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STATICS OF THIN-WALLED SHELLS OF REVOLUTION

by

V. S. Chernina

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13. ABSTRACT

This book is devoted to the theory and design of thin-walled shells of revolution. The basic equations of the theory are derived in an elementary form for a shell of revolution subjected to arbitrary landing. The deformations of cylindrical, conical, spherical, and toroidal shells are analyzed in detail under axisymmetrical and bending loads. The book is intended for design engineers and scientific workers engaged in designing machines and structures for strength. Orig. art. has: 52 figures, 9 tables which illustrate the text, in which many samples analyses are included. [AM9008876]

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Translator's note: On several occasions, symbols found in formulae and calculations appear to have been rendered incorrectly in the original document. They will be shown exactly as they appear in the original.

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FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH

DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}
rot	curl
lg	log

EDITOR'S FOREWARD

This work does not purport to give a survey of the multiplicity of directions of contemporary shell theory. It is devoted to only one section of this theory – to the stressed state of a shell of revolution, which historically earlier than other applications formed and has the largest domain in the problems of heavy and chemical machine building, ship building and construction.

In a comparatively small space V. S. Chernina managed to give an account of this subject with sufficient completeness. The contents of this book do not conform to the traditional problem of axisymmetric loading of a shell of revolution; much space is allotted to the problem of flexure, in the development of which a great contribution was made by the works of V. S. Chernina herself. The difficulties, which were anticipated here, have more of a technical than theoretical character, since the procedures of asymptotic integration of basic equations already developed for the case of axisymmetric loading are applicable.

The restriction to the case of a shell of revolution made it possible to simplify the presentation of Chapter I, devoted to establishing the initial geometric and static dependences. In Chapter II the reduction of the problem to systems of conventional differential equations of the eighth order was carried out. Cases of axisymmetric and flexural deformation were subjected to a detailed discussion, when the use of the first integrals make it possible to reduce the order of the systems to the fourth order and with the aid of a certain procedure to arrive at the problem of asymptotic integration of one (complex) differential equation of the second order – to a Meissner equation and to an equation of the "Meissner type". Much space in Chapter II was allotted to the problem of temperature stresses in a shell of revolution; its presentation to a considerable extent is also based on the work of V. S. Chernina.

Chapters III-VI contain solutions of problems pertaining to shells of revolution of discrete geometric shapes – circular cylindrical, conical, spherical, torus-shaped. It is natural that much space is allotted to the circular cylindrical shell as the most common type of shell designs in mechanical engineering. This problem was and continues to be the theme of numerous works, but the author of the foreward is not aware of so simple, and moreover successful examination of the important problem of the flexure of a cylindrical shell.

In the final chapters formulations of the problems of conical, spherical, and torus-shaped shells are completely presented; expressions of the particular solutions for methods of loading encountered in practice are given and asymptotic presentations of the solutions of homogeneous Meissner equations are throughly developed.

In the final chapter a method of calculating dislocational stresses in a shell of revolution, rapidly leading to a solution, is demonstrated. The problem of flexure of a circular plate with a small initial curvature, which occupies a considerable part of the chapter on the spherical shell, is enriched with new results, which will find a place in the practice of strength ratings.

The examples illustrating the general methods have a special value; each of them has an independent significance, as a scheme invariably arising in a strength rating. Many of the examples presented were drawn by V. S. Chernina from her personal experience.

It is possible to anticipate with confidence that the work of V. S. Chernina will find its place as a reference manual of design

U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

engineers in the design offices of factories and scientific-research institutes.

A. I. Lur'ye

INTRODUCTION

A shell of revolution is a common element of mechanical engineering design, precision instruments, and construction engineering. In order to design an operationally effective structure, it is necessary to know how to calculate the stressed state of the elements, under a given load, which are included in it.

The present book, as is clear from its title, is devoted to the rating of shells of revolution for a static load. All the problems are solved in linear formulation of the basis of the technical theory of shells assuming ideal elasticity of the material and smallness of the deformations (strains).

The derivation of the basic equations of the theory directly for a shell of revolution with an arbitrary shape of the meridan is given in Chapter I. It is analogous to the conventional derivation of the basic equations for an arbitrary shell, which can be found in many books and monographs, devoted to this theme [3], [5], [21]; however it makes it possible to avoid the excesses of cumbersome notation and it does not require from the reader great knowledge in the field of differential geometry, since the geometry of a surface of revolution and accordingly the geometry of a shell of revolution are comparatively simple.

All the equations are written in a geographical coordinate system (Θ, ϕ) and only the shells of revolution enclosed in a circumferential direction are examined, (the shells) limited by two boundaries, coinciding with the coordinate lines θ = const. The complete system of equations, describing the equilibrium of a shell of revolution, is a system of differential equations in partial derivatives (§ 6).

Chapter II gives an account of the method of separating variables (§ 8) and a system of conventional differential equations is extracted to the solution of which the problem of determining the stressed state is reduced, having in the circumferential direction the rule of variation of the type cos $k\phi$, sin $k\phi$. In the general case, (k – is any whole number) this system has an eighth order. Eight boundary conditions are attached to it – four on each of the parallel circles limiting the shell. When k = 0.1 the order of the system can be reduced by one half (due to the obtaining of the two first static integrals and the two integrals of the equations of compatibility of the deformations) and the solution of the problem is considerably simplified. The basic contents of this book are devoted to an examination of these two cases: 1) the load on a shell is axisymmetric (k = 0), 2) the shell is deformed under the effect of a flexural load (k = 1). A profound analogy is traced between both cases.

In §§ 10-13 of Chapter II the axisymmetric deformation of a shell of revolution with an arbritrary shape of the meridan is examined, in §§ 14-18 – the deformation under a flexural load.

The solution of the axisymmetric problem reduces to a system of Meissner resolvent equations. For the case of the problem k = 1 analogous equations are obtained, which are subsequently called equations of the Meissner type. The presentation results in conventional variables of the theory of shells (forces, moments, deformations), without reverting to complex combinations of these magnitudes. The complex combination of the desired unknown quantities is introduced only in the final stage of the solution, i.e., after obtaining the two Meissner resolvent equations (or of the Meissner type), possessing a specific symmetry.

Sections 10-13 and 14-18 of Chapter II are the main point of the book. In reading any of the subsequent chapters, devoted to conical (Chapter IV), spherical (Chapter V), or torus-shaped (Chapter VI), shells, it is necessary to turn to the basic equations obtained in these sections. Chapter III is an exception. It can be read independently, since in view of the comparative simplicity of the geometry of the cylindrical shell and the already formed tradition, the derivation of the basic resolvent equations for $k = 0.1$ is given in it directly for a cylindrical shell without turning to the corresponding sections of Chapter II.

Chapter VII is devoted to internal stresses. In it are examined internal stressed states of the type cos $k\phi$, sin $k\phi$ ($k = 0.1$). The dislocation parameters, which characterize these states, are constants of integration in the first integrals of the system of differential equation connecting the components of "elastic" deformation.

By it contents and method of presentation the present book is very close to the well-known monograph of A. I. Lur'ye "Statics of thin-walled elastic shells". Since the time when it was issued approximately twenty years have passed. The mentioned monograph, in which with comprehensive clarity the theory of axisymmetric deformation of thin-walled shells of revolution is examined, had great effect on the author of these lines and aroused interest in this theme. This interest has not subsequently diminished in connection with the abundance of problems, which have confronted the author in his chosen profession.

The present book was conceived as a certain analog of A. I. Lur'ye's monograph, in which, from a unique point of view, the deformation of shells of revolution under axisymmetric and flexural loads is examined, since both the indicated cases are identically and frequently encountered in practice, and the methods of solving the problems arising here possess a great deal in common.

Over a period of many years the corresponding member of the Academy of Sciences of the USSR, Professor A. I. Lur'ye manifested constant attention to the works of the author, and now has agreed to assume the task of editing this book and has rendered the author the honor of introducing this work to the reader. I now consider it my earnest duty to express to the dear reader – Anatoly Isakovich Lur'ye – my profound gratitude.

The author wishes to express his sincere appreciation to A. K. Kibyanskaya for her assistance in preparing the manuscript for printing.

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C H A P T E R   I

FUNDAMENTAL EQUATIONS OF THE THEORY OF SHELLS OF REVOLUTION

§ 1. Geometry of a Surface of Revolution

The position of a point on the surface can be assigned by two fundamentally different methods: by the first method it is determined by assignment of coordinates of the point in a certain coordinate system, connected with the surrounding space; by the second – the position of the same point is determined by two numbers (α, β), where lines α = const, β = const form a system of curvilinear coordinate lines, located on the surface itself. It is natural that both these methods can be carried out in a countless quantity of variants. Let us examine the surface of revolution, formed by rotation of some curve Γ_1 around axis OO_1, along which axis OZ is directed (Fig. 1). The position of point M is determined by radius vector $\mathbf{r}$, drawn from some point O of space to the given point. If we connect Cartesian system of rectangular coordinates $OXYZ$ to selected point O, then

$$\mathbf{r} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}. \tag{1.1}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ – unit vectors, directed along axis X, Y, Z. The equation of the surface of revolution in coordinate system X, Y, Z can be written in the form

$$X = v\cos\varphi,\ Y = v\sin\varphi,\ Z = f(v). \tag{1.2}$$

The geometrical visualization of parameters v and ϕ is clear; v – radius of the circumference, which is obtained as a result of intersection of the surface by a plane, perpendicular to the axis of rotation, ϕ – the angle, read along the arc of this circumference, starting from the radius, parallel to axis OX. Parameters v and ϕ can serve as curvilinear coordinates on the surface; in this case lines v = const and ϕ = const will be parallels and meridians respectively, which form an orthogonal network of curves on the surface of revolution. The position of a point on the surface of revolution is convenient to determine also in cylindrical coordinate system v ϕ, Z with the origin of coordinates at point O. In this case the radius vector of point M can be represented in the form

$$\boldsymbol{r} = v\boldsymbol{e} + Z\boldsymbol{k}. \tag{1.3}$$

where

$$\boldsymbol{e} = \boldsymbol{i}\cos\varphi + \boldsymbol{j}\sin\varphi. \tag{1.4}$$

Unit vector $\boldsymbol{e}$ is directed along the radius of a parallel circle to the considered point.

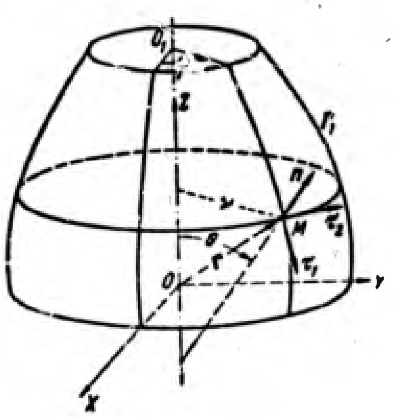


Fig. 1. The surface, formed by rotation of curve Γ_1 around axis OZ.

Let us introduce unit vectors of tangents to the meridian and to parallel circle at the given point

$$\tau_1 = \frac{\partial r}{\partial_1 s}, \quad \tau_2 = \frac{\partial r}{\partial_2 s}. \tag{1.5}$$

where $d_1 s$ — element of length of the arc of meridian, $d_2 s$ — element of length of the arc of parallel circle. Vectors τ_1 and τ_2 are mutually perpendicular, inasmuch as the meridians and parallels form an orthogonal network of curves on the surface of revolution. To the movement of the end of radius vector r from the given point M to the point of surface infinitely close to it M' corresponds to quantity

$$dr = \frac{\partial r}{\partial_1 s} d_1 s + \frac{\partial r}{\partial_2 s} d_2 s. \tag{1.6}$$

From (1.6) it is simple to conclude that relation $\frac{d_1 s}{d_2 s}$ determines the direction of such movement. When $d_1 s = 0$, $d_2 s \neq 0$ we obtain movement along parallels $d_2 r = t_2 d_2 s$, when $d_1 s \neq 0$, $d_2 s = 0$ movement of the end of radius vector occurs along meridian $d_1 r = \tau_1 d_1 s$. The unit vector of the tangent at point M to some curve Γ on the surface is equal to

$$t = \frac{dr}{ds} = \tau_1 \frac{d_1 s}{ds} + \tau_2 \frac{d_2 s}{ds}. \tag{1.7}$$

where ds — element of length of the arc of line Γ. Vectors t at the given point are arranged in a tangent plane to the surface at this point. The position of the tangent plane is entirely determined by the assignment of two nonparallel tangent vectors, for example τ_1 and τ_2. At point M let us construct a normal to the surface, having determined the unit vector of normal n as the vector product of vectors τ_1 and τ_2

$$n = \tau_1 \times \tau_2 \qquad (1.8)$$

Let us agree to always use a right-handed coordinate system. Three vectors τ_1, τ_2 n form a trihedron of orthogonal axes. In view of the symmetry of rotation all the normals to the surface at points located on one parallel intersect at one point on the axis of rotation and form a cone with angle of opening 2θ. Being limited to the examination of only such surfaces, on which the setting of angle θ uniquely determines the parallel circumference, just as the setting of angle ϕ determines the meridian, let us take system (θ, ϕ) as the basic system of curvilinear coordinates on our surface. In accordance with the terminology accepted in the theory of surfaces [25] we call the curve, which is obtained as a result of intersection of the surface by a plane, passing through the normal at point M, the normal section of the surface. Through any point of the surface it is possible to draw an infinite set of normal sections, to each of which corresponds its vector of tangent t.

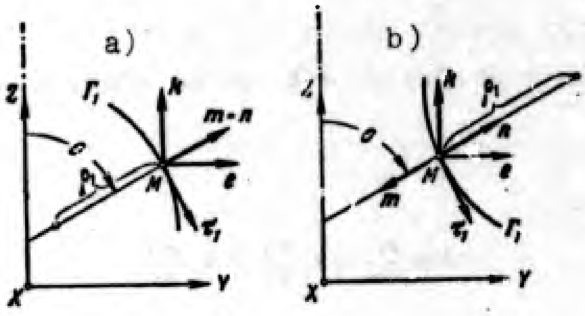


Fig. 2. The meridional section of the surface of revolution:
a) normal to the surface and normal to curve Γ_1 coincide,
b) normals have opposite directions.

Let us examine from the beginning a normal section with tangent τ_1, i.e., meridian Γ_1. It is a plane curve, the normal to which at point M (its unit vector m is directed opposite the principal normal – toward convexity Γ_1) either coincides with the normal to the surface (Fig. 2a), or is opposite it in direction (Fig. 2b). In the first case during motion along the meridian in positive direction $d\theta > 0$, in the second – $d\theta < 0$. If through ρ_1 we designate the radius of curvature of the meridian, then the element of length of the arc of meridian will be equal to

$$d_1 s = R_1 d\theta, \text{ where } \left.\begin{array}{lll} R_1 = \rho_1 & \text{when} & m = n, \\ R_1 = -\rho_1 & \text{when} & m = -n. \end{array}\right\} \tag{1.9}$$

In the case, shown on Fig. 2a,

$$\left.\begin{array}{l} \tau_1 = e\cos\theta - k\sin\theta, \\ m = e\sin\theta + k\cos\theta. \end{array}\right\} \tag{1.10}$$

By differentiating (1.10) with respect to $d_1 s$, we obtain known Frenet formulas for plane curve:

$$\left.\begin{array}{l} \dfrac{d\tau_1}{d_1 s} = (-e\sin\theta - k\cos\theta)\dfrac{d\theta}{d_1 s} = -\dfrac{m}{\rho_1}, \\ \dfrac{dm}{d_1 s} = (e\cos\theta - k\sin\theta)\dfrac{d\theta}{d_1 s} = \dfrac{\tau_1}{\rho_1}. \end{array}\right\} \tag{1.11}$$

By comparing (1.9) and (1.11), it is simple to write the formulas for derivatives of vectors τ_1 and n along the length of the arc of meridian

$$\frac{d\tau_1}{d_1 s} = -\frac{n}{R_1}, \quad \frac{dn}{d_1 s} = \frac{\tau_1}{R_1}. \tag{1.12}$$

or

$$\frac{\partial \tau_1}{\partial \theta} = -n, \quad \frac{\partial n}{\partial \theta} = \tau_1. \tag{1.13}$$

In formulas (1.13) there are introduced partial derivatives, since on the surface vectors τ_1, τ_2, n are functions of coordinates θ, ϕ. Let us set a goal to obtain the remaining formulas of differentiation of vectors τ_1, τ_2, n with respect to coordinates. For this let us examine the parallel circumference of radius $\nu(\theta)$ (Fig. 3), which is an inclined section of the surface from tangent τ_2 at the given point. The plane of the parallel circumference forms an angle, equal to $(\frac{\pi}{2} - \theta)$, with the plane of the normal section, which has a common tangent with it. The element of length of the arc of parallel is equal to

$$d_2 s = \nu d\varphi. \tag{1.14}$$

On the basis of Frenet formulas

$$\frac{d\tau_2}{d_2 s} = -\frac{e}{\nu}, \quad \frac{de}{d_2 s} = \frac{\tau_2}{\nu} \tag{1.15}$$

and (1.14) we obtain

$$\frac{\partial \tau_2}{\partial \varphi} = -e, \quad \frac{\partial e}{\partial \varphi} = \tau_2. \tag{1.16}$$

Since

$$e = \tau_1 \cos \theta + n \sin \theta. \tag{1.17}$$

then the second relationship (1.16) can be rewritten so:

$$\frac{\partial \tau_1}{\partial \varphi}\cos\theta + \frac{\partial n}{\partial \varphi}\sin\theta = \tau_2. \qquad (1.18)$$

By scalar multiplication of both sides of equality (1.18) by n and taking into consideration that $\left(\frac{\partial n}{\partial \phi}\cdot n\right) = 0$ (which is simple to check by differentiation of equality $(n\cdot n) = 1$), we obtain that $\left(\frac{\partial \tau_1}{\partial \phi}\cdot n\right) = 0$, i.e., vector $\frac{\partial \tau_1}{\partial \phi}$ does not have a component along axis n. Since $\frac{\partial \tau_1}{\partial \phi}$ also does not have a component along axis τ_1, then $\frac{\partial \tau_1}{\partial \phi} = a\tau_2$. Analogously, by scalar multiplication of (1.18) by τ_1, we ensure that $\left(\frac{\partial n}{\partial \phi}\cdot \tau_1\right) = 0$ and consequently, $\frac{\partial n}{\partial \phi} = b\tau_2$. By turning again to equality (1.18), we find that it can be performed identically only with $a = \cos\theta$, $b = \sin\theta$ and therefore

$$\frac{\partial \tau_1}{\partial \varphi} = \tau_2\cos\theta, \quad \frac{\partial n}{\partial \varphi} = \tau_2\sin\theta. \qquad (1.19)$$

By differentiating scalar products $(\tau_2\cdot n) = 0$ and $(\tau_2\cdot\tau_1) = 0$ with respect to θ and taking into account formula (1.13), we ensure that

$$\left(\frac{\partial \tau_2}{\partial \theta}\cdot n\right) = 0 \quad \text{and} \quad \left(\frac{\partial \tau_2}{\partial \theta}\cdot \tau_1\right) = 0,$$

whence follows obvious formula $\frac{\partial \tau_2}{\partial \theta} = 0$. As a result the following derivation formulas are obtained:

$$\left.\begin{array}{l} \frac{\partial \tau_1}{\partial \theta} = -n, \quad \frac{\partial \tau_2}{\partial \theta} = 0, \quad \frac{\partial n}{\partial \theta} = \tau_1, \\ \frac{\partial \tau_1}{\partial \varphi} = \tau_2\cos\theta, \quad \frac{\partial \tau_2}{\partial \varphi} = -\tau_1\cos\theta - n\sin\theta, \quad \frac{\partial n}{\partial \varphi} = \tau_2\sin\theta. \end{array}\right\} \qquad (1.20)$$

Taking into account relationships (1.9), (1.14), formulas (1.20) can be rewritten still in the following form:

$$\left.\begin{aligned}&\frac{\partial\tau_1}{\partial_1 s}=-\frac{n}{R_1},\quad \frac{\partial\tau_2}{\partial_1 s}=0,\quad \frac{\partial n}{\partial_1 s}=\frac{\tau_1}{R_1},\\ &\frac{\partial\tau_1}{\partial_2 s}=\tau_2\frac{\cos\theta}{v},\quad \frac{\partial\tau_2}{\partial_2 s}=-\tau_1\frac{\cos\theta}{v}-n\frac{\sin\theta}{v},\quad \frac{\partial n}{\partial_2 s}=\tau_2\frac{\sin\theta}{v}.\end{aligned}\right\}\qquad (1.20^*)$$

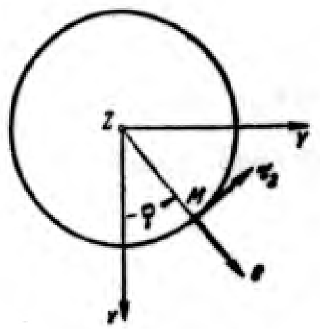


Fig. 3. The circumference, which is the inclined section of the surface from tangent τ_2 at given point M.

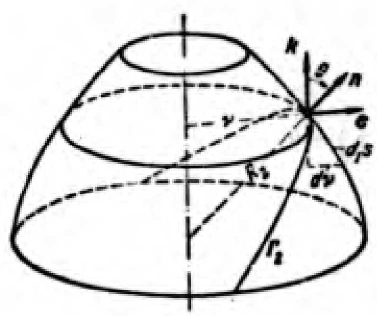


Fig. 4. Curve Γ_2 – normal section of the surface from tangent τ_2.

Let us now turn to the examination of normal section with tangent τ_2 (curve Γ_2 in Fig. 4). Curve Γ_2 and the parallel circle at point M have common tangent τ_2, consequently,

$$\tau_{\varphi} = \frac{\partial r}{\partial_2 s} = \frac{\partial r}{\partial \tilde{s}}. \tag{1.21}$$

where through $\partial\tilde{s}$ there is designated the differential of arc Γ_2. By using the first Frenet formula for the parallel and for curve Γ_2, we obtain

$$\frac{\partial^2 r}{\partial_2 s^2} = -\frac{\varepsilon}{v}, \quad \frac{\partial^2 r}{\partial \tilde{s}^2} = -\frac{n}{R_2}. \tag{1.22}$$

where R_2 — the radius of curvature of normal section Γ_2. By scalar multiplication of both relationships (1.22) by n, we will have

$$\left(\frac{\partial^2 r}{\partial_2 s^2} \cdot n\right) = -\frac{\sin\theta}{v}, \quad \left(\frac{\partial^2 r}{\partial \tilde{s}^2} \cdot n\right) = -\frac{1}{R_2}. \tag{1.23}$$

From (1.21) and (1.23) follows

$$\frac{\sin\theta}{v} = \frac{1}{R_2}, \quad \text{or} \quad v = R_2 \sin\theta. \tag{1.24}$$

Relationship (1.24) indicates that the center of curvature of normal section Γ_2, which at the given point has common tangent with a parallel, is projected to the center of the parallel circumference and, consequently, is located on the axis of rotation. This assumption is a consequence of the Meusnier theorem, known in the theory of surfaces, according to which the curvature of the curve on the surface and the curvature of the normal section, which has a common tangent with the given curve, are connected according to the following law: the curvature of a curve is equal to the curvature of a normal section, multiplied by the cosine of the angle between the osculating plane of the curve and the plane of the normal section.

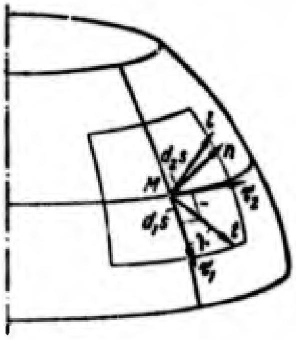


Fig. 5. Unit vectors τ_1, τ_2, t, located on the tangent plane to the surface at point M.

Let us note that from (1.23) and the first formula (1.20) taking into account (1.9), (1.5) the following formulas are obtained for curvatures $1/R_1$ and $1/R_2$

$$\left.\begin{aligned} \frac{1}{R_1} &= -\left(\frac{\partial \tau_1}{\partial_1 s} \cdot n\right), \\ \frac{1}{R_2} &= -\left(\frac{\partial \tau_2}{\partial_2 s} \cdot n\right). \end{aligned}\right\} \tag{1.25}$$

Let us now determine the curvature of the surface, considering the normal section with unit vector of tangent t, forming angle λ with direction τ_1 (Fig. 5). With change of λ from 0 to π the whole set of normal sections at the given point turns out to be exhausted. By designating the element of length of the arc of curve Γ_t through $d\sigma$, let us write

$$\left.\begin{aligned} \cos\lambda = \frac{d_1 s}{d\sigma}, \quad \sin\lambda = \frac{d_2 s}{d\sigma}, \\ t = \tau_1 \frac{d_1 s}{d\sigma} + \tau_2 \frac{d_2 s}{d\sigma}. \end{aligned}\right\} \tag{1.26}$$

Let us compute quantity

$$\frac{\partial t}{\partial \sigma} = \frac{\partial t}{R_1 \partial \theta} \frac{d_1 s}{d\sigma} + \frac{\partial t}{\nu \partial \varphi} \frac{d_2 s}{d\sigma}. \tag{1.27}$$

By using derivation formulas (1.20) and formulas (1.26), we obtain

$$\frac{dt}{d\sigma}=\left(-\frac{\cos^2\lambda}{R_1}-\frac{\sin^2\lambda}{R_2}\right)n+l\left(\frac{\sin\lambda\cos\theta}{v}+\frac{d\lambda}{d\sigma}\right), \tag{1.28}$$

where $l = -\tau_1 \sin\lambda + \tau_2 \cos\lambda$ – the unit vector, perpendicular to vector t and located in tangent plane.

