "\alpha_{\mathcal{W}}"] & \mathcal{V}_{n,A} \arrow[r] \arrow[d, "\alpha_{\mathcal{V}}"] & 0 \\
1 \arrow[r] & (\mu_{p^n})_A \arrow[r] & (\mathbb{G}_{m,A})^n \arrow[r, "\Theta"] & (\mathbb{G}_{m,A})^n \arrow[r] & 0.
\end{tikzcd}$$

Here the map $(\mathbf{Z}/p^n) \to \mu_{p^n}$ is given by $a \mapsto \zeta_n^a$; in particular it restricts to the trivial morphism on the special fibre.

2): Since the special fibre of $\mathcal{W}_{n,A}$ and $\mathcal{V}_{n,A}$ is isomorphic to $\mathbf{A}^n_{A/\mathfrak{m}}$, we see that $F_j(X_1, ..., X_j) \equiv 1 \pmod{\mathfrak{m}}$ for each j $(1 \leq j \leq n-1)$.

7.1. Formalities. We will need a formal scheme version of the KASW theory. As a matter of notation, for any A-scheme X, we will denote by $\widehat{X}$ the formal completion of X, with respect to the $\mathfrak{m}$-adic topology on A. Then $\widehat{X}$ is a Spf$\widehat{A}$-formal scheme, where $\widehat{A}$ is the $\mathfrak{m}$-adic completion of A, with fraction field $\widehat{K}$. Similarly, any morphism of A-schemes $\phi : X \to Y$ defines a map $\widehat{\phi} : \widehat{X} \to \widehat{Y}$. If E is any complete field containing $\widehat{K}$, with ring of integers E°, then a base change functor is defined, $\mathcal{X} \mapsto \mathcal{X}_{E^\circ}$ from the category of Spf$\widehat{A}$-formal schemes, to the category of SpfE°-formal schemes. As usual, for any such field E we will normalize its norm $|\cdot|$ so that $|p| = p^{-1}$. The symbol $\mathfrak{m}$ will denote either the maximal ideal of A, $\widehat{A}$ or E°, depending on the context.

To ease notation, set $\widetilde{F}_j = F_j(x_1, ..., x_j) + \lambda x_{j+1}$ and $\widetilde{G}_j = G_j(x_1, ..., x_j) + \lambda^p x_{j+1}$. The following identifications are immediate:

$$\begin{aligned}
\widehat{\mathbb{G}}_{m,E^\circ} &= \mathrm{Spf}E^\circ\langle x, y\rangle/(xy-1) \\
\widehat{\mathcal{W}}_{n,E^\circ} &= \mathrm{Spf}E^\circ\langle x_1, y_1, ..., x_n, y_n\rangle/(y_1(1+\lambda x_1)-1, y_2\widetilde{F}_1 - 1, ..., y_n\widetilde{F}_{n-1}-1) \\
\widehat{\mathcal{V}}_{n,E^\circ} &= \mathrm{Spf}E^\circ\langle x_1, y_1, ..., x_n, y_n\rangle/(y_1(1+\lambda^p x_1)-1, y_2\widetilde{G}_1 - 1, ..., y_n(G_{n-1}-1).
\end{aligned}$$

Given a SpfE°-formal scheme $\mathcal{X}$, we will denote by $\mathcal{X}_\eta$ (resp. $\mathcal{X}_s$) its generic fibre, which is a E-analytic space (see [B3]) (resp. its special fibre, which is a scheme over Spec$E^\circ/\mathfrak{m}$). We can then form the analytic space $(\widehat{\mathbb{G}}_{m,E^\circ})_\eta$, which can be described as an annulus C of equation $|x| = 1$, $x \in (\mathbf{A}^1_{E^\circ})^{an}$. Similarly, the space $(\widehat{\mathcal{W}}_{n,E^\circ})_\eta$ is the set of all $(x_1, y_1, ..., x_n, y_n)$ with $|x_i|, |y_i| \leq 1$ and such that

$$\begin{aligned} y_1(1+\lambda x_1) &= 1 \\ y_i\widetilde{F}_{i-1}(x_1, ..., x_i) &= 1 \quad i = 2, ..., n-1. \end{aligned}$$

These conditions imply that $|y_i| = 1$ for all i, and moreover y_i is determined by $x_1, ..., x_i$. Since $\lambda \in \mathfrak{m}$, we deduce from remark (2) after proposition 25, that $\widetilde{F}_i \equiv 1 \pmod{\mathfrak{m}}$, that is $|\widetilde{F}_i(x_1, .., x_{i+1})| = 1$ whenever $|x_1|, ..., |x_{i+1}| \leq 1$. Therefore the generic fibre of $\widehat{\mathcal{W}}_{n,E^\circ}$ is an n-fold product of closed discs $D_1 \times ... \times D_1$ of radius 1.

The same argument and the same conclusion apply to $(\widehat{\mathcal{V}}_{n,E^\circ})_\eta$.

We make the obvious remark that the map Σ_n extends to a morphism $\mathbb{G}^n_{m,E^\circ} \to \mathbb{G}_{m,E^\circ}$ and we consider the following diagram of formal schemes:

$$\text{(25)} \qquad \begin{array}{ccc} \widehat{\mathcal{W}}_{n,E^\circ} & \xrightarrow{\widehat{\chi}} & \widehat{\mathcal{V}}_{n,E^\circ} \\ {\scriptstyle\widehat{\pi\alpha_{\mathcal{W}}}}\downarrow & & \downarrow{\scriptstyle\widehat{\Sigma_n\alpha_{\mathcal{V}}}} \\ \widehat{\mathbb{G}}_{m,E^\circ} & \xrightarrow{\widehat{\theta}} & \widehat{\mathbb{G}}_{m,E^\circ}. \end{array}$$

The group scheme structures on our schemes are handed down to their formal completions and even to the associated analytic spaces, so the above is a diagram of homomorphisms of formal group schemes, which restrict, on the generic fibre, to homomorphisms of analytic groups.

In particular, we obtain certain group structures on $D_1 \times ... \times D_1$ (resp. on C) and a map of analytic groups $(\widehat{\pi\alpha_{\mathcal{W}}})_\eta : D_1 \times ... \times D_1 \to C$. In coordinates, it is given by the polynomial $\widetilde{F}_n(x_1, ..., x_n) = \lambda x_n + F_n(x_1, ..., x_{n-1})$. According to our normalization, we have $|\lambda| = p^{-1/(p-1)}$. More generally, for all $j \leq n$, a standard calculation gives

$$|1 - \zeta_j| = \rho_j := p^{\frac{-1}{(p-1)p^{j-1}}}.$$

From remark (2) after proposition 25 it then follows that the image of $(\widehat{\pi\alpha_{\mathcal{W}}})_\eta$ is contained in the closed disc $D_{\rho_n} \subset C$ which is centered at the point $1 \in C$ and has radius ρ_n. The inclusion $D_{\rho_n} \hookrightarrow C$ can be lifted explicitly to a map of formal schemes $j : \mathrm{Spf}E^\circ\langle x\rangle \to \mathrm{Spf}E^\circ\langle x, y\rangle/(xy-1)$ defined as $x \mapsto 1+(1-\zeta_n)x$. This suggests to introduce a morphism ϕ which makes the following diagram commute:

$$\begin{tikzcd} & \widehat{\mathcal{W}}_{n,E^\circ} \arrow[dl, "\phi"'] \arrow[d, "\widehat{\pi\alpha_{\mathcal{W}}}"] \\ \mathrm{Spf}E^\circ\langle x\rangle \arrow[r, "j"] & \mathrm{Spf}E^\circ\langle x, y\rangle(xy-1). \end{tikzcd}$$

Clearly, ϕ is determined by sending

$$x \mapsto \frac{\lambda}{1-\zeta_n}x_n + \frac{1}{1-\zeta_n}(F_n(x_1, ..., x_{n-1}) - 1).$$

Lemma 17. *The map* $\phi_\eta : (\widehat{\mathcal{W}}_{n,E^\circ})_\eta \to (\mathrm{Spf}E^\circ\langle x\rangle)_\eta \simeq D_1$ *is surjective.*

Proof. Let $(\mathbf{Z}/p^n)_\eta$ be the general fibre of the subgroup $(\mathbf{Z}/p^n)_A$ of $\mathcal{W}_{n,A}$, provided by theorem 14. This is a finite subgroup of $(\widehat{\mathcal{W}}_{n,A})_\eta$. We see from remark (1) after proposition 25 and from our explicit description of KerΘ, that $(\mathbf{Z}/p^n)_\eta$ is mapped by $(\widehat{\pi\alpha}_{\mathcal{W}})_\eta$ onto the set $\mu_{p^n} \subset E$. Since the primitive p^n-roots of unit have absolute value equal to ρ_n, it follows that the reduction modulo the maximal ideal of the set $\phi((\mathbf{Z}/p^n)_\eta) \subset D_\rho$ covers all the $A/\mathfrak{m}$-rational residue classes in $(\mathrm{Spf}\widehat{A}\langle x\rangle)_s = \mathbf{A}^1_{A/\mathfrak{m}}$. In particular, after base changing to SpfE° we see that the induced map on the special fibres $\phi_s : (\widehat{\mathcal{W}}_{n,E^\circ})_s \simeq \mathbf{A}^1_{E^\circ/\mathfrak{m}} \times ... \times \mathbf{A}^1_{E^\circ/\mathfrak{m}} \to \mathbf{A}^1_{E^\circ/\mathfrak{m}}$ is not constant.

Take two points $a, b \in (\mathbf{Z}/p^n)_\eta$ which have distinct residue classes in $A^1_{A/\mathfrak{m}}$, and define a map $i : \mathrm{Spf}E^\circ\langle t\rangle \to \widehat{\mathcal{W}}_{n,E^\circ}$ by $t \mapsto at + (1-t)b$. By construction, the composition $\phi i : \mathrm{Spf}E^\circ\langle t\rangle \to \mathrm{Spf}E^\circ\langle x\rangle$ is a morphism of formal schemes, represented by some polynomial $P(t)$ with the property that its reduction $\overline{P}(t) \in A/\mathfrak{m}[t]$ is not a constant. Clearly it suffices to show that $(\phi i)_\eta$ is surjective. Luckily, this is an elementary statement.

Let $p \in D_1$ be any point. We want to show that p is in the image of $(\phi i)_\eta$. To start with, let $\mathcal{H}(p)$ be the completed residue field of the point p. After base changing to $\mathcal{H}(p)$, we may assume that p is rational over our base field. Then the claim amounts to showing that the polynomial $\widetilde{P}(t) = P(t) - p$ has a root t_0 such that $|t_0| \le 1$. Let $\widetilde{P}(t) = \sum_{i=0}^m a_i t^i$ (with $a_m \neq 0$); by hypothesis we know that $|a_i| \le 1$ for all i, and there exists $j > 0$ such that $|a_j| = 1$. Let $b_i = a_i/a_m$; the b_i's are the elementary symmetric polynomials in the roots $t_0, ..., t_{m-1}$ of $\widetilde{P}$. Since $|a_j| = 1$, there exists a subset $I \subset \{0, ..., m-1\}$ with $|I| = m - j$ and such that $t^I = \prod_{i\in I} t_i \ge |a_m|^{-1}$. Then $b_0/t^I = \prod_{i\notin I} t_i = a_0/a_j$ has norm $|a_0| \le 1$, which says that $|t_i| \le 1$ for at least one $i \notin I$. □

Proposition 25 and theorem 15 give us a commutative diagram

$$\begin{tikzcd} \mathcal{W}_{n,E^\circ} \arrow[r] \arrow[d, "\alpha^{(n)}_{\mathcal{W}}"'] & \mathcal{W}_{n-1,E^\circ} \arrow[d, "\alpha^{(n-1)}_{\mathcal{W}}"] \\ \mathbf{G}^n_{m,E^\circ} \arrow[r] & \mathbf{G}^{n-1}_{m,E^\circ}. \end{tikzcd} \tag{26}$$

Therefore, we can iterate the construction above, to obtain a factorization

$$\begin{tikzcd} & \widehat{\mathcal{W}}_{n,E^\circ} \arrow[dl, "\phi_n"'] \arrow[d, "\widehat{\alpha}_{\mathcal{W}}"] \\ \mathrm{Spf}E^\circ\langle x_1, ..., x_n\rangle \arrow[r, "j_n"] & \widehat{\mathbf{G}}^n_{m,E^\circ} \end{tikzcd}$$

such that the image of $(\mathrm{Spf}E^\circ\langle x_1, ..., x_n\rangle)_\eta$ is a product of discs $D_{\rho_1} \times ... \times D_{\rho_n} \subset (\widehat{\mathbf{G}}^n_{m,E^\circ})_\eta$, with specified radii.

Proposition 26. *The map $\phi_{n,\eta}$ is an isomorphism of analytic spaces.*

Proof. Taking into account that all the maps in diagram (26) are group homomorphisms, a simple induction argument, using lemma 17, shows that $\phi_{n,\eta}$ is surjective. Injectivity follows easily from theorem 14(b), which also implies that $\phi_{n,\eta}$ induces isomorphisms on the local rings at each point. □

The map $\phi_{n,\eta}$ is even an analytic group isomomorphism, if we endow $(\mathrm{Spf}E^\circ\langle x_1,...,x_n\rangle)_\eta$ with the group structure which makes of $j_{n,\eta}$ the imbedding of a subgroup.

We turn again to diagram (25): consider the composition $\hat{\theta}j : \mathrm{Spf}E^\circ\langle x\rangle \to \widehat{\mathbb{G}}_{m,E^\circ}$. In coordinates, this is the map $x \mapsto (1+(1-\zeta_n)x)^{p^n}$. An easy computation shows that the image of the induced map of analytic spaces is the closed disc $D \subset (\widehat{\mathbb{G}}_{m,E^\circ})_\eta$ which is centered at the point $1 \in (\widehat{\mathbb{G}}_{m,E^\circ})_\eta$ and with radius $|\lambda|^p$. Hence we can proceed as above and find a factorization

$$\text{(27)} \qquad \begin{tikzcd} \mathrm{Spf}E^\circ\langle x\rangle \arrow[d, "\tilde{\theta}"'] \arrow[dr, "\hat{\theta}j"] & \\ \mathrm{Spf}E^\circ\langle x\rangle \arrow[r, "j'"] & \widehat{\mathbb{G}}_{m,E^\circ} \end{tikzcd}$$

where j' is the map $x \mapsto 1+\lambda^p x$. We can combine diagrams (25) and (27) to get a new one

$$\text{(28)} \qquad \begin{tikzcd} \widehat{\mathcal{W}}_{n,E^\circ} \arrow[r, "\hat{\chi}"] \arrow[d, "\phi"'] & \widehat{\mathcal{V}}_{n,E^\circ} \arrow[d, "\sigma"] \\ \mathrm{Spf}E^\circ\langle x\rangle \arrow[r, "\tilde{\theta}"] & \mathrm{Spf}E^\circ\langle x\rangle \end{tikzcd}$$

where $\sigma : \widehat{\mathcal{V}}_{n,E^\circ} \to \mathrm{Spf}E^\circ\langle x\rangle$ is determined by requiring that $j'\sigma = \widehat{\Sigma_n\alpha_{\mathcal{V}}}$. The following is an immediate consequence of the definitions and of lemma 17.