Let us compute the curvature of the surface in direction t by formula

$$\frac{1}{R_t}=-\left(\frac{\partial t}{\partial\sigma}\cdot n\right). \tag{1.29}$$

By using in this case the expression for $\frac{\partial t}{\partial\sigma}$ according to (1.28) and taking into consideration that $(l\cdot n) = 0$, we obtain

$$\frac{1}{R_t}=\frac{\cos^2\lambda}{R_1}+\frac{\sin^2\lambda}{R_2}. \tag{1.30}$$

From (1.30) and (1.26) follows important formula

$$\frac{1}{R_t}=\frac{1}{R_1}\left(\frac{d_1s}{d\sigma}\right)^2+\frac{1}{R_2}\left(\frac{d_2s}{d\sigma}\right)^2. \tag{1.31}$$

By investigating curvature $1/R_t$ as a function of parameter λ, we find that it takes extermal values when $\lambda = 0$, $\lambda = \pi/2$, moreover in the first case it is equal to $1/R_1$, in the second $1/R_2$. In the language of geometry of surfaces this means that the examined normal sections Γ_1 and Γ_2 are principal, R_1, R_2 are the principal radii of curvature, and meridians and parallels form a network of lines of curvature on the surface of revolution. Let us recall that the lines, the tangents to which at each point coincide with principal directions, are called the lines of curvature.

It is not possible to form the surface of revolution with randomly assigned radii of curvature R_1 and R_2. In this case it is simple to ensure, comprising the condition of independence of the second derivative $\frac{\partial^2 r}{\partial\theta\partial\phi}$ from the order of differentiation

$$\frac{\partial}{\partial\varphi}(R_1\tau_1)=\frac{\partial}{\partial\theta}(\nu\tau_2), \tag{1.32}$$

which, after the utilization of derivation formulas (1.20) and formula (1.29), gives the following relationship between R_1 and R_2:

$$\frac{d\,(R_2\sin\theta)}{d\theta}=R_1\cos\theta. \tag{1.33}$$

Formula (1.33) can be obtained by another way, namely: the unit vector of tangent τ_1 can be presented in the form

$$\tau_1=\frac{e}{R_1}\frac{d\nu}{d\theta}+\frac{k}{R_1}\frac{dZ}{d\theta}, \tag{1.34}$$

but, from another side,

$$\tau_1=e\cos\theta-k\sin\theta. \tag{1.35}$$

From comparison of (1.34) and (1.35) we conclude

$$\frac{d\nu}{d\theta}=R_1\cos\theta,\quad \frac{dZ}{d\theta}=-R_1\sin\theta. \tag{1.36}$$

The first of these formulas repeats relationship (1.33).

Let us give another formula, which will subsequently be useful:

$$\frac{d}{d\theta}\left(\frac{1}{R_2}\right)=\frac{\cos\theta}{v}\left(1-\frac{R_1}{R_2}\right). \qquad (1.37)$$

To all the aforesaid we should add that generally, when the surface is not related to the lines of curvature and coordinate lines α = const, β = const are not orthogonal, for the computation of curvature a more complex formula than (1.31) is obtained. In this case

$$dr = t_\alpha d\sigma_\alpha + t_\beta d\sigma_\beta, \qquad (1.38)$$

where, as earlier, $t_\alpha = \frac{\partial r}{\partial \sigma_\alpha}$, $t_\beta = \frac{\partial r}{\partial \sigma_\beta}$ — unit vectors of tangents to coordinate lines, moreover

$$(t_\alpha \cdot t_\beta) \neq 0, \quad (t_\alpha \cdot n) = (t_\beta \cdot n) = 0. \qquad (1.39)$$

The direction of normal section Γ_t is characterized by unit tangent vector

$$t = \frac{dr}{d\sigma} = t_\alpha \frac{d\sigma_\alpha}{d\sigma} + t_\beta \frac{d\sigma_\beta}{d\sigma}$$

and curvature $1/R_t$ is equal to

$$\frac{1}{R_t} = -\left(\frac{\partial t}{\partial \sigma}\cdot n\right) = -\left(\frac{\partial^2 r}{\partial \sigma_\alpha^2}\cdot n\right)\left(\frac{d\sigma_\alpha}{d\sigma}\right)^2 - 2\left(\frac{\partial^2 r}{\partial \sigma_\alpha \partial \sigma_\beta}\cdot n\right)\frac{d\sigma_\alpha}{d\sigma}\frac{d\sigma_\beta}{d\sigma} - \left(\frac{\partial^2 r}{\partial \sigma_\beta^2}\cdot n\right)\left(\frac{d\sigma_\beta}{d\sigma}\right)^2. \qquad (1.40)$$

By analogy with formulas (1.25) by introducing designations

$$\left.\begin{aligned}\frac{1}{R_\alpha}&=-\left(\frac{\partial^2 r}{\partial\sigma_\alpha^2}\cdot n\right)=-\left(\frac{\partial t_\alpha}{\partial\sigma_\alpha}\cdot n\right),\\ \frac{1}{R_\beta}&=-\left(\frac{\partial^2 r}{\partial\sigma_\beta^2}\cdot n\right)=-\left(\frac{\partial t_\beta}{\partial\sigma_\beta}\cdot n\right),\\ \frac{1}{R_{\alpha\beta}}&=-\left(\frac{\partial^2 r}{\partial\sigma_\alpha\partial\sigma_\beta}\cdot n\right)=-\left(\frac{\partial t_\alpha}{\partial\sigma_\beta}\cdot n\right)=-\left(\frac{\partial t_\beta}{\partial\sigma_\alpha}\cdot n\right),\end{aligned}\right\} \tag{1.41}$$

let us write (1.40) in the form

$$\frac{1}{R_t}=\frac{1}{R_\alpha}\left(\frac{d\sigma_\alpha}{d\sigma}\right)^2+\frac{2}{R_{\alpha\beta}}\frac{d\sigma_\alpha}{d\sigma}\frac{d\sigma_\beta}{d\sigma}+\frac{1}{R_\beta}\left(\frac{d\sigma_\beta}{d\sigma}\right)^2. \tag{1.42}$$

Formula (1.42) differs from formula (1.31) by the presence of a term, containing the product of $d\sigma_\alpha\ d\sigma_\beta$. $\frac{1}{R_\alpha}$, $\frac{1}{R_\beta}$ – curvatures of the surface in directions t_α, t_β; $\frac{1}{R_{\alpha\beta}}$ – quantity, which it is accepted to call torsion in the theory of shells. As a result of (1.39) formulas (1.41) can be rewritten in another form:

$$\left.\begin{aligned}\frac{1}{R_\alpha}&=\left(t_\alpha\cdot\frac{\partial n}{\partial\sigma_\alpha}\right),\ \frac{1}{R_\beta}=\left(t_\beta\cdot\frac{\partial n}{\partial\sigma_\beta}\right),\\ \frac{1}{R_{\alpha\beta}}&=\left(t_\beta\cdot\frac{\partial n}{\partial\sigma_\alpha}\right)=\left(t_\alpha\cdot\frac{\partial n}{\partial\sigma_\beta}\right).\end{aligned}\right\} \tag{1.43}$$

By returning to previous coordinate lines θ, ϕ, let us assume $d\sigma_\sigma = R_1 d\theta$, $d\sigma_\beta = \nu d\phi$, $t_\alpha = \tau_1$, $t_\beta = \tau_2$. By using derivation formulas (1.20) and formula (1.43), we ensure that in this case $\frac{1}{R_{12}} = 0$, i.e., torsion is equal to zero, if the surface is related to the lines of curvature.

§ 2. Deformation of a Surface of Revolution

With deformation of a surface point M, obtaining displacement $\boldsymbol{U}$, changes into some point M^*, the radius vector of which is equal to

$$\boldsymbol{r}^* = \boldsymbol{r} + \boldsymbol{U}, \tag{2.1}$$

$$\boldsymbol{U} = u\boldsymbol{\tau}_1 + v\boldsymbol{\tau}_2 + w\boldsymbol{n}, \tag{2.2}$$

where u, v, w – components of the vector of displacement $\boldsymbol{U}$ along axes $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\boldsymbol{n}$. In a particular case of axisymmetric deformation the component of vector $\boldsymbol{U}$ along axis $\boldsymbol{\tau}_2$ is equal to zero ($v = 0$) and all points of the surface, located initially on one meridian, continue to remain in the same meridional plane after deformation. In this case the displacement of points on all meridians is the same, i.e., does not depend on coordinate ϕ. Subsequently we will consider the general case of deformation of a surface of revolution ($v \neq 0$). Let us examine infinitesimal displacement with respect to a deformed surface

$$\left.\begin{aligned} d\boldsymbol{r}^* &= d\boldsymbol{r} + d\boldsymbol{U}, \\ d\boldsymbol{U} &= \frac{\partial \boldsymbol{U}}{\partial_1 s} d_1 s + \frac{\partial \boldsymbol{U}}{\partial_2 s} d_2 s. \end{aligned}\right\} \tag{2.3}$$

Recalling that $d_1 s = R_1 d\theta$, $d_2 s = \nu d\phi$ and using derivation formulas (1.20), we obtain

$$\left.\begin{aligned} \frac{\partial \boldsymbol{U}}{\partial_1 s} &= \varepsilon_1 \boldsymbol{\tau}_1 + \gamma_1 \boldsymbol{\tau}_2 + \vartheta_1 \boldsymbol{n}, \\ \frac{\partial \boldsymbol{U}}{\partial_2 s} &= \gamma_2 \boldsymbol{\tau}_1 + \varepsilon_2 \boldsymbol{\tau}_2 + \vartheta_2 \boldsymbol{n}. \end{aligned}\right\} \tag{2.4}$$

where there are introduced designations

$$\begin{aligned} \varepsilon_1 &= \frac{1}{R_1}\frac{\partial u}{\partial \theta} + \frac{w}{R_1}, & \gamma_2 &= \frac{1}{\nu}\frac{\partial u}{\partial \varphi} - \frac{v\cos\theta}{\nu}, \\ \gamma_1 &= \frac{1}{R_1}\frac{\partial v}{\partial \theta}, & \varepsilon_2 &= \frac{1}{\nu}\frac{\partial v}{\partial \varphi} + \frac{u\cos\theta}{\nu} + \frac{w\sin\theta}{\nu}, \\ \vartheta_1 &= \frac{1}{R_1}\frac{\partial w}{\partial \theta} - \frac{u}{R_1}, & \vartheta_2 &= \frac{1}{\nu}\frac{\partial w}{\partial \varphi} - \frac{v\sin\theta}{\nu}. \end{aligned} \tag{2.5}$$

and, thus, $dr^* = d_1 r^* + d_2 r^*$, where

$$\begin{aligned} d_1 r^* &= \frac{\partial r^*}{\partial_1 s} d_1 s, & \frac{\partial r^*}{\partial_1 s} &= \tau_1(1+\varepsilon_1) + \tau_2\gamma_1 + n\vartheta_1, \\ d_2 r^* &= \frac{\partial r^*}{\partial_2 s} d_2 s, & \frac{\partial r^*}{\partial_2 s} &= \tau_1\gamma_2 + \tau_2(1+\varepsilon_2) + n\vartheta_2. \end{aligned} \tag{2.6}$$

Let us explain the geometrical visualization of introduced quantities ε_1, ε_2, γ_1, γ_2, ϑ_1, ϑ_2. With displacement in meridional direction ($d_1 s \neq 0$, $d_2 s = 0$) we find the element of length of the arc of meridian after deformation

$$d_1 s^* = |d_1 r^*| = \sqrt{(1+\varepsilon_1)^2 + \gamma_1^2 + \vartheta_1^2}\, d_1 s. \tag{2.7}$$

Analogously with displacement along parallel ($d_1 s = 0$, $d_2 s \neq 0$)

$$d_2 s^* = |d_2 r^*| = \sqrt{\gamma_2^2 + (1+\varepsilon_2)^2 + \vartheta_2^2}\, d_2 s. \tag{2.8}$$

Being limited to the examination of small deformations, i.e., disregarding the squares of quantities ε_1, γ_1, ϑ_1, ε_2, γ_2, ϑ_2 in comparison with one, we obtain that

$$\begin{aligned} d_1 s^* &= (1+\varepsilon_1) d_1 s, & \varepsilon_1 &= \frac{d_1 s^* - d_1 s}{d_1 s}, \\ d_2 s^* &= (1+\varepsilon_2) d_2 s, & \varepsilon_2 &= \frac{d_2 s^* - d_2 s}{d_2 s}. \end{aligned} \tag{2.9}$$

whence it is clear that ε_1, ε_2 are relative elongations in meridional and circumferential directions. The vectors of tangents to coordinate lines on a deformed surface have the form

$$\left.\begin{aligned} \tau_1^* &= \frac{\partial r^*}{\partial_1 s^*} = \frac{1}{1+\varepsilon_1}[\tau_1(1+\varepsilon_1)+\tau_2\gamma_1+n\theta_1] \approx \tau_1+\tau_2\gamma_1+n\theta_1, \\ \tau_2^* &= \frac{\partial r^*}{\partial_2 s^*} = \frac{1}{1+\varepsilon_2}[\tau_1\gamma_2+\tau_2(1+\varepsilon_2)+n\theta_2] \approx \tau_1\gamma_2+\tau_2+n\theta_2. \end{aligned}\right\} \quad (2.10)$$

Vectors τ_1^* and τ_2^* are not orthogonal to each other, which is simple to check, by calculating their scalar product

$$(\tau_1^* \cdot \tau_2^*) = \gamma_2 + \gamma_1 = \gamma. \quad (2.11)$$

If deformations are small, then quantity γ is equal to the variation of initially right angle between vectors τ_1 and τ_2 and is called shear. In fact, if γ is small, then $\cos(\tau_1^*, \tau_2^*) = \gamma \approx \cos\left(\frac{\pi}{2} - \gamma\right)$.

Analogical $\cos(\tau_1^*, \tau_2) = \gamma_1$, $\cos(\tau_2^*, \tau_1) = \gamma_2$, i.e., γ_1 and γ_2 are the angles between vectors τ_1^*, τ_1 and τ_2^*, τ_2 respectively (Fig. 6). Thus, the aggregate of three quantities ε_1, ε_2, γ characterizes elongations and changes of angles between coordinate lines during deformation.

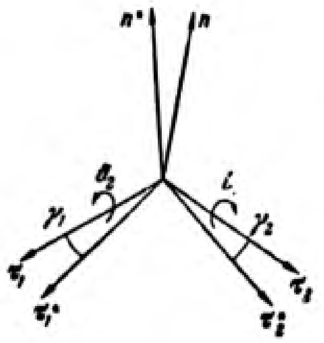


Fig. 6. Tangents to coordinate lines and normal to the surface before and after deformation.

Let us explain the geometrical visualization of quantities ϑ_1 and ϑ_2 (formula (2.5)). Let us determine the vector of the normal to deformed surface $\mathbf{n}^*$ as vector product

$$\mathbf{n}^* = \boldsymbol{\tau}_1^* \times \boldsymbol{\tau}_2^*; \tag{2.12}$$

then, using formulas (2.10), we obtain

$$\mathbf{n}^* = \mathbf{n} - \vartheta_1 \boldsymbol{\tau}_1 - \vartheta_2 \boldsymbol{\tau}_2 \tag{2.13}$$

and

$$\cos(\mathbf{n}^*, \boldsymbol{\tau}_1) = -\vartheta_1 \approx \cos\left(\frac{\pi}{2} + \vartheta_1\right),$$
$$\cos(\mathbf{n}^*, \boldsymbol{\tau}_2) = -\vartheta_2 \approx \cos\left(\frac{\pi}{2} + \vartheta_2\right),$$

i.e., $-\vartheta_1$ and ϑ_2 represent the angles of rotation of the normal to the surface around axes $\boldsymbol{\tau}_2$ and $\boldsymbol{\tau}_1$ respectively (see Fig. 6). Let us introduce vector

$$\boldsymbol{\Omega} = \vartheta_2 \boldsymbol{\tau}_1 - \vartheta_1 \boldsymbol{\tau}_2 + \delta \mathbf{n}, \tag{2.14}$$

where

$$\delta = \frac{1}{2}(\gamma_1 - \gamma_2). \tag{2.15}$$

then relationship (2.13) can be rewritten in this form:

$$\mathbf{n}^* - \mathbf{n} = \boldsymbol{\Omega} \times \mathbf{n}.$$

Moving ahead, let us call $\boldsymbol{\Omega}$ the rotation vector. If we take into account the meaning of quantities $-\vartheta_1$, ϑ_2 and take quantity δ for the characteristic of rotation of the element of the middle surface around the normal, then such a name is natural. In § 3 there will be shown that with acceptance of Kirchhoff-Love geometric hypotheses vector $\boldsymbol{\Omega}$ is equal to the value on the middle surface of the rotation vector, common in the theory of deformations of continuous medium.

Let us now turn to the study of the curvature of a deformed surface. For this, by using formulas (2.10), (1.25) and (2.13), and also formulas of differentiation of unit vectors $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\boldsymbol{n}$ (1.20), let us compute quantities

$$\left.\begin{aligned} \frac{\partial \boldsymbol{\tau}_1^{\bullet}}{\partial_1 s^{\bullet}} &= \frac{1}{1+\varepsilon_1}\left[\frac{\vartheta_1}{R_1}\boldsymbol{\tau}_1 + \frac{1}{R_1}\frac{\partial \gamma_1}{\partial \theta}\boldsymbol{\tau}_2 + \left(-\frac{1}{R_1} + \frac{1}{R_1}\frac{\partial \vartheta_1}{\partial \theta}\right)\boldsymbol{n}\right], \\ \frac{\partial \boldsymbol{\tau}_2^{\bullet}}{\partial_2 s^{\bullet}} &= \frac{1}{1+\varepsilon_2}\left[\left(-\frac{\cos\theta}{\nu} + \frac{1}{\nu}\frac{\partial \gamma_2}{\partial \varphi}\right)\boldsymbol{\tau}_1 + \left(\frac{\gamma_2\cos\theta}{\nu} + \frac{\vartheta_2\sin\theta}{\nu}\right)\boldsymbol{\tau}_2 + \right. \\ &\qquad \left. + \left(\frac{1}{\nu}\frac{\partial \vartheta_2}{\partial \varphi} - \frac{\sin\theta}{\nu}\right)\boldsymbol{n}\right]. \end{aligned}\right\} \quad (2.16)$$

$$\left.\begin{aligned} \frac{1}{R_1^{\bullet}} &= -\left(\frac{\partial \boldsymbol{\tau}_1^{\bullet}}{\partial_1 s^{\bullet}}\cdot \boldsymbol{n}^{\bullet}\right) = \frac{1-\varepsilon_1}{R_1} - \frac{1}{R_1}\frac{\partial \vartheta_1}{\partial \theta}, \\ \frac{1}{R_2^{\bullet}} &= -\left(\frac{\partial \boldsymbol{\tau}_2^{\bullet}}{\partial_2 s^{\bullet}}\cdot \boldsymbol{n}^{\bullet}\right) = \frac{(1-\varepsilon_2)\sin\theta}{\nu} - \frac{1}{\nu}\frac{\partial \vartheta_2}{\partial \varphi} - \frac{\vartheta_1\cos\theta}{\nu}. \end{aligned}\right\} \quad (2.17)$$

By introducing designations

$$\varkappa_1 = -\frac{1}{R_1}\frac{\partial \vartheta_1}{\partial \theta}, \quad \varkappa_2 = -\frac{\vartheta_1\cos\theta}{\nu} - \frac{1}{\nu}\frac{\partial \vartheta_2}{\partial \varphi},$$

let us copy (2.17) again

$$\frac{1}{R_1^*}-\frac{1}{R_1}=-\frac{\varepsilon_1}{R_1}+\varkappa_1, \quad \frac{1}{R_2^*}-\frac{1}{R_2}=-\frac{\varepsilon_2}{R_2}+\varkappa_2. \tag{2.19}$$

From (2.19) it is clear that quantities κ_1 and κ_2 characterize changes of principal curvatures during deformation, moreover the first terms in the right sides of (2.19) are connected with the change of curvatures due to extensions or compressions of coordinate lines ε_1, ε_2. Since after deformation of the surface directions τ_1^*, τ_2^* are no longer principle, then it is necessary to even calculate torsion $1/R_{12}^*$, which in this case will be nonzero. On the basis of formula (1.43), (2.10), (2.13) and derivation formulas (1.20) we obtain

$$\left.\begin{aligned}
\frac{\partial n^*}{\partial_1 s^*}&=\frac{1}{1+\varepsilon_1}\left[\left(\frac{1}{R_1}-\frac{1}{R_1}\frac{\partial\vartheta_1}{\partial\theta}\right)\tau_1-\frac{1}{R_1}\frac{\partial\vartheta_2}{\partial\theta}\tau_2+\frac{\vartheta_1}{R_1}n\right],\\
\frac{\partial n^*}{\partial_2 s^*}&=\frac{1}{1+\varepsilon_2}\left[\left(-\frac{1}{v}\frac{\partial\vartheta_1}{\partial\varphi}+\frac{\vartheta_2\cos\theta}{v}\right)\tau_1+\right.\\
&\qquad\left.+\left(-\frac{1}{v}\frac{\partial\vartheta_2}{\partial\varphi}+\frac{\sin\theta}{v}\right)\tau_2+\frac{\vartheta_2\sin\theta}{v}n\right].
\end{aligned}\right\} \tag{2.20}$$

$$\left.\begin{aligned}
\frac{1}{R_{12}^*}&=\left(\tau_2^*\cdot\frac{\partial n^*}{\partial_1 s^*}\right)=\tau_1+\frac{\gamma_2}{R_1},\\
\frac{1}{R_{12}^*}&=\left(\tau_1^*\cdot\frac{\partial n^*}{\partial_2 s^*}\right)=\tau_2+\frac{\gamma_1}{R_2}.
\end{aligned}\right\} \tag{2.21}$$

In (2.21) there are introduced designations

$$\tau_1=-\frac{1}{R_1}\frac{\partial\vartheta_2}{\partial\theta}, \quad \tau_2=-\frac{1}{v}\frac{\partial\vartheta_1}{\partial\varphi}+\frac{\vartheta_2\cos\theta}{v}. \tag{2.22}$$

By recalling the expressions for ϑ_1, ϑ_2, γ_1 and γ_2 through displacements (formulas (2.5)), directly by checking we ensure that the right sides of (2.21) are identical, i.e., there takes place identity

$$\tau_1 + \frac{\gamma_2}{R_1} = \tau_2 + \frac{\gamma_1}{R_2}. \tag{2.23}$$

Let us also note incidentally tne following identities, which will be subsequently useful:

$$\frac{1}{R_1}\frac{\partial \gamma_2}{\partial \theta} + \frac{\vartheta_2}{R_1} = \frac{1}{\nu}\frac{\partial \varepsilon_1}{\partial \varphi} - \frac{\gamma \cos\theta}{\nu}, \tag{2.24}$$

$$\frac{\partial \gamma_1}{\partial \varphi} + \vartheta_1 \sin\theta = -\varepsilon_1 \cos\theta + \frac{1}{R_1}\frac{\partial (\nu\varepsilon_2)}{\partial \theta}. \tag{2.25}$$

By designating torsion $1/R^*_{12}$ by the letter τ, for it we have two equivalent expressions

$$\tau = \tau_1 + \frac{\gamma_2}{R_1} \quad \text{or} \quad \tau = \tau_2 + \frac{\gamma_1}{R_2}. \tag{2.26}$$

Thus, the change of surface curvature with deformation is characterized by three quantities κ_1, κ_2 and τ.

In conclusion let us compute the derivatives from the rotation vector $\mathbf{\Omega}$., On the basis of (2.14), by using derivation formulas (1.20) and taking into account (2.?2), we will have

$$\left.\begin{aligned} \frac{\partial \mathbf{\Omega}}{\partial_1 s} &= \left(-\tau_1 + \frac{\vartheta}{R_1}\right)\mathbf{t}_1 + \varkappa_1 \mathbf{t}_2 + \zeta_1 \mathbf{n}, \\ \frac{\partial \mathbf{\Omega}}{\partial_2 s} &= -\varkappa_2 \mathbf{t}_1 + \left(\tau_2 + \frac{\vartheta}{R_2}\right)\mathbf{t}_2 + \zeta_2 \mathbf{n}. \end{aligned}\right\} \tag{2.27}$$

where through ζ_1 and ζ_2 there are designated quantities

$$\zeta_1 = \frac{1}{R_1}\frac{\partial \vartheta}{\partial \theta} - \frac{\vartheta_2}{R_1}, \quad \zeta_2 = \frac{1}{\nu}\frac{\partial \vartheta}{\partial \varphi} + \frac{\vartheta_1 \sin\theta}{\nu}. \tag{2.28}$$

By using identities (2.23), (2.24), (2.25), and also taking into consideration that $\delta = \frac{1}{2}(\gamma_1 - \gamma_2)$, $\gamma = \gamma_1 + \gamma_2$, let us convert the right sides of (2.27) so that only quantities ε_1, ε_2, γ, κ_1, κ_2, τ and their derivatives would enter them. For example,

$$-\tau_1 + \frac{\delta}{R_1} = -\tau_1 - \frac{\gamma_2}{R_1} + \frac{\gamma_2}{R_1} + \frac{\gamma_1}{2R_1} - \frac{\gamma_2}{2R_1} = -\tau + \frac{\gamma}{2R_1},$$

$$\zeta_1 = \frac{1}{R_1}\frac{\partial\delta}{\partial\theta} - \frac{\vartheta_2}{R_1} = \frac{1}{2R_1}\frac{\partial\gamma}{\partial\theta} - \frac{1}{R_1}\frac{\partial\gamma_2}{\partial\theta} - \frac{\vartheta_2}{R_1} =$$
$$= \frac{1}{2R_1}\frac{\partial\gamma}{\partial\theta} + \frac{\gamma\cos\theta}{\nu} - \frac{1}{\nu}\frac{\partial\varepsilon_1}{\partial\varphi},$$

etc.

Finally we obtain

$$\left.\begin{aligned} \frac{\partial\Omega}{\partial_1 s} &= \left(-\tau + \frac{\gamma}{2R_1}\right)\tau_1 + \varkappa_1\tau_2 + \zeta_1 n, \\ \frac{\partial\Omega}{\partial_2 s} &= -\varkappa_2\tau_1 + \left(\tau - \frac{\gamma}{2R_2}\right)\tau_2 + \zeta_2 n, \end{aligned}\right\} \quad (2.29)$$

where ζ_1, ζ_2 are converted to the form

$$\left.\begin{aligned} \zeta_1 &= \frac{1}{2R_1}\frac{\partial\gamma}{\partial\theta} + \frac{\gamma\cos\theta}{\nu} - \frac{1}{\nu}\frac{\partial\varepsilon_1}{\partial\varphi}, \\ \zeta_2 &= -\frac{1}{2\nu}\frac{\partial\gamma}{\partial\varphi} - \frac{\varepsilon_1\cos\theta}{\nu} + \frac{1}{\nu R_1}\frac{\partial(\nu\varepsilon_2)}{\partial\theta}. \end{aligned}\right\} \quad (2.30)$$

§ 3. Deformation of a Shell of Revolution

The position of a point on a surface of revolution is governed by two curvilinear coordinates θ, ϕ. To determine the position of a point not on the involved surface, it is necessary to give three numbers or three curvilinear spatial coordinates. If point N of the space is not far from the surface, then the position of N

relative to the latter is simply determined by a section of the normal to the surface drawn to the point. The surface relative to which the position of the given point is determined we call the reference surface. Coordinates θ, ϕ, ζ where ζ - distance along the normal from the reference surface to the given points, are now curvilinear coordinates of a point in space. While ζ is considered positive if point N is on the positive side of the normal, in the opposite case ζ is a negative number. The locus of points ζ = const forms an equidistant surface, all points of which are equidistant from the reference. A body bound by two equidistant surfaces $\zeta = \pm\frac{h}{2}$ and by two cones $\theta = \theta_1, \theta = \theta_2$, is closed in the circumference $(0 \leqslant \varphi \leqslant 2\pi)$ of a shell of revolution of constant thickness h. We can also imagine a shell of variable thickness along the meridian; in this instance the surfaces bounding it will not be equidistant from the reference surface, and we have the relationships $\zeta = -\frac{1}{2}h(\theta)$, where h is a known function of θ. In both cases (h = const and $h = h(\theta)$) the reference surface goes in the middle between the bounding surfaces and is called the middle surface. Subsequently only thin-walled shells will be considered, i.e., those for which the ratio of wall depth h to a certain characteristic of shell dimension, for example h/R_1 or h/R_2, is smaller than one. A parameter characterizing the thin-wall aspect can also be the ratio of thickness to the total meridian arc length or to the radius of a parallel circle of the extreme section of the shell.

We designate through $\boldsymbol{R}$ the radius-vector of point N of the shell

$$\boldsymbol{R} = \boldsymbol{r} + \zeta\boldsymbol{n}, \tag{3.1}$$

where $\boldsymbol{r}$ is the radius-vector of the corresponding point M on the reference surface (M and N are on one normal).

Let us examine an infinitesimally displacement on equidistant surface ζ = const

$$d\boldsymbol{R} = d\boldsymbol{r} + \zeta d\boldsymbol{n}. \tag{3.2}$$

With the aid of formulas (1.6), (1.20) we calculate

$$dR = \frac{\partial R}{\partial_1 s} d_1 s + \frac{\partial R}{\partial_2 s} d_2 s = \tau_1 \left(1 + \frac{\zeta}{R_1}\right) d_1 s + \tau_2 \left(1 + \frac{\zeta}{R_2}\right) d_2 s. \tag{3.3}$$

Designating through $d_1\sigma$, $d_2\sigma$ elements of arcs of coordinate lines on an equidistant surface after comparison of (3.3) with (1.6), we find

$$\left.\begin{aligned} d_1\sigma &= \left(1 + \frac{\zeta}{R_1}\right) d_1 s \\ d_2\sigma &= \left(1 + \frac{\zeta}{R_2}\right) d_2 s. \end{aligned}\right\} \tag{3.4}$$

Let us note, furthermore, that unit vectors of the tangentials to meridians and parallels on equidistant surface T_1, T_2 and the vector of normal N are equal to

$$T_1 = \frac{\partial R}{\partial_1 \sigma} = \tau_1, \quad T_2 = \frac{\partial R}{\partial_2 \sigma} = \tau_2, \quad N = n. \tag{3.5}$$

Introduced system of curvilinear coordinates θ, ϕ, ζ is orthogonal, and that is why an area element of the equidistant surface and a volume element of the shell are defined easily as

$$d\Sigma = d_1\sigma\, d_2\sigma = \left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right) R_1 \nu\, d\theta\, d\varphi. \tag{3.6}$$

$$dU = d_1\sigma\, d_2\sigma\, d\zeta = \left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right) R_1 \nu\, d\theta\, d\varphi\, d\zeta. \tag{3.7}$$

Let us turn now to shell deformation. It is assumed that the shell is so thin-walled that during deformation: 1) all points which before deformation were on one normal to the middle surface will be on the normal to a deformed middle surface; 2) there is no extension or compression of the normals. These hypotheses are the basis of the theory of thin-walled plates and shells by Kirchhoff and Love. Here we give only the kinematic ocmponent of the Kirchhoff-Love hypothesis. Usually added is the static assumption about the smallness of normal stress σ_3 on the areas ζ = const. The last assumption means that during calculation of deformations e_1, e_2 in terms of values of stress of σ_1, σ_2, σ_3 the quantity σ_3 can be neglected (see formulas (5.7). The introduction of kinematic hypotheses allows describing the deformation of a three-dimensional continuum, such as a shell, with the aid of quantities characterizing the deformation of a middle surface, i.e., reducing a three-dimensional problem to a two-dimensional. Let us note that for the given kinematic picture of shell deformation all equations which are obtained in this section are accurate within the framework of the linear theory of small deformations. Therefore it is possible to speak of inaccuracies of the Kirchhoff-Love hypotheses only for the following reasons: 1) neglect of the quantity σ_3 in deriving elasticity relationships and 2) the distributed and boundary load on the shell can have such a character that the accepted picture of deformations is not satisfactory.

The amount of error in the Kirchhoff-Love hypotheses has been studied in [6], [16], [24], [1] and others.

With several stipulations it can be considered that in most cases acceptance of these hypotheses leads an error of the order of h/R in comparison with unity. In any case, this gives to us the right to make all practical calculations dropping terms of the order h/R in comparison with unity, inasmuch as the error of the basic hypotheses is not less, but sometimes can be even considerably greater [6].

Thus, according to the Kirchhoff-Love hypotheses the radius-vector of point N^*, which point N of the shell becomes during deformation, has the form

$$R^* = r^* + \zeta n^*. \tag{3.8}$$

where r^* - radius-vector of a point on the deformed middle surface, by (2.1) equal to $r + U$. Designating by $U^{(\zeta)}$ the displacement vector of point N of the shell at deformation, from (3.1) and (3.8) we have

$$U^{(\zeta)} = R^* - R = U + \zeta (n^* - n). \tag{3.9}$$

From (3.9) and (2.13) it follows that components of displacement vector $\boldsymbol{L}^{(\zeta)}=u^{(\zeta)}\boldsymbol{\tau}_1+v^{(\zeta)}\boldsymbol{\tau}_2+w^{(\zeta)}\boldsymbol{n}$ along axes $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{n}$ are

$$u^{(\zeta)}=u-\vartheta_1\zeta, \quad v^{(\zeta)}=v-\vartheta_2\zeta, \quad w^{(\zeta)}=w. \tag{3.10}$$

On the basis of (3.8), using equations (2.6) and (2.20), we compute

$$\left.\begin{aligned} d_1R^*&=\frac{\partial(r^*+\zeta n^*)}{\partial_1 s}d_1s=\Big[\tau_1\Big(1+\varepsilon_1+\frac{\zeta}{R_1}-\frac{\zeta}{R_1}\frac{\partial\vartheta_1}{\partial\theta}\Big)+\\ &\quad+\tau_2\Big(\gamma_1-\frac{\zeta}{R_1}\frac{\partial\vartheta_2}{\partial\theta}\Big)+n\Big(1+\frac{\zeta}{R_1}\Big)\vartheta_1\Big]d_1s,\\ d_2R^*&=\frac{\partial(r^*+\zeta n^*)}{\partial_2 s}d_2s=\Big[\tau_1\Big(\gamma_2-\frac{\zeta}{\nu}\frac{\partial\vartheta_1}{\partial\varphi}+\frac{\vartheta_2\zeta\cos\theta}{\nu}\Big)+\\ &+\tau_2\Big(1+\frac{\zeta}{R_2}+\varepsilon_2-\frac{\zeta\vartheta_1\cos\theta}{\nu}-\frac{\zeta}{\nu}\frac{\partial\vartheta_2}{\partial\varphi}\Big)+n\Big(1+\frac{\zeta}{R_2}\Big)\vartheta_2\Big]d_2s.\end{aligned}\right\} \tag{3.11}$$

Remembering the expressions for $\varkappa_1, \varkappa_2, \tau_1$ and τ_2 (formulas (2.18), (2.22)) and introducing instead of d_1s, d_2s elements of arcs $d_1\sigma, d_2\sigma$ using formulas (3.4), we will rewrite (3.11) again

$$\left.\begin{aligned} d_1R^*&=[\tau_1(1+e_1)+\tau_2\omega_1+n\vartheta_1]\,d_1\sigma,\\ d_2R^*&=[\tau_1\omega_2+\tau_2(1+e_2)+n\vartheta_2]\,d_2\sigma,\end{aligned}\right\} \tag{3.12}$$

where

$$\left.\begin{aligned} e_1&=\frac{\varepsilon_1+\zeta\varkappa_1}{1+\zeta/R_1}, \quad e_2=\frac{\varepsilon_2+\zeta\varkappa_2}{1+\zeta/R_2},\\ \omega_1&=\frac{\gamma_1+\zeta\tau_1}{1+\zeta/R_1}, \quad \omega_2=\frac{\gamma_2+\zeta\tau_2}{1+\zeta/R_2}.\end{aligned}\right\} \tag{3.13}$$

Elements of arcs $d_1\sigma^*, d_2\sigma^*$ on an equidistant surface after deformation are equal to

$$\left.\begin{aligned} d_1\sigma^*&=\sqrt{(1+e_1)^2+\omega_1^2+\vartheta_1^2}\,d_1\sigma\approx(1+e_1)\,d_1\sigma,\\ d_2\sigma^*&=\sqrt{\omega_2^2+(1+e_2)^2+\vartheta_2^2}\,d_2\sigma\approx(1+e_2)\,d_2\sigma.\end{aligned}\right\} \tag{3.14}$$

Expressions (3.14) indicate that quantities e_1, e_2 are the relative elongations of elements of arcs of meridians and parallels of the equidistant surface ζ = const at deformation. We form vectors of the tangentials to coordinate lines and normals to an equidistant surface after deformation

$$\left.\begin{aligned} T_1^* &= \frac{\partial R^*}{\partial_1 \sigma^*} = \tau_1 + \tau_2 \omega_1 + n \vartheta_1, \\ T_2^* &= \frac{\partial R^*}{\partial_2 \sigma^*} = \tau_1 \omega_2 + \tau_2 + n \vartheta_2, \\ N^* &= n^* = n - \vartheta_1 \tau_1 - \vartheta_2 \tau_2. \end{aligned}\right\} \qquad (3.15)$$

Using (3.15), it is simple to see that the changes in initially right angles between vectors T_1, T_2 and N at deformation (shears), are

$$\left.\begin{aligned} e_{12} &= (T_1^* \cdot T_2^*) = \omega_2 + \omega_1 = \omega, \\ e_{13} &= (T_1^* \cdot N^*) = 0, \\ e_{23} &= (T_2^* \cdot N^*) = 0, \end{aligned}\right\} \qquad (3.16)$$

the last two equalities of (3.16) are of course corollaries of the accepted kinematic hypotheses. Elongation of e_3 in the direction of the normal is also equal to zero because of the inextensibility of the normals. Let us show also that acceptance of the Kirchhoff-Love hypotheses leads to the equality

$$\frac{1}{2}\left(\operatorname{rot} U^{(\zeta)}\right)_{\zeta=0} = \Omega,$$

where $\operatorname{rot} U^{(\zeta)}$ designates doubled vector of rotation of an element of continuous medium at deformation, and Ω - vector of rotation, introduced in § 2.

Projections of the vector $\operatorname{rot} U^{(\zeta)}$ onto orthogonal directions T_1, T_2, N is computed using known equations of vectorial calculus [11], in which for a selected coordinate grid θ, ϕ, ζ it is necessary to set Lame' coefficients equal to $(R_1+\zeta)$, $(1+\zeta/R_2)v$, 1. Taking into account equation (3.10), we obtain

$$(\mathrm{rot}\, U^{(\zeta)})_1 = \frac{1}{\left(1+\frac{\zeta}{R_2}\right)\nu}\left\{\frac{\partial w}{\partial \varphi} - \frac{\partial}{\partial \zeta}\left[\nu\left(1+\frac{\zeta}{R_2}\right)(v-\vartheta_2\zeta)\right]\right\},$$

$$(\mathrm{rot}\, U^{(\zeta)})_2 = \frac{1}{R_1+\zeta}\left\{\frac{\partial}{\partial \zeta}[(R_1+\zeta)(u-\vartheta_1\zeta)] - \frac{\partial w}{\partial \theta}\right\},$$

$$(\mathrm{rot}\, U^{(\zeta)})_3 = \frac{1}{(R_1+\zeta)\left(1+\frac{\zeta}{R_2}\right)\nu}\left\{\frac{\partial}{\partial \theta}\left[\left(1+\frac{\zeta}{R_2}\right)\nu(v-\vartheta_2\zeta)\right] - \frac{\partial}{\partial \varphi}[(R_1+\zeta)(u-\vartheta_1\zeta)]\right\}.$$

Differentiating and remembering the designations of (2.5), we have

$$\frac{1}{2}(\mathrm{rot}\, U^{(\zeta)})_{1(\zeta=0)} = \vartheta_2, \quad \frac{1}{2}(\mathrm{rot}\, U^{(\zeta)})_{2(\zeta=0)} = -\vartheta_1,$$

$$\frac{1}{2}(\mathrm{rot}\, U^{(\zeta)})_{3(\zeta=0)} = \frac{1}{2}(\gamma_1-\gamma_2) = \delta.$$

In the right sides of these equalities are Ω projections of (2.14).

Thus, of the six components of deformation of a three-dimensional elastic medium $e_1, e_2, e_3, e_{12}, e_{13}, e_{23}$ in our case only three differ from zero: $e_1, e_2, e_{12}=\omega$. Using (3.16), (3.13) we write the expression for the shear

$$\omega = \frac{\gamma_1+\zeta\tau_1}{1+\zeta/R_1} + \frac{\gamma_2+\zeta\tau_2}{1+\zeta/R_2} \tag{3.17}$$

and transforming it with the aid of (2.11) and (2.26) to the form

$$\omega = \frac{\gamma\left(1-\frac{\zeta^2}{R_1R_2}\right) + \zeta 2\tau\left[1+\zeta\frac{(R_1+R_2)}{2R_1R_2}\right]}{(1+\zeta/R_1)(1+\zeta/R_2)}, \tag{3.18}$$

then elongations e_1, e_2 and shear ω prove to be expressed through six quantities $\varepsilon_1, \varepsilon_2, \gamma, \varkappa_1, \varkappa_2, \tau$, characterizing the deformed middle surface. Hence follows the conclusion that with the accepted hypotheses the deformation of a shell is determined through the deformation of the middle surface. The quantities $\varepsilon_1, \varepsilon_2, \gamma, \varkappa_1, \varkappa_2, \tau$ will be called the components of deformation of the middle surface.

The deformation components $\varepsilon_1, \varepsilon_2, \gamma, \varkappa_1, \varkappa_2, \tau$ cannot be arbitrarily assigned functions of coordinates θ, ϕ. In order that the deformed surface which corresponds to these functions can exist, they should satisfy certain differential relationships - so-called conditions of continuity of deformation - the necessity of which and the number become apparent if we remember expressions for components of deformation through displacements of the middle surface u, v, w (formulas (2.5), (2.18), (2.26)):

$$
\left.
\begin{aligned}
\varepsilon_1 &= \frac{1}{R_1}\frac{\partial u}{\partial \theta} + \frac{w}{R_1},\\
\varepsilon_2 &= \frac{1}{\nu}\frac{\partial v}{\partial \varphi} + \frac{u\cos\theta + w\sin\theta}{\nu},\\
\gamma &= \frac{1}{R_1}\frac{\partial v}{\partial \theta} + \frac{1}{\nu}\frac{\partial u}{\partial \varphi} - \frac{v\cos\theta}{\nu},\\
\varkappa_1 &= -\frac{1}{R_1}\frac{\partial}{\partial \theta}\left(\frac{1}{R_1}\frac{\partial w}{\partial \theta} - \frac{u}{R_1}\right),\\
\varkappa_2 &= -\frac{\cos\theta}{\nu}\left(\frac{1}{R_1}\frac{\partial w}{\partial \theta} - \frac{u}{R_1}\right) - \frac{1}{\nu}\frac{\partial}{\partial \varphi}\left(\frac{1}{\nu}\frac{\partial w}{\partial \varphi} - \frac{v\sin\theta}{\nu}\right),\\
2\tau &= -\frac{1}{R_1}\frac{\partial}{\partial \theta}\left(\frac{1}{\nu}\frac{\partial w}{\partial \varphi} - \frac{v\sin\theta}{\nu}\right) + \frac{1}{R_1}\left(\frac{1}{\nu}\frac{\partial u}{\partial \varphi} - \frac{v\cos\theta}{\nu}\right) -\\
&\quad - \frac{1}{\nu}\frac{\partial}{\partial \varphi}\left(\frac{1}{R_1}\frac{\partial w}{\partial \theta} - \frac{u}{R_1}\right) +\\
&\quad + \frac{\cos\theta}{\nu}\left(\frac{1}{\nu}\frac{\partial w}{\partial \varphi} - \frac{v\sin\theta}{\nu}\right) + \frac{1}{R_1 R_2}\frac{\partial v}{\partial \theta}.
\end{aligned}
\right\} \qquad (3.19)
$$

Really, the six quantities $\varepsilon_1, \varepsilon_2, \gamma, \varkappa_1, \varkappa_2, \tau$ are expressed through three functions u, v, w and, consequently, they themselves should be connected by three relationships, which are conditions of the integrability of system (3.19) relative to displacements. In this way the search for conditions of the continuity of deformations proves to be closely connected to the problem of determining components of the vector of displacement $U(u, v, w)$ in terms of assigned components of deformation. This problem was examined generally by A. I. Lure [14]. The conditions of continuity themselves for shells of arbitrary shape were obtained by A. L. Gol'denveyzer from purely geometric considerations [8].

The problem of determining vector of displacements $\boldsymbol{U}$ in terms of assigned deformation components can be considered solved if it is possible to express derivatives $\frac{\partial U}{\partial_1 s}$, $\frac{\partial U}{\partial_2 s}$ through assigned functions $\varepsilon_1, \varepsilon_2, \gamma, \varkappa_1, \varkappa_2, \tau$. Then

$$U=\int\left(\frac{\partial U}{\partial_1 s}d_1 s+\frac{\partial U}{\partial_2 s}d_2 s\right)+U_0, \tag{3.20}$$

where U_0 is a vectorial constant of integration. From formulas (2.4) it follows that for this it is necessary to find expressions for $\vartheta_1, \vartheta_2, \gamma_1, \gamma_2$ through assigned functions or, which is the same, to find components of the vector of rotation $\Omega\left[\vartheta_2, \cdot \vartheta_1, \delta=\frac{1}{2}(\gamma_1-\gamma_2)\right]$ through $\varepsilon_1, \varepsilon_2, \gamma, \varkappa_1, \varkappa_2, \tau$. The latter is done easily since in accordance with (2.29) derivatives $\frac{\partial \Omega}{\partial_1 s}$, $\frac{\partial \Omega}{\partial_2 s}$ have been already expressed through assigned deformation components. Thus,

$$\Omega=\int\left(\frac{\partial \Omega}{\partial_1 s}d_1 s+\frac{\partial \Omega}{\partial_2 s}d_2 s\right)+\Omega_0. \tag{3.21}$$

This reasoning shows that displacements of a shell in terms of assigned components of deformation are determined accurate to within the displacements of a solid: U_0, Ω_0 - vectors of the displacement and rotation of a solid.

The condition of independence of line integral $\int\left(\frac{\partial \Omega}{\partial_1 s}d_1 s+\frac{\partial \Omega}{\partial_2 s}d_2 s\right)$ from the course of integration has the form

$$\frac{\partial^2 \Omega}{\partial \vartheta\, \partial \varphi}=\frac{\partial^2 \Omega}{\partial \varphi\, \partial \vartheta}. \tag{3.22}$$

Taking into account that according to (2.29) and (1.9), (1.14),(1.24),

$$\left.\begin{aligned}\frac{\partial \Omega}{\partial \vartheta}&=\left(-\tau R_1+\frac{\gamma}{2}\right)\tau_1+\varkappa_1 R_2\tau_2+\zeta_1 R_1 n,\\ \frac{\partial \Omega}{\partial \varphi}&=-\varkappa_2\nu\tau_1+\left(\tau\nu-\frac{\gamma}{2}\sin\vartheta\right)\tau_2+\zeta_2\nu n,\end{aligned}\right\} \tag{3.23}$$

and using derivation formulas (1.20), we find that to vector condition (3.22) are equivalent the following three equations:

$$
\left.\begin{aligned}
\frac{\partial}{\partial\theta}(\nu\varkappa_2) - \varkappa_1 R_1 \cos\theta - R_1\frac{\partial\tau}{\partial\varphi} + \frac{1}{2}\frac{\partial\gamma}{\partial\varphi} - \zeta_2\nu = 0,\\
R_1\frac{\partial\varkappa_1}{\partial\varphi} - \frac{\partial}{\partial\theta}(\tau\nu) - R_1\tau\cos\theta + \gamma\cos\theta + \frac{1}{2}\sin\theta\frac{\partial\gamma}{\partial\theta} + \zeta_1 R_1\sin\theta = 0,\\
\varkappa_2\nu + \varkappa_1 R_1\sin\theta + \frac{\partial}{\partial\theta}(\nu\zeta_2) - \frac{\partial}{\partial\varphi}(R_1\zeta_1) = 0.
\end{aligned}\right\} \tag{3.24}
$$

The condition of independence from the course of integration integral (3.20)

$$
\frac{\partial^2 U}{\partial\theta\,\partial\varphi} = \frac{\partial^2 U}{\partial\varphi\,\partial\theta} \tag{3.25}
$$

We rewrite in another form. Introducing auxiliary vectors

$$
\left.\begin{aligned}
V_{(1)} &= \frac{\partial U}{\partial\theta} - \Omega\times\tau_1 R_1,\\
V_{(2)} &= \frac{\partial U}{\partial\varphi} - \Omega\times\tau_2\nu,
\end{aligned}\right\} \tag{3.26}
$$

instead of (3.25) we will have

$$
\frac{\partial V_{(1)}}{\partial\varphi} + \frac{\partial\Omega}{\partial\varphi}\times\tau_1 R_1 = \frac{\partial V_{(2)}}{\partial\theta} + \frac{\partial\Omega}{\partial\theta}\times\tau_2\nu, \tag{3.27}
$$

since on the basis of (1.20)

$$
\frac{\partial}{\partial\varphi}(\tau_1 R_1) = \frac{\partial}{\partial\theta}(\tau_2\nu).
$$

Turning to equations (1.9), (1.14), (1.20), (2.4), (2.13), (2.15), it is simple to explain that vectors $V_{(1)}$, $V_{(2)}$ are located in a tangential plane and contain only deformation components ε_1, ε_2, γ:

$$
\left.\begin{aligned}
V_{(1)} &= R_1\varepsilon_1\tau_1 + \frac{1}{2}R_1\gamma\tau_2,\\
V_{(2)} &= \frac{1}{2}\nu\gamma\tau_1 + \nu\varepsilon_2\tau_2.
\end{aligned}\right\} \tag{3.28}
$$

Taking into account (3.23) and (3.28), instead of (3.27) we obtain three conditions:

$$\left.\begin{aligned} R_1 \frac{\partial \varepsilon_1}{\partial \varphi} - R_1 \gamma \cos\theta - \frac{1}{2} \nu \frac{\partial \gamma}{\partial \theta} + R_1 \nu \zeta_1 = 0, \\ R_1 \varepsilon_1 \cos\theta + \frac{1}{2} R_1 \frac{\partial \gamma}{\partial \varphi} - \frac{\partial}{\partial \theta}(\nu \varepsilon_2) + R_1 \nu \zeta_2 = 0, \\ -R_1 \nu \tau + \frac{1}{2} \gamma \left(-R_1 \sin\theta + \frac{\nu R_1}{R_2}\right) + R_1 \nu \tau = 0. \end{aligned}\right\} \qquad (3.29)$$

The last condition (3.29) is an identity. Excluding from (3.24) and (3.29) the quantities ζ_1, ζ_2, we arrive at three differential relationships relative to the quantities $\varepsilon_1, \varepsilon_2, \gamma, \varkappa_1, \varkappa_2, \tau$:

$$\left.\begin{aligned} \frac{\partial}{\partial \theta}(\nu \varkappa_2) - R_1 \varkappa_1 \cos\theta - R_1 \frac{\partial \tau}{\partial \varphi} + \frac{\partial \gamma}{\partial \varphi} - \frac{1}{R_1} \frac{\partial (\nu \varepsilon_2)}{\partial \theta} + \varepsilon_1 \cos\theta = 0, \\ R_1 \frac{\partial \varkappa_1}{\partial \varphi} - \frac{\partial (\nu \tau)}{\partial \theta} - R_1 \cos\theta\, \tau + \gamma \cos\theta + \frac{\partial \gamma}{\partial \theta} \sin\theta + \\ + \frac{\gamma R_1 \cos\theta \sin\theta}{\nu} - \frac{R_1 \sin\theta}{\nu} \frac{\partial \varepsilon_1}{\partial \varphi} = 0, \\ \nu \varkappa_2 + \varkappa_1 R_1 \sin\theta + \frac{\partial}{\partial \theta}\left[-\frac{1}{2} \frac{\partial \gamma}{\partial \varphi} - \varepsilon_1 \cos\theta + \frac{1}{R_1} \frac{\partial (\nu \varepsilon_2)}{\partial \theta}\right] - \\ - \frac{\partial}{\partial \varphi}\left[\frac{1}{2} \frac{\partial \gamma}{\partial \theta} + \frac{\gamma R_1 \cos\theta}{\nu} - \frac{R_1}{\nu} \frac{\partial \varepsilon_1}{\partial \varphi}\right] = 0. \end{aligned}\right\} \qquad (3.30)$$

These equations are the desired conditions of deformation continuity.