Lemma 18. *The map σ_η is surjective.* $\square$

Again, with the obvious group structures, diagram (28) induces group homomorphisms on the generic fibres. Finally, we consider the composition $\widehat{\Theta}j_n : \mathrm{Spf}E^\circ\langle x_1,...,x_n\rangle \to \widehat{\mathbb{G}}^n_{m,E^\circ}$. The usual argument tells us that $\widehat{\Theta}j_n$ factors through a map $\widetilde{\Theta} : \mathrm{Spf}E^\circ\langle x_1,...,x_n\rangle \to \mathrm{Spf}E^\circ\langle x_1,...,x_n\rangle$ and a linear map $j'_n : \mathrm{Spf}E^\circ\langle x_1,...,x_n\rangle \to \widehat{\mathbb{G}}^n_{m,E^\circ}$ which induces on the generic fibres an imbedding $D_{\rho_1^p} \times ... \times D_{\rho_n^p} \hookrightarrow (\widehat{\mathbb{G}}^n_{m,E^\circ})_\eta$. We endow $(\mathrm{Spf}E^\circ\langle x_1,...,x_n\rangle)_\eta$ with the analytic group structure which turns j'_n into a group homomorphism. Putting this together with remark (1) after proposition 25, we see that diagram (28) factors through a diagram

$$\text{(29)} \qquad \begin{tikzcd} \widehat{\mathcal{W}}_{n,E^\circ} \arrow[r, "\hat{\chi}"] \arrow[d, "\phi_n"'] & \widehat{\mathcal{V}}_{n,E^\circ} \arrow[d, "\sigma_n"] \\ \mathrm{Spf}E^\circ\langle x_1,...,x_n\rangle \arrow[r, "\widetilde{\Theta}"] & \mathrm{Spf}E^\circ\langle x_1,...,x_n\rangle. \end{tikzcd}$$

Proposition 27. *The map $\sigma_{n,\eta} : (\widehat{\mathcal{V}}_{n,E^\circ})_\eta \to D_{\rho_1^p} \times ... \times D_{\rho_n^p}$ is an isomorphism of analytic spaces. With the assigned group structure on $D_{\rho_1^p} \times ... \times D_{\rho_n^p}$, it is even an analytic group isomorphism.*

Proof. Diagram (29) presents $(\widehat{\mathcal{V}}_{n,E^\circ})_\eta$ (resp. $D_{\rho_1^p} \times ... \times D_{\rho_n^p}$) as quotient of the analytic group $(\widehat{\mathcal{W}}_{n,E^\circ})_\eta$ (resp. of $(\mathrm{Spf}E^\circ\langle x_1,...,x_n\rangle)_\eta$) by the action of the subgroup $(\mathbb{Z}/p^n)_\eta$. Seen this way, $\sigma_{n,\eta}$ is the map induced by the isomorphism $\phi_{n,\eta}$ on the quotient spaces. Hence it is an isomorphism as well. $\square$

8. Global results

We continue with the notation of section 6. In particular, everything is defined over some complete base field $E \supset k_\infty$. Up to a finite base change, we can even assume that E contains the field $\widehat{K}$ introduced in section 7. Since here we are interested in geometric questions only, we can and do assume throughout that our Lubin-Tate group is $\mathbb{G}_m$ and then $\psi : \mu_{p^\infty} \to \mathbb{O}_n^\times$ will be a character of the group of p-power roots of unit.

Let $f(x) \in E[x]$ be any polynomial. It defines a finite morphism $f : \mathbf{A}^1_E \to \mathbf{A}^1_E$ and we want to study the cohomology of $\mathcal{L}(f) := f^*\mathcal{L}$. Unless f is a constant, we have non-vanishing cohomology only in degree one, hence we are really interested in determining $H^1_c(\mathbf{A}^1_{\widehat{E}_a}, \mathcal{L}(f))$.

Our strategy consists in subdividing $\mathbf{A}^1$ into two regions $\mathbf{A}^1 = D_r \cup (\mathbf{A}^1 - D_r)$, where D_r is a closed disc centered at the origin, with some big radius r. Then we will apply the standard short exact sequence

$$(30) \qquad 0 \longrightarrow H^1_c(\mathbf{A}^1 - D_r, \mathcal{L}(f)) \longrightarrow H^1_c(\mathbf{A}^1, \mathcal{L}(f)) \longrightarrow H^1(D_r, \mathcal{L}(f)) \longrightarrow 0.$$

8.1. Integration inside and outside the disc. Set $f(x) = \sum_{j=0}^m a_j x^j$ $(a_m \neq 0)$ and define $f_0(x) = a_m x^m$. Clearly, for $r_0 >> 0$ we have $|x| \geq r_0 \Rightarrow |f(x)| = |a_m x^m|$. Select a real number r_0 with such a property. Then, for any $r \geq r_0$ the restriction $f_{|\mathbf{A}^1 - D_r}$ has image contained into $\mathbf{A}^1 - D_{r'}$, with $r' = |a_m| r^m$.

Write $f(x) = a_m x^m \cdot \left(1 + \sum_{j=0}^{m-1} \frac{a_j}{a_m} x^{j-m}\right)$. By our choice of r_0, the factor in parenthesis has norm equal to 1 when $|x| \geq r_0$. After maybe replacing r_0 by a larger number, we can even assume that this factor is arbitrarily close to 1, uniformily for $|x| \geq r_0$. In particular, for r_0 sufficiently large and any $r \geq r_0$, the power series $f_1(x) = x \cdot \left(1 + \sum_{j=0}^{m-1} \frac{a_j}{a_m} x^{j-m}\right)^{1/m}$ converges in the region $\mathbf{A}^1 - D_r$, and in fact defines an automorphism of this region. It is clear that $f_1(x)$ and $f_0(x)$ give a factorization of f in their domain of definition:

$$\mathbf{A}^1 - D_r \xrightarrow{f_1} \mathbf{A}^1 - D_r \xrightarrow{f_0} \mathbf{A}^1 - D_{r'}.$$

Therefore $H^1_c(\mathbf{A}^1 - D_r, \mathcal{L}(f)) \simeq H^1_c(\mathbf{A}^1 - D_r, f_1^*\mathcal{L}(f_0)) \simeq H^1_c(\mathbf{A}^1 - D_r, \mathcal{L}(f_0))$ and, given $r \geq r_0$, we can rewrite the exact sequence (30) as

$$(31) \qquad 0 \longrightarrow H^1_c(\mathbf{A}^1 - D_r, \mathcal{L}(f_0)) \longrightarrow H^1_c(\mathbf{A}^1, \mathcal{L}(f)) \longrightarrow H^1(D_r, \mathcal{L}(f)) \longrightarrow 0.$$

Let $[p]$ denote the action of p determined by the $\mathbf{Z}_p$-module structure on the formal group $\mathbb{G}_m$. From the proof of lemma 10, we obtain for every $n > 0$ a commutative diagram of analytic maps

$$(32) \qquad \begin{array}{ccc} D_{\rho_n} & = & D_{\rho_n} \\ \lambda \downarrow & & \downarrow [p]^n \\ D_{\rho_1 p^{n-1}} & \xrightarrow{\alpha} & D_{\rho_1/p} \end{array}$$

where α is given by the power series $\alpha(x) = \exp(p^n x)$. The radiuses attached to the discs appearing in the diagram, are arrived at via an elementary calculation, which we omit.

Definition 19. *The n-th analytic Kummer torsor $\mathcal{K}^{(n)}$ is the sheaf on $D_{\rho_1/p}$ of étale local sections of the morphism $[p]^n$ endowed with the natural translation action of the group $Ker[p]^n = Ker\lambda \cap D_{\rho_n}$. When there is no danger of confusion, we will usually omit the superscript n.*

Let ψ_n be the restriction of ψ to $\mathrm{Ker}[p]^n$. We form the associated bundle

$$\mathcal{K}_\psi := \mathcal{K} \times_{\mathrm{Ker}[p]^n} \psi_n.$$

Then diagram (32) says that there is an isomorphism

$$\alpha^* \mathcal{K}_\psi \simeq \mathcal{L}_{\psi | D_{\rho_1 p^{n-1}}}. \tag{33}$$

Therefore $f^*\alpha^*\mathcal{K}_\psi \simeq \mathcal{L}(f)_{|f^{-1}(D_{\rho_1 p^{n-1}})}$. Take n_0 big enough, so that $\rho_1 p^{n_0-1}$ is greater than $|a_m| r_0^m$. Then, for any $n \geq n_0$, we can select $r = (|a_m|^{-1}\rho_1 p^{n-1})^{1/m}$ in the exact sequence (31), and according to (33) we can rewrite the former as

$$0 \longrightarrow H^1_c(\mathbf{A}^1 - D_r, \mathcal{L}(f_0)) \longrightarrow H^1_c(\mathbf{A}^1, \mathcal{L}(f)) \longrightarrow H^1(D_r, f^*\alpha^*\mathcal{K}_\psi) \longrightarrow 0. \tag{34}$$

Next, recall that the logarithm λ is an isometry on the open disc of radius ρ_1. It follows that $[p]^n$ is an isomorphism on the open disc of radius $\rho_1 p^{-n}$. Let us write $\alpha(f(x)) = \exp(p^n f(x)) = \sum_{i\geq 1} \frac{1}{i!}(p^n f(x))^i$. By our choice of the radius r, we have: $|x| \leq r \Rightarrow |p^n f(x)| \leq \rho_1/p$. Hence we see that for i_0 sufficiently large, and $|x| \leq r$, the rest power series $h(x) = \sum_{i > i_0} \frac{1}{i!}(p^n f(x))^i$ has norm less than $\rho_1 p^{-n}$. Set $g(x) = \alpha(f(x)) - h(x)$. By the yoga of torsors

$$f^*\alpha^*\mathcal{K}_\psi \simeq g^*\mathcal{K}_\psi \otimes h^*\mathcal{K}_\psi. \tag{35}$$

By the choice of m_0, $h^*\mathcal{K}_\psi$ is a constant sheaf (of rank one), therefore $f^*\alpha^*\mathcal{K}_\psi \simeq g^*\mathcal{K}_\psi$.

8.2. Cohomological trivialities. We want to construct a section $s_\eta : D_{\rho_1/p} \to (\widehat{\mathcal{V}}_{n,k^\circ})_\eta$ for the analytic group homomorphism σ_η. We can proceed as follows. First, notice that the map $x \mapsto (x, 1, ..., 1)$ defines a section s_0 for the morphism $\Sigma_n : \mathbb{G}^n_{m,A} \to \mathbb{G}_{m,A}$. After formally completing, this is still a section $\widehat{s}_0$ for $\widehat{\Sigma}_n$. We have seen that the map σ factors as $\widehat{\mathcal{V}}_{n,E^\circ} \xrightarrow{\sigma_n} \mathrm{Spf}E^\circ\langle x_1, ..., x_n\rangle \xrightarrow{\epsilon} \mathrm{Spf}E^\circ\langle x\rangle$, and unwinding the definitions one checks easily that, after base change to $\mathrm{Spf}E^\circ$, $\widehat{s}_0$ induces a section $\widetilde{s}$ for ϵ. Thanks to proposition 27 we can set $s_\eta = \sigma_{n,\eta}^{-1}\widetilde{s}_\eta$. For later use, we point out that $s_\eta : (\mathrm{Spf}E^\circ\langle x\rangle)_\eta \to (\mathrm{Spf}E^\circ\langle x_1, ...x_n\rangle)_\eta$ is given by a sequence of n power series $(f_1(x), ..., f_n(x))$ all of which, by the maximum principle, have coefficients in E°. Therefore, s_η extends to a morphism of formal schemes $s : \mathrm{Spf}E^\circ\langle x\rangle \to \mathrm{Spf}E^\circ\langle x_1, ...x_n\rangle$ defined just by taking the same power series.

Let H be the kernel of σ, i.e. the preimage of the zero section $0_{E^\circ} : \mathrm{Spf}E^\circ \hookrightarrow \mathrm{Spf}E^\circ\langle x\rangle$. We obtain an isomorphism of analytic spaces over $D_{\rho_1/p}$

$$\begin{tikzcd}
H_\eta \times D_{\rho_1/p} \arrow[rr, "\gamma"] \arrow[dr, "p_2"'] & & (\widehat{\mathcal{V}}_{n,E^\circ})_\eta \arrow[dl, "\sigma_\eta"] \\
& D_{\rho_1/p}. &
\end{tikzcd}$$

Explicitly: $\gamma(h, x) = h + s(y)$ if $+$ denotes the group law in $\mathcal{V}_{n,E^\circ}$.

Let $\mathbb{O}_{n,\mathcal{V}}$ be the constant sheaf of $\mathbb{O}_n$-modules on $(\widehat{\mathcal{V}}_{n,E^\circ})_\eta$. We derive:

$$R\sigma_{\eta *}\mathbb{O}_{n,\mathcal{V}} \simeq Rp_{2*}R\gamma_*^{-1}\mathbb{O}_{n,\mathcal{V}} \simeq Rp_{2*}\mathbb{O}_{n,\mathcal{V}}.$$

Moreover, $R^i p_{2*}\mathbb{O}_{n,\mathcal{V}}$ is the constant sheaf $\mathcal{G}^i$ with stalk $\mathcal{G}^i_x \simeq H^i(H_\eta, \mathbb{O}_n)$ at all points $x \in D_{\rho_1/p}$. Therefore, the E_2-term of the Leray spectral sequence for p_2 can be computed as follows: $H^i(D_{\rho_1/p}, R^j p_{2*}\mathbb{O}_{n,\mathcal{V}}) = 0$ if $i > 0$ and $H^0(D_{\rho_1/p}, p_{2*}\mathbb{O}_{n,\mathcal{V}}) \simeq \mathcal{G}^i_0$. In particular the spectral sequence degenerates and we have $H^i((\widehat{\mathcal{V}}_{n,E^\circ})_\eta, \mathbb{O}_{n,\mathcal{V}}) \simeq \mathcal{G}^i_0$.