§ 4. The Stressed State in a Shell

Let us examine the stressed state of an element of a shell bounded by sections $\theta, \theta + d\theta$, by planes $\varphi, \varphi + d\varphi$ and by surfaces $\zeta = \pm h/2$. From the side of the rejected part of a shell on a selected element act forces which reduce to the following system of stresses. On area element $d_2\sigma\, d\zeta$ of a normal section of a shell perpendicular to τ_1, act normal stresses σ_1 and tangential stresses τ_{12} and τ_{13}, on area element $d_1\sigma\, d\zeta$ of a meridian section of the shell act normal stresses σ_2 and tangential stresses τ_{21} and τ_{23}. In this way the vector of forces on area $d_2\sigma\, d\zeta$ is

$$k_1 \left(1 + \frac{\zeta}{R_2}\right) \nu\, d\varphi\, d\zeta = (\sigma_1 \tau_1 + \tau_{12} \tau_2 + \tau_{13} n) \left(1 + \frac{\zeta}{R_2}\right) \nu\, d\varphi\, d\zeta. \qquad (4.1)$$

and on area $d_1\sigma\, d\zeta$ acts vector of forces

$$k_2 \left(1 + \frac{\zeta}{R_1}\right) R_1\, d\theta\, d\zeta = (\sigma_2 \tau_2 + \tau_{21} \tau_1 + \tau_{23} n) \left(1 + \frac{\zeta}{R_1}\right) R_1\, d\theta\, d\zeta. \qquad (4.2)$$

Total forces and moments on the boundaries of the involved element of the shell are respectively

$$\left.\begin{aligned} K_1\nu\,d\varphi &= \int_{-h/2}^{+h/2} k_1\left(1+\frac{\zeta}{R_2}\right)\nu\,d\varphi\,d\zeta = (T_1\tau_1 + S_{12}\tau_2 + Q_1 n)\,\nu\,d\varphi, \\ K_2 R_1\,d\theta &= \int_{-h/2}^{+h/2} k_2\left(1+\frac{\zeta}{R_1}\right) R_1\,d\theta\,d\zeta = (T_2\tau_2 + S_{21}\tau_1 + Q_2 n) R_1\,d\theta, \end{aligned}\right\} \tag{4.3}$$

$$\left.\begin{aligned} M_1\nu\,d\varphi &= \int_{-h/2}^{+h/2} [\zeta n \times k_1]\left(1+\frac{\zeta}{R_2}\right)\nu\,d\varphi\,d\zeta = (M_1\tau_2 - H_{12}\tau_1)\,\nu\,d\varphi, \\ M_2 R_1\,d\theta &= \int_{-h/2}^{+h/2} [\zeta n \times k_2]\left(1+\frac{\zeta}{R_1}\right) R_1\,d\theta\,d\zeta = \\ &= (-M_2\tau_1 + H_{21}\tau_2) R_1\,d\theta, \end{aligned}\right\} \tag{4.4}$$

where the following designations have been introduced:

$$\left.\begin{aligned} &\int_{-h/2}^{+h/2} \sigma_1\left(1+\frac{\zeta}{R_2}\right) d\zeta = T_1, && \int_{-h/2}^{+h/2} \tau_{12}\left(1+\frac{\zeta}{R_2}\right) d\zeta = S_{12}, \\ &\int_{-h/2}^{+h/2} \sigma_2\left(1+\frac{\zeta}{R_1}\right) d\zeta = T_2, && \int_{-h/2}^{+h/2} \tau_{21}\left(1+\frac{\zeta}{R_1}\right) d\zeta = S_{21}, \\ &\int_{-h/2}^{+h/2} \tau_{13}\left(1+\frac{\zeta}{R_2}\right) d\zeta = Q_1, && \int_{-h/2}^{+h/2} \tau_{23}\left(1+\frac{\zeta}{R_1}\right) d\zeta = Q_2, \\ &\int_{-h/2}^{+h/2} \sigma_1\left(1+\frac{\zeta}{R_2}\right)\zeta\,d\zeta = M_1, && \int_{-h/2}^{+h/2} \sigma_2\left(1+\frac{\zeta}{R_1}\right)\zeta\,d\zeta = M_2, \\ &\int_{-h/2}^{+h/2} \tau_{12}\left(1+\frac{\zeta}{R_2}\right)\zeta\,d\zeta = H_{12}, && \int_{-h/2}^{+h/2} \tau_{21}\left(1+\frac{\zeta}{R_1}\right)\zeta\,d\zeta = H_{21}. \end{aligned}\right\} \tag{4.5}$$

By definition introduced quantities $T_1, S_{12}, Q_1, M_1, H_{12}$ are referred to a unit length of an arc of the parallel circle of the middle surface of the force and moment statically equivalent to the stresses in the normal section of the shell perpendicular to the direction of the meridian. The meaning of quantities $T_2, S_{21}, Q_2, M_2, H_{21}$ is the same - referred to a unit length of an arc of the meridian of the middle surface of the force and moment statically equivalent to stress in the normal section perpendicular to vector τ_2. Correspondingly quantities T_1, M_1 and T_2, M_2 are called the meridian and neighboring tensions and the bending moments. The quantities Q_1 and Q_2 are the shearing forces; $S_{12}, S_{21}, H_{12}, H_{21}$ - tangential forces and torsional moments. Their directions are shown in Fig. 7.

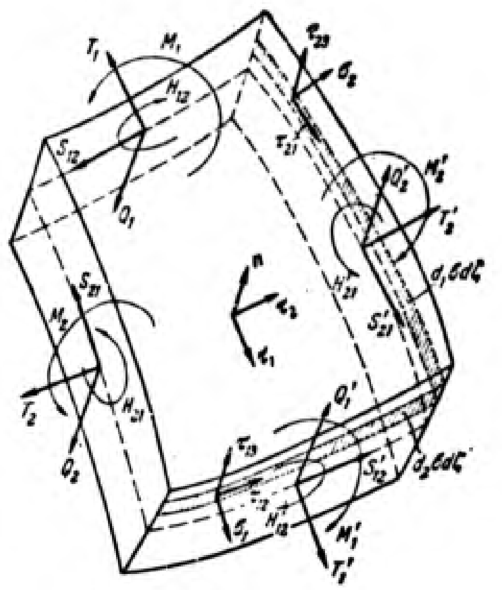


Fig. 7. Positive directions of forces and moments acting on an element from the rejected part of the shell.

Replacing the system of stresses acting on the shell element, by a statically equivalent system of forces and moments actually allows reducing the three-dimensional problem about the equilibrium of a volume element of the shell to the two-dimensional problem about the equilibrium of an element of the middle surface. Such an approach to the study of the stressed state of a shell completely agrees with the earlier assumptions about the nature of deformation, as a result of which it turns out that the deformation of a shell is described by six components of deformations of the middle surface. However, in the given case instead of six quantities there are ten static characteristics (4.5). First of all, one ought to note that although the study of shell deformation assumed that shifts e_{13}, e_{23}

are equal to zero, the corresponding stresses τ_{13}, τ_{23} differ from zero and should be taken into account in making up conditions of equilibrium, hence the appearance of the two characteristics Q_1 and Q_2. This contradition is because of the accepted method of developing a theory of shells, which in this point is completely analogous to the theory of beams and the theory of plates: The study of deformation uses the hypothesis of the nondeformable normals, i.e., it is held that shifts e_{13}, e_{23} are negligibly small in comparison with other components of deformation, but the shearing forces are determined after solving the problem from conditions of equilibrium and, generally speaking, do not prove to be small. Besides the quantities Q_1, Q_2 there are eight more characteristics (4.5), which should correspond to six deformation components. However, from (4.5) it is easy to see that four quantities S_{12}, S_{21}, H_{12}, H_{21} are not independent, since

$$S_{12}+\frac{H_{12}}{R_1}=S_{21}+\frac{H_{21}}{R_2}. \qquad (4.6)$$

A. I. Lur'ye [15] noted that holding four integral characteristics S_{12}, S_{21}, H_{12}, H_{21} instead of one tangential stress $\tau_{12} = \tau_{21}$ is not completely necessary, inasmuch as in the creation of subsequent relationships of the theory only combinations of these quantities appear, namely:

$$S=S_{12}-\frac{H_{21}}{R_2}=S_{21}-\frac{H_{12}}{R_1}=\int_{-h/2}^{+h/2}\tau_{12}\left(1-\frac{\zeta^2}{R_1R_2}\right)d\zeta. \qquad (4.7)$$

$$H=\frac{H_{12}+H_{21}}{2}=\int_{-h/2}^{+h/2}\tau_{12}\left[1+\frac{\zeta}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right]\zeta\,d\zeta. \qquad (4.8)$$

Thus, to six components of deformation ε_1, ε_2, γ, $\varkappa_1$, $\varkappa_2$, τ correspond six static quantities T_1, T_2, S, M_1, M_2, H. This fact considerably facilitates formulation of elasticity relationships, which determine the connection between static and geometric quantities.

Before passing to the composition of equilibrium conditions of the chosen shell element, let us refer the external forces to the middle surface, i.e., replace the external load acting on the shell by statically equivalent load on the middle surface. Let us designate through $\boldsymbol{F}$ the vector of volume forces referred to a volume unit, and through $\boldsymbol{p}+$ and $\boldsymbol{p}-$ the vectors of surface forces referred to a unit of area and acting on the limiting surfaces $\zeta = \pm h/2$.

$$\left.\begin{aligned} \boldsymbol{F} &= F_1\boldsymbol{\tau}_1 + F_2\boldsymbol{\tau}_2 + F_n\boldsymbol{n}, \\ \boldsymbol{p}^+ &= p_1^+\boldsymbol{\tau}_1 + p_2^+\boldsymbol{\tau}_2 + p_n^+\boldsymbol{n}, \\ \boldsymbol{p}^- &= p_1^-\boldsymbol{\tau}_1 + p_2^-\boldsymbol{\tau}_2 + p_n^-\boldsymbol{n}. \end{aligned}\right\} \tag{4.9}$$

Then the main vector of all external forces applied to the considered element and the moment of their relative center on the middle surface of the element are equal to

$$\left.\begin{aligned} \boldsymbol{E}\nu R_1\, d\theta\, d\varphi &= \boldsymbol{p}^+\left(1+\frac{h}{2R_1}\right)\left(1+\frac{h}{2R_2}\right)\nu R_1\, d\theta\, d\varphi + \\ &+ \boldsymbol{p}^-\left(1-\frac{h}{2R_1}\right)\left(1-\frac{h}{2R_2}\right)\nu R_1\, d\theta\, d\varphi + \\ &+ \int_{-h/2}^{+h/2} \boldsymbol{F}\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right) R_1\nu\, d\theta\, d\varphi\, d\zeta, \\ \boldsymbol{L}\nu R_1\, d\theta\, d\varphi &= \frac{h}{2}[\boldsymbol{n}\times\boldsymbol{p}^+]\left(1+\frac{h}{2R_1}\right)\left(1+\frac{h}{2R_2}\right)\nu R_1\, d\theta\, d\varphi - \\ &- \frac{h}{2}[\boldsymbol{n}\times\boldsymbol{p}^-]\left(1-\frac{h}{2R_1}\right)\left(1-\frac{h}{2R_2}\right)\nu R_1\, d\theta\, d\varphi + \\ &+ \int_{-h/2}^{+h/2} [\zeta\boldsymbol{n}\times\boldsymbol{F}]\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right) R_1\nu\, d\theta\, d\varphi\, d\zeta, \end{aligned}\right\} \tag{4.10}$$

where $\boldsymbol{E}$, $\boldsymbol{L}$ - corrected vectors of forces and moments of the external load, referred to a unit of area of the middle surface. After canceling by the common factor and vectorial multiplication in (4.10), we have

$$\left.\begin{aligned} \boldsymbol{E} &= E_1\boldsymbol{\tau}_1 + E_2\boldsymbol{\tau}_2 + E_n\boldsymbol{n}, \\ \boldsymbol{L} &= L_1\boldsymbol{\tau}_1 + L_2\boldsymbol{\tau}_2 + L_n\boldsymbol{n}, \end{aligned}\right\} \tag{4.11}$$

$$
\left.
\begin{aligned}
E_1 &= p_1^+\left(1+\frac{h}{2R_2}\right)\left(1+\frac{h}{2R_1}\right)+p_1^-\left(1-\frac{h}{2R_2}\right)\left(1-\frac{h}{2R_1}\right)+\\
&\quad+\int_{-h/2}^{+h/2}F_1\left(1+\frac{\zeta}{R_2}\right)\left(1+\frac{\zeta}{R_1}\right)d\zeta,\\
E_2 &= p_2^+\left(1+\frac{h}{2R_2}\right)\left(1+\frac{h}{2R_1}\right)+p_2^-\left(1-\frac{h}{2R_2}\right)\left(1-\frac{h}{2R_1}\right)+\\
&\quad+\int_{-h/2}^{+h/2}F_2\left(1+\frac{\zeta}{R_2}\right)\left(1+\frac{\zeta}{R_1}\right)d\zeta,\\
E_n &= p_n^+\left(1+\frac{h}{2R_2}\right)\left(1+\frac{h}{2R_1}\right)+p_n^-\left(1-\frac{h}{2R_2}\right)\left(1-\frac{h}{2R_1}\right)+\\
&\quad+\int_{-h/2}^{+h/2}F_n\left(1+\frac{\zeta}{R_2}\right)\left(1+\frac{\zeta}{R_1}\right)d\zeta,\\
L_1 &= \left[-p_2^+\left(1+\frac{h}{2R_2}\right)\left(1+\frac{h}{2R_1}\right)+p_2^-\left(1-\frac{h}{2R_2}\right)\left(1-\frac{h}{2R_1}\right)\right]\frac{h}{2}-\\
&\quad-\int_{-h/2}^{+h/2}F_2\zeta\left(1+\frac{\zeta}{R_2}\right)\left(1+\frac{\zeta}{R_1}\right)d\zeta,\\
L_2 &= \left[p_1^+\left(1+\frac{h}{2R_2}\right)\left(1+\frac{h}{2R_1}\right)-p_1^-\left(1-\frac{h}{2R_2}\right)\left(1-\frac{h}{2R_1}\right)\right]\frac{h}{2}-\\
&\quad-\int_{-h/2}^{+h/2}F_1\zeta\left(1+\frac{\zeta}{R_2}\right)\left(1+\frac{\zeta}{R_1}\right)d\zeta,\\
L_n &= 0.
\end{aligned}
\right\}
\qquad (4.12)
$$

Let us compose vector conditions of equilibrium of an element of the middle surface under the action of corrected external forces (4.11) and systems of internal forces and moments (4.3), (4.4) (see Fig. 7)

$$
\begin{aligned}
&-K_1\nu\,d\varphi+K_1\nu\,d\varphi+\frac{\partial}{\partial\theta}(K_1\nu)\,d\theta\,d\varphi-K_2R_1\,d\theta+\\
&\quad+K_2R_1\,d\theta+\frac{\partial}{\partial\varphi}(K_2)\,R_1\,d\varphi\,d\theta+ER_1\nu\,d\theta\,d\varphi=0.
\end{aligned}
\qquad (4.13)
$$

$$
\begin{aligned}
&-M_1\nu\,d\varphi+M_1\nu\,d\varphi+\frac{\partial}{\partial\theta}(M_1\nu)\,d\theta\,d\varphi-M_2R_1\,d\theta+\\
&\quad+M_2R_1\,d\theta+\frac{\partial}{\partial\varphi}(M_2)\,R_1\,d\varphi\,d\theta-[R_1\,d\theta\,\tau_1\times(-K_1\nu\,d\varphi)]+\\
&\quad+[r_1\times K_2R_1\,d\theta]+[r_2\times(-K_2R_1\,d\theta)]+LR_1\nu\,d\varphi\,d\theta=0
\end{aligned}
\qquad (4.14)
$$

Here $\mathbf{r}_1$ and $\mathbf{r}_2$ are the radii vectors connecting point O, relative to the equation of moments (4.14) was composed, with centers of application of forces K_2 on the sides of the element ϕ = const, $\phi + d\phi$ = const, while

$$
r_1-r_2=\nu\,d\varphi\,\tau_2. \qquad (4.15)
$$

Taking into account (4.15) and of drive similar terms, we reduce (4.13), (4.14) to the form

$$\frac{\partial}{\partial\theta}(K_1\nu)+R_1\frac{\partial}{\partial\varphi}(K_2)+E\nu R_1=0. \qquad (4.16)$$

$$\frac{\partial}{\partial\theta}(M_1\nu)+R_1\frac{\partial}{\partial\varphi}(M_2)+(\tau_1\times K_1)\nu R_1+(\tau_2\times K_2)\nu R_1+L\nu R_1=0. \qquad (4.17)$$

Vectorial equations (4.16), (4.17) after substituting into them expressions (4.3), (4.4) and differentiating with formulas (1.20) lead to six conditions of equilibrium of the element:

$$\left.\begin{aligned}
\frac{\partial}{\partial\theta}(T_1\nu)+Q_1\nu-T_2R_1\cos\theta+R_1\frac{\partial S_{21}}{\partial\varphi}+E_1\nu R_1=0.\\
\frac{\partial}{\partial\theta}(S_{12}\nu)+R_1\frac{\partial T_2}{\partial\varphi}+S_{21}R_1\cos\theta+Q_2R_1\sin\theta+E_2\nu R_1=0.\\
-T_1\nu+\frac{\partial}{\partial\theta}(Q_1\nu)+R_1\frac{\partial Q_2}{\partial\varphi}-T_2R_1\sin\theta+E_n\nu R_1=0.\\
-\frac{\partial}{\partial\theta}(H_{12}\nu)-R_1\frac{\partial M_2}{\partial\varphi}-H_{21}R_1\cos\theta+Q_2\nu R_1+L_1\nu R_1=0.\\
\frac{\partial}{\partial\theta}(M_1\nu)-M_2R_1\cos\theta+R_1\frac{\partial H_{21}}{\partial\varphi}-Q_1\nu R_1+L_2\nu R_1=0.\\
\frac{H_{12}}{R_1}-\frac{H_{21}}{R_2}+S_{12}-S_{21}=0.
\end{aligned}\right| \qquad (4.18)$$

The sixth equation of (4.18) because of (4.6) is an identity. It is easy to show that equilibrium equations (4.18) contain forces and moments S_{12}, S_{21}, H_{12}, H_{21} only in combinations (4.7). Let us take as an example the first equation and transform it in the following manner:

$$\frac{\partial}{\partial\theta}(T_1\nu)-T_2R_1\cos\theta+R_1\frac{\partial}{\partial\varphi}\left(S+\frac{H}{R_1}\right)+\nu N_1+E_1\nu R_1=0. \qquad (4.19)$$

where we designate

$$\nu N_1=\nu Q_1-\frac{1}{2}\frac{\partial}{\partial\varphi}(H_{21}-H_{12}); \qquad (4.20)$$

however, because of the fourth equation of (4.18) and (4.7), (4.8) νN_1 is expressed also only through S and H

$$R_1\nu N_1=\frac{\partial}{\partial\theta}(\nu M_1)-M_2R_1\cos\theta+R_1\frac{\partial H}{\partial\varphi}+L_2\nu R_1. \qquad (4.21)$$

Transforming similarly the remaining equations of system (4.18), we arrive at the following three equilibrium equations relative to the six quantities T_1, T_2, S, M_1, M_2, H:

$$
\left.
\begin{aligned}
&\frac{\partial}{\partial\theta}(\nu T_1) - T_2 R_1 \cos\theta + R_1 \frac{\partial}{\partial\varphi}\left(S + \frac{H}{R_1}\right) + \nu N_1 + E_1 \nu R_1 = 0,\\
&\frac{\partial}{\partial\theta}\left[\nu\left(S + \frac{H}{R_2}\right)\right] + R_1 \frac{\partial T_2}{\partial\varphi} + \left(S + \frac{H}{R_1}\right) R_1 \cos\theta +\\
&\qquad\qquad + N_2 R_1 \sin\theta + E_2 \nu R_1 = 0,\\
&\frac{\partial}{\partial\theta}(\nu N_1) + R_1 \frac{\partial N_2}{\partial\varphi} - T_1 \nu - T_2 R_1 \sin\theta + E_n \nu R_1 = 0.
\end{aligned}
\right\}
\qquad (4.22)
$$

where

$$
\left.
\begin{aligned}
&N_1 = \frac{1}{\nu R_1}\left[\frac{\partial}{\partial\theta}(\nu M_1) - M_2 R_1 \cos\theta + R_1 \frac{\partial H}{\partial\varphi}\right] + L_2,\\
&N_2 = Q_2 + \frac{1}{R_1}\frac{\partial}{\partial\theta}(H_{21} - H_{12}) =\\
&\qquad = \frac{1}{\nu R_1}\left[\nu \frac{\partial H}{\partial\theta} + 2HR_1 \cos\theta + R_1 \frac{\partial M_2}{\partial\varphi}\right] - L_1.
\end{aligned}
\right\}
\qquad (4.23)
$$

Forces N_1 and N_2 in system (4.22) are analogous to shearing forces Q_1, Q_2 in system (4.18).

The description of the stressed state with the aid of integral (4.5) is not contradicted by the following assumption about the distribution of stresses in terms of coordinate ζ:

$$
\left.
\begin{aligned}
&\left(1 + \frac{\zeta}{R_2}\right)\sigma_1 = \frac{T_1}{h} + \frac{6M_1}{h^2}\,\frac{\zeta}{h/2},\\
&\left(1 + \frac{\zeta}{R_1}\right)\sigma_2 = \frac{T_2}{h} + \frac{6M_2}{h^2}\,\frac{\zeta}{h/2}.
\end{aligned}
\right\}
\qquad (4.24)
$$

$$
\left.
\begin{aligned}
&\left(1 + \frac{\zeta}{R_2}\right)\tau_{12} = \frac{S_{12}}{h} + \frac{6H_{12}}{h^2}\,\frac{\zeta}{h/2},\\
&\left(1 + \frac{\zeta}{R_1}\right)\tau_{21} = \frac{S_{21}}{h} + \frac{6H_{21}}{h^2}\,\frac{\zeta}{h/2}.
\end{aligned}
\right\}
\qquad (4.25)
$$

Really, stresses, which will be presented in the form of (4.24), (4.25), identically satisfy (4.5). if we reject in the formulas quantities of the order of h/R in comparison with unity then for the stresses we will have simpler expressions, for example:

$$\tau_{12} = \tau_{21} = \frac{S}{h} + \frac{6H}{h^2}\,\frac{\zeta}{h/2},$$

while with the accepted accuracy $S_{12} \approx S_{21} \approx S$, $H_{12} \approx H_{21} \approx H$. However, subsequently we will require in equilibrium equations the quantities,

S, H, introduced using equations (4.7), (4.8), since this gives greater order to the basic equations.

The nature of stress distribution τ_{13}, τ_{23} through the thickness of the shell can be explained when one considers the equilibrium of a shell element bound by sections $\theta = \text{const}$, $\theta + d\theta = \text{const}$, by planes $\varphi = \text{const}$, $\varphi + d\varphi = \text{const}$ and by surface $\zeta = \text{const}$, $\zeta + d\zeta = \text{const}$, i.e., elements of a layer of thickness $d\zeta$. To this element are applied external force $F\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\nu R_1\, d\theta\, d\varphi\, d\zeta$ and internal forces

$$\left.\begin{aligned} &-k_1\left(1 + \frac{\zeta}{R_2}\right)\nu\, d\varphi\, d\zeta, \\ &+k_1\left(1 + \frac{\zeta}{R_2}\right)\nu\, d\varphi\, d\zeta + \frac{\partial}{\partial\theta}\left[k_1\left(1 + \frac{\zeta}{R_2}\right)\nu\right] d\varphi\, d\zeta\, d\theta, \end{aligned}\right\} \tag{4.26}$$

$$\left.\begin{aligned} &-k_2\left(1 + \frac{\zeta}{R_1}\right)R_1\, d\theta\, d\zeta, \\ &+k_2\left(1 + \frac{\zeta}{R_1}\right)R_1\, d\theta\, d\zeta + \frac{\partial}{\partial\varphi}\left[k_2\left(1 + \frac{\zeta}{R_1}\right)R_1\right] d\theta\, d\zeta\, d\varphi, \end{aligned}\right\} \tag{4.27}$$

$$\left.\begin{aligned} &-k_3\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\nu R_1\, d\theta\, d\varphi, \\ &k_3\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\nu R_1\, d\theta\, d\varphi + \\ &\quad + \frac{\partial}{\partial\zeta}\left[k_3\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\nu R_1\right] d\theta\, d\varphi\, d\zeta, \end{aligned}\right\} \tag{4.28}$$

where

$$k_3 = \sigma_3 n + \tau_{13}\tau_1 + \tau_{23}\tau_2. \tag{4.29}$$

σ_3 - normal stress acting on area $d\Sigma = d_1\sigma d_2\sigma$ of an element of surface ζ = const.

The vectorial condition of equilibrium of an element of a layer of the shell has the form

$$\begin{aligned} &\frac{\partial}{\partial\theta}\left[k_1\left(1 + \frac{\zeta}{R_2}\right)\nu\right] + \frac{\partial}{\partial\varphi}\left[k_2\left(1 + \frac{\zeta}{R_1}\right)R_1\right] + \\ &+ \frac{\partial}{\partial\zeta}\left[k_3\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\nu R_1\right] + F\nu R_1\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right) = 0. \end{aligned} \tag{4.30}$$

Substituting into (4.30) expressions for k_1, k_2, k_3 according to (4.1), (4.2), (4.29) and differentiating with formulas (1.20), we obtain three equilibrium equations of the medium making up the shell in projections onto directions τ_1, τ_2, n:

$$\left.\begin{aligned}
&\frac{\partial}{\partial\theta}\left[\sigma_1\left(1+\frac{\zeta}{R_2}\right)\nu\right]+\frac{\partial}{\partial\varphi}\left[\tau_{12}\left(1+\frac{\zeta}{R_1}\right)R_1\right]+\\
&\quad+\frac{\partial}{\partial\zeta}\left[\tau_{13}\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)\nu R_1\right]+\\
&\quad+\tau_{13}\left(1+\frac{\zeta}{R_2}\right)\nu-\sigma_2\left(1+\frac{\zeta}{R_1}\right)R_1\cos\theta+\\
&\qquad+F_1\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)\nu R_1=0.\\
&\frac{\partial}{\partial\theta}\left[\tau_{12}\left(1+\frac{\zeta}{R_2}\right)\nu\right]+\frac{\partial}{\partial\varphi}\left[\sigma_2\left(1+\frac{\zeta}{R_1}\right)R_1\right]+\\
&\quad+\frac{\partial}{\partial\zeta}\left[\tau_{23}\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)\nu R_1\right]+\\
&\quad+\tau_{12}\left(1+\frac{\zeta}{R_1}\right)R_1\cos\theta+\tau_{23}\left(1+\frac{\zeta}{R_1}\right)R_1\sin\theta+\\
&\qquad+F_2\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)\nu R_1=0.\\
&\frac{\partial}{\partial\theta}\left[\tau_{13}\left(1+\frac{\zeta}{R_2}\right)\nu\right]+\frac{\partial}{\partial\varphi}\left[\tau_{23}\left(1+\frac{\zeta}{R_1}\right)R_1\right]+\\
&\quad+\frac{\partial}{\partial\zeta}\left[\sigma_3\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)\nu R_1\right]-\\
&\quad-\sigma_1\left(1+\frac{\zeta}{R_2}\right)\nu-\sigma_2\left(1+\frac{\zeta}{R_1}\right)R_1\sin\theta+\\
&\qquad+F_n\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)\nu R_1=0.
\end{aligned}\right\} \tag{4.31}$$

The stress components in (4.31) are functions of coordinates θ, ϕ, ζ, while the dependences of stress σ_1, σ_2, τ_{12} on the coordinate ζ have been predetermined by relationships (4.24), (4.25). It is obvious then that from three equations of (4.31) containing derivatives of τ_{13}, τ_{23}, σ_3 during coordinate ζ, by integrating over this coordinate from ζ to $h/2$ we can find τ_{13}, τ_{23}, σ_3 as functions of ζ. In this case one ought to have in view that on surfaces $\zeta=\pm h/2$ the following boundary conditions must be executed:

$$\zeta=+\frac{h}{2}\quad \tau_{13}=p_1^+,\quad \tau_{23}=p_2^+,\quad \sigma_3=p_n^+. \tag{4.32}$$

$$\zeta=-\frac{h}{2}\quad \tau_{13}=-p_1^-,\quad \tau_{23}=-p_2^-,\quad \sigma_3=-p_n^-. \tag{4.33}$$

First we transform the first and second equation of (4.31), noticing that

$$
\left.\begin{aligned}
&\frac{\partial}{\partial\zeta}\left[\left(1+\frac{\zeta}{R_1}\right)R_1\right]=1, \quad \frac{\partial}{\partial\zeta}\left[\left(1+\frac{\zeta}{R_2}\right)\nu\right]=\sin\theta,\\
&\left(1+\frac{\zeta}{R_1}\right)R_1\frac{\partial}{\partial\zeta}\left[\tau_{13}\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)\nu R_1\right]+\\
&\qquad+\tau_{13}\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)\nu R_1=\\
&\qquad\qquad=\frac{\partial}{\partial\zeta}\left[\tau_{13}\left(1+\frac{\zeta}{R_1}\right)^2\left(1+\frac{\zeta}{R_2}\right)\nu R_1^2\right],\\
&\left(1+\frac{\zeta}{R_2}\right)\nu\frac{\partial}{\partial\zeta}\left[\tau_{23}\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)\nu R_1\right]+\\
&\qquad+\tau_{23}\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)\nu R_1\sin\theta=\\
&\qquad\qquad=\frac{\partial}{\partial\zeta}\left[\tau_{23}\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)^2\nu^2 R_1\right].
\end{aligned}\right\} \tag{4.34}
$$

then

$$
\begin{aligned}
&R_1\left(1+\frac{\zeta}{R_1}\right)\frac{\partial}{\partial\theta}\left[\sigma_1\left(1+\frac{\zeta}{R_2}\right)\nu\right]+R_1\left(1+\frac{\zeta}{R_1}\right)\frac{\partial}{\partial\varphi}\left[\tau_{12}\left(1+\frac{\zeta}{R_1}\right)R_1\right]-\\
&\quad-\sigma_2\left(1+\frac{\zeta}{R_1}\right)^2R_1^2\cos\theta+\frac{\partial}{\partial\zeta}\left[\tau_{13}\left(1+\frac{\zeta}{R_1}\right)^2\left(1+\frac{\zeta}{R_2}\right)\nu R_1^2\right]+\\
&\qquad+F_1\left(1+\frac{\zeta}{R_1}\right)^2\left(1+\frac{\zeta}{R_2}\right)\nu R_1^2=0.
\end{aligned} \tag{4.35}
$$

$$
\begin{aligned}
&\nu\left(1+\frac{\zeta}{R_2}\right)\frac{\partial}{\partial\theta}\left[\tau_{12}\left(1+\frac{\zeta}{R_2}\right)\nu\right]+\nu\left(1+\frac{\zeta}{R_2}\right)\frac{\partial}{\partial\varphi}\left[\sigma_2\left(1+\frac{\zeta}{R_1}\right)R_1\right]+\\
&+\tau_{12}\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)\nu R_1\cos\theta+\frac{\partial}{\partial\zeta}\left[\tau_{23}\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)^2\nu^2R_1\right]+\\
&\qquad+F_2\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)^2\nu^2R_1=0.
\end{aligned} \tag{4.36}
$$

Integrating (4.35) and (4.36) over coordinate ζ from ζ to $h/2$ and taking into account (4.24), (4.25) and (4.32) we obtain

$$
\begin{aligned}
&-\tau_{13}\left(1+\frac{\zeta}{R_1}\right)^2\left(1+\frac{\zeta}{R_2}\right)\nu R_1^2+p_1^+\left(1+\frac{h}{2R_1}\right)^2\left(1+\frac{h}{2R_2}\right)\nu R_1^2+\\
&\quad+\int_{\zeta}^{h/2}F_1\left(1+\frac{\zeta}{R_1}\right)^2\left(1+\frac{\zeta}{R_2}\right)\nu R_1^2\,d\zeta+\\
&\quad+\left(\frac{1}{2}-\frac{\zeta}{h}\right)\left[R_1\frac{\partial}{\partial\theta}(\nu T_1)+R_1\frac{\partial}{\partial\varphi}(R_1S_{21})-T_2R_1^2\cos\theta\right]+\\
&+\frac{3}{2h}\left(1-\frac{4\zeta^2}{h^2}\right)\left[R_1\frac{\partial}{\partial\theta}(\nu M_1)+R_1\frac{\partial}{\partial\varphi}(R_1H_{21})-M_2R_1^2\cos\theta\right]=0.
\end{aligned} \tag{4.37}
$$

$$
\begin{aligned}
&-\tau_{23}\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)^2 \nu^2 R_1 + p_2^+\left(1+\frac{h}{2R_1}\right)\left(1+\frac{h}{2R_2}\right)^2 \nu^2 R_1 + \\
&+\int_{\zeta}^{h/2} F_2\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)^2 \nu^2 R_1\, d\zeta + \\
&+\left(\frac{1}{2}-\frac{\zeta}{h}\right)\left[\nu\frac{\partial}{\partial\theta}(\nu S_{12})+\nu\frac{\partial}{\partial\varphi}(T_2R_1)+S_{21}R_1\nu\cos\theta\right]+ \\
&+\frac{3}{2h}\left(1-\frac{4\zeta^2}{h^2}\right)\left[\nu\frac{\partial}{\partial\theta}(\nu H_{12})+\nu\frac{\partial}{\partial\varphi}(M_2R_1)+H_{21}\nu R_1\cos\theta\right]=0.
\end{aligned}
\qquad (4.38)
$$

Here during integration terms of the order of h/R have been rejected as smaller than one, for example $\int_{\zeta}^{h/2}\left(1+\frac{\zeta}{R_1}\right)d\zeta \approx \left(\frac{h}{2}-\zeta\right)$. Being limited further to the same accuracy and using equilibrium equations (4.18), and also expressions for components of the applied external load E_1, E_2 and L_1, L_2, on the basis of (4.37), (4.38) we will have

$$
\begin{aligned}
\tau_{13}=&\frac{3}{2}\frac{Q_1}{h}\left(1-\frac{4\zeta^2}{h^2}\right)-\frac{1}{4}p_1^+\left[1-2\frac{\zeta}{(h/2)}-3\frac{\zeta^2}{(h/2)^2}\right]+ \\
&+\frac{1}{4}p_1^-\left[1+2\frac{\zeta}{(h/2)}-3\frac{\zeta^2}{(h/2)^2}\right]+\int_{\zeta}^{h/2}F_1\,d\zeta- \\
&-\left(\frac{1}{2}-\frac{\zeta}{h}\right)\int_{-h/2}^{+h/2}F_1\,d\zeta-\frac{3}{2h}\left(1-\frac{4\zeta^2}{h^2}\right)\int_{-h/2}^{+h/2}F_1\zeta\,d\zeta.
\end{aligned}
\qquad (4.39)
$$

$$
\begin{aligned}
\tau_{23}=&\frac{3}{2}\frac{Q_2}{h}\left(1-\frac{4\zeta^2}{h^2}\right)-\frac{1}{4}p_2^+\left[1-\frac{2\zeta}{(h/2)}-3\frac{\zeta^2}{(h/2)^2}\right]+ \\
&+\frac{1}{4}p_2^-\left[1+\frac{2\zeta}{(h/2)}-3\frac{\zeta^2}{(h/2)^2}\right]+ \\
&+\int_{\zeta}^{h/2}F_2\,d\zeta-\left(\frac{1}{2}-\frac{\zeta}{h}\right)\int_{-h/2}^{+h/2}F_2\,d\zeta-\frac{3}{2h}\left(1-\frac{4\zeta^2}{h^2}\right)\int_{-h/2}^{+h/2}F_2\zeta\,d\zeta.
\end{aligned}
\qquad (4.40)
$$

Volume forces on the shell - in most cases either forces of weight

$$
F=\gamma l=\gamma(l_1\tau_1+l_2\tau_2+l_n n). \qquad (4.41)
$$

where l - unit vector indicating the direction of gravity, γ - specific gravity of the material from which the shell is made; or forces of inertia appearing during rotation of the shell around its axis

$$F = \frac{\gamma\omega^2}{g} v\left(1+\frac{\zeta}{R_2}\right) e \approx \frac{\gamma\omega^2 v}{g}(a\sin\theta + z_1\cos\theta). \qquad (4.42)$$

In the first case F_1, F_2, F_n generally depend on coordinate ζ, in the second case this relationship also can be neglected. Assuming that volume forces depend on coordinate ζ, we observe that terms containing volume forces in (4.39) and (4.40) cancel out, and finally we obtain

$$\tau_{13} = \frac{3}{2}\frac{Q_1}{h}\left(1-\frac{4\zeta^2}{h^2}\right) - \frac{1}{4}p_1^+\left[1-2\frac{\zeta}{(h/2)} - 3\frac{\zeta^2}{(h/2)^2}\right] + \\ + \frac{1}{4}p_1^-\left[1+2\frac{\zeta}{(h/2)} - 3\frac{\zeta^2}{(h/2)^2}\right]. \qquad (4.43)$$

$$\tau_{23} = \frac{3}{2}\frac{Q_2}{h}\left(1-\frac{4\zeta^2}{h^2}\right) - \frac{1}{4}p_2^+\left[1-2\frac{\zeta}{(h/2)} - 3\frac{\zeta^2}{(h/2)^2}\right] + \\ + \frac{1}{4}p_2^-\left[1+2\frac{\zeta}{(h/2)} - 3\frac{\zeta^2}{(h/2)^2}\right]. \qquad (4.44)$$

It is simple to verify that the obtained expressions satisfy conditions (4.32), (4.33). Integrating similarly the third equation of (4.31), it would be possible to obtain the expression for stress $\sigma_3(\zeta)$, also satisfying conditions (4.32). We will not give the appropriate calculations, since the assumption of the smallness of this stress in comparison with the others

$$\sigma_3 \approx 0 \qquad (4.45)$$

is basic in the theory of thin shells and essentially is used subsequently.

In conclusion let us note that rejecting terms of the order of h/R in comparison with unity and considering volume forces to be independent of ζ, we can simplify the load terms in equilibrium equations (4.22), namely

$$E_1 \approx p_1^+ + p_1^- + F_1 h = q_1,$$
$$E_2 \approx p_2^+ + p_2^- + F_2 h = q_2,$$
$$E_n \approx p_n^+ + p_n^- + F_n h = q_n.$$

Furthermore, confined to the same accuracy, we can reject moments of external loading L_1, L_2 in expressions (4.23), since after the substitution of (4.23) into (4.22) the terms corresponding to them will have the order of h/R in comparison with the loading terms (4.22). After determining forces and moments acting on an element of the shell, stresses $\sigma_1, \sigma_2, \tau_{12}$ are calculated using simplified formulas (4.24), (4.25).

Stresses τ_{13}, τ_{23} are secondary and rarely calculated in practice; however, they can also be calculated according to formulas (4.43), (4.44), where without hurting the accuracy in this instance we can set $Q_1 \approx N_1$ and $Q_2 \approx N_2$. Maximum bending stresses occur at $\zeta = \pm h/2$ and are equal to

$$\sigma_1 = \pm \frac{6M_1}{h^2}, \quad \sigma_2 = \pm \frac{6M_2}{h^2}. \tag{4.46}$$

§ 5. Potential Energy of Deformation and Elasticity Relationships

Compose the variation of the potential energy of deformation of a shell, keeping in mind in this case that on the basis of the accepted kinematic hypotheses $e_{13} = e_{23} = e_3 = 0$. Then

$$\delta U = \int\int_0\int (\sigma_1 \delta e_1 + \sigma_2 \delta e_2 + \tau_{12} \delta\omega)\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right) \nu R_1 \, d\varphi \, d\theta \, d\zeta. \tag{5.1}$$

Remembering expressions for e_1, e_2, ω through components of the deformation of the middle surface, on the basis of equations (3.13), (3.18) we have

$$\delta e_1 = \frac{\delta \varepsilon_1 + \zeta \delta \varkappa_1}{1 + \zeta/R_1}, \quad \delta e_2 = \frac{\delta \varepsilon_2 + \zeta \delta \varkappa_2}{1 + \zeta/R_2}, \tag{5.2}$$

$$\delta\omega = \frac{1}{(1 + \zeta/R_1)(1 + \zeta R_2)} \left\{ \delta\gamma \left(1 - \frac{\zeta^2}{R_1 R_2}\right) + \zeta 2 \, \delta\tau \left[1 + \frac{\zeta(R_1 + R_2)}{2R_1 R_2}\right] \right\}. \tag{5.3}$$

Substituting (5.2), (5.3) into (5.1) and integrating over coordinate ζ taking into account formulas (4.5), (4.7), (4.8), we obtain

$$\delta U = \int_{\Sigma}\int \delta V \nu R_1 \, d\varphi \, d\theta, \tag{5.4}$$

where

$$\delta V = T_1 \delta\varepsilon_1 + T_2 \delta\varepsilon_2 + M_1 \delta\varkappa_1 + M_2 \delta\varkappa_2 + S \delta\gamma + 2H \delta\tau. \tag{5.5}$$

Requiring that δV be the total differential

$$\delta V = \frac{\partial V}{\partial \varepsilon_1}\delta\varepsilon_1 + \frac{\partial V}{\partial \varepsilon_2}\delta\varepsilon_2 + \frac{\partial V}{\partial \gamma}\delta\gamma + \frac{\partial V}{\partial \varkappa_1}\delta\varkappa_1 + \frac{\partial V}{\partial \varkappa_2}\delta\varkappa_2 + \frac{\partial V}{\partial \tau}\delta\tau,$$

we obtain equations analogous to the Green equations in the theory of elasticity:

$$\left.\begin{aligned} T_1 = \frac{\partial V}{\partial \varepsilon_1}, \quad T_2 = \frac{\partial V}{\partial \varepsilon_2}, \quad S = \frac{\partial V}{\partial \gamma}, \\ M_1 = \frac{\partial V}{\partial \varkappa_1}, \quad M_2 = \frac{\partial V}{\partial \varkappa_2}, \quad 2H = \frac{\partial V}{\partial \tau}. \end{aligned}\right\} \tag{5.6}$$

Thus far there have been no assumptions about the character of the physical connection between stresses and deformations in a shell. This connection within the framework of the theory of shells should be expressed in the form of relationship between power characteristics $(T_1, T_2, S, M_1, M_2, H)$, on one hand and deformation components $(\varepsilon_1, \varepsilon_2, \gamma, \varkappa_1, \varkappa_2, \tau)$, on the other. Meanwhile the introduction of such relationship is necessary, since without them the problem of calculating the shell is statically indefinable: the three of equilibrium equations (4.22) use six unknown power factors. Assuming that the shell is made from elastic isotropic material, and ignoring stress σ_3 in comparison with other stresses, we write Hoake's law in the form

$$\left.\begin{aligned} e_1 &= \frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)] \approx \frac{1}{E}(\sigma_1 - \mu\sigma_2), \\ e_2 &= \frac{1}{E}[\sigma_2 - \mu(\sigma_1 + \sigma_3)] \approx \frac{1}{E}(\sigma_2 - \mu\sigma_1), \\ \omega &= \frac{2(1+\mu)}{E}\tau_{12}, \end{aligned}\right\} \tag{5.7}$$

whence follow the relationships between stresses and deformations

$$\left.\begin{aligned} \sigma_1 = \frac{E}{(1-\mu^2)}(e_1 + \mu e_2), \quad \sigma_2 = \frac{E}{(1-\mu^2)}(e_2 + \mu e_1), \\ \tau_{12} = \frac{E}{2(1+\mu)}\omega. \end{aligned}\right\} \tag{5.8}$$

Using the obtained expressions, we calculate

$$\begin{aligned} \delta V &= \int_{-h/2}^{+h/2} (\sigma_1 \delta e_1 + \sigma_2 \delta e_2 + \tau_{12}\delta\omega)\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right) d\zeta = \\ &= \delta\left\{\frac{E}{2(1-\mu^2)} \int_{-h/2}^{+h/2} \left[e_1^2 + e_2^2 + 2\mu e_1 e_2 + \frac{(1-\mu)}{2}\omega^2\right]\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right) d\zeta\right\} \end{aligned} \tag{5.9}$$

$$\begin{aligned} V = \frac{E}{2(1-\mu^2)} \int_{-h/2}^{+h/2} \left[e_1^2 + e_2^2 + 2\mu e_1 e_2 + \frac{(1-\mu)}{2}\omega^2\right] \times \\ \times\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right) d\zeta. \end{aligned} \tag{5.10}$$

With the aid of formulas (3.13), (3.17) we present subintegral expression (5.10) in the form of a series in powers of ζ, rejecting in this case all terms containing powers of ζ higher than square.

$$\left[e_1^2 + e_2^2 + 2\mu e_1 e_2 + \frac{(1-\mu)}{2}\omega^2\right]\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right) = A_0 + A_1\zeta + A_2\zeta^2. \tag{5.11}$$

where

$$\left.\begin{aligned} A_0 &= (\varepsilon_1 + \varepsilon_2)^2 - 2(1-\mu)\left(\varepsilon_1\varepsilon_2 - \frac{\gamma^2}{4}\right), \\ A_2 &= \varkappa_1^2 + \varkappa_2^2 + 2\mu\varkappa_1\varkappa_2 + 2(1-\mu)\tau^2 + \\ &+ 2(\varepsilon_1\varkappa_1 - \varepsilon_2\varkappa_2)\left(\frac{1}{R_2} - \frac{1}{R_1}\right) - (1-\mu)\gamma\tau\left(\frac{1}{R_2} + \frac{1}{R_1}\right) + \\ &+ \left(\frac{\varepsilon_2^2}{R_2} - \frac{\varepsilon_1^2}{R_1}\right)\left(\frac{1}{R_2} - \frac{1}{R_1}\right) + \frac{(1-\mu)}{2}\gamma^2\left(\frac{1}{R_1^2} + \frac{1}{R_2^2} - \frac{1}{R_1R_2}\right). \end{aligned}\right\} \tag{5.12}$$

The expression for A_1 is not written out, since in integrating over ζ from $-\frac{h}{2}$ to $+\frac{h}{2}$ the corresponding term in the right side of (5.10) disappears. After integration we have

$$
\left.\begin{aligned}
&V = V_1 + V_2 + V_3 + V_4, \\
&V_1 = \frac{Eh}{2(1-\mu^2)}\left[(\varepsilon_1+\varepsilon_2)^2 - 2(1-\mu)\left(\varepsilon_1\varepsilon_2 - \frac{\gamma^2}{4}\right)\right], \\
&V_2 = \frac{E}{2(1-\mu^2)}\frac{h^3}{12}\left[(\varkappa_1+\varkappa_2)^2 - 2(1-\mu)(\varkappa_1\varkappa_2 - \tau^2)\right], \\
&V_3 = \frac{E}{2(1-\mu^2)}\frac{h^3}{12}\Big[2(\varepsilon_1\varkappa_1 - \varepsilon_2\varkappa_2)\left(\frac{1}{R_2} - \frac{1}{R_1}\right) - \\
&\qquad\qquad - (1-\mu)\gamma\tau\left(\frac{1}{R_2} + \frac{1}{R_1}\right)\Big], \\
&V_4 = \frac{E}{2(1-\mu^2)}\frac{h^3}{12}\Big[\left(\frac{\varepsilon_2^2}{R_2} - \frac{\varepsilon_1^2}{R_1}\right)\left(\frac{1}{R_2} - \frac{1}{R_1}\right) + \\
&\qquad\qquad + \frac{(1-\mu)}{2}\gamma^2\left(\frac{1}{R_1^2} + \frac{1}{R_2^2} - \frac{1}{R_1R_2}\right)\Big].
\end{aligned}\right\} \qquad (5.13)
$$

V. V. Novozhilov [21] made a comparative evaluation of the terms in (5.13), introducing auxiliary quantities

$$
\varepsilon_1' = \varkappa_1\frac{h}{2}, \quad \varepsilon_2' = \varkappa_2\frac{h}{2}, \quad \gamma' = \tau h, \qquad (5.14)
$$

which have the sense of elongations and shifts in the surface layer of the shell caused by the bending and twisting of the middle surface, while $\varepsilon_1', \varepsilon_2', \gamma'$ are dimensionless quantities, just as $\varepsilon_1, \varepsilon_2, \gamma$. With these introduced quantities the expressions for V_2, V_3, V_4, can be rewritten thus:

$$
\left.\begin{aligned}
&V_2 = \frac{Eh}{2(1-\mu^2)}\frac{1}{3}\left[(\varepsilon_1'+\varepsilon_2')^2 - 2(1-\mu)(\varepsilon_1'\varepsilon_2' - \gamma'^2)\right], \\
&V_3 = \frac{Eh}{2(1-\mu^2)}\frac{1}{6}\Big[2(\varepsilon_1'\varepsilon_1 - \varepsilon_2'\varepsilon_2)\left(\frac{h}{R_2} - \frac{h}{R_1}\right) - \\
&\qquad\qquad - (1-\mu)\gamma\gamma'\left(\frac{h}{R_2} + \frac{h}{R_1}\right)\Big], \\
&V_4 = \frac{Eh}{2(1-\mu^2)}\frac{1}{12}\Big[\left(\varepsilon_2^2\frac{h}{R_2} - \varepsilon_1^2\frac{h}{R_1}\right)\left(\frac{h}{R_2} - \frac{h}{R_1}\right) + \\
&\qquad\qquad + \frac{(1-\mu)}{2}\gamma^2h^2\left(\frac{1}{R_1^2} + \frac{1}{R_2^2} - \frac{1}{R_1R_2}\right)\Big].
\end{aligned}\right\} \qquad (5.15)
$$

After this, comparing components V_1, V_2, V_3, V_4, it is easy to see that V_4 has the order of h^2/R^2, and V_3 the order of h/R if we say V_1 is of the order of unity and assume that deformations $\varepsilon_1, \varepsilon_2, \gamma, \varepsilon_1', \varepsilon_2', \gamma'$ are quantities of one order. In the same case, when $(\varepsilon_1, \varepsilon_2, \gamma) \ll (\varepsilon_1', \varepsilon_2', \gamma')$, the inequality $V_3 \ll V_2$, exists; if, however, it is the reverse $(\varepsilon_1', \varepsilon_2', \gamma') \ll (\varepsilon_1, \varepsilon_2, \gamma)$, then $V_3 \ll V_1$. In all following assumptions relative to quantities $\varepsilon_1, \varepsilon_2, \gamma$ and $\varepsilon_1', \varepsilon_2', \gamma'$ and $V_4 \ll V_1$ and $V_3 \ll (V_1 + V_2)$ and in (5.13) we can reject the terms V_3, V_4.