On the other hand, recall (proposition 27) that $(\widehat{\mathcal{V}}_{n,E^\circ})_\eta$ is isomorphic to a product of n discs of different radiuses. Thus $H^i((\widehat{\mathcal{V}}_{n,E^\circ})_\eta, \mathbb{O}_{n,\mathcal{V}})$ vanishes for $i \neq 0$ and is free of rank one for $i = 0$. We conclude

$$R^i\sigma_{\eta *}\mathbb{O}_{n,\mathcal{V}} \simeq R^i p_{2*}\mathbb{O}_{n,\mathcal{V}} \simeq \begin{cases} \mathbb{O}_n & i = 0 \\ 0 & i > 0 \end{cases} \tag{36}$$

The situation so far is summarized by the following diagram, in which the square is fibred

$$\begin{tikzcd} H_\eta \times D_{\rho_1/p} \arrow[r, "\sim"] \arrow[dr, "p_2"'] & (\widehat{\mathcal{V}}_{n,E^\circ})_\eta \arrow[d, "\sigma_\eta"] & H_\eta \times D_r \arrow[l, "\mathfrak{g}"'] \arrow[d, "\overline{\sigma}"] \\ & D_{\rho_1/p} & D_r. \arrow[l, "g"] \end{tikzcd} \tag{37}$$

Our target is to compute $H^1(D_r, g^*\mathcal{K}_\psi)$. We write

$$H^1(H_\eta \times D_r, \overline{\sigma}^* g^*\mathcal{K}_\psi) \simeq H^1(D_r, R\overline{\sigma}_*\overline{\sigma}^* g^*\mathcal{K}_\psi) \simeq H^1(D_r, g^*\mathcal{K}_\psi \otimes R\overline{\sigma}_*\mathbb{O}_n)$$

and notice that the last term is isomorphic to $H^1(D_r, g^*\mathcal{K}_\psi)$ by virtue of (36) and of the proper base change theorem. Hence we are reconduced to the study of $H^1(H_\eta \times D_r, \overline{\sigma}^* g^*\mathcal{K}_\psi) \simeq H^1(H_\eta \times D_r, \mathfrak{g}^*\sigma_\eta^*\mathcal{K}_\psi)$.

To this purpose, we go back to diagram (28). It is not hard to decide that the $\mathbf{Z}/p^n$-torsor $\mathcal{K}$ can be recovered as the sheaf of local sections of the morphism $\widetilde{\theta}_\eta$ in (28). We introduce another $\mathbf{Z}/p^n$-torsor, over the étale site of $(\widehat{\mathcal{V}}_{n,E^\circ})_\eta$, by taking the sheaf of local sections of $\widehat{\chi}_\eta$. Call $\mathbf{W}$ this torsor. It is clear that $\mathbf{W} \simeq \sigma_\eta^*\mathcal{K}$, and after taking the associated locally constant sheaves we have also $\mathbf{W}_\psi \simeq \sigma_\eta^*\mathcal{K}_\psi$.

8.3. Resolution of a singularity. In the last section we derived an isomorphism

$$H^1(D_r, f^*\alpha^*\mathcal{K}_\psi) \simeq H^1(H_\eta \times D_r, \mathfrak{g}^*\mathbf{W}_\psi). \tag{38}$$

It remains to compute the right-hand side of this equation, and for this we revert to formal schemes.

The polynomial map $g : D_r \to D_{\rho_1/p}$ extends to a morphism of formal schemes $\widehat{g} : \mathrm{Spf}E^\circ\langle x\rangle \to \mathrm{Spf}E^\circ\langle x\rangle$. To describe $\widehat{g}$ in detail, we find it convenient to rescale the coordinates:

$$\begin{tikzcd} D_r \arrow[r, "g"] & D_{\rho_1/p} \arrow[d, "\beta_2"] \\ D_1 \arrow[u, "\beta_1"] \arrow[r, "\widetilde{g}"] & D_1. \end{tikzcd}$$

Here $\beta_i(x) = b_i \cdot x$ $(i = 1, 2)$ are linear maps which identify the discs of radiuses r, respectively ρ_1/p, with discs of radius 1, and hence $\widetilde{g}(x) = b_2 g(b_1 x)$. Notice that it may be necessary to pass to some finite ramified extension in order to define β_i; since we only want geometric results, this is harmless. Write $f(x) = \sum_{j=0}^m a_j x^j$ and $g(x) = p^n f(x) + \sum_{i=2}^{i_0} \frac{1}{i!}(p^n f(x))^i$. Then

$$\widetilde{g}(x) = b_2 p^n f(b_1 x) + b_2 \sum_{i=2}^{i_0} \frac{1}{i!}(p^n f(b_1 x))^i.$$

Let us examine this expression: we have $b_2p^n f(b_1x) = b_2p^n a_m b_1^m x^m + \sum_{j=0}^{m-1} b_2 a_j b_1^j x^j$; due to our choice of the radius r, one sees easily that

$$(39) \qquad |b_2p^n a_m b_1^m| = 1 \qquad \text{and} \qquad |b_2 a_j b_1^j| < 1 \quad \text{for all } j < m.$$

We also need to bound the term $\frac{1}{i!}(p^n f(b_1x))^i$. Recall the following elementary estimate:

$$(40) \qquad |x| \leq \rho_1 \Rightarrow \left|\frac{1}{i!}x^i\right| \leq \rho_1 \qquad i \geq 1.$$

We know that $|x| \leq 1$ implies $|p^n f(b_1x)| \leq \rho_1/p$; select any $c \in E^a$ with $|c| = p$. Then, directly from (40) we derive: $|x| \leq \rho_1/p \Rightarrow |\frac{1}{i!}x^i c^i| \leq \rho_1 \Rightarrow |\frac{1}{i!}x^i| \leq \rho_1|c|^{-i} = (\rho_1/p)p^{1-i}$.

By the maximum principle, this says that all the coefficients of the polynomials $b_2\frac{1}{i!}(p^n f(b_1x))^i$ have norm strictly less than 1. We sum up our findings in the following

Lemma 19. *The polynomial $\widetilde{g}$ defines a map $\widehat{g} : SpfE^\circ\langle x\rangle \to SpfE^\circ\langle x\rangle$ which provides a formal model for the morphism g of analytic spaces. Moreover, the induced map $\widehat{g}_s$ on the special fibre depends only on the leading coefficient of the polynomial f which enters in the definition of g.*

Proof. : In order to give a map of formal schemes, we only need to know that the coefficients of $\widetilde{g}$ are in E°, and this we know by the argument above, and by (39). The same estimates show that only the leading term of $f(x)$ determines the reduction of $\widetilde{g}(x)$ modulo the maximal ideal, since the other terms only contribute to form coefficients which have norm strictly less than 1. □

Next we define $\mathcal{Y}$ to be the fibre product in the following fibre diagram

$$(41) \qquad \begin{array}{ccc} \widehat{\mathcal{V}}_{n,E^\circ} & \xleftarrow{\widehat{\theta}} & \mathcal{Y} \\ {\scriptstyle\sigma}\downarrow & & \downarrow \\ \mathrm{Spf}E^\circ\langle x\rangle & \xleftarrow{\widehat{g}} & \mathrm{Spf}E^\circ\langle x\rangle. \end{array}$$

Notice that on the generic fibres, diagram (41) coincides with the square in diagram (37).

Definition 20. *The n-th formal Witt torsor $\widehat{\mathbf{W}}^{(n)}$ on the étale site of $\widehat{\mathcal{V}}_{n,E^\circ}$ is the sheaf of local sections of the map $\widehat{\chi}$. As usual we drop the superscript when no confusion is likely to arise.*

Again we can form the associated locally constant sheaf of Λ-modules $\widehat{\mathbf{W}}_\psi$. On the one hand $\widehat{\mathbf{W}}_\psi$ restricts to a sheaf $\widehat{\mathbf{W}}_{\psi,s}$ on the special fibre $(\widehat{\mathcal{V}}_{n,E^\circ})_s \simeq W_{n,E^\circ/\mathfrak{m}}$; on the other hand, there exist morphisms of sites

$$(\widehat{\mathcal{V}}_{n,E^\circ})_{\eta,et} \xleftarrow{\mu} (\widehat{\mathcal{V}}_{n,E^\circ})_{\eta,qet} \xrightarrow{\nu} (\widehat{\mathcal{V}}_{n,E^\circ})_{et}$$

from the so-called quasi-étale topology of the analytic space $(\widehat{\mathcal{V}}_{n,E^\circ})_\eta$ to the étale site of the formal scheme $\widehat{\mathcal{V}}_{n,E^\circ}$ and respectively, to the étale site of $(\widehat{\mathcal{V}}_{n,E^\circ})_\eta$; the latter morphism μ is just the natural restriction map. See [B3] for details, where it also shown that the cohomology computed in the quasi-étale site is compatible with étale cohomology, i.e. that for any formal scheme $\mathcal{X}$ and any sheaf F on $\mathcal{X}_{\eta,et}$ one has

$$H^\bullet(\mathcal{X}_{\eta,qet}, \mu^\bullet F) \simeq H^\bullet(\mathcal{X}_{\eta,et}, F).$$

With this notation, it is easy to check that

$$\mu_* \nu^* \widehat{\mathbf{W}}_\psi \simeq \mathbf{W}_\psi. \tag{42}$$

We would like to apply the theory of vanishing cycles for formal schemes of [B3]. For the case at hand, Berkovich's theory gives a canonical isomorphism

$$H^1(H_\eta \times D_r, \mathfrak{g}^* \mathbf{W}_\psi) \simeq H^1(\mathcal{Y}_s, R\Psi_\eta \mathfrak{g}^* \mathbf{W}_\psi).$$

However, this isomorphism would be really useful, only if we knew that $\mathcal{Y}$ is smooth. In that case, a simple argument would tell us that $R\Psi_\eta \mathfrak{g}^* \mathbf{W}_\psi \simeq \widehat{\mathfrak{g}}_s^* \widehat{\mathbf{W}}_{\psi,s}$, and this would allow us to conclude quickly. Since we do not know whether $\mathcal{Y}$ is smooth, we will find instead a morphism $\tau : \mathcal{Z} \to \mathcal{Y}$ from a smooth $\mathcal{Z}$, which induces an isomorphism on the generic fibres. Then we will replace $\widehat{\mathfrak{g}}^* \widehat{\mathbf{W}}_\psi$ by $\tau^* \widehat{\mathfrak{g}}^* \widehat{\mathbf{W}}_\psi$.

Here is how we do it. Composing the chain of epimorphisms $\widehat{\mathcal{V}}_{n,E^\circ} \to \widehat{\mathcal{V}}_{n-1,E^\circ} \to \dots \to \widehat{\mathcal{V}}_{1,E^\circ}$ we obtain a map $q : \widehat{\mathcal{V}}_{n,E^\circ} \to \widehat{\mathcal{V}}_{1,E^\circ}$. Let $0_{E^\circ} : \mathrm{Spf} E^\circ \to \widehat{\mathcal{V}}_{1,E^\circ}$ be the zero section of the formal group scheme $\widehat{\mathcal{V}}_{1,E^\circ}$ and denote by H_0 the preimage $q^{-1}(0_{E^\circ}) \subset \widehat{\mathcal{V}}_{n,E^\circ}$. H_0 is a smooth formal scheme; in fact, by the remark after proposition 25, the map q is just a linear projection in certain coordinates.

Gathering our scattered constructions we can put together a diagram

$$\begin{tikzcd}
\widehat{\mathcal{V}}_{n,E^\circ} \arrow[r, "\sigma_n"] \arrow[d, equal] & \mathrm{Spf} E^\circ\langle x_1, \dots, x_n\rangle \arrow[r, shift left, "\epsilon"] \arrow[d, "j'_n"'] & \mathrm{Spf} E^\circ\langle x\rangle \arrow[l, shift left, "\widetilde{s}"] \arrow[d, "j'"'] \\
\widehat{\mathcal{V}}_{n,E^\circ} \arrow[r, "\widehat{\alpha}_{\mathcal{V}}"] & \widehat{\mathbb{G}}^n_{m,E^\circ} \arrow[r, shift left, "\widehat{\Sigma}_n"] & \widehat{\mathbb{G}}_{m,E^\circ} \arrow[l, shift left, "\widehat{s}_0"]
\end{tikzcd} \tag{43}$$

where $\widehat{s}_0$ (resp. $\widetilde{s}$) is a section for $\widehat{\Sigma}_n$ (resp. ϵ) and we have defined σ as the composition $\widehat{\sigma}_n \epsilon$. Moreover, $s : \mathrm{Spf} E^\circ\langle x\rangle \to \widehat{\mathcal{V}}_{n,E^\circ}$ is a morphism with the property that $\sigma_{n,\eta} s_\eta = \widetilde{s}_\eta$ and the vertical morphisms j' and j'_n restrict on the generic fibres to imbeddings of analytic spaces.

We define a morphism of formal group schemes $\delta : H_0 \to H$ by

$$\delta(h) = h - s\sigma(h)$$

where of course the $-$ is given by the group structure of $\widehat{\mathcal{V}}_{n,E^\circ}$.