Thus, we say that the potential energy of deformation of a shell per unit area of the middle surface is expressed through the components of deformation of the middle surface in the following manner:

$$V = \frac{Eh}{2(1-\mu^2)}\left[(\varepsilon_1+\varepsilon_2)^2 - 2(1-\mu)\left(\varepsilon_1\varepsilon_2 - \frac{\gamma^2}{4}\right)\right] + \\ + \frac{E}{2(1-\mu^2)}\frac{h^3}{12}[(\varkappa_1+\varkappa_2)^2 - 2(1-\mu)(\varkappa_1\varkappa_2 - \tau^2)].$$

Calculating partial derivatives of function $V(\varepsilon_1, \varepsilon_2, \gamma, \varkappa_1, \varkappa_2, \tau)$, on the basis of formulas (5.16) and the Green equations (5.6) we obtain the elasticity relationships:

$$\left.\begin{aligned} T_1 &= B(\varepsilon_1 + \mu\varepsilon_2), & M_1 &= D(\varkappa_1 + \mu\varkappa_2), \\ T_2 &= B(\varepsilon_2 + \mu\varepsilon_1), & M_2 &= D(\varkappa_2 + \mu\varkappa_1), \\ S &= B\frac{1-\mu}{2}\gamma, & H &= D(1-\mu)\tau, \end{aligned}\right\} \qquad (5.17)$$

where the designations

$$B = \frac{Eh}{1-\mu^2}, \quad D = \frac{Eh^3}{12(1-\mu^2)} \qquad (5.18)$$

have been introduced, where B is called the cylindrical rigidity of elongation, and D is the cylindrical rigidity of bend for a shell.

In conclusion one ought to say that recently in literature again attention has returned to the question of evaluating the terms in formula (5.13). Preferring a more accurate notation of the expression for the potential energy of a shell and formulating, consequently, elasticity relationships authors are guided by the following considerations : 1) the requirement of invariance of the relationships of elasticity on replacing curvilinear coordinates on the middle surface [39], 2) possibility of decreasing the inaccuracy in the technical theory of shells, which, generally speaking, depends on the nature of the change in loads on the shell [28].

§ 6. Total System of Equations Describing Shell Equilibrium. Boundary Conditions

Usually the problem of shell calculation involves finding displacements and the stressed state in a shell for a preassigned external load (surface and volume forces) and preassigned method of fixing the edges of the shell. The total system of equations which describe shell equilibrium is three differential equilibrium equations in six static quantities $(T_1, T_2, S, M_1, M_2, H)$, six elasticity relationships, which connect the static quantities and deformation components $(\varepsilon_1, \varepsilon_2, \gamma, \varkappa_1, \varkappa_2, \tau)$, and six difference relationships, with the aid of which the deformation components are expressed through displacements u, v, w. The number of equations is fifteen and coincides with the number of unknowns. Of all possible solutions of this system it is necessary to find such which satisfies preassigned conditions on the shell edges. Hence we see the necessity of formulating the conditions of fixing of the edges of the shell in terms which correspond to introduced static and geometric quantities. This can best be done if we use the principle of possible displacements. We agree to consider subsequently shells of revolution, closed in coordinate ϕ and bounded by parallel circles $\Gamma_1\,(\theta=\theta_1)$ and $\Gamma_2\,(\theta=\theta_2)$. The equation which expresses the principle of possible displacements has the form

$$\delta U-\delta A=0. \qquad (6.1)$$

where δU is the work of internal forces on variations of displacements, δA is the work of external forces on the same variations. Only those displacements which will not contradict geometr constraints imposed on the shell are compared. In view of the closure of the shell, the displacements and variations should be periodic functions of ϕ. The expression for δU was already obtained in the previous secti (formulas (5.3), (5.5)), so that designating through $\sigma_1^{(1)}, \tau_{12}^{(1)}, \tau_{13}^{(1)}$ and $\sigma_1^{(2)}, \tau_{12}^{(2)}, \tau_{13}^{(2)}$ the stresses on the extreme sections Γ_1 and Γ_2 respectively, we can write equation (6.1) in detail.

$$\begin{aligned}
&\iint_{\Sigma_0} [T_1\delta\varepsilon_1 + T_2\delta\varepsilon_2 + S\delta\gamma + M_1\delta\varkappa_1 + M_2\delta\varkappa_2 + 2H\delta\tau] R_1\nu\, d\theta\, d\varphi - \\
&- \iint_{\Sigma_+} \left[p_1^+ \delta u^{(\zeta)}\left(\frac{h}{2}\right) + p_2^+ \delta v^{(\zeta)}\left(\frac{h}{2}\right) + p_n^+ \delta w^{(\zeta)}\left(\frac{h}{2}\right)\right]\left(1+\frac{h}{2R_1}\right)\times \\
&\times\left(1+\frac{h}{2R_2}\right) R_1\nu\, d\varphi\, d\theta - \iint_{\Sigma_-}\left[p_1^- \delta u^{(\zeta)}\left(-\frac{h}{2}\right) + p_2^- \delta v^{(\zeta)}\left(-\frac{h}{2}\right) + \right.\\
&\qquad \left. + p_n^- \delta w^{(\zeta)}\left(-\frac{h}{2}\right)\right]\left(1-\frac{h}{2R_1}\right)\left(1-\frac{h}{2R_2}\right) R_1\nu\, d\varphi\, d\theta - \\
&- \iiint_{\delta} [F_1\delta u^{(\zeta)} + F_2\delta v^{(\zeta)} + F_n\delta w^{(\zeta)}]\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)\times \\
&\times R_1\nu\, d\varphi\, d\theta\, d\zeta - \left\{\int_{\Gamma_2}\int_{-h/2}^{+h/2} [\sigma_1^{(2)}\delta u^{(\zeta)} + \tau_{12}^{(2)}\delta v^{(\zeta)} + \tau_{13}^{(2)}\delta w^{(\zeta)}]\times \right.\\
&\times\left(1+\frac{\zeta}{R_2}\right) d\zeta\,\nu\, d\varphi - \int_{\Gamma_1}\int_{-h/2}^{+h/2} [\sigma_1^{(1)}\delta u^{(\zeta)} + \tau_{12}^{(1)}\delta v^{(\zeta)} + \tau_{13}^{(1)}\delta w^{(\zeta)}]\times \\
&\qquad\qquad \left.\times\left(1+\frac{\zeta}{R_2}\right) d\zeta\,\nu\, d\varphi\right\} = 0. \qquad (6.2)
\end{aligned}$$

where Σ_0, Σ_+ and Σ_- designate integration over the middle surface and surfaces $\zeta = \pm h/2$ respectively, the positive direction of the pass of contours Γ_1 and Γ_2 coincides with the direction of positive gain in coordinates ϕ, so that $\int_{\Gamma_1}\{\ldots\}\nu\, d\varphi = \int^{2\pi}\{\ldots\}\nu\, d\varphi$. Taking into consideration that by (3.10)

$$\left.\begin{aligned}
&\delta u^{(\zeta)} = \delta u - \zeta\delta\vartheta_1, \quad \delta v^{(\zeta)} = \delta v - \zeta\delta\vartheta_2, \\
&\delta w^{(\zeta)} = \delta w,
\end{aligned}\right\} \qquad (6.3)$$

and introducing the designations

$$\left.\begin{aligned}
&T_1^{(i)} = \int_{-h/2}^{+h/2} \sigma_1^{(i)}\left(1+\frac{\zeta}{R_2}\right) d\zeta, \quad M_1^{(i)} = \int_{-h/2}^{+h/2} \sigma_1^{(i)}\left(1+\frac{\zeta}{R_2}\right)\zeta\, d\zeta, \\
&S_{12}^{(i)} = \int_{-h/2}^{+h/2} \tau_{12}^{(i)}\left(1+\frac{\zeta}{R_2}\right) d\zeta, \quad H_{12}^{(i)} = \int_{-h/2}^{+h/2} \tau_{12}^{(i)}\left(1+\frac{\zeta}{R_2}\right)\zeta\, d\zeta, \\
&Q_1^{(i)} = \int_{-h/2}^{+h/2} \tau_{13}^{(i)}\left(1+\frac{\zeta}{R_2}\right) d\zeta, \quad i = 1,\ 2.
\end{aligned}\right| \qquad (6.4)$$

we transform the contour integrals in (6.2) to the form

$$\delta A_2^{(i)} = (-1)^i \int\limits_{\Gamma_i} \int\limits_{-h/2}^{+h/2} [\sigma_1^{(i)} \delta u^{(\zeta)} + \tau_{12}^{(i)} \delta v^{(\zeta)} + \tau_{13}^{(i)} \delta w^{(\zeta)}] \times$$
$$\times \left(1 + \frac{\zeta}{R_2}\right) d\zeta \nu\, d\varphi = (-1)^i \int\limits_{\Gamma_i} [T_1^{(i)} \delta u + S_{12}^{(i)} \delta v + Q_1^{(i)} \delta w -$$
$$- M_1^{(i)} \delta\vartheta_1 - H_{12}^{(i)} \delta\vartheta_2]\, \nu\, d\varphi. \qquad (6.5)$$

On edge θ_i = const variation $\delta\vartheta_2$ is not independent; really, from (2.5) it is easy to see that the form of functions $u(\varphi)$, $v(\varphi)$, $w(\varphi)$ at this edge entirely determines $\vartheta_2(\varphi)$. consequently,

$$\delta\vartheta_2 = \frac{1}{\nu} \frac{\partial(\delta w)}{\partial\varphi} - \frac{\sin\theta}{\nu} \delta v \qquad (6.6)$$

(6.5) one ought to rewrite thus:

$$\delta A_2^{(i)} = (-1)^i \int\limits_{\Gamma_i} \Big[T_1^{(i)} \delta u + \left(S_{12}^{(i)} + \frac{H_{12}^{(i)}}{R_2}\right) \delta v - M_1^{(i)} \delta\vartheta_1 +$$
$$+ \left(Q_1^{(i)} + \frac{1}{\nu} \frac{\partial H_{12}^{(i)}}{\partial\varphi}\right) \delta w \Big] \nu\, d\varphi, \qquad i = 1,\ 2. \qquad (6.7)$$

The terms in (6.2) which are the work of surface and volume external forces on variations of displacements, with the aid of equations (6.3) and designations (4.12) are transformed to

$$\delta A_1 = \int\limits_{\Sigma_+}\int [p_1^+ \delta u^{(\zeta)} + p_2^+ \delta v^{(\zeta)} + p_n^+ \delta w^{(\zeta)}] \left(1 + \frac{h}{2R_1}\right)\left(1 + \frac{h}{2R_2}\right) \times$$
$$\times R_1 \nu\, d\varphi\, d\theta + \int\limits_{\Sigma_-}\int [p_1^- \delta u^{(\zeta)} + p_2^- \delta v^{(\zeta)} + p_n^- \delta w^{(\zeta)}] \left(1 - \frac{h}{2R_1}\right) \times$$
$$\times \left(1 - \frac{h}{2R_2}\right) R_1 \nu\, d\varphi\, d\theta + \int\int\limits_{\Omega}\int [F_1 \delta u^{(\zeta)} + F_2 \delta v^{(\zeta)} + F_n \delta w^{(\zeta)}] \times$$
$$\times \left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right) R_1 \nu\, d\varphi\, d\theta\, d\zeta = \int\limits_{\Sigma_0}\int [E_1 \delta u + E_2 \delta v +$$
$$+ E_n \delta w - L_2 \delta\vartheta_1 + L_1 \delta\vartheta_2] R_1 \nu\, d\varphi\, d\theta. \qquad (6.8)$$

But on the basis of (2.5) variations $\delta\vartheta_1$, $\delta\vartheta_2$ are expressed through three independent variations δu, δv, δw

$$\left.\begin{aligned}\delta\theta_1 &= \frac{1}{R_1}\frac{\partial}{\partial\theta}(\delta w) - \frac{1}{R_1}\delta u,\\ \delta\theta_2 &= \frac{1}{\nu}\frac{\partial}{\partial\varphi}(\delta w) - \frac{\sin\theta}{\nu}\delta v,\end{aligned}\right\} \qquad (6.9)$$

therefore the last two terms in the right side of (6.8) must be rewritten in the following manner:

$$\left.\begin{aligned}-\iint_{\Sigma_0} L_2\delta\theta_1 R_1\nu\, d\varphi\, d\theta &= -\iint_{\Sigma_0} L_2\frac{\partial}{\partial\theta}(\delta w)\,\nu\, d\theta\, d\varphi + \\ &\quad + \iint_{\Sigma_0} L_2\delta u\nu\, d\theta\, d\varphi.\\ \iint_{\Sigma_0} L_1\delta\theta_2 R_1\nu\, d\varphi\, d\theta &= \iint_{\Sigma_0} L_1\frac{\partial}{\partial\varphi}(\delta w)\, R_1\, d\theta\, d\varphi - \\ &\quad - \iint_{\Sigma_0} L_1\sin\theta\,\delta v R_1\, d\theta\, d\varphi.\end{aligned}\right\} \qquad (6.10)$$

Integrating by parts and noticing that

$$\iint_{\Sigma_0}\frac{\partial}{\partial\theta}(L_2\nu\,\delta w)\, d\theta\, d\varphi = \int_{\theta_1}^{\theta_2}\int_0^{2\pi}\frac{\partial}{\partial\theta}(L_2\nu\,\delta w)\, d\theta\, d\varphi =$$

$$= \int_{\Gamma_2} L_2\nu\,\delta w\, d\varphi - \int_{\Gamma_1} L_2\nu\,\delta w\, d\varphi,$$

$$\iint_{\Sigma_0}\frac{\partial}{\partial\varphi}(L_1 R_1\,\delta w)\, d\theta\, d\varphi = \int_{\theta_1}^{\theta_2}[L_1 R_1\,\delta w]_0^{2\pi}\, d\theta = 0$$

(in view of the periodicity of the external loading and variations of displacements in coordinate ϕ), we will have

$$\left.\begin{aligned}-\iint_{\Sigma_0} L_2\delta\theta_1 R_1\nu\, d\varphi\, d\theta &= \int_{\Gamma_2} L_2\delta w\nu\, d\varphi - \int_{\Gamma_1} L_2\delta w\nu\, d\varphi + \\ &\quad + \iint_{\Sigma_0}\left[\frac{1}{R_1\nu}\frac{\partial}{\partial\theta}(L_2\nu)\delta w + \frac{L_2}{R_1}\delta u\right] R_1\nu\, d\theta\, d\varphi.\\ \iint_{\Sigma_0} L_1\delta\theta_2 R_1\nu\, d\theta\, d\varphi &= \\ &= -\iint_{\Sigma_0}\left[\frac{1}{\nu}\frac{\partial L_1}{\partial\varphi}\delta w + \frac{\sin\theta\, L_1}{\nu}\delta v\right] R_1\nu\, d\theta\, d\varphi.\end{aligned}\right\} \qquad (6.11)$$

Taking into account of the calculations, we find that the right side of (6.8) is equal to

$$\delta A_1 = \iint_{\Sigma_0} \Big\{ \Big(E_1 + \frac{L_2}{R_1}\Big)\delta u + \Big(E_2 - \frac{\sin\theta L_1}{\nu}\Big)\delta v + \\ + \Big[E_3 + \frac{1}{R_1\nu}\frac{\partial}{\partial\theta}(L_2\nu) - \frac{1}{\nu R_1}\frac{\partial(L_1R_1)}{\partial\varphi}\Big]\delta w \Big\} R_1\nu\, d\theta\, d\varphi. \tag{6.12}$$

We transform also the expression for $\delta U = \iint_{\Sigma_0} (T_1\delta\varepsilon_1 + T_2\delta\varepsilon_2 + S\delta\gamma + M_1\delta\varkappa_1 + M_2\delta\varkappa_2 + 2H\delta\tau)\, d\Sigma_0$, going from variations of deformation components to δu, δv, δw, using formulas (2.5), (2.18), (2.26). Make the necessary calculations:

$$\iint_{\Sigma_0} T_1\delta\varepsilon_1\, d\Sigma_0 = \int_{\theta_1}^{\theta_2}\int_0^{2\pi} T_1\Big[\frac{1}{R_1}\frac{\partial}{\partial\theta}(\delta u) + \frac{\delta w}{R_1}\Big] R_1\nu\, d\theta\, d\varphi = \\ = \int_{\theta_1}^{\theta_2}\int_0^{2\pi} \frac{\partial}{\partial\theta}(T_1\nu\delta u)\, d\theta\, d\varphi - \int_{\theta_1}^{\theta_2}\int_0^{2\pi}\frac{\partial}{\partial\theta}(T_1\nu)\delta u\, d\varphi\, d\theta + \\ + \int_{\theta_1}^{\theta_2}\int_0^{2\pi} T_1\delta w\nu\, d\theta\, d\varphi = \int_{\Gamma_2} T_1\delta u\nu\, d\varphi - \int_{\Gamma_1} T_1\delta u\nu\, d\varphi + \\ + \iint_{\Sigma_0}\Big[\frac{T_1}{R_1}\delta w - \frac{1}{R_1\nu}\frac{\partial}{\partial\theta}(T_1\nu)\delta u\Big] d\Sigma_0.$$

$$\iint_{\Sigma_0} M_1\delta\varkappa_1\, d\Sigma_0 = -\int_{\theta_1}^{\theta_2}\int_0^{2\pi} M_1\frac{1}{R_1}\frac{\partial}{\partial\theta}(\delta\theta_1) R_1\nu\, d\theta\, d\varphi = \\ = -\int_{\Gamma_2} M_1\nu\delta\theta_1\, d\varphi + \int_{\Gamma_1} M_1\nu\delta\theta_1\, d\varphi + \int_{\theta_1}^{\theta_2}\int_0^{2\pi}\frac{\partial}{\partial\theta}(M_1\nu)\times \\ \times\Big(\frac{1}{R_1}\frac{\partial\delta w}{\partial\theta} - \frac{\delta u}{R_1}\Big) d\theta\, d\varphi = -\int_{\Gamma_2}\Big[M_1\delta\theta_1 - \frac{1}{R_1\nu}\frac{\partial}{\partial\theta}(M_1\nu)\delta w\Big]\nu\, d\varphi + \\ + \int_{\Gamma_1}\Big[M_1\delta\theta_1 - \frac{1}{R_1\nu}\frac{\partial}{\partial\theta}(M_1\nu)\delta w\Big]\nu\, d\varphi - \\ - \iint_{\Sigma_0}\Big\{\frac{1}{R_1\nu}\frac{\partial}{\partial\theta}\Big[\frac{1}{R_1}\frac{\partial}{\partial\theta}(M_1\nu)\Big]\delta w + \frac{1}{R_1\nu}\frac{\partial}{\partial\theta}(M_1\nu)\delta u\Big\} d\Sigma_0.$$

After analogous calculations we have

$$\iint_{\Sigma_0} S\delta\gamma\, d\Sigma_0 = \int_{\Gamma_2} S\delta v\nu\, d\varphi - \int_{\Gamma_1} S\delta v\nu\, d\varphi - \\ - \iint_{\Sigma_0}\Big[\frac{1}{R_1\nu}\frac{\partial}{\partial\theta}(S\nu)\delta v + \frac{1}{R_1\nu}\frac{\partial}{\partial\varphi}(SR_1)\delta u + \frac{S\cos\theta}{\nu}\delta v\Big] d\Sigma_0.$$

$$\iint_{\Sigma_0} T_2\,\delta\varepsilon_2\,d\Sigma_0 = \iint_{\Sigma_0}\left[\frac{1}{R_1\nu}\frac{\partial}{\partial\varphi}(T_2R_1)\delta v + \frac{T_2\cos\theta}{\nu}\delta u + \frac{T_2}{R_2}\delta w\right]d\Sigma_0.$$

$$\iint_{\Sigma_0} M_2\,\delta\varkappa_2\,d\Sigma_0 = -\int_{\Gamma_2} M_2\cos\theta\,\delta w\nu\,d\varphi + \int_{\Gamma_1} M_2\cos\theta\,\delta w\nu\,d\varphi +$$

$$+\iint_{\Sigma_0}\left\{\frac{1}{\nu R_1}\frac{\partial}{\partial\theta}(M_2\cos\theta)\delta w - \frac{1}{R_1\nu}\frac{\partial}{\partial\varphi}\left[\frac{1}{\nu}\frac{\partial}{\partial\varphi}(M_2R_1)\right]\delta w -\right.$$

$$\left.-\frac{\sin\theta}{\nu^2R_1}\frac{\partial}{\partial\varphi}(M_2R_1)\delta v + \frac{M_2\cos\theta}{R_1\nu}\delta u\right\}d\Sigma_0.$$

$$\iint_{\Sigma_0} 2H\,\delta\tau\,d\Sigma_0 = \iint_{\Sigma_0} H\left[-\frac{1}{R_1}\frac{\partial(\delta\theta_2)}{\partial\theta} + \frac{1}{R_1\nu}\frac{\partial(\delta u)}{\partial\varphi} - \frac{\cos\theta}{\nu R_1}\delta v\right]d\Sigma_0 +$$

$$+\iint_{\Sigma_0} H\left[\frac{\cos\theta}{\nu}\delta\theta_2 - \frac{1}{\nu}\frac{\partial(\delta\theta_1)}{\partial\varphi} + \frac{1}{R_1R_2}\frac{\partial(\delta v)}{\partial\theta}\right]d\Sigma_0 =$$

$$= 2\int_{\Gamma_2}\left(\frac{1}{\nu}\frac{\partial H}{\partial\varphi}\delta w + \frac{H\sin\theta}{\nu}\delta v\right)\nu\,d\varphi -$$

$$-2\int_{\Gamma_1}\left(\frac{1}{\nu}\frac{\partial H}{\partial\varphi}\delta w + \frac{H\sin\theta}{\nu}\delta v\right)\nu\,d\varphi -$$

$$-\iint_{\Sigma_0}\left\{\frac{\sin\theta}{\nu^2R_1}\frac{\partial}{\partial\theta}(\nu H)\delta v + \frac{1}{\nu R_1}\frac{\partial}{\partial\varphi}\left[\frac{1}{\nu}\frac{\partial(H\nu)}{\partial\theta}\right]\delta w + \frac{1}{R_1\nu}\frac{\partial H}{\partial\varphi}\delta u +\right.$$

$$\left.+\frac{\cos\theta}{R_1\nu}H\delta v\right\}d\Sigma_0 - \iint_{\Sigma_0}\left\{\frac{H\cos\theta\sin\theta}{\nu^2}\delta v + \frac{\partial}{\partial\varphi}\left(\frac{HR_1\cos\theta}{\nu}\right)\delta w +\right.$$

$$\left.+\frac{1}{R_1\nu}\frac{\partial H}{\partial\varphi}\delta u + \frac{1}{R_1\nu}\frac{\partial}{\partial\theta}\left(\frac{\partial H}{\partial\varphi}\right)\delta w + \frac{1}{\nu R_1}\frac{\partial}{\partial\theta}(H\sin\theta)\delta v\right\}d\Sigma_0.$$

Now we can write equation (6.2) in the following form:

$$\iint_{\Sigma_0}\left\{\left[-\frac{\partial}{\partial\theta}(T_1\nu) + T_2R_1\cos\theta - R_1\frac{\partial S}{\partial\varphi} - 2\frac{\partial H}{\partial\varphi} + M_2\cos\theta -\right.\right.$$

$$\left.-\frac{1}{R_1}\frac{\partial}{\partial\theta}(M_1\nu) - L_2\nu - E_1\nu R_1\right]\delta u + \left[-R_1\frac{\partial T_2}{\partial\varphi} - SR_1\cos\theta -\right.$$

$$-\frac{\partial(S\nu)}{\partial\theta} - \frac{\partial}{\partial\theta}(H\sin\theta) - H\cos\theta - H\frac{R_1}{\nu}\cos\theta\sin\theta -$$

$$\left.-\frac{\sin\theta}{\nu}\frac{\partial}{\partial\theta}(H\nu) - \frac{\sin\theta}{\nu}R_1\frac{\partial M_2}{\partial\varphi} + L_1R_1\sin\theta - E_2R_1\nu\right]\delta v +$$

$$+\left[-\frac{\partial}{\partial\theta}\left(\frac{\partial H}{\partial\varphi} + \frac{1}{R_1}\frac{\partial(M_1\nu)}{\partial\theta} - M_2\cos\theta - L_2\nu\right) -\right.$$

$$-\frac{\partial}{\partial\varphi}\left(\frac{1}{\nu}\frac{\partial}{\partial\theta}(H\nu) + H\frac{R_1\cos\theta}{\nu} + \frac{R_1}{\nu}\frac{\partial M_2}{\partial\varphi} + L_1R_1\right) +$$

$$\left.\left.+T_1\nu + T_2R_1\sin\theta - E_nR_1\nu\right]\delta w\right\}\frac{d\Sigma_0}{R_1\nu} + \int_{\Gamma_1}\left\{(-T_1 + T_1^{(1)})\delta u +\right.$$

$$+\left[-\left(S + \frac{2H}{R_2}\right) + \left(S_{12}^{(1)} + \frac{H_{12}^{(1)}}{R_2}\right)\right]\delta v + \left[-\frac{2}{\nu}\frac{\partial H}{\partial\varphi} + \frac{M_2\cos\theta}{\nu} -\right.$$

$$\left.\left.-\frac{1}{R_1\nu}\frac{\partial(M_1\nu)}{\partial\theta} - L_2 + \left(Q_1^{(1)} + \frac{1}{\nu}\frac{\partial H_{12}^{(1)}}{\partial\varphi}\right)\right]\delta w + (M_1 - M_1^{(1)})\delta\theta_1\right\}\nu\,d\varphi -$$

$$-\int_{\Gamma_2}\left\{(-T_1 + T_1^{(2)})\delta u + \left[-\left(S + \frac{2H}{R_2}\right) + \left(S_{12}^{(2)} + \frac{H_{12}^{(2)}}{R_2}\right)\right]\delta v +\right. \qquad (6.13)$$

$$+\left[\left(-\frac{2}{\nu}\frac{\partial H}{\partial\varphi} + \frac{M_2\cos\theta}{\nu} - \frac{1}{R_1\nu}\frac{\partial(M_1\nu)}{\partial\theta} - L_2\right) +\right.$$

$$\left.\left.+\left(Q_1^{(2)} + \frac{1}{\nu}\frac{\partial H_{12}^{(2)}}{\partial\varphi}\right)\right]\delta w + (M_1 - M_1^{(2)})\delta\theta_1\right\}\nu\,d\varphi = 0.$$

The three $\delta u, \delta v, \delta w$ variations are independent throughout the domain Σ_0. On shell boundaries Γ_1, Γ_2 there are four independent variations $\delta u, \delta v, \delta w, \delta\vartheta_1$. Equation (6.13) can be identically satisfied only when corresponding factors for independent variations in surface and contour integrals turn into zero. In this case the subintegral expression of the surface integral supplies the already known equilibrium equations (4.22), and the contour integrals supply boundary conditions. Rewrite the contour integrals taking into account formulas (4.23). Then we will have

$$\int_{\Gamma_i} \Big\{(-T_1 + T_1^{(i)})\delta u + \left[-\left(S + \frac{2H}{R_2}\right) + \left(S_{12}^{(i)} + \frac{H_{12}^{(i)}}{R_2}\right)\right]\delta v + \\ + \left[-\left(N_1 + \frac{1}{v}\frac{\partial H}{\partial \varphi}\right) + \left(Q_1^{(i)} + \frac{1}{v}\frac{\partial H_{12}^{(i)}}{\partial \varphi}\right)\right]\delta w + \\ + (M_1 - M_1^{(i)})\delta\vartheta_1\Big\} v\,d\varphi, \qquad i = 1, 2. \tag{6.14}$$

If shell edges Γ_1, Γ_2 can freely move (quantities $\delta u, \delta v, \delta w, \delta\vartheta_1$ are independent and can assume arbitrary values), then for vanishing contour integrals (6.14) execution of the following conditions on Γ_i is necessary:

$$\left.\begin{aligned} &T_1 = T_1^{(i)}, \quad \left(S + \frac{2H}{R_2}\right) = S_{12}^{(i)} + \frac{H_{12}^{(i)}}{R_2}, \\ &N_1 + \frac{1}{v}\frac{\partial H}{\partial \varphi} = Q_1^{(i)} + \frac{1}{v}\frac{\partial H_{12}^{(i)}}{\partial \varphi}, \quad M_1 = M_1^{(i)}. \end{aligned}\right\} \tag{6.15}$$

In the same case, when on one or both edges displacements

$$u = u^{(i)}, \quad v = v^{(i)}, \quad w = w^{(i)}, \quad \vartheta_1 = \vartheta_1^{(i)}, \tag{6.16}$$

are assigned, corresponding variations turn into zero and contour integrals (6.14) cancel out. We can imagine also combined boundary conditions when on one edge partially displacements, partially force are assigned, for example:

$$\left.\begin{aligned} &u = u^{(i)}, \quad w = w^{(i)}, \\ &S + \frac{2H}{R_2} = S_{12}^{(i)} + \frac{H_{12}^{(i)}}{R_2}, \quad M_1 = M_1^{(i)} \end{aligned}\right\} \tag{6.17}$$

etc. One ought to note that although the stressed state of the edge is characterized by five static quantities $T_1^{(i)}, S_{12}^{(i)}, Q_1^{(i)}, M_1^{(i)}, H_{12}^{(i)}$, torsional moment $H_{12}^{(i)}$ is in (6.15) only in combination with shearing force and tangential force and correspondingly the number of boundary conventions is not five, but four. A lack of an independent boundary condition for H_{12} means that on the section of the boundary $\Gamma_i(\varphi_0, \varphi_1)$ system of forces

$$\left.\begin{aligned}\int_{\varphi_0}^{\varphi_1} K_1^{(i)} \nu\, d\varphi &= \int_{\varphi_0}^{\varphi_1} (T_1^{(i)}\tau_1 + S_{12}^{(i)}\tau_2 + Q_1^{(i)} n)\, \nu\, d\varphi, \\ \int_{\varphi_0}^{\varphi_1} M_1^{(i)} \nu\, d\varphi &= \int_{\varphi_0}^{\varphi_1} (M_1^{(i)}\tau_2 - H_{12}^{(i)}\tau_1)\, \nu\, d\varphi\end{aligned}\right\} \qquad (6.18)$$

is replaced by a statically equivalent system consisting of

$$\int_{\varphi_0}^{\varphi_1} K_1^{(i)*} \nu\, d\varphi = \int_{\varphi_0}^{\varphi_1} \left[T_1^{(i)}\tau_1 + \left(S_{12}^{(i)} + \frac{H_{12}^{(i)}}{R_2}\right)\tau_2 + \left(Q_1^{(i)} + \frac{1}{\nu}\frac{\partial H_{12}^{(i)}}{\partial\varphi}\right) n\right] \nu\, d\varphi,$$

$$\int_{\varphi_0}^{\varphi_1} M_1^{(i)*} \nu\, d\varphi = \int_{\varphi_0}^{\varphi_1} M_1^{(i)}\tau_2 \nu\, d\varphi \qquad (6.19)$$

two concentrated forces on the ends of the interval (φ_0, φ_1), directed along the normal to the shell and equal in magnitude to $[H_{12}^{(i)}]_{\varphi_0}$, $[-H_{12}^{(i)}]_{\varphi_1}$ (Fig 8). The static equivalency of both systems is checked easily directly. First let us note that

$$\int_{\varphi_0}^{\varphi_1} K_1^{(i)*} \nu\, d\varphi = \int_{\varphi_0}^{\varphi_1} \left[K_1^{(i)} + \frac{H_{12}^{(i)}}{R_2}\tau_2 + \frac{1}{\nu}\frac{\partial H_{12}^{(i)}}{\partial\varphi} n\right] \nu\, d\varphi =$$

$$= \int_{\varphi_0}^{\varphi_1} \left[K_1^{(i)} + \frac{H_{12}^{(i)}}{R_2}\tau_2 + \frac{1}{\nu}\frac{\partial}{\partial\varphi}(H_{12}^{(i)} n) - \frac{H_{12}^{(i)}}{\nu}\frac{\partial n}{\partial\varphi}\right] \nu\, d\varphi.$$

and taking into account formulas (1.20) and (1.24), we obtain derive

$$\int_{\varphi_0}^{\varphi_1} K_1^{(i)*} \nu\, d\varphi = \int_{\varphi_0}^{\varphi_1} \left[K_1^{(i)} + \frac{1}{\nu}\frac{\partial}{\partial\varphi}(H_{12}^{(i)} n)\right] \nu\, d\varphi. \qquad (6.20)$$

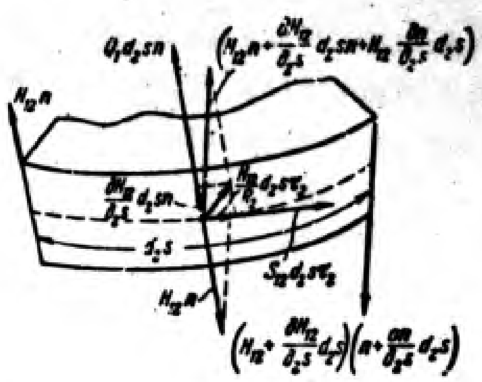


Fig. 8. System of forces and moments on the edge of a shell.

then

$$\int_{\varphi_0}^{\varphi_1} K_1^{(l)*}\nu\,d\varphi + [H_{12}^{(l)}n]_{\varphi_1} - [H_{12}^{(l)}n]_{\varphi_0} = \int_{\varphi_0}^{\varphi_1} K_1^{(l)}\nu\,d\varphi. \qquad (6.21)$$

$$\int_{\varphi_0}^{\varphi_1} r\times K_1^{(l)*}\nu\,d\varphi + \int_{\varphi_0}^{\varphi_1} M_1^{(l)*}\nu\,d\varphi + [r\times H_{12}^{(l)}n]_{\varphi_1} - [r\times H_{12}^{(l)}n]_{\varphi_0} =$$
$$= \int_{\varphi_0}^{\varphi_1} r\times K_1^{(l)}\nu\,d\varphi + \int_{\varphi_0}^{\varphi_1} r\times \frac{\partial(H_{12}^{(l)}n)}{\partial\varphi}\,d\varphi + \int_{\varphi_0}^{\varphi_1} (M_1^{(l)} + H_{12}^{(l)}\tau_1)\nu\,d\varphi +$$
$$+ [r\times H_{12}^{(l)}n]_{\varphi_1} - [r\times H_{12}^{(l)}n]_{\varphi_0}. \qquad (6.22)$$

Calculating the second integral of the right side of (6.22) with using the integration by parts and taking into consideration that $\frac{1}{\nu}\frac{\partial r}{\partial\varphi} = \tau_2$ and $\tau_2\times n = \tau_1$, we obtain

$$\int_{\varphi_0}^{\varphi_1} [r\times K_1^{(l)*} + M_1^{(l)*}]\,\nu\,d\varphi + [r\times H_{12}^{(l)}n]_{\varphi_1} - [r\times H_{12}^{(l)}n]_{\varphi_0} =$$
$$= \int_{\varphi_0}^{\varphi_1} [r\times K_1^{(l)} + M_1^{(l)}]\,\nu\,d\varphi. \qquad (6.23)$$

The static equivalency of the systems is proved. With a complete pass around of the contour Γ, $\varphi_1 = \varphi_0 + 2\pi$ and, because of the periodicity of the stressed state, there exists the equality

$$[H_{12}^{(l)}n]_{\varphi_0} = [H_{12}^{(l)}n]_{\varphi_1}.$$

Therefore on the basis of (6.21), (6.23) we can write

$$\left.\begin{aligned}\int\limits_{\Gamma_i} K_1^{(i)*}\nu\,d\varphi &= \int\limits_{\Gamma_i} K_1^{(i)}\nu\,d\varphi,\\ \int\limits_{\Gamma_i}\left[r\times K_1^{(i)*}+M_1^{(i)*}\right]\nu\,d\varphi &= \int\limits_{\Gamma_i}\left[r\times K_1^{(i)}+M_1^{(i)}\right]\nu\,d\varphi.\end{aligned}\right\} \quad (6.24)$$

§ 7. Static-Geometric Analogy. Stress Functions

System of equilibrium equations (4.22), (4.23) connects six static quantities $(T_1, T_2, S, M_1, M_2, H)$ and components of load E_1, E_2, E_n, L_1, L_2. In the absence of a distributed external load the corresponding uniform system of equations has

$$\left.\begin{aligned}&L_1(T_1, T_2, S, M_1, M_2, H)\equiv\frac{\partial}{\partial\theta}(\nu T_1)-R_1T_2\cos\theta+\\&\quad+R_1\frac{\partial S}{\partial\varphi}+2\frac{\partial H}{\partial\varphi}+\frac{1}{R_1}\frac{\partial(\nu M_1)}{\partial\theta}-M_2\cos\theta=0,\\&L_2(T_1, T_2, S, M_1, M_2, H)\equiv R_1\frac{\partial T_2}{\partial\varphi}+\frac{\partial}{\partial\theta}(\nu S)+SR_1\cos\theta+\\&\quad+2H\cos\theta+2\sin\theta\frac{\partial H}{\partial\theta}+2H\frac{R_1\cos\theta}{R_2}+\frac{R_1}{R_2}\frac{\partial M_2}{\partial\varphi}=0,\\&L_3(T_1, T_2, S, M_1, M_2, H)\equiv\frac{\partial}{\partial\theta}\left[\frac{1}{R_1}\frac{\partial(\nu M_1)}{\partial\theta}-M_2\cos\theta+\frac{\partial H}{\partial\varphi}\right]+\\&\quad+\frac{\partial}{\partial\varphi}\left[\frac{\partial H}{\partial\varphi}+\frac{2HR_1\cos\theta}{\nu}+\frac{R_1}{\nu}\frac{\partial M_2}{\partial\varphi}\right]-\nu T_1-R_1\sin\theta T_2=0.\end{aligned}\right\} \quad (7.1)$$

Comparing (7.1) with equations (3.30), it is easy to see that the left sides of the equations of continuity and uniform equations of statics contain the same differential operators L_1, L_2, L_3 and the quantities

$$\left.\begin{matrix}(T_1, & \varkappa_2), & (2H, & \gamma),\\(T_2, & \varkappa_1), & (M_1, & -\varepsilon_2),\\(S, & -\tau), & (M_2, & -\varepsilon_1)\end{matrix}\right\} \quad (7.2)$$

are in these equations identically. This fact bears name static-geometric analogy. In a new form of notation the equations of continuity (3.30) look thus:

$$L_i\left(\varkappa_2, \varkappa_1, -\tau, -\varepsilon_2, -\varepsilon_1, \frac{\gamma}{2}\right)=0, \quad i=1, 2, 3. \quad (7.3)$$

By the very meaning of the equations of continuity it is clear that the components of deformation, while expressed through displacements u, v and w using relationships (3.13), satisfy these equations identically. But then from static-geometric analogy it follows that static quantities $(T_1, T_2, S, M_1, M_2, H)$ also can be expressed through three stress functions A, B, C using formulas analogous to (3.13):

$$\left.\begin{aligned}
T_1 &= \varkappa_2(A, B, C) = -\frac{1}{\nu}\frac{\partial}{\partial\varphi}\left(\frac{1}{\nu}\frac{\partial C}{\partial\varphi} - \frac{B\sin\theta}{\nu}\right) - \\
&\qquad - \frac{\cos\theta}{\nu}\left(\frac{1}{R_1}\frac{\partial C}{\partial\theta} - \frac{A}{R_1}\right), \\
T_2 &= \varkappa_1(A, B, C) = -\frac{1}{R_1}\frac{\partial}{\partial\theta}\left(\frac{1}{R_1}\frac{\partial C}{\partial\theta} - \frac{A}{R_1}\right), \\
S &= -\tau(A, B, C) = \frac{1}{R_1}\frac{\partial}{\partial\theta}\left(\frac{1}{\nu}\frac{\partial C}{\partial\varphi} - \frac{B\sin\theta}{\nu}\right) - \\
&\qquad - \frac{1}{R_1}\left(\frac{1}{\nu}\frac{\partial A}{\partial\varphi} - \frac{B\cos\theta}{\nu}\right), \\
M_2 &= -\varepsilon_1(A, B, C) = -\frac{1}{R_1}\frac{\partial A}{\partial\theta} - \frac{C}{R_1}, \\
M_1 &= -\varepsilon_2(A, B, C) = -\frac{1}{\nu}\frac{\partial B}{\partial\varphi} - \frac{1}{\nu}(A\cos\theta + C\sin\theta), \\
H &= \frac{1}{2}\gamma(A, B, C) = \frac{1}{2R_1}\frac{\partial B}{\partial\theta} + \frac{1}{2\nu}\frac{\partial A}{\partial\varphi} - \frac{B\cos\theta}{\nu},
\end{aligned}\right\} \qquad (7.4)$$

and in this case uniform equations of statics (7.1) will be identically satisfied. Functions of stress were introduced into shell theory by A. I. Lur'ye [13] and A. I. Gol'denveyser [6], while initially there were four stress functions, using which the expressions were written out for the eight quantities $T_1, T_2, S_{12}, S_{21}, M_1, M_2, H_{12}, H_{21}$, which satisfied uniform equilibrium in the form of (4.18). However, during the transition to generalized forces $T_1, T_2, S, \ldots, H$ and the corresponding replacements of shear forces Q_1, Q_2 by the quantities N_1, N_2 the fourth function disappeared. In (7.4) there are only three functions A, B, C. It is easy to see that the combinations of forces $S + \frac{2H}{R_2}$ and $N_1 + \frac{1}{\nu}\frac{\partial H}{\partial\varphi}$, which are necessary for boundary conditions (6.15) are expressed also through the three functions A, B, C. Really, using the first relationship of (4.23) and expressions (7.4), after the simplifications which are involved in executing differentiation with respect to coordinate θ in expressions $\frac{\partial}{\partial\theta}\left(\frac{1}{\nu}\frac{\partial C}{\partial\varphi}\right)$, $\frac{\partial}{\partial\theta}\left(\frac{B\sin\theta}{\nu}\right)$, $\frac{\partial}{\partial\theta}(\nu M_1)$ and the others, with the aid of formulas (1.33), (1.24), (1.37), we obtain

$$\left.\begin{aligned} S+\frac{2H}{R_2}&=\frac{1}{R_1\nu}\frac{\partial}{\partial\varphi}\left(\frac{\partial C}{\partial\theta}-A\right)-\frac{\cos\theta}{\nu^2}\frac{\partial C}{\partial\varphi}+\frac{\sin\theta}{\nu^2}\frac{\partial A}{\partial\varphi},\\ N_1+\frac{1}{\nu}\frac{\partial H}{\partial\varphi}&=-\frac{\sin\theta}{\nu R_1}\left(\frac{\partial C}{\partial\theta}-A\right)+\frac{1}{\nu^2}\frac{\partial^2 A}{\partial\varphi^2}-\frac{\cos\theta}{\nu^2}\frac{\partial B}{\partial\varphi}.\end{aligned}\right\} \qquad (7.5)$$

After the aforesaid two ways of solving the uniform problem are outlined naturally. The first way involves the substitution into equilibrium equations (7.1) of expressions for generalized forces ($T_1, T_2, S, \ldots, H$ through components of deformation, i.e., using elasticity relationships (5.17) and the subsequent expression of deformation components through displacements. As a result we obtain a system of three equations of equilibrium containing the Poisson coefficient

$$L_i^{\bullet}(u, v, w, \mu)=0, \qquad i=1, 2, 3. \qquad (7.6)$$

The second way consists of using expressions for generalized forces through stress functions. In this case, as already was said above, the equations of equilibrium will be satisfied, and it is necessary to care only about execution of relationships of continuity of deformations (7.3). Actually, reversing relationships of elasticit (3.17), i.e., expressing the components of deformation through generalized forces

$$\left.\begin{aligned} \varepsilon_1&=\frac{1}{Eh}(T_1-\mu T_2), & \varkappa_1&=\frac{12}{Eh^3}(M_1-\mu M_2),\\ \varepsilon_2&=\frac{1}{Eh}(T_2-\mu T_1), & \varkappa_2&=\frac{12}{Eh^3}(M_2-\mu M_1),\\ \frac{1}{2}\gamma&=\frac{(1+\mu)}{Eh}S, & \tau&=\frac{12}{Eh^3}(1+\mu)H.\end{aligned}\right\} \qquad (7.7)$$

we see that the components of deformation also can be expressed through functions of stress using formulas (7.4) and there remains only to subbordinate them to equations of compatibility (7.3), which leads to three differential equations in functions A, B, C:

$$L_i^{\bullet}(A, B, C, -\mu)=0, \qquad i=1, 2, 3. \qquad (7.8)$$

of completely the same form as (7.6), with the only difference that instead of μ they contain precisely the same $-\mu$. The change of sign of μ becomes obvious if we compare equations (5.17) and (7.7). If on the shell edges we have the preassigned displacements

$$u^{(j)},\ v^{(j)},\ w^{(j)},\ \vartheta_1^{(j)}=\left[\frac{1}{R_1}\left(\frac{\partial w}{\partial \theta}-u\right)\right]^{(j)},\qquad j=1,\ 2. \tag{7.9}$$

i.e., on both edges of θ_j displacements u, v, w and angle of rotation ϑ_1 are known functions of coordinate ϕ, then it is necessary to find a solution of equations (7.6) which would satisfy these conditions. Let us assume that such a solution is obtained. Then it is simple to understand that the solution the problem in displacements under conditions (7.9), after replacing μ by $(-\mu)$, is also the solution of equations (7.8) at values of stress functions assigned on the edges of the shell

$$A^{(j)},\ B^{(j)},\ C^{(j)},\ \left[\frac{1}{R_1}\left(\frac{\partial C}{\partial \theta}-A\right)\right]^{(j)},\qquad j=1,\ 2. \tag{7.10}$$

while (7.9) and (7.10) are the same functions ϕ. But assigning conditions (7.10) is equivalent to the imposition of conditions on the static quantities

$$T_1^{(j)},\ M_1^{(j)},\ \left(S+\frac{2H}{R_2}\right)^{(j)} \text{ and } \left(N_1+\frac{1}{v}\frac{\partial H}{\partial \varphi}\right)^{(j)} \tag{7.11}$$

on the edges of the shell; really, from (7.4), (7.5) it is immediately evident that if A, B, C and $\frac{\partial C}{\partial \theta}-A$ on the edges assigned functions of ϕ, then the renumerated combinations of static quantities also are known functions of ϕ. In this way, in the absence of distributed loads the problems of calculating a shell with assigned displacements on the edges (conditions (7.9)) and assigned forces (7.10) on the edges are mathematically equivalent.