Lemma 20. *The morphism $\delta_\eta : H_{0,\eta} \to H_\eta$ is an isomorphism.*

Proof. Let us introduce the auxiliary morphism $\widetilde{\delta} : \widehat{\mathbb{G}}^n_{m,E^\circ} \to \widehat{\mathbb{G}}^n_{m,E^\circ}$ defined as $x \mapsto x - \widehat{s}_0 \widehat{\Sigma}_n(x)$. (This time "$-$" is given in terms of the group law on $\widehat{\mathbb{G}}^n_{m,E^\circ}$). As $\sigma_{n,\eta}$ is an isomorphism (proposition 27), to prove the claim it suffices show that $\sigma_{n,\eta} \delta_\eta \sigma_{n,\eta}^{-1}$ is an isomorphism $\sigma_{n,\eta}(H_{0,\eta}) \to \sigma_{n,\eta}(H_\eta)$. Using diagram (43), this is in turns equivalent to showing that $\widetilde{\delta}_\eta$ restricts to an isomorphism $\widehat{\alpha}_{\mathcal{V},\eta}(H_{0,\eta}) \to \widehat{\alpha}_{\mathcal{V},\eta}(H_\eta)$. From theorem 15 one can easily check that

$$\widehat{\alpha}_{\mathcal{V},\eta}(H_{0,\eta}) = \widehat{\alpha}_{\mathcal{V},\eta}(\widehat{\mathcal{V}}_{n,E^\circ})_\eta \cap \{(x_1, \dots, x_n) \in (\widehat{\mathbb{G}}^n_{m,E^\circ})_\eta \quad | \quad x_1 = 1\}.$$

We will show that $\widetilde{\delta}_\eta$ restricts to an isomorphism of the space $\{(1, x_2, \dots, x_n) \in (\widehat{\mathbb{G}}^n_{m,E^\circ})_\eta\}$ onto its image; this clearly implies the lemma. To this purpose we can write down the coordinate expression for $\widetilde{\delta}_\eta$:

$$\widetilde{\delta}_\eta : (1, x_2, \dots, x_n) \mapsto (1, x_2, \dots, x_n) - (\widehat{\Sigma}_{n,\eta}(1, x_2, \dots, x_n), 1, \dots, 1) = (x_2^{-p} \cdot \dots x_n^{-p^{n-1}}, x_2, \dots, x_n)$$

from which the claim is immediate. $\square$

Let $\omega : H_0 \times \mathrm{Spf}E^\circ\langle x\rangle \to \widehat{\mathcal{V}}_{n,E^\circ}$ be given by $(h,x) \mapsto \delta(h)+s(x)$ (addition taken in $\mathcal{V}_{n,E^\circ}$). From lemma 20 it follows easily that ω_η is an isomorphism. Finally define $\mathcal{Z}$ as the fibre product in the following fibre diagram

$$\begin{array}{ccccc} \mathcal{Z} & \xrightarrow{\tau} & \mathcal{Y} & \longrightarrow & \mathrm{Spf}E^\circ\langle x\rangle \\ \downarrow & & \downarrow{\scriptstyle\widehat{\mathfrak{g}}} & & \downarrow{\scriptstyle\widehat{g}} \\ H_0 \times \mathrm{Spf}E^\circ\langle x\rangle & \xrightarrow{\omega} & \widehat{\mathcal{V}}_{n,E^\circ} & \xrightarrow{\sigma} & \mathrm{Spf}E^\circ\langle x\rangle. \end{array}$$

Notice that the composition $\sigma\omega$ is just the projection on the second factor. Thus, we see that $\mathcal{Z}$ is isomorphic to $H_0 \times \mathrm{Spf}E^\circ\langle x\rangle$, and in terms of this isomorphism the composition $\mathcal{Z} \to \mathcal{Y} \to \mathrm{Spf}E^\circ\langle x\rangle$ becomes the projection on the second factor. In particular, $\mathcal{Z}$ is a smooth formal scheme, as required.

8.4. Repatching.

Lemma 21. *With the notation above we have*

$$R^i\Psi_\eta\tau_\eta^*\mathfrak{g}^*\mathbf{W}_\psi = \begin{cases} \tau_s^*\widehat{\mathfrak{g}}_s^*\widehat{\mathbf{W}}_{\psi,s} & i=0 \\ 0 & i>0. \end{cases}$$

Proof. Define $\mathcal{Z}_1$ as the fibre product in the following fibre diagram

$$\begin{array}{ccc} \mathcal{Z}_1 & \xrightarrow{\zeta_2} & \widehat{\mathcal{W}}_{n,E^0} \\ {\scriptstyle\zeta_1}\downarrow & & \downarrow{\scriptstyle\widehat{\chi}} \\ \mathcal{Z} \xrightarrow{\tau} & \mathcal{Y} & \xrightarrow{\widehat{\mathfrak{g}}} \widehat{\mathcal{V}}_{n,E^0}. \end{array}$$

Then clearly

$$\zeta_1^*\tau_\eta^*\widehat{\mathfrak{g}}^*\widehat{\mathbf{W}}_\psi \simeq \zeta_2^*\widehat{\chi}^*\widehat{\mathbf{W}}_\psi \simeq \mathbf{O}_{n,\mathcal{Z}_1}.$$

According to corollary 4.5 of [B3] we have

$$R^i\Psi_\eta\zeta_{2,\eta}^*\widehat{\chi}_\eta^*\mathbf{W}_\psi \simeq \zeta_{1,s}^*R^i\Psi_\eta\tau_\eta^*\mathfrak{g}^*\mathbf{W}_\psi$$

and since $\mathcal{Z}_1$ is smooth, by [B3] Corollary 5.4 we obtain

$$R^i\Psi_\eta\mathbf{O}_{n,\mathcal{Z}_{1,\eta}} = 0 \qquad i>0.$$

Since $\zeta_1 : \mathcal{Z}_1 \to \mathcal{Z}$ is an étale covering, the assertion follows. $\square$

Lemma 22. *Let $f_0 : \mathbf{A}^1 \to \mathbf{A}^1$ be the morphism $x \mapsto x^m$. Then $H_c^1(\mathbf{A}^1_{\widehat{E}^a}, \mathcal{L}_\psi(f_0))$ is a free $\mathbf{O}_n$-module of rank $m-1$.*

Proof. After a finite extension, we can assume that $\mathbf{O}_n$ contains all the m-th roots of unity. Let $j : \mathbb{G}_m \hookrightarrow \mathbf{A}^1$ be the obvious imbedding. Then we have

$$H_c^1(\mathbf{A}^1_{\widehat{E}^a}, \mathcal{L}_\psi(f_0)) \simeq H_c^1(\mathbf{A}^1_{\widehat{E}^a}, f_{0!}\mathcal{L}_\psi(f_0)) \simeq \bigoplus_{\chi:\mathbf{Z}/m\mathbf{Z}\to\mathbf{O}_n} H_c^1(\mathbf{A}^1_{\widehat{E}^a}, j_!\mathcal{K}_\chi \otimes \mathcal{L}_\psi) \tag{44}$$

where the sum is indexed by the characters of the group

$$\pi_1(\mathbb{G}_m,\overline{x})/f_{0*}\pi_1(\mathbb{G}_m,\overline{x}) \simeq \mathbf{Z}/m\mathbf{Z}.$$

Now, by proposition 17 the summand corresponding to the trivial charcter gives no contribution in (44). Moreover, by proposition 29, each of the remaining terms is a free $\mathbb{O}_n$-module of rank one. The assertion follows. □

Theorem 16. *The cohomology group $H^1_c(\mathbf{A}^1_{\widehat{E}^a}, \mathcal{L}(f))$ is a free $\mathbb{O}_n$-module whose rank is equal to $deg(f) - 1$.*

Proof. From lemma 21 we derive a canonical isomorphism

$$H^1(\mathcal{Z}_\eta, \tau_\eta^* \mathfrak{g}^* \mathbf{W}_\psi) \simeq H^1(\mathcal{Z}_s, \tau_s^* \widehat{\mathfrak{g}}_s^* \widehat{\mathbf{W}}_{\psi,s}).$$

Since ω_η is an isomorphism, so is τ_η, hence the left-hand side in the above equation computes $H^1(\mathcal{Y}, \mathfrak{g}^*\mathbf{W}_\psi)$ and by (38) this is isomorphic to $H^1(D_r, f^*\alpha^*\mathcal{K}_\psi)$.

On the other hand, the right-hand side depends only on the special fibre of the map $\widehat{\mathfrak{g}}$. By lemma 19, we know that $\widehat{\mathfrak{g}}_s$ is determined by the leading coefficient of f. With the notation of section 8.1, this implies that there exists an isomorphism

$$H^1(D_r, f^*\alpha^*\mathcal{K}_\psi) \simeq H^1(D_r, f_0^*\alpha^*\mathcal{K}_\psi).$$

Comparing with (34) we obtain a short exact sequence

$$0 \longrightarrow H^1_c(\mathbf{A}^1 - D_r, \mathcal{L}(f_0)) \longrightarrow H^1_c(\mathbf{A}^1, \mathcal{L}(f)) \longrightarrow H^1(D_r, f_0^*\alpha^*\mathcal{K}_\psi) \longrightarrow 0$$

which says that $\dim_{\mathbf{B}_\ell} H^1_c(\mathbf{A}^1, \mathcal{L}(f)) = \dim_{\mathbf{B}_\ell} H^1_c(\mathbf{A}^1, \mathcal{L}(f_0))$; after a linear change of coordinate on $\mathbf{A}^1$ we can assume that $f_0(x) = x^m$, and then the claim follows from lemma 22. □

Corollary 6. *For any positive real number r let $U_r = \mathbf{A}_1 - D_r$. Then for all $r >> 0$ we have*

$$H^1_c(U_{r,\widehat{E}^a}, \mathcal{L}_\psi(f)) = 0.$$

Proof. Let $\Delta_r \subset D_r$ be the open disc of radius r centered at the origin. Then by [B1] Proposition 5.2.9 it follows

$$H^1_c(\mathbf{A}^1_{\widehat{E}^a}, \mathcal{L}_\psi(f)) \simeq \varinjlim_{r>0} H^1_c(\Delta_{r,\widehat{E}^a}, \mathcal{L}_\psi(f)).$$

Since the cohomology of $\mathcal{L}(f)$ is a finitely generated module, the limit is already attained for some $r >> 0$. By the usual short exact sequence we derive $H^1_c(\mathbf{A}^1 - \Delta_r, \mathcal{L}(f)) = 0$ for $r >> 0$. Since $H^1_c(U_{r,\widehat{E}^a}, \mathcal{L}_\psi(f)) \hookrightarrow H^1_c(\mathbf{A}^1 - \Delta_r, \mathcal{L}(f))$, the conclusion follows. □

Definition 21. *Let C be an open curve defined over E and let s be any E-rational point on the smooth compactification $\overline{C}$ of C. Let F be a locally constant sheaf of $\mathbb{O}_n$-modules of finite rank on C. We say that F has meromorphic ramification at the point s if the sheaf F_{η_s} over η_s is trivialized over some covering $\mathcal{G} \in \underline{Cov}^{mer}(\eta_s)$.*

Proposition 28. *Let F be a locally constant sheaf of finitely generated $\mathbb{O}_n$-modules on $\mathbb{G}_{m,E}$ which is trivialized on some meromorphic covering of $\mathbb{G}_{m,E}$. Then $H^1_c(\mathbf{A}^1_{\widehat{E}^a}, F)$ is a finitely generated $\mathbb{O}_n$-module.*

Proof. It follows from theorem 8 that we can find an integer N such that $\phi_N^* F$ extends to a locally constant sheaf on $\mathbf{A}^1_E$ with meromorphic ramification at the point $\infty \in \mathbb{P}^1_E$. Then, after a finite extension of $\mathbb{O}_n$, by corollary 3 in section 4.3, $\phi_N^* F$ is a direct sum of sheaves of the type $\mathcal{L}(f_i) \otimes M_i$ for various polynomials f_i and finitely generated $\mathbb{O}_n$-modules M_i. Moreover F is a direct summand in the sheaf $\phi_{N*}\phi_N^* F \simeq \phi_{N*}(\oplus_i \mathcal{L}(f_i) \otimes M_i)$. Hence the claim follows easily from theorem 16. □

An argument as in the proof of corollary 6 yields

Corollary 7. *Let F be a sheaf like in proposition 28. Then for all $r >> 0$ we have*

$$H_c^1(U_{r,\widehat{E}^a}, F) = 0.$$

□

Lemma 23. *Let X be a compact analytic variety and F a locally constant sheaf of finitely generated $\mathbb{O}_n$-modules on X. Then we can find a finite subring $A \subset \mathbb{O}_n$ and a locally constant sheaf F' of A-modules on X such that $F \simeq F' \otimes_A \mathbb{O}_n$.*

Proof. First of all, since X is compact, we can find a finite covering $\bigcup_i U_i = X$ by open subsets, and for each i a finite étale morphism $V_i \to U_i$ such that $G = F_{|V_i}$ is the constant sheaf associated to a certain finitely generated $\mathbb{O}_n$-module M_i. . The descent data for F from V_i to U_i is then essentially a finite set of automorphisms of M_i. These automorphisms are then defined already over some finite subring $A_i \subset \mathbb{O}_n$. Hence we can find a locally constant sheaf F_i of A_i-modules on U_i such that $F_{|U_i} = F_i \otimes_{A_i} \mathbb{O}_n$.

Similarly, let $U_{ij} = U_i \cap U_j$, so that F is defined by a cocycle system of morphisms $\phi_{ij} : (F_i \otimes_{A_i} \mathbb{O}_n)_{|U_{ij}} \to (F_j \otimes_{A_j} \mathbb{O}_n)_{|U_{ij}}$. Again, these morphisms are already defined on some big finite subring $A_{ij} \supset A_i + A_j$ and the claim follows easily. □

We come now to the main result of this chapter.

Theorem 17. *Let C be an open curve over E and F a locally constant sheaf of $\mathbb{O}_n$-modules of finite rank on C. Suppose that F is meromorphically ramified at all the points in $\overline{C} - C$. Then $H_c^1(C_{\widehat{E}^a}, F)$ is a finitely generated $\mathbb{O}_n$-module.*

Proof. Let $B_1, .., B_n$ be n small discs around each of the points $s_1, ..., s_n$ of $\overline{C} - C$. For each disc, take an imbedding

$$j_i : B_i \hookrightarrow \mathbb{P}_E^1$$

such that the image of s_i is ∞. Then it follows from proposition 8 that we can find sheaves F_i on $\mathbb{G}_m$ such that: 1) $j_i^* F_i \simeq F_{|B_i - \{s_i\}}$ and 2) F_i trivializes on a meromorphic covering of $\mathbb{G}_m$.

Now, it follows easily from corollary 7 that, after replacing B_i by some smaller discs, we obtain

$$H_c^1(B_i - \{s_i\}, F) = 0.$$

On the other hand, the complement $X = C - \bigcup_i B_i$ is a compact affinoid domain. Take a finite subring $A \subset \mathbb{O}_n$ and a sheaf F' of A-modules on X as in lemma 23, so that $F_{|X} = F' \otimes_A \mathbb{O}_n$. Then we have

$$H_c^1(X, F) \simeq H_c^1(X, F') \otimes_A \mathbb{O}_n.$$

But it follows from [B3] Corollary 5.6 that the right-hand side is a finite A-module, and this implies the theorem. □

Remark: One may wonder whether the condition on the ramification on F is really necessary for the finiteness of the cohomology. We will not attempt here a precise analysis, but we give an example to demonstrate the general situation.

We construct inductively a sequence of polynomials in one variable $f_i(t)$ ($i = 1, 2, ...$) and positive real numbers $r_1 < r_2 < ...$ such that $\lim_{i\to\infty} r_i = \infty$ and $\lim_{i\to\infty} f_i = f$ is an entire power series on the affine line $\mathbf{A}_k^1$. Suppose f_i of degree i and r_i have already been constructed, with the property that $H_c^1(\mathbf{A}^1, \mathcal{L}(f_i)) = H_c^1(\Delta_{r_i}, \mathcal{L}(f_i))$ is a free $\mathbb{O}_n$-module of rank equal to $i-1$.

Choose an element $\delta \in k^\times$ of norm small enough so that $|\delta| \cdot r_i < \rho_1$. Set $f_{i+1}(t) = (1+\delta t)f_i(t)$. Then it is clear that

$$\mathcal{L}(f_{i+1})_{|\Delta_{r_i}} \simeq \mathcal{L}(f_i)_{|\Delta_{r_i}}$$

and as a consequence we get an imbedding

$$H^1_c(\mathbf{A}^1, \mathcal{L}(f_i)) \hookrightarrow H^1_c(\mathbf{A}^1, \mathcal{L}(f_{i+1})).$$

On the other hand, the polynomial $f_{i+1}(t)$ has degree $i+1$, hence by the usual argument (and by theorem 16) we find $r_{i+1} > 0$ such that $H^1_c(\Delta_{r_{i+1}}, \mathcal{L}(f_{i+1}))$ is free of rank i. Clearly the sequence $f_1(t), f_2(t), \ldots$ converges to some $f(t)$ and the cohomology of $\mathcal{L}(f)$ cannot be finitely generated.

What we learn from the above counterexample is that the finiteness properties of analytic étale cohomology have much to do with the ramification of the coefficient sheaf. This should be contrasted with the case of algebraic étale cohomology, where the finiteness properties are completely unrelated to ramification. I do not know whether this phenomenon has analogues in any one of the various sheaf theories currently available.

As for the case where finiteness does hold, we should remark that actually we expect a much more precise statement than theorem 17. Recall that in positive characteristic, the Euler-Poincaré characteristic of an étale sheaf is predicted by the Grothendieck-Ogg-Shafarevich formula.

Noticing that, inside the class of meromorphically ramified sheaves, the natural analogue of the Grothendieck-Ogg-Shafarevich formula makes perfect good sense, we are led to the following

Conjecture 2. *Let C be an open curve defined over E, and F a locally constant sheaf of $\mathbb{O}_n$-modules of finite rank on C. Suppose that F has at most meromorphic ramification at each of the points of $\overline{C} - C$. Then the Euler-Poincaré characteristic of F is given by the formula*

$$\chi_c(C_{\widehat{E}^a}, F) = rk(F)\chi(C) - \sum_{s \in \overline{C} - C} sw_s(F_{\eta_s})$$

where $\chi(C)$ is the Euler-Poincaré characteristic of C.

We remark that for a sheaf of the form $\mathcal{L}(f)$ (f some polynomial map on $\mathbf{A}^1$), proposition 9 gives $sw_\infty(\mathcal{L}(f)_{\eta_\infty}) = \deg f$. Then theorem 16 can be restated as

Corollary 8. *Conjecture* 2 *holds true for all the sheaves of the form $\mathcal{L}(f)$ as in theorem* 16. □

This is the reason why we spent considerable effort in computing the exact rank of the cohomology of $\mathcal{L}(f)$, while a simpler argument would have sufficed to prove its finiteness. In a sequel to this paper we plan to show how to derive the full conjecture 2 from corollary 8 and the principle of the stationary phase which is object of the next chapter.

9. Stationary phase

9.1. Vanishing at infinity. In this section we prove a vanishing result which will be used in the next section. The principle of the stationary phase captures special features of the Fourier transform on rank one vector spaces. Hence here the base variety S is reduced to a point and both E and its dual E' are affine spaces of dimension one, identified with $\mathbf{A}^1_E$.

Fix some integer $n > 0$ and let $\psi : G_\infty \to \mathbb{O}_n^\times$ be a non-trivial character. $\mathcal{L} = \mathcal{L}_\psi$ denotes the rank one locally constant sheaf of $\mathbb{O}_n$-modules on $\mathbb{G}_a = \mathbb{G}_{a,E}$ attached to the Lubin-Tate torsor and the non-trivial character ψ.

We choose linear coordinates x and y on the first and second factor of $\mathbf{A}^1_E \times \mathbf{A}^1_E$ and a linear coordinate t on $\mathbb{G}_a$. Then the dual pairing $\langle , \rangle$ of the previous section reduces to a map $\mu : \mathbf{A}^1 \times \mathbf{A}^1 \to \mathbb{G}_a$ defined by the ring homomorphism $E[t] \to E[x,y]$ which sends t to xy. For a complex $K^{\cdot}$ of $\mathbb{O}_n$-modules on $\mathbf{A}^1$, the Fourier transform in degree q is then the functor

$$\mathcal{F}^q(K^{\cdot}) = R^{q+1}\mathrm{pr}'_!(\mathrm{pr}^* K^{\cdot} \otimes \mu^* \mathcal{L})$$

where $\mathrm{pr}, \mathrm{pr}'$ are the two projections of $\mathbf{A}^1 \times \mathbf{A}^1$ on the two factors.

We apply the constructions of section 4.2 to the germ of analytic space $(C, s) = (\mathbb{P}^1_E, \infty)$. We denote by $\mathbf{S}$ the pro-analytic space associated to this germ. Also, let $\mathbf{X}$ denote the pro-analytic space $(\mathbf{A}^1_E \times \mathbb{P}^1_E) \times_{\mathbf{P}^1_E} \mathbf{S}$. The sheaf $\mu^*\mathcal{L}$ induces a sheaf on $\mathbf{X}_{\eta_\infty}$, which we will denote by the same name. Then for each $q \geq 0$ we may form $R^q\Psi^{mer}_{\eta_\infty}(\mu^*\mathcal{L})$, which is a sheaf on $\mathbf{X}_{\overline{\infty}} = \mathbf{A}^1_{\widehat{E}^a}$.

A bit more generally, suppose that E is the completion of an algebraic extension of a complete subfield E_0 containing k_∞. All the varieties and sheaves introduced above are obtained by base change from corresponding objects defined over E_0, and we can consider the functor $R\Psi^{mer}_{\eta_\infty, E_0}$.

Theorem 18. *With reference to the notation above,* $R^q\Psi^{mer}_{\eta_\infty, E_0}(\mu^*\mathcal{L}) = 0$ *for all* $q \geq 0$.

Proof. The proof is basically a variation of the proof of theorem 12 (with the two affine axes swapped in $\mathbf{A}^1 \times \mathbf{P}^1 \times \mathbf{A}^1$). Thanks to proposition 13, it suffices to consider the case $E = E_0$, and hence we need only to study $R^q\Psi^{mer}_{\eta_\infty}(\mu^*\mathcal{L})$. We will show that the stalk of $R^q\Psi^{mer}_{\eta_\infty}(\mu^*\mathcal{L})$ vanishes at all points $p \in \mathbf{A}^1$.

Let $\mathbf{Y} = (\mathbf{A}^1 \times \mathbf{P}^1 \times \mathbf{A}^1) \times_{\mathbf{P}^1} \mathbf{S}$. We define a map $\tau : \mathbf{A}^1 \times \mathbb{P}^1 \times \mathbf{A}^1 \to \mathbf{A}^1 \times \mathbf{P}^1$ by letting $(x, y, z) \mapsto (x + z, y)$. Then τ induces a smooth map of pro-analytic spaces $\mathbf{Y} \to \mathbf{X}$ which we denote again by τ. Proposition 11 applies and we obtain

$$\tau^*_{\bar{s}}(R^q\Psi^{mer}_{\eta_\infty}(\mu^*\mathcal{L})) \simeq R^q\Psi^{mer}_{\eta_\infty}(\tau^*_{\eta_\infty}\mu^*\mathcal{L}).$$

In particular

$$(R^q\Psi^{mer}_{\eta_\infty}(\mu^*\mathcal{L}))_p \simeq (R^q\Psi^{mer}_{\eta_\infty}(\tau^*_{\eta_\infty}\mu^*\mathcal{L}))_{(0,p)}.$$

To determine the stalk at $(0,p)$ of the right-hand side, we will use the formula 14 of section 4.2. With reference to the notation from section 4.2 we have

$$(R^q\Psi^{mer}_{\eta_\infty}(\tau^*_{\eta_\infty}\mu^*\mathcal{L}))_{(0,p)} \simeq \varinjlim_{T \in \mathcal{I}_\infty} \varinjlim_{U_T} H^q(\mathrm{j}_T^{-1}U_T, (\tau^*\mu^*\mathcal{L})_T)$$

where $(\tau^*\mu^*\mathcal{L})_T$ denotes the restriction of $\tau^*\mu^*\mathcal{L}$ to $\mathbf{A}^1 \times \mathcal{G}_T \times \mathbf{A}^1$ and U_T ranges on all the étale neighborhoods of $(0, \infty, p)$ inside $\mathbf{A}^1_{E_T} \times \mathbf{P}^1_{E_T} \times \mathbf{A}^1_{E_T}$. Let $\mathcal{C}_T$ be the partially ordered set consisting of all such U_T and let $\mathcal{C} = \bigcup_{T \in \mathcal{I}_\infty} \mathcal{C}_T$.

Next we introduce the family $\mathcal{C}^\delta_T$ consisting of all the varieties of the form $B_T \times W_T$ such that

1) B_T is an open disc in $\mathbf{A}^1_{E_T}$, of radius r_B and centered at 0, and $W_T \xrightarrow{\phi} \mathbf{P}^1_{E_T} \times \mathbf{A}^1_{E_T}$ is an étale neighborhood of $(\infty, p) \in \mathbf{P}^1_{E_T} \times \mathbf{A}^1_{E_T}$;

2) the image $\phi(W_T)$ is contained in an open subset of the form $B' \times N(p)$, with B' an open disc of radius r_W around ∞ and $N(p)$ some fixed open neighborhood of p;

3) the ratio r_B/r_W is equal to the constant δ.

Lemma 24. *For any real number* $\delta > 0$ *the family* $\mathcal{C}^\delta = \bigcup_{T \in \mathcal{I}_\infty} \mathcal{C}^\delta_T$ *is cofinal in* $\mathcal{C}$.

Proof. This is of course just a special case (up to swapping the axes) of lemma 15 of section 6.1, with $S = \mathrm{Spec}E_T$. $\square$

Fix a real number δ strictly greater than $\rho(\psi, t)$. Let $B_T \times W_T \in \mathcal{C}_T^\delta$ be any neighborhood as above, and set $\mathcal{W}_T = W_T \times_{\mathbf{P}^1} \mathcal{G}_T$. In view of the lemma, the theorem will follow if we show that

$$H^q(B_T \times \mathcal{W}_T, (\tau^*\mu^*\mathcal{L})_T) = 0 \qquad (q \geq 0). \tag{45}$$

Let $\mathrm{pr}_{23} : B_T \times \mathcal{W}_T \to \mathcal{W}_T$ be the projection. Define $\mu' : \mathbf{A}^1 \times \mathbf{A}^1 \to \mathbb{G}_a$ by setting $(y, z) \mapsto yz$. An application of the Yoga of torsors gives us the isomorphism

$$(\tau^*\mu^*\mathcal{L})_T \simeq \mathrm{pr}_{12}^*\mathcal{L}(\mu) \otimes \mathrm{pr}_{23}^*\mathcal{L}(\mu').$$

Now we can proceed exactly as in the proof of theorem 12 and conclude that $R\mathrm{pr}_{23*}(\tau^*\mu^*\mathcal{L})_T = 0$, which, by virtue of the Leray spectral sequence for pr_{23}, implies (45). □

9.2. p-adic stationary phase. We continue with the notation of section 9.1. Let $\overline{\mathbf{X}}$ be the $\mathbf{S}$-space $(\mathbf{P}_E^1 \times \mathbf{P}_E^1) \times_{\mathbf{P}_E^1} \mathbf{S} = \mathbf{P}_E^1 \times \mathbf{S}$ so that there is an embedding of $\mathbf{S}$-spaces $\mathbf{X} \to \overline{\mathbf{X}}$. We have two natural projections

$$\begin{tikzcd} & \overline{\mathbf{X}} \arrow[dl, "\overline{\mathrm{pr}}"'] \arrow[dr, "\overline{\mathrm{pr}}'"] & \\ \mathbf{P}_E^1 & & S. \end{tikzcd}$$

Given an E-rational point $s \in \mathbf{A}_E^1$, we will consider also the germ $(\mathbf{A}_E^1, s)$ and the associated analytic spaces $\mathbf{A}_E^1(s)$ and η_s. For any sheaf F of $\mathbb{O}_n$-modules on $\mathbf{A}_E^1$ we will let $F(s) = H^0(\mathcal{G}_s^{mer}, F_{\eta_s})$ which, according to proposition 10, carries a natural structure of $\pi_1^{mer}(\eta_s)$-module.

For a given sheaf G on $\mathbf{A}_E^1 \times \mathbf{A}_E^1$ we denote by $G_!$ the extension by zero of G to $\mathbf{P}_E^1 \times \mathbf{A}_E^1$; then $G_!$ determines a unique sheaf on $\overline{\mathbf{X}}_{\eta_\infty}$. We are interested in studying complexes of the form

$$\mathbb{K}_F = R\Psi_{\eta_\infty}^{mer}((\mathrm{pr}^* F)_! \otimes (\mu^*\mathcal{L})_!)$$

where F is a sheaf on $\mathbf{A}_E^1$.

Lemma 25. *Suppose that F is locally constant on the complement of a finite set $S \subset \mathbf{A}_E^1$ and that the stalks of F at all points are finitely generated $\mathbb{O}_n$-modules. Then $\mathbb{K}_F$ vanishes on the complement of $S \cup \{\infty\}$. If, moreover, $S \subset \mathbf{A}_E^1(E)$ and F is the extension by zero of $F_{|\mathbf{A}^1 - S}$ then $\mathcal{F}(F)$ is a complex concentrated in degrees 0 and 1, and $\mathcal{F}^1(F)$ is supported on a finite set.*

Proof. With the notation of section 4.2, let $\mathbf{Y}$ be an $\mathbf{S}$-analytic space, $j : \mathbf{Y}_{\eta_\infty} \to \mathbf{Y}$ the open imbedding and $i : \mathbf{Y}_{\overline{\infty}} \to \mathbf{Y}$ the imbedding of the special fibre. Let G be a sheaf on $\mathbf{Y}_\eta$ and H a locally constant sheaf on $\mathbf{Y}$. Then one has the standard general formula

$$R\Psi_{\eta_\infty}^{mer}(j^*H \otimes G) \simeq i^*H \otimes R\Psi_{\eta_\infty}^{mer} G.$$

Let $F_!$ be the extension by zero of F to $\mathbf{P}_E^1$ and set $U = \mathbf{A}_E^1 - S$; clearly $\overline{\mathrm{pr}}^* F_!$ is locally constant on $U \times \mathbf{S}$. Then from theorem 18 and the above remark we derive

$$\mathbb{K}_{F|U} \simeq i^* F_{|U} \otimes (R\Psi_{\eta_\infty}^{mer}(\mu^*\mathcal{L})_!)_{|U} = 0$$

which proves the first claim.

Assume now that $S \subset \mathbf{A}_E^1(E)$ and F is extended by zero from U. By Poincaré duality and proper base change, it is clear that $\mathcal{F}^i(F)$ can be non-zero only for $-1 \leq i \leq 1$. Since F is extended by zero from a locally constant sheaf on U, it is also obvious that $\mathcal{F}^{-1}(F) = 0$.

Let $T = \{t_1, \ldots t_n\}$ be any finite collection of points in $\mathbf{A}^1$, with the property that $\mathcal{F}^1(F)_{t_i} \neq 0$ for all $t_i \in T$. Let K be a complete extension of E big enough to contain the residue fields of all the points t_i. Let $\pi : \mathbf{A}^1_K \to \mathbf{A}^1_{\bar{k}^a}$ be the base change morphism. Define $\mu_{t_i} : \mathbf{A}^1_K \to \mathbf{A}^1_K$ as $x \mapsto t_i x$. By Poincaré duality we obtain

$$H^0(U_K, \pi^*\mathcal{H}om(F,\Lambda) \otimes \mu^*_{t_i}\pi^*\mathcal{L}_{\psi^{-1}}) \neq 0$$

for all t_i. This implies that π^*F contains $\oplus_i \mu^*_{t_i}(\pi^*\mathcal{L})$ as a direct summand. Since F has finitely generated stalks, it follows immediately that the cardinality of T is bounded, *i.e.* $\mathcal{F}^1(F)$ has punctual support. □

Suppose that for a certain point $s \in \mathbf{P}^1_E(E)$ the stalk F_s vanishes. The definition of $R\Psi^{mer}_{\eta_\infty}$ being purely local, it is clear that the stalk of $\mathbb{K}^q_F$ at s only depends on F_{η_s}. This prompts us to make the following

Definition 22. *For any point $s \in \mathbf{P}^1_E(E)$ let $pr_{\eta_s} : \eta_s \times \eta_\infty \to \eta_s$ be the projection on the first factor. For a topological group G, denote by $\underline{Rep}(G, \mathbb{O}_n)$ the category of $\mathbb{O}_n$-modules with continuous G-action. The local Fourier transform at the point s is the functor*

$$\begin{aligned} \mathcal{F}^{(s,\infty)}_{loc,\psi} : S(\eta_s, \mathbb{O}_n) &\longrightarrow \underline{Rep}(\pi_1^{mer}(\eta_\infty), \mathbb{O}_n) \\ F &\longmapsto R^1\Psi^{mer}_{\eta_\infty}(pr^*_{\eta_s}F \otimes (\mu^*\mathcal{L}_\psi)_{|\eta_s \times \eta_\infty}). \end{aligned}$$

We are now ready to state the main result of this chapter.

Theorem 19 (Principle of Stationary Phase). *Let F be a sheaf on $\mathbf{A}^1_E$, which is the extension by zero of a locally constant sheaf with finitely generated stalks, defined on the complement of a finite subset $S \subset \mathbf{A}^1_E(E)$. Then there is a canonical equivariant direct sum decomposition*

$$\mathcal{F}^0_\psi(F)(\infty) \simeq \oplus_{s \in S \cup \{\infty\}} \mathcal{F}^{(s,\infty)}_{loc,\psi}(F_{\eta_s})).$$

Proof. Let $s \in S \cup \{\infty\}$ and define $\phi : \eta_s \times \mathbf{S} \to \mathbf{X}$ as the map of S-spaces induced by the obvious imbedding. Notice that ϕ is a smooth morphism. Thus, from proposition 11 we derive

$$\mathcal{H}^1(\phi^*_{\overline{\infty}}\mathbb{K}_F) \simeq R^1\Psi^{mer}_{\eta_\infty}(\phi^*_{\eta_\infty}((\mathrm{pr}^*F)_! \otimes (\mu^*\mathcal{L}_\psi)_!)) \simeq \mathcal{F}^{(s,\infty)}_{loc}(F_{\eta_s})).$$

It follows from lemma 25 that, under the stated hypotheses, $\mathcal{F}^1_\psi(F)_{\eta_\infty} = 0$, *i.e.* $\mathcal{F}_\psi(F)_{\eta_\infty}$ reduces to a single sheaf placed in degree zero. Hence the spectral sequence of corollary 2 gives

$$\mathcal{F}^0_\psi(F)(\infty) = H^0(\overline{\infty}, R^0\Psi^{mer}_{\eta_\infty}\mathcal{F}_\psi(F)_{\eta_\infty}).$$

On the other hand, consider the compact morphism $\mathrm{pr}' : \mathbf{X} \to \mathbf{S}$ induced by the projection onto the second factor. From proposition 12 we derive

$$R\Psi^{mer}_{\eta_\infty}(\mathcal{F}(F)_{\eta_\infty}) \simeq R\Psi^{mer}_{\eta_\infty}R\mathrm{pr}'_{\eta_\infty *}((\mathrm{pr}^*F)_! \otimes (\mu^*\mathcal{L}_\psi)_!)[1] \simeq R\mathrm{pr}'_{\overline{\infty}*}(\mathbb{K}_F)[1].$$

From lemma 25 we know that the complex $\mathbb{K}_F$ is concentrated on the set $S \cup \{\infty\}$, therefore $R^q\mathrm{pr}'_{\overline{\infty}*}(\mathbb{K}_F)$ vanishes for $q > 0$ and the claim of the theorem follows. □

Remark: the proof also shows that $R^i\Psi^{mer}_{\eta_\infty}((\mathrm{pr}^*F)_! \otimes (\mu^*\mathcal{L}_\psi)_!)$ vanishes for $i \neq 1$.

9.3. Basic study of the local Fourier transform. In this section we propose to show how our local Fourier transforms honour their name with a behaviour which, as much as possible, mimicks that of their namesakes introduced by Laumon.

It is my belief that the meromorphic quotient of the analytic local fundamental group provides the right framework in which to place the theory of the local Fourier transforms. In other words, I expect that, for any sheaf F on $\mathbf{A}^1_E$ with only meromorphic ramification, the monodromy at infinity of $\mathcal{F}^0_\psi(F)$ can be completely described in terms of this meromorphic fundamental group.

Currently I do not know yet how to prove such a claim: I hope to return to this problem in a future paper.

To start with, we construct a functor

$$\omega : \underline{\mathrm{Rep}}(\pi_1^{mer}(\eta_s), \mathbb{O}_n) \to \mathrm{S}(\eta_s, \mathbb{O}_n).$$

This can be obtained as follows. Given an $\mathbb{O}_n$-module V with a $\pi_1^{mer}(\eta_s)$-action, we can use the isomorphism of corollary 1 (or better of its "arithmetic" variant, as at the end of section 4.1) to induce an action of $\pi_1^{mer}(\mathbb{G}_{m,E}, \overline{x})$ on V. This depends on the choice of a geometric point $\overline{x} \in \mathbb{G}_{m,E}$. Then a standard argument yields a locally constant $\mathbb{O}_n$-sheaf $\omega_{\overline{x}}(V)$ whose stalk at the point $\overline{x}$ is canonically identified with V. Then we define

$$\omega(V) = \omega_{\overline{x}}(V)_{|\eta_s}.$$

In what follows we will bound ourselves to the study of sheaves which are in the essential image of ω, and consequently we will regard the local Fourier transforms as functors

$$\mathcal{F}^{(s,\infty)}_{\psi,loc} : \underline{\mathrm{Rep}}(\pi_1^{mer}(\eta_s), \mathbb{O}_n) \to \underline{\mathrm{Rep}}(\pi_1^{mer}(\eta_\infty), \mathbb{O}_n).$$

Lemma 26. *1) Let $V \in \underline{Rep}(\pi_1^{mer}(\eta_s), \mathbb{O}_n)$ be unramified, i.e. suppose that the $\pi_1^{mer}(\eta_s)$-action on V factors through the quotient $Gal(E^a/E)$. Then*

$$\mathcal{F}^{(\infty,\infty)}_{\psi,loc}(V) = 0.$$

2) If we denote by $\mathbb{O}_n$ the trivial representation of rank one, then

$$\mathcal{F}^{(0,\infty)}_{\psi,loc}(\mathbb{O}_n) = \mathbb{O}_n.$$

Proof. For (1), we observe that

$$\mathcal{F}^{(\infty,\infty)}_{\psi,loc}(V) \simeq \mathcal{F}^{(\infty,\infty)}_{\psi,loc}(\mathbb{O}_n) \otimes V$$

which allows us to reduce to the case $V = \mathbb{O}_n$; from lemma 14 we derive $\mathcal{F}_\psi(\mathbb{O}_n)(\infty) = 0$ and the claim follows from theorem 19. Part (2) is dealt with in a similar way, by considering the (global) Fourier transform of the extension by zero of the trivial sheaf $\mathbb{O}_{n,\mathbb{G}_m}$, and using theorem 19 to analyse the local contributions at infinity. □

Definition 23. *Let $\chi : \pi_1^{alg}(\mathbb{G}_{m,E}, \overline{x}) \to \mathbb{O}_n^\times$ be a non-trivial character. It defines a locally constant $\mathbb{O}_n$-sheaf $\mathcal{K}_\chi$ on $\mathbb{G}_{m,E}$ which we call the Kummer sheaf associated to the character χ.*

Proposition 29. *Let $G(\chi, \psi)$ be the $\mathbb{O}_n$-module with continue $Gal(E^a/E)$-action defined as*

$$G(\chi, \psi) = H^1_c(\mathbb{G}_{m,E^a}, \mathcal{K}_\chi \otimes \mathcal{L}_\psi).$$

Then: 1) $G(\chi, \psi)$ is a free $\mathbb{O}_n$-module of rank one and the $H^i_c(\mathbb{G}_{m,E^a}, \mathcal{K}_\chi \otimes \mathcal{L}_\psi)$ vanish for $i \neq 1$;
2) if j is the imbedding of $\mathbb{G}_{m,E}$ in $\mathbf{A}^1_E$, there is a canonical isomorphism

$$\mathcal{F}_\psi(j_*\mathcal{K}_\chi) \simeq j_*\mathcal{K}_{\chi^{-1}} \otimes G(\chi, \psi).$$

Proof. The second statement can be infered, *mutatis mutandis*, from the proof of proposition 1.4.3.2 of [Lau2]. It is easy to verify that the cohomology of $\mathcal{K}_\chi \otimes \mathcal{L}_\psi$ vanishes in degrees $i \neq 1$. To show that $G(\chi, \psi)$ has rank one, we can use (2) and the involutivity theorem 11 to obtain

$$(-1)^* j_* \mathcal{K}_\chi(-1) \simeq j_* \mathcal{K}_\chi \overset{\mathbf{L}}{\otimes} G(\chi, \psi) \overset{\mathbf{L}}{\otimes} G(\chi^{-1}, \psi).$$

This implies that $G(\chi, \psi) \overset{\mathbf{L}}{\otimes} G(\chi^{-1}, \psi)$ must be free of rank one, hence the claim. □

Definition 24. *We say that a representation of $\pi_1^{mer}(\eta_s)$ is tame if it factors through the quotient $\pi_1^{alg}(\eta_s)$.*

Lemma 27. *For any $V \in \underline{Rep}(\pi_1^{mer}(\eta_0), \mathbb{O}_n)$ of finite rank there exists a locally constant sheaf $\mathcal{V}$ over $\mathbb{G}_{m,E}$ such that $\mathcal{V}_{\eta_0} \simeq V$ and $\mathcal{V}_{\eta_\infty}$ is a tame representation of $\pi_1^{mer}(\eta_\infty)$.*

Proof. This is a direct consequence of theorem 8. □

For $s \in \mathbb{G}_{m,E}(E)$, let $\mu_s : \mathbf{A}_E^1 \to \mathbf{A}_E^1$ be the map $x \mapsto sx$ and set $L(s) = (\mu_s^* \mathcal{L}_\psi)_{\eta_\infty}$. This $\mathbb{O}_n$-module is a rank 1 representation of $\pi_1^{mer}(\eta_\infty)$ of Swan conductor one.

The translation map $\tau_s : \mathbf{A}_E^1 \to \mathbf{A}_E^1$ defined by $x \mapsto x + s$ induces a morphism $\eta_0 \to \eta_s$ and hence a group homomorphism

$$\tau_{s*} : \pi_1^{mer}(\eta_0) \to \pi_1^{mer}(\eta_s)$$

as well as a functor

$$\tau_s^* : \underline{\mathrm{Rep}}(\pi_1^{mer}(\eta_s), \mathbb{O}_n) \to \underline{\mathrm{Rep}}(\pi_1^{mer}(\eta_0), \mathbb{O}_n).$$

Proposition 30. *1) For all $s \in \mathbb{P}_E^1(E)$ the functors $\mathcal{F}_{\psi,loc}^{(s,\infty)}$ are exact.*

2) If $V \in \underline{Rep}(\pi_1^{mer}(\eta_\infty), \mathbb{O}_n)$ is a tame representation, then $\mathcal{F}_{\psi,loc}^{(\infty,\infty)}(V) = 0$.

3) If $V \in \underline{Rep}(\pi_1^{mer}(\eta_s), \mathbb{O}_n)$ and $s \in \mathbf{A}_E^1(E)$ then

$$\mathcal{F}_{\psi,loc}^{(s,\infty)}(V) \simeq \mathcal{F}_{\psi,loc}^{(0,\infty)}(\tau_s^* V) \otimes L(-s).$$

Proof. The first claim follows immediately from the remark after the proof of theorem 19. For the proof of the second claim, thanks to proposition 13 we can base change everything to $\widehat{E}^a$, at the cost of replacing everywhere the vanishing cycle functor with its generalization $R\Psi_{\eta_\infty,E}^{mer}$. We leave to the reader to state the obvious variants of the principle of the stationary phase for the more general functor. Basically, all the statements remain formally unchanged. Therefore we only show the proof for the case $E = E^a$, in which case

$$\pi_1^{alg}(\eta_\infty) \simeq \pi_1^{alg}(\mathbb{G}_{m,E}, \overline{x}) \simeq \widehat{\mathbf{Z}}(1).$$

Thanks to part (1), we can also assume that V is a simple representation. After replacing $\mathbb{O}_n$ by some finite extension, any representation of $\widehat{\mathbf{Z}}(1)$ diagonalizes. Hence we can assume that V is a rank one $\mathbb{O}_n$-module, attached to some character $\chi : \widehat{\mathbf{Z}}(1) \to \mathbb{O}_n^\times$. The case of a trivial character has already been taken care of in lemma 26. Let χ be a non-trivial character; we consider the associated Kummer sheaf $\mathcal{K}_\chi$ on $\mathbb{G}_{m,E}$ and its extension $j_* \mathcal{K}_\chi$ to $\mathbf{A}_E^1$. Now, let Δ be an open disc in $\mathbf{A}_E^1$, centered at 0. Denote by $\mathcal{K}'_\chi$ the extension by zero of $j_* \mathcal{K}_{\chi|\Delta}$. Then $\mathcal{K}'_\chi$ imbeds in $j_* \mathcal{K}_\chi$ and there is a short exact sequence

$$0 \to \mathcal{K}'_\chi \to j_* \mathcal{K}_\chi \to \mathcal{K}''_\chi \to 0.$$

An argument as in the proof of lemma 25 shows that $\mathcal{F}(\mathcal{K}'_X)$ is a complex concentrated in degree zero, and hence we obtain a short exact sequence:

$$0 \to \mathcal{F}^0(\mathcal{K}'_X) \to \mathcal{F}^0(j_*\mathcal{K}_X) \to \mathcal{F}^0(\mathcal{K}''_X) \to 0.$$

Let $s \in \mathbf{G}_m(E)$ be any point. It is easy to check that $\mu_s^*\mathcal{K}'_X$ is isomorphic to the extension by zero of $j_*\mathcal{K}_{X|\mu_s^{-1}(\Delta)}$. It follows:

$$H_c^i(\mathbf{A}_E^1, \mathcal{K}'_X \otimes \mu_s^*\mathcal{L}_\psi) \simeq H_c^i(\mu_s^{-1}(\Delta), j_*\mathcal{K}_X \otimes \mathcal{L}_\psi).$$

From proposition 5.2.9 of [B1] we know that

$$H_c^i(\mathbf{A}_E^1, \mathcal{K}_X \otimes \mathcal{L}_\psi) \simeq \lim_{|s|\to 0} H_c^i(\mu_s^{-1}(\Delta), \mathcal{K}_X \otimes \mathcal{L}_\psi).$$

Proposition 29 says that the left-hand side of this equation has rank one, therefore the limit is already attained for some value $|s_0|$. This means that on the complement $U = \mathbf{A}_E^1 - \mu_{s_0}^{-1}(\Delta)$ we have $\mathcal{F}^0(\mathcal{K}'_X)_{|U} \simeq \mathcal{F}^0(\mathcal{K}_X)_{|U}$, and therefore $\mathcal{F}^0(\mathcal{K}''_X)_{|U} = 0$; in particular $\mathcal{F}^0(\mathcal{K}''_X)_{\eta_\infty} = 0$. Next, notice that the sheaf $\mathcal{K}''_X$ is locally constant on the complement of a single point p (of type (2) in the notation of [B1], paragraph 3.6) in $\mathbf{A}_E^1$, namely the point corresponding to the sup-norm on the disc Δ (see [B1], remark 6.3.4). Therefore lemma 25 applies, and shows that $\mathbb{K}_{\mathcal{K}''_X}$ is concentrated on $\{p, \infty\}$. It is also clear that the stalk of $\mathbb{K}^1_{\mathcal{K}''_X}$ over ∞ is isomorphic to the stalk of $\mathbb{K}^1_{\mathcal{K}_X}$ over the same point. Now, the same argument which was used in the proof of theorem 19 shows that $\mathcal{F}^0(\mathcal{K}''_X)(\infty) \simeq H^0(\mathbf{P}_E^1, \mathbb{K}^1_{\mathcal{K}''_X})$. This implies $\mathbb{K}^1_{\mathcal{K}''_X} = 0$. It follows that also the stalk of $\mathbb{K}^1_{\mathcal{K}_X}$ vanishes over ∞, and therefore $\mathcal{F}_{\psi,loc}^{(\infty,\infty)}(\mathcal{K}_{X,\eta_\infty})$ vanishes, as stated.

For (3), let $\tau_s^*\mathcal{V}$ be a global extension of τ_s^*V, as provided by lemma 27. According to part (2) and theorem 19, the only contribution to $\mathcal{F}(\mathcal{V})(\infty)$ (resp. $F(\tau_s^*\mathcal{V})(\infty)$) comes from $\mathcal{F}_{\psi,loc}^{(s,\infty)}(\mathcal{V}_{\eta_s})$ (resp. $\mathcal{F}_{\psi,loc}^{(0,\infty)}(\tau_s^*\mathcal{V}_{\eta_0})$). Proposition 22 allows to compare the two terms and yields the claim. □

Proposition 30 says that it suffices to study the functors $\mathcal{F}_{\psi,loc}^{(s,\infty)}$ for the values $s = 0$ and $s = \infty$ to know all of them. From this point on, the theory proceeds formally as in the finite field case. We leave the task of making a detailed study of this theory to a sequel of this paper.

Remark: If we take the formal multiplicative group $\mathbf{G}_m$ as the underlying Lubin-Tate group, then the theory above can be refined by using the constructions of section 5.3. Suppose that a sheaf F is defined over (the completion of) any algebraic extension E_0 of $\mathbf{Q}_p$. In this case the principle of stationary phase gives a canonical decomposition of the *semilinear* $\pi_1^{mer}(\eta_\infty)$-representation which describes the asymptotic behaviour of $\mathcal{F}(F)$, in terms of local contributions. In particular the local Fourier transforms land in the category of these semilinear representations.

10. The homomorphism Γ

10.1. Definition and basic properties. From now on we restrict for simplicity to the Lubin-Tate torsor arising from the multiplicative group $\mathbf{G}_m$. Accordingly, the value ρ_1 equals $p^{-1/(p-1)}$. Also, G_n equals the group μ_{p^n} of p^n-th roots of unity. We pick a non-trivial character ψ of the group $G_\infty = \mu_{p^\infty}$ with values in the ring of integers $\mathbf{O}$ of $\mathbf{B}_\ell$. Then, by composing with the natural projections we obtain a compatible sequence of characters $\psi_n : \mu_{p^\infty} \to \mathbf{O}/\ell^n$.

Let V be a k-vector space, $\sigma : V \to V'$ a symmetric isomorphism and $f : V \to k$ the associated non-degenerate quadratic form. We take inspiration from formula (1) of the introduction to make the following definition:

$$\Gamma(f) = \varprojlim_n H_c^{\dim V}(V \times_k \widehat{k}^a, f^*\mathcal{L}_{\psi_n}) \otimes_{\mathbb{O}} \mathbf{E}_\lambda(\dim V/2).$$

In this chapter we will be concerned with the study of the $\mathrm{Gal}(k^a/k_\infty)$-module $\Gamma(f)$, seen as a function of f. With the present setup, this cohomology group carries also a semilinear action of $\mathrm{Gal}(k^a/k)$, as explained in section 5.3. Even though it may be interesting and worth exploring, we will not deal here with this extra structure.

As usual, to make sense of the "half Tate twist" we extend the coefficient field: the $\mathbf{E}_\lambda$ appearing above is the extension of $\mathbb{B}_\ell$ containing $q^{1/2}$. Then the Tate module $\mathbb{E}_\lambda(1/2)$ is the unramified Galois representation on which Frobenius acts as multiplication by $q^{-1/2}$.

The next two results establish the elementary properties of Γ.

Lemma 28. *For any f as above, $\Gamma(f)$ is a Gal(k^a/k_∞)-module of rank one, which depends only on the isomorphism class of f.*

Proof. It suffices to prove the corresponding result for the torsion modules $\Gamma_n(f) = H_c^{\dim V}(V \times_k \widehat{k}^a, f^*\mathcal{L}_{\psi_n})$. Let g be another non-degenerate quadratic form, in the same isomorphism class as f. Then we have $g = f \circ h$ for some automorphism $h : V \to V$. We get

$$H_c^{\dim V}(V \times_k \widehat{k}^a, g^*\mathcal{L}_{\psi_n}) \simeq H_c^{\dim V}(V \times_k \widehat{k}^a, h^* f^*\mathcal{L}_{\psi_n}) \simeq H_c^{\dim V}(V \times_k \widehat{k}^a, f^*\mathcal{L}_{\psi_n})$$

which proves the second assertion. Since the characteristic of k is different from 2, we can always find a basis $\{\mathbf{e}_1, ..., \mathbf{e}_m\}$ of V, such that the quadratic form f diagonalizes in this basis. Let V_i for $i = 1, ..., m$ be the span of $\mathbf{e}_i$, and let $p_i : V \to V_i$ be the projection such that $p_i(\mathbf{e}_j) = \delta_{ij}\mathbf{e}_i$. Denote also by f_i the restriction of f to V_i. The yoga of torsors (for which we refer to [SGA4$\frac{1}{2}$]) implies the formula

$$f^*\mathcal{L} \simeq p_i^* f_1^*\mathcal{L} \otimes ... \otimes p_i^* f_m^*\mathcal{L}.$$

Since $H_c^0(V_i \times_k \widehat{k}^a, f_i^*\mathcal{L}) = H^0(V_i \times_k \widehat{k}^a, f_i^*\mathcal{L}) = 0$, it follows that $H_c^j(V_i \times_k \widehat{k}^a, f_i^*\mathcal{L}) \neq 0$ if and only if $j = 1$. Then, by Kunneth formula we have:

$$\Gamma_n(f) \simeq \Gamma(f_1) \overset{\mathbf{L}}{\otimes} ... \overset{\mathbf{L}}{\otimes} \Gamma_n(f_m).$$

Hence, to prove the first assertion it suffices to assume $\dim V = 1$. Let f' be the inverse transpose of f, defined as in proposition 23. Combining proposition 23 and the involutivity theorem 11 we obtain

$$\mathcal{L}(f) \simeq \mathcal{L}(f) \overset{\mathbf{L}}{\otimes} \Gamma_n(f) \overset{\mathbf{L}}{\otimes} \Gamma_n(f')$$

which implies that $\Gamma_n(f)$ is free of rank rank one. □

Remark: the proof also shows that the groups $H_c^i(V \times_k \widehat{k}^a, f^*\mathcal{L})$ vanish for $i \neq \dim V$.

Proposition 31. *The map $f \mapsto \Gamma(f)$ descends to a group homomorphism from the Witt group $W(k)$ of k to the group of isomorphism classes of rank one Gal(k^a/k_∞)-modules (with multiplication given by tensor product).*

Proof. Again, we reduce easily to the corresponding statement for torsion coefficients. Let $f : V \to k$, $g : W \to k$ be two nondegenerate quadratic forms, and let $f \oplus g : V \oplus W \to k$ be their sum. Denote also by p_V (resp. p_W) the projection of $V \oplus W$ onto V (resp. onto W). From another application of the yoga of torsors, one obtains

$$(f \oplus g)^*\mathcal{L} \simeq p_V^* f^*\mathcal{L} \otimes p_W^* g^*\mathcal{L}. \tag{46}$$

Using (46) and the Kunneth formula it follows

$$\Gamma_n(f)\otimes\Gamma_n(g)\simeq H_c^{\dim V+\dim W}((V\oplus W)\times_k\widehat{k}^a, p_V^*f^*\mathcal{L}\otimes p_W^*g^*\mathcal{L})\simeq\Gamma_n(f\oplus g)$$

which says that Γ_n induces a homomorphism from the monoid of isomorphism classes of quadratic forms, to the group of isomorphism classes of $\mathrm{Gal}(k^a/k_\infty)$-modules of rank one. Let $f_V : V\oplus V'\to k$ be the standard quadratic form induced by the dual pairing: $f_V(x,\xi)=\langle x,\xi\rangle$ for all $x\in V,\xi\in V'$. We want to show that $\Gamma_n(f_V)$ is the trivial $\mathrm{Gal}(k^a/k_\infty)$-representation. But this is nothing else than a special case of lemma 14 in section 6.1. Since the relations in the Witt group are generated by all the isotropic quadratic forms of the form f_V, the claim follows. □

10.2. Computation of $\Gamma(f)$. In this section we obtain some information on the Galois structure of $\Gamma(f)$.

For $a\in k^\times$, let M_a denote the ℓ-adic representation of $\mathrm{Gal}(k^a/k_\infty)$ corresponding to the character $\sigma\mapsto\sigma(\sqrt{a})/\sqrt{a}=\pm1$ and let $f_a : k\to k$ be the quadratic form $x\mapsto ax^2$.

Lemma 29. *With the notation above*

$$\Gamma(f_a)\simeq\Gamma(f_1)\otimes M_a.$$

Proof. Define a projective system of sheaves $\mathcal{M}_a=\{\mathcal{M}_{a,n}\}_{n\in\mathbf{N}}$ on $\mathbf{A}^1_k$, by requiring $f_{a*}(\mathbf{E}_\lambda/\lambda^n)=(\mathbf{E}_\lambda/\lambda^n)\oplus\mathcal{M}_{a,n}$. Then we have

$$H_c^1(\mathbf{A}^1,f_a^*\mathcal{L}_{\psi_n})\simeq H_c^1(\mathbf{A}^1,\mathcal{L}_{\psi_n}\otimes f_{a*}(\mathbf{E}_\lambda/\lambda^n))\simeq H_c^1(\mathbf{A}^1,\mathcal{L}_{\psi_n}\otimes\mathcal{M}_{a,n}).$$

By $\mathcal{M}_a\simeq\mathcal{M}_1\otimes M_a$, the assertion follows. □

Given a general non-degenerate quadratic form $f:V\to k$ on a vector space of dimension n, denote by $D(f)$ the discriminant of f. Set $H_c^n(V,f^*\mathcal{L}_\psi)=\varprojlim_n H_c^n(V,f^*\mathcal{L}_{\psi_n})\otimes_{\mathbf{O}}\mathbf{E}_\lambda$.

Proposition 32. *With the notation above, let $n=2m$ (resp. $=2m+1$) and $d=(-1)^mD(f)$. Then we have*

$$H_c^n(V,f^*\mathcal{L}_\psi)\simeq\begin{cases}M_d(-m) & n\ \textit{even}\\ H_c^1(\mathbf{A}^1,\mathcal{L}_\psi\otimes\mathcal{M}_d)(-m) & n\ \textit{odd.}\end{cases}$$

Proof. Let $U=f^{-1}(\mathbf{G}_m)$ and $W=f^{-1}(0)$. Then from Theoreme 3.3 and Table 3.7 of [SGA7] Exp. XII Quadriques, we derive

$$R^qf_{|U!}\mathbf{E}_\lambda\simeq\begin{cases}M_d(-(m-1)) & q=n-1, n\text{ even}\\ \mathcal{M}_d(-m) & q=n-1, n\text{ odd}\\ \mathbf{E}_\lambda(-(n-1)) & q=2n-2\\ 0 & \text{otherwise.}\end{cases}$$

From this and the projection formula we obtain

$$H_c^q(U,f^*\mathcal{L}_\psi)\simeq\varprojlim_n H_c^1(\mathbf{G}_m,\mathcal{L}_{\psi_n}\otimes R^{q-1}f_{|U!}(\mathbf{O}/\ell^n))\otimes_{\mathbf{O}}\mathbf{E}_\lambda.$$

Since W is the affine cone over the non-singular quadric $Q\subset\mathbf{P}(V)$ defined by f, we can compute $H_c^q(W,f^*\mathcal{L}_\psi)=H_c^q(W,\mathbf{E}_\lambda)$ by using [SGA7] Exp. XV Formule de Picard-Lefschetz. We have $H_c^q(W,\mathbf{E}_\lambda)=H^q_{\{0\}}(W,\mathbf{E}_\lambda)$ by Prop. 2.1.2(ii) *loc.cit.* In the long exact sequence

$$\ldots\to H^q_{\{0\}}(W,\mathbf{E}_\lambda)\to H^q(W,\mathbf{E}_\lambda)\to H^q(W-\{0\},\mathbf{E}_\lambda)\to\ldots$$

we have $H^q(W, \mathbb{E}_\lambda) = \mathbb{E}_\lambda$ for $q = 0$ and $= 0$ otherwise by Prop. 2.1.2(i) *loc.cit.* Finally, since $W - \{0\}$ is a $\mathbb{G}_m$-bundle over Q, we obtain

$$H^q_c(W, \mathbb{E}_\lambda) \simeq \begin{cases} M_d(-(m-1)) & q = n-1, n \text{ even} \\ M_d(-m) & q = n, n\text{even} \\ \mathbb{E}_\lambda(-(n-1)) & q = 2n-2 \\ 0 & \text{otherwise.} \end{cases}$$

From these computation we can easily deduce the claim. (Warning: in this proof we have used somewhat freely an ℓ-adic language: this is only a harmless abbreviation for some more cumbersome notation, and does not imply that we rely on a formalism of analytic ℓ-adic sheaves). □

Corollary 9. *With the notation above*

$$\Gamma(f_a)^{\otimes 2} \simeq M_{-1}$$

and the $Gal(k^a/k_\infty)$*-action on* $\Gamma(f)$ *factors through* μ_4*.*

Proof. It follows immediately from proposition 32 and proposition 31. □

As an example we consider the classical case of the norm of the quaternion algebras. Recall that for any pair of elements $a, b \in k$, one obtains an associative k-algebra $\left(\frac{a,b}{k}\right)$ of dimension 4, with basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, and multiplication fixed by the rules:

$$\mathbf{i}^2 = a \quad \mathbf{j}^2 = b \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

Let π be a uniformizing parameter for k. If $a \in \mathcal{O}_F^*$ is not a quadratic residue modulo π, then the algebra $\left(\frac{a,\pi}{k}\right)$ is a division algebra and any two division algebras arising in this way are isomorphic. We denote by $\mathbb{H}$ this division algebra: it is the quaternion algebra over k. The algebra $\mathbb{H}$ is endowed with a *norm map* $N : \mathbb{H} \to k$. The norm map induces a homomorphism from the multiplicative group $\mathbb{H}^*$ to k^*. In terms of the basis given above, one has

$$N(x \cdot 1 + y \cdot \mathbf{i} + z \cdot \mathbf{j} + w \cdot \mathbf{k}) = x^2 - ay^2 - \pi z^2 + a\pi w^2. \tag{47}$$

The following result is now a straightforward consequence of proposition 32 and corollary 9.

Theorem 20. *The action of* $Gal(k^a/k_\infty)$ *on* $\Gamma(N)$ *is trivial.* □

In [We] it is proved that, with the notation of the introduction, the constant $\gamma(N)$ equals -1. This shows that Weil's invariant is not a homomorphic image of ours.

10.3. Quadratic Gauss sums. In this final section we obtain an explicit description of the Galois action on $\Gamma(x^2)$, thus complementing proposition 32. Unfortunately our method works only when the residue characteristic is different from 2. Therefore in this chapter we assume throughout that p is odd.

Let $f_1 : \mathbf{A}^1 \to \mathbf{A}^1$ be the quadratic form in one variable $x \mapsto x^2$.

Let D_r be the closed disc of radius r in $\mathbf{A}^1$, centered at the origin and $j : \mathbf{A}^1 - D_r \to A$ the imbedding of the complement of D_r. Suppose that the restriction of $f_1^*\mathcal{L}$ to D_r is not the constant sheaf. Then the pair $(\mathbf{A}^1, D_r)$ gives an exact sequence in cohomology

$$H^1_c(\mathbf{A}^1, f_1^*\mathcal{L}) \to H^1(D_r, f_1^*\mathcal{L}) \to H^2_c(\mathbf{A}^1 - D_r, f_1^*\mathcal{L}).$$

By Poincaré duality $H^2_c(\mathbf{A}^1 - D_r, f_1^*\mathcal{L}) \simeq \mathrm{Hom}(H^0(\mathbf{A}^1 - D_r, f_1^*\mathcal{L}^{-1}), \mathbf{B}_\ell) = 0$, and therefore $H^1_c(\mathbf{A}^1, f_1^*\mathcal{L}) \simeq H^1(D_r, f_1^*\mathcal{L})$ if and only if $H^1(D_r, f_1^*\mathcal{L}) \neq 0$. Set $r_1 = \rho_1^{1/2}$. We will show that indeed $H^1(D_{r_1}, f_1^*\mathcal{L})$ does not vanish.

Basically we follow the arguments of chapter 8, except that now we try to keep track of the Galois action. First of all, we call back on stage the Kummer torsor $\mathcal{K}^{(n)}$ defined in section 8.1. Throughout this section we will need only $\mathcal{K}^{(1)}$, which therefore we will simply denote by $\mathcal{K}$.

The restriction of the character ψ to $\mu_p = \mathrm{Ker}[p]$ is a character of the latter group, and we can form the associated local system $\mathcal{K}_{\psi_n}$ of $\mathbb{O}_n$-modules on $D_{\rho_1/p}$. Then by formula 33 we obtain

$$\alpha^*\mathcal{K}_{\psi_n} \simeq \mathcal{L}_{\psi_n|D_{\rho_1}}.$$

Comparing with section 8.1, and keeping in mind that $p > 2$ by assumption, one checks easily that in formula 35 we can take $g(x) = apx^2$. We deduce an isomorphism

$$f_1^*\mathcal{L}_{\psi_n} \simeq f_1^*\alpha^*\mathcal{K}_{\psi_n} \simeq g^*\mathcal{K}_{\psi_n}.$$

In particular

$$H^1(D_{r_1}, f_1^*\mathcal{L}_\psi) \simeq \varprojlim_n H^1(D_{r_1}, g^*\mathcal{K}_{\psi_n}) \otimes_{\mathbb{O}} \mathbb{E}_\lambda.$$

In order to study $g^*\mathcal{K}_{\psi_n}$ we will use the Witt torsor $\widehat{\mathbf{W}} = \widehat{\mathbf{W}}^{(1)}$ defined in section 8.3. Recall that this is a sheaf on $\widehat{\mathcal{V}}_{1,k^\circ}$. Notice that the theory of $\mathcal{V}_{1,A}$ is considerably simpler than the general case, and was already developed in [O-S-S], where it was called $\mathcal{G}^{(\lambda^p)}$. We form the associated $\mathbb{O}_n$-local system $\widehat{\mathbf{W}}_{\psi_n} = \widehat{\mathbf{W}} \times_{\mathbf{O}_n} \psi_n$.

With the notation of chapter 8, formula 42 gives an isomorphism

$$\mu_*\nu^*\widehat{\mathbf{W}}_{\psi_n} \simeq \sigma_\eta^*\mathcal{K}_{\psi_n}.$$

Next we have to replace g by its integral model $\widehat{g} : \mathrm{Spf}k^\circ\langle x\rangle \to \mathrm{Spf}k^\circ\langle x\rangle$, as in lemma 19. For this it is necessary to extend the base field to $k(\mu_{p^\infty}, \lambda^{1/2})$, since the map β_1 of section 8.3 are defined only after passing to this overfield. After that, we can choose the constants b_1, b_2 in such a way that $\widehat{g}$ becomes the map

$$\mathrm{Spf}k^\circ\langle x\rangle \xrightarrow{\widehat{g}} \mathrm{Spf}k^\circ\langle x\rangle \qquad x \mapsto x^2.$$

Moreover, peeking at [O-S-S] one can check that for the special case of $\mathcal{V}_{1,A}$, the map σ is linear, hence it is an isomorphism. and we can identify the formal scheme $\mathcal{Y}$ of diagram 41 with $\widehat{\mathcal{V}}_1$ itself. In particular, $\mathcal{Y}$ is smooth and after the identification we also get $\widehat{\mathfrak{g}} = \widehat{g}$.

Let $\mathcal{H}$ be the subgroup of $\mathrm{Gal}(k^a/k_\infty)$ which fixes $k(\mu_{p^\infty}, \lambda^{1/2})$. Then from the diagram 41 we derive an $\mathcal{H}$-equivariant isomorphism

$$H^*(D_{r_1}, g^*\mathcal{K}_{\psi_n}) \simeq H^*(\mathcal{Y}_\eta, \mathfrak{g}^*\mathbf{W}_{\psi_n}).$$

We can now state the main result of this section.

Theorem 21. *The action of $\mathcal{H}$ on $H_c^1(\mathbb{A}_k^1, f_1^*\mathcal{L}_\psi)$ is unramified. Let Fr be any lifting of Frobenius in $\mathcal{H}$. Then we have the formula*

$$Tr(Fr, H_c^1(\mathbb{A}_k^1, f_1^*\mathcal{L}_\psi)) = \sum_{x\in\mathbb{F}} \psi(tr_{\mathbb{F}/\mathbb{F}_p}(x^2))$$

where $tr_{\mathbb{F}/\mathbb{F}_p}$ is the trace map of the field extension $\mathbf{F}/\mathbf{F}_p$ (here $\mathbb{F}$ is the residue field of k).

Proof. By the arguments above, it suffices to prove the same claim for $H^1(\mathcal{Y}_\eta, \mathfrak{g}^*\mathbf{W}_{\psi_n})$. Here we can use Berkovich's vanishing cycles, as in section 8.4. The situation is simpler than *loc.cit.*, since $\mathcal{Y}$ is already smooth. Hence the same argument of lemma 21 yields

$$H^1(\mathcal{Y}_\eta, \mathfrak{g}^*\mathbf{W}_{\psi_n}) \simeq H^1(\mathcal{Y}_s, \mathfrak{g}_s^*\widehat{\mathbf{W}}_{\psi_n,s}).$$

The first consequence is that the action of $\mathcal{H}$ is indeed unramified. Finally we recall that the pair $(\widehat{\mathcal{V}}_{1,s}, \widehat{\mathbf{W}}_s)$ is isomorphic to the pair $(\mathbf{A}^1_{\mathbf{F}}, \mathcal{L})$ where $\mathcal{L}$ is the Artin-Schreier torsor on $\mathbf{A}^1_{\mathbf{F}}$, which provides the kernel for the Deligne-Fourier transform. Moreover the map $\mathfrak{g}_s$ is given by $x \mapsto x^2$.

A standard application of the Grothendieck-Lefschetz fixed point formula yields

$$\mathrm{Tr}(Fr, H^1(\mathbf{A}^1_{\mathbf{F}}, \mathfrak{g}_s^*\mathcal{L}_{\psi_n})) = \sum_{x\in\mathbf{F}} \psi(tr_{\mathbf{F}/\mathbf{F}_p}(x^2)).$$

Together with the remarks above, this concludes the proof of the theorem □

Remark: Saibi has defined and studied in his thesis [Sa] a Fourier transform over general unipotent groups in positive characteristic, complete with a sheaves-to-functions dictionary, and his theory applies especially to the Witt group schemes W_n. One could hope to combine this construction into the line of thought developed in this chapter, and thereby extend the range of its usefulness. For instance, one would expect to be able to remove our assumption on the residue characteristic of k, just by pushing the analysis to the next level $n = 2$. Unfortunately, already for W_2 the calculations involved become overwhelmingly complicated. It is clear that, if the deformation argument has to play any role in future developments, a more systematic approach will have to be found.

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