CHAPTER II

EQUILIBRIUM OF AN ELASTIC SHELL OF REVOLUTION SUBJECTED TO AXISYMMETRIC AND BENDING LOADS

§ 8. Method of Separation of Variables

Let us examine a shell of resolution closed in circular direction. The stressed state, deformations and displacements in such a shell should be periodic functions of angular coordinate ϕ. We will look for a solution of system (6.1) in the form of trigonometric series

$$\left.\begin{aligned}
T_1 &= T_{1(0)} + \sum_{k=1}^{\infty}\left(T_{1(k)}\cos k\varphi + T_1^{(k)}\sin k\varphi\right),\\
T_2 &= T_{2(0)} + \sum_{k=1}^{\infty}\left(T_{2(k)}\cos k\varphi + T_2^{(k)}\sin k\varphi\right),\\
S &= \sum_{k=0}^{\infty}\left(S_{(k)}\sin k\varphi + S^{(k)}\cos k\varphi\right),\\
N_1 &= \sum_{k=0}^{\infty}\left(N_{1(k)}\cos k\varphi + N_1^{(k)}\sin k\varphi\right),\\
N_2 &= \sum_{k=0}^{\infty}\left(N_{2(k)}\sin k\varphi + N_2^{(k)}\cos k\varphi\right),\\
M_1 &= \sum_{k=0}^{\infty}\left(M_{1(k)}\cos k\varphi + M_1^{(k)}\sin k\varphi\right),\\
M_2 &= \sum_{k=0}^{\infty}\left(M_{2(k)}\cos k\varphi + M_2^{(k)}\sin k\varphi\right),\\
H &= \sum_{k=0}^{\infty}\left(H_{(k)}\sin k\varphi + H^{(k)}\cos k\varphi\right).
\end{aligned}\right\} \quad (8.1)$$

$$\left.\begin{aligned}
\varepsilon_1 &= \sum_{k=0}^{\infty}\left(\varepsilon_{1(k)}\cos k\varphi + \varepsilon_1^{(k)}\sin k\varphi\right),\\
\varepsilon_2 &= \sum_{k=0}^{\infty}\left(\varepsilon_{2(k)}\cos k\varphi + \varepsilon_2^{(k)}\sin k\varphi\right),\\
\gamma &= \sum_{k=0}^{\infty}\left(\gamma_{(k)}\sin k\varphi + \gamma^{(k)}\cos k\varphi\right),\\
\varkappa_1 &= \sum_{k=0}^{\infty}\left(\varkappa_{1(k)}\cos k\varphi + \varkappa_1^{(k)}\sin k\varphi\right),\\
\varkappa_2 &= \sum_{k=0}^{\infty}\left(\varkappa_{2(k)}\cos k\varphi + \varkappa_2^{(k)}\sin k\varphi\right),\\
\tau &= \sum_{k=0}^{\infty}\left(\tau_{(k)}\sin k\varphi + \tau^{(k)}\cos k\varphi\right),\\
\vartheta_1 &= \sum_{k=0}^{\infty}\left(\vartheta_{1(k)}\cos k\varphi + \vartheta_1^{(k)}\sin k\varphi\right),\\
\vartheta_2 &= \sum_{k=0}^{\infty}\left(\vartheta_{2(k)}\sin k\varphi + \vartheta_2^{(k)}\cos k\varphi\right),
\end{aligned}\right\} \quad (8.2)$$

$$\left.\begin{aligned}
u &= \sum_{k=0}^{\infty}\left(u_{(k)}\cos k\varphi + u^{(k)}\sin k\varphi\right),\\
v &= \sum_{k=0}^{\infty}\left(v_{(k)}\sin k\varphi + v^{(k)}\cos k\varphi\right),\\
w &= \sum_{k=0}^{\infty}\left(w_{(k)}\cos k\varphi + w^{(k)}\sin k\varphi\right).
\end{aligned}\right\} \quad (8.3)$$

Coefficients of series (8.1)-(8.3) are the subject attribute functions of coordinate θ. It is natural to assume that a solution in such form can be found if the load on the shell and boundary conditions contain functions also representable by Fourier series. And namely, components of the distributed load can be represented by an expansion of the form

$$\left.\begin{aligned}
q_1(\theta, \varphi) &= \sum_{k=0}^{\infty}\left(q_{1(k)}\cos k\varphi + q_1^{(k)}\sin k\varphi\right),\\
q_2(\theta, \varphi) &= \sum_{k=0}^{\infty}\left(q_{2(k)}\sin k\varphi + q_2^{(k)}\cos k\varphi\right),\\
q_n(\theta, \varphi) &= \sum_{k=0}^{\infty}\left(q_{n(k)}\cos k\varphi + q_n^{(k)}\sin k\varphi\right),
\end{aligned}\right\} \quad (8.4)$$

where $q_{1(k)}, q_{2(k)}, q_{n(k)}, q_1^{(k)}, q_2^{(k)}, q_n^{(k)}$ are known functions of coordinate θ, and on the edges $\theta = \theta_i$ $(i = 1, 2)$ are preassigned displacements $u^{(i)}, v^{(i)}, w^{(i)}, \vartheta_1^{(i)}$ or force of $T_1^{(i)}, \left(S + \frac{2H}{R_2}\right)^{(i)}, \left(N_1 + \frac{1}{\nu}\frac{\partial H}{\partial \varphi}\right)^{(i)}, M_1^{(i)}$, or their combinations, also representable in the form

$$\left.\begin{aligned}
u^{(l)} &= \sum_{k=0}^{\infty}\left(u^{l}_{(k)}\cos k\varphi + u^{(k)l}\sin k\varphi\right),\\
v^{(l)} &= \sum_{k=0}^{\infty}\left(v^{l}_{(k)}\sin k\varphi + v^{(k)l}\cos k\varphi\right),\\
w^{(l)} &= \sum_{k=0}^{\infty}\left(w^{l}_{(k)}\cos k\varphi + w^{(k)l}\sin k\varphi\right),\\
\vartheta_1^{(l)} &= \sum_{k=0}^{\infty}\left(\vartheta^{l}_{1(k)}\cos k\varphi + \vartheta_1^{(k)l}\sin k\varphi\right),
\end{aligned}\right\} \qquad (8.5)$$

$$\left.\begin{aligned}
T_1^{(l)} &= \sum_{k=0}^{\infty}\left(T^{l}_{1(k)}\cos k\varphi + T_1^{(k)l}\sin k\varphi\right),\\
\left(S+\frac{2H}{R_2}\right)^{(l)} &= \\
&= \sum_{k=0}^{\infty}\left[\left(S_{(k)}+\frac{2H_{(k)}}{R_2}\right)^{l}\sin k\varphi + \left(S^{(k)}+\frac{2H^{(k)}}{R_2}\right)^{l}\cos k\varphi\right],\\
M_1^{(l)} &= \sum_{k=0}^{\infty}\left(M^{l}_{1(k)}\cos k\varphi + M_1^{(k)l}\sin k\varphi\right),\\
\left(N_1+\frac{1}{\nu}\frac{\partial H}{\partial\varphi}\right)^{(l)} &= \\
&= \sum_{k=0}^{\infty}\left[\left(N_{1(k)}+\frac{kH_{(k)}}{\nu}\right)^{l}\cos k\varphi + \left(N_1^{(k)}-\frac{kH^{(k)}}{\nu}\right)^{l}\sin k\varphi\right],
\end{aligned}\right\} \qquad (8.6)$$

where $u^{l}_{(k)}$, $u^{(k)l}$ and the remaining series of coefficients are assigned numbers.

Then the total system of differential equations, which describe the equilibrium of a shell of revolution closed in the circular direction falls into an infinite number of systems of equations in usual derivatives of the eighth degree each. On the bases of (8.5), (8.6) boundary conditions are decomposed. In this way the stressed state corresponding to the k-th harmonic of the form cos $k\phi$ (or sin $k\phi$) in the decomposition of load is looked for independently of the others by solving a system of usual differential equations of the eighth degree under assigned boundary conditions (four on each edge of the shell).

To the zero harmonic ($k = 0$) corresponds axisymmetric of deformation shell. The total system of equations which describe the axisymmetric deformation consists of equilibrium equations

$$\left.\begin{aligned}&\frac{d}{d\theta}(\nu T_{1(0)})-T_{2(0)}R_1\cos\theta+\nu N_{1(0)}+q_{1(0)}\nu R_1=0,\\&\frac{d}{d\theta}(\nu N_{1(0)})-T_{1(0)}\nu-T_{2(0)}R_1\sin\theta+q_{n(0)}\nu R_1=0,\\&N_{1(0)}\nu R_1=\frac{d}{d\theta}(\nu M_{1(0)})-M_{2(0)}R_1\cos\theta.\end{aligned}\right\}\quad(8.7)$$

$$\left.\begin{aligned}&\frac{d}{d\theta}\left[\nu\left(S^{(0)}+\frac{H^{(0)}}{R_2}\right)\right]+\left(S^{(0)}+\frac{H^{(0)}}{R_1}\right)R_1\cos\theta+\\&\qquad+N_2^{(0)}R_1\sin\theta+q_2^{(0)}\nu R_1=0,\\&N_2^{(0)}\nu R_1=\nu\frac{dH^{(0)}}{d\theta}+2H^{(0)}R_1\cos\theta.\end{aligned}\right\}\quad(8.8)$$

elasticity relationships

$$\left.\begin{aligned}&T_{1(0)}=B(\varepsilon_{1(0)}+\mu\varepsilon_{2(0)}),\quad M_{1(0)}=D(\varkappa_{1(0)}+\mu\varkappa_{2(0)}),\\&T_{2(0)}=B(\varepsilon_{2(0)}+\mu\varepsilon_{1(0)}),\quad M_{2(0)}=D(\varkappa_{2(0)}+\mu\varkappa_{1(0)}).\end{aligned}\right\}\quad(8.9)$$

$$S^{(0)}=B\frac{1-\mu}{2}\gamma^{(0)},\quad H^{(0)}=D(1-\mu)\tau^{(0)}\quad(8.10)$$

expressions connecting deformations and displacements,

$$\left.\begin{aligned}&\varepsilon_{1(0)}=\frac{1}{R_1}\left(\frac{du_{(0)}}{d\theta}+w_{(0)}\right),\\&\varepsilon_{2(0)}=\frac{1}{\nu}\left(u_{(0)}\cos\theta+w_{(0)}\sin\theta\right),\\&\varkappa_{1(0)}=-\frac{1}{R_1}\frac{d}{d\theta}\left(\frac{1}{R_1}\frac{dw_{(0)}}{d\theta}-\frac{u_{(0)}}{R_1}\right),\\&\varkappa_{2(0)}=-\frac{\cos\theta}{\nu R_1}\left(\frac{dw_{(0)}}{d\theta}-u_{(0)}\right),\end{aligned}\right\}\quad(8.11)$$

$$\left.\begin{aligned}&\gamma^{(0)}=\frac{1}{R_1}\frac{dv^{(0)}}{d\theta}-\frac{v^{(0)}\cos\theta}{\nu},\\&\tau^{0}=\frac{1}{R_1}\frac{d}{d\theta}\left(\frac{v^{(0)}\sin\theta}{\nu}\right)-\frac{v^{(0)}\cos\theta}{\nu R_1}.\end{aligned}\right\}\quad(8.12)$$

$$\vartheta_{1(0)}=\frac{1}{R_1}\left(\frac{dw_{(0)}}{d\theta}-u_{(0)}\right),\quad(8.13)$$

$$\vartheta_2^{(0)}=-\frac{v^{(0)}\sin\theta}{\nu}.\quad(8.14)$$

The six deformation components $\varepsilon_{1(0)}$, $\varepsilon_{2(0)}$, $\varkappa_{1(0)}$, $\varkappa_{2(0)}$, $\gamma^{(0)}$, $\tau^{(0)}$ are interconnected by three relationships of continuity, which turn into identities upon the substitution of expressions (8.11), (8.12):

$$\left.\begin{aligned}&\frac{d}{d\theta}(\nu\varkappa_{2(0)})-R_1\varkappa_{1(0)}\cos\theta-\frac{1}{R_1}\frac{d}{d\theta}(\varepsilon_{2(0)}\nu)+\varepsilon_{1(0)}\cos\theta=0,\\&\nu\varkappa_{2(0)}+\varkappa_{1(0)}R_1\sin\theta+\frac{d}{d\theta}\left[\frac{1}{R_1}\frac{d}{d\theta}(\nu\varepsilon_{2(0)})-\varepsilon_{1(0)}\cos\theta\right]=0,\end{aligned}\right\}\quad(8.15)$$

$$
-\frac{d}{d\theta}(\nu\tau^{(0)}) - R_1 \cos\theta\,\tau^{(0)} + \gamma^{(0)}\cos\theta + \frac{d\gamma^{(0)}}{d\theta}\sin\theta + \\
+ \frac{R_1}{\nu}\sin\theta\cos\theta\,\gamma^{(0)} = 0. \quad (8.16)
$$

The obtained system of equations should be integrated taking into account boundary conditions, which correspond to the null term of decompositions (8.5), (8.6). For example, at $\theta = \theta_i$ the following conditions should hold:

$$
\left.\begin{aligned} u_{(0)} &= u^i_{(0)}, & v^{(0)} &= v^{(0)\,i}, \\ w_{(0)} &= w^i_{(0)}, & \vartheta_{1(0)} &= \vartheta^i_{1(0)} \end{aligned}\right\} \quad (8.17)
$$

or

$$
\left.\begin{aligned} T_{1(0)} &= T^i_{1(0)}, & S^{(0)} + \frac{2H^{(0)}}{R_2} &= \left(S^{(0)} + \frac{2H^{(0)}}{R_2}\right)^i, \\ N_{1(0)} &= N^i_{1(0)}, & M_{1(0)} &= M^i_{1(0)}, \end{aligned}\right\} \quad (8.18)
$$

or four others, that are combinations of (8.17), (8.18). Ways of solving this system of equations, and also the problem of how the right sides of (8.17), (8.18) should appear in order that the formulation of the problem will be correct, will be considered in the following sections (§§ 9-14).

At $k \neq 0$ two independent systems of equations are obtained, in the first of which are quantities noted by a lower index (k), and in the second by an upper. For the lower index, i.e., for loads of the form

$$
q_{1(k)}\cos k\varphi, \quad q_{2(k)}\sin k\varphi, \quad q_{n(k)}\cos k\varphi, \quad (8.19)
$$

we have:

equilibrim of equations

$$
\left.\begin{aligned} &\frac{d}{d\vartheta}(\nu T_{1(k)}) - T_{2(k)}R_1\cos\theta \underset{(-)}{+} kR_1\left(S_{(k)} + \frac{H_{(k)}}{R_1}\right) + \\ &\qquad + \nu N_{1(k)} + q_{1(k)}\nu R_1 = 0, \\ &\frac{d}{d\vartheta}\left[\nu\left(S_{(k)} + \frac{H_{(k)}}{R_2}\right)\right] \underset{(+)}{-} kR_1 T_{2(k)} + \left(S_{(k)} + \frac{H_{(k)}}{R_1}\right)R_1\cos\theta + \\ &\qquad + N_{2(k)}R_1\sin\theta + q_{2(k)}\nu R_1 = 0, \end{aligned}\right\} \quad (8.20)
$$

$$\frac{d}{d\theta}(\nu N_{1(k)}) \overset{+}{(-)} kR_1N_{2(k)} - T_{1(k)}\nu - \\ - T_{2(k)}R_1\sin\theta + q_{n(k)}\nu R_1 = 0.$$

$$N_{1(k)}\nu R_1 = \frac{d}{d\theta}(\nu M_{1(k)}) - M_{2(k)}R_1\cos\theta \overset{+}{(-)} kR_1H_{(k)}.$$

$$N_{2(k)}\nu R_1 = \nu\frac{dH_{(k)}}{d\theta} + 2H_{(k)}R_1\cos\theta \overset{-}{(+)} kR_1M_{2(k)}. \qquad (8.20)\ \text{(cont.)}$$

relationship of elasticity

$$\begin{aligned} T_{1(k)} &= B(\varepsilon_{1(k)} + \mu\varepsilon_{2(k)}). & M_{1(k)} &= D(\varkappa_{1(k)} + \mu\varkappa_{2(k)}). \\ T_{2(k)} &= B(\varepsilon_{2(k)} + \mu\varepsilon_{1(k)}). & M_{2(k)} &= D(\varkappa_{2(k)} + \mu\varkappa_{1(k)}). \\ S_{(k)} &= B\frac{(1-\mu)}{2}\gamma_{(k)}. & H_{(k)} &= D(1-\mu)\tau_{(k)}. \end{aligned} \qquad (8.21)$$

equations connecting deformations with displacements

$$\varepsilon_{1(k)} = \frac{1}{R_1}\left(\frac{du_{(k)}}{d\theta} + w_{(k)}\right).$$

$$\varepsilon_{2(k)} = \overset{+}{(-)}\frac{kv_{(k)}}{\nu} + \frac{1}{\nu}(u_{(k)}\cos\theta + w_{(k)}\sin\theta).$$

$$\gamma_{(k)} = \frac{1}{R_1}\frac{dv_{(k)}}{d\theta} \overset{-}{(+)} \frac{ku_{(k)}}{\nu} - \frac{v_{(k)}\cos\theta}{\nu}.$$

$$\varkappa_{1(k)} = -\frac{1}{R_1}\frac{d}{d\theta}\left(\frac{1}{R_1}\frac{dw_{(k)}}{d\theta} - \frac{u_{(k)}}{R_1}\right).$$

$$\varkappa_{2(k)} = -\frac{\cos\theta}{\nu R_1}\left(\frac{dw_{(k)}}{d\theta} - u_{(k)}\right) + \frac{k^2}{\nu^2}w_{(k)} \overset{+}{(-)} \frac{1}{\nu^2}kv_{(k)}\sin\theta. \qquad (8.22)$$

$$\tau_{(k)} = \overset{+}{(-)}\frac{k}{R_1}\frac{d}{d\theta}\left(\frac{w_{(k)}}{\nu}\right) + \frac{1}{R_1}\frac{d}{d\theta}\left(\frac{v_{(k)}\sin\theta}{\nu}\right) \overset{-}{(+)} \frac{ku_{(k)}}{\nu R_1} - \\ - \frac{v_{(k)}\cos\theta}{\nu R_1} = \overset{+}{(-)}\frac{k}{\nu R_1}\frac{dw_{(k)}}{d\theta} \overset{-}{(+)} \frac{ku_{(k)}}{\nu R_1} - \\ - \frac{\cos\theta}{\nu^2}\left(\overset{+}{(-)}kw_{(k)} + v_{(k)}\sin\theta\right) + \frac{\sin\theta}{\nu R_1}\frac{dv_{(k)}}{d\theta}.$$

$$\begin{aligned} \vartheta_{1(k)} &= \frac{1}{R_1}\left(\frac{dw_{(k)}}{d\theta} - u_{(k)}\right). \\ \vartheta_{2(k)} &= \overset{-}{(+)}\frac{kw_{(k)}}{\nu} - \frac{v_{(k)}\sin\theta}{\nu} \end{aligned} \qquad (8.23)$$

equations of continuity

$$\left.\begin{aligned}
&\frac{d}{d\theta}(\nu\varkappa_{2(k)}) - R_1\varkappa_{1(k)}\cos\theta \underset{(+)}{-} R_1 k\tau_{(k)} \underset{(-)}{+} k\gamma_{(k)} - \\
&\qquad - \frac{1}{R_1}\frac{d(\nu\varepsilon_{2(k)})}{d\theta} + \varepsilon_{1(k)}\cos\theta = 0, \\
&\underset{(+)}{-} R_1 k\varkappa_{1(k)} - \frac{d(\nu\tau_{(k)})}{d\theta} - R_1\cos\theta\,\tau_{(k)} + \frac{d\gamma_{(k)}}{d\theta}\cos\theta + \\
&\qquad + \gamma_{(k)}\frac{R_1}{\nu}\sin\theta\cos\theta \underset{(-)}{+} k\,\frac{R_1\sin\theta}{\nu}\varepsilon_{1(k)} = 0, \\
&\nu\varkappa_{2(k)} + \varkappa_{1(k)}R_1\sin\theta + \frac{d}{d\theta}\left[\underset{(+)}{-}\frac{1}{2}k\gamma_{(k)} - \varepsilon_{1(k)}\cos\theta + \right. \\
&\left. + \frac{1}{R_1}\frac{d(\nu\varepsilon_{2(k)})}{d\theta}\right] \underset{(+)}{-} k\,\frac{1}{2}\frac{d\gamma_{(k)}}{d\theta} \underset{(+)}{-} k\gamma_{(k)}\frac{R_1\cos\theta}{\nu} - k^2\frac{R_1}{\nu}\varepsilon_{1(k)} = 0.
\end{aligned}\right\} \qquad (8.24)$$

Boundary conditions which are purely power or purely geometric have the form

$$\left.\begin{aligned}
&T_{1(k)} = T^l_{1(k)}, \quad S_{(k)} + \frac{H_{(k)}}{R_2} = \left(S_{(k)} + \frac{H_{(k)}}{R_2}\right)^l, \\
&N_{1(k)} \pm \frac{kH_{(k)}}{\nu} = \left(N_{1(k)} \pm \frac{kH_{(k)}}{\nu}\right)^l, \quad M_{1(k)} = M^l_{1(k)}.
\end{aligned}\right\} \qquad (8.25)$$

$$u_{(k)} = u^l_{(k)}, \quad v_{(k)} = v^l_{(k)}, \quad w_{(k)} = w^l_{(k)}, \quad \vartheta_{1(k)} = \vartheta^l_{1(k)} \quad (l = 1,\ 2). \qquad (8.26)$$

The corresponding system for a load of the form

$$q_1^{(k)}\sin k\varphi, \quad q_2^{(k)}\cos k\varphi, \quad q_n^{(k)}\sin k\varphi \qquad (8.27)$$

is obtained from equations (8.20)-(8.26) by raising the index (k) to the top and replacing (k) by ($-k$), which makes the signs of the individual terms change to the signs in parentheses. In each of these cases the number of equations of the obtained system can be decreased in two ways. The first way involves excluding from the equations of equilibrium the quantities $N_{1(k)}, N_{2(k)}$ (or $N_1^{(k)}, N_2^{(k)}$) and expressing forces and moments with elasticity of relationships and equations (8.22) through displacements, writing equations of equilibrium in displacements, i.e., obtaining three equations in three unknown functions $u_{(k)}, v_{(k)}, w_{(k)}$ (or $u^{(k)}, v^{(k)}, w^{(k)}$). The second way is solving the problem in forces and moments. The basic unknowns are six static quantites $T_{1(k)}, T_{2(k)}, S_{(k)}, M_{1(k)}, M_{2(k)}, H_{(k)}$ (or $T_1^{(k)}, T_2^{(k)}, S^{(k)}, M_1^{(k)}, M_2^{(k)}, H^{(k)}$), determination of which requires six equations. The first group of three equations is obtained from equilibrium of (8.20) after excluding $N_{1(k)}, N_{2(k)}$ $(N_1^{(k)}, N_2^{(k)})$: the second group of three equations can be obtained by writing equations of continuity (8.24) in forces and moments, which is easily done with elasticity of relationships.

In both cases a system of usual differential equations of the eighth degree is obtained in rather complex form. However, at $k = 0.1$ there is no reason to solve the problem in these ways, since it turns out that in these cases the degree of system of equilibrium of equations (8.7), (8.20) and continuity of equations (8.15) and (8.24) can be lowered, whereupon obtaining resolvent equations is facilitated by introducing special unknown functions [49], [50], [12], [42], [44], [21]. Decreasing the order of the system of differential equations by m units is possible if m first integrals of the system can be found. The first integrals of the system of equilibrium equations have a simple mechanical sense: they are the conditions of equilibrium of the finite part of the shell contained between extreme section θ_i ($i = 0$ or 1) and flow section θ. At $k = 0$ the load on the shell and the condition of fastening are axisymmetric. For the finite part of the shell contained between extreme section θ_i and flow section θ, two conditions of statics should be executed: 1) the sum of the projections of all interval force onto axis of revolution OZ should be equal to zero, 2) the sum of the moments of all forces causing twisting around axis OZ also should be equal to zero. External forces refer to all loads applied to the shell and its edge θ_i; the internal refer to forces T_1, S_{12}, Q_1 and moments M_1, H_{12} acting in section $\theta = \text{const}$.

Loads of the form

$$q_{1(1)} \cos\varphi, \quad q_{2(1)} \sin\varphi, \quad q_{n(1)} \cos\varphi, \tag{8.28}$$

$$q_1^{(1)} \sin\varphi, \quad q_2^{(1)} \cos\varphi, \quad q_n^{(1)} \sin\varphi \tag{8.29}$$

subsequently will be called bending, since they tend to bend the shell like a beam. Load (8.28) give rise to a bend in plane $\phi = 0$ and a distribution of forces and moments symmetric in this plane. Load (8.29) give rise to a bend of the shell in the plane perpendicular to $\theta = 0$, and stress distribution of antisymmetric in this plane. Considering the bend of a bounded shell element (θ_i, θ), which is under a load symmetric in $\phi = 0$, we arrive at the following two conditions of statics: 1) the sum of projections of all internal and external forces onto axis OX should be zero, 2) the sum of moments of all forces relative to axis OY should be zero. Analogously in the case of a bend in the plane perpendicular to $\phi = 0$, we obtain two equilibrium conditions, expressed by the equation of forces in projections onto axis OY and by the equation of moments in projections onto axis OX.

Thus, in all three examined cases ($k = 0$ and two cases that correspond to two types of load at $k = 1$) there are two finite relationships which connect unknown static quantities T_1, S_{12}, N_1, M_1, H_{12} and assigned load components which are the first integrals of equilibrium equations (8.7), (8.8) and (8.20). Because of the static-geometric analogy we can verify that as soon as there is a first integral of the system of equilibrium equations, then there must also be a corresponding integral of the continuity of equations. In this way, in the problem about axisymmetric of shell deformation ($k = 0$) and the problem of bend ($k = 1$) it is possible to decrease the degree of the basic system of equations to four units: at $k \geq 2$ as the internal forces, as also the external load in any section of the shell form a self-balanced system, i.e., a system of forces statically equivalent to zero. In accordance with this such an external load ($k \geq 2$) we will call "self-balanced." In contrast to this the load represented by terms of series (8.4), corresponding to $k = 0.1$, we call "nonself-balanced."

At $k \geq 2$ there is not one first integral of the system of equilibrium equations, since the equations of equilibrium of any part of the shell contained between two segments θ_i, θ are automatically executed because of steadiness of a load in any segment.

§ 9. System of Basic Equations in Canonic Form

The total system of equations describing the deformation of a shell under load (8.19) consists of fifteen equations: three differential equilibrium of equations (8.20), six finite relationships (8.21) and six differential equations (8.22), and contain a corresponding number of unknowns: six static quantities, six components of deformation and three displacements. Above we dealt with methods of reducing it to a system containing only displacements or only forces and moments. However, we can convert system (8.20) to (8.22) in another way, selecting as the basic unknowns four static and four geometric quantites. In this case the system can be represented in the form of eight differential equations of the first degree – brought to "canonical form" [38], [37].

From considerations of convenience of the integration of the system by numerical methods for the basic unknowns are usually combinations of the forces which figure in the boundary conditions, namely:

$$T_{1(k)},\quad S^{\bullet}_{(k)} = S_{(k)} + \frac{2H_{(k)}}{R_2},\quad N^{\bullet}_{(k)} = N_{1(k)} \pm \frac{kH_{(k)}}{v},\quad M_{1(k)}, \tag{9.1}$$

and displacements

$$u_{(k)},\quad v_{(k)},\quad w_{(k)},\quad \vartheta_{1(k)}. \tag{9.2}$$

The system is written in canonical form in Table 1.

§ 10. Twisting the Shell

It is easy to see that system of equations (8.7)-(8.16), describing the equilibrium of an axisymmetrically loaded shell of revolution, breaks into two individual systems, while the first contain quantities with a lower zero index, and the second contains quantities with an upper index. They correspond to two different forms of shell deformation: bending with elongation (first system) and twisting (second system). The problem of the twisting of a thin-walled shell of revolution is solved elementary. Reall , the first equation of (8.8) after excluding from it $N_2^{(0)}$ (subsequently within the limits of this section the sign (0) is everywhere omitted) and using formula (1.24) can be written thus:

$$\frac{d}{d\theta}\left[v\left(S + \frac{H}{R_2}\right)\right] + \left(S + \frac{H}{R_2}\right)R_1\cos\theta + H\cos\theta +$$
$$+ \sin\theta\frac{dH}{d\theta} + H\frac{R_1\cos\theta}{R_2} + q_2 v R_1 = 0. \tag{10.1}$$

Table 1.

	$X_1=u_{(k)}$	$X_2=v_{(k)}$	$X_3=w_{(k)}$	$X_4=\theta_{1(k)}$	$X_5=T_{1(k)}$	$X_6=S^*_{(k)}$	$X_7=N^*_{(k)}$	$X_8=M_{1(k)}$	$a_{1(k)}$	$a_{2(k)}$	$a_{3(k)}$
$\frac{dX_1}{d\theta}=$	$-\frac{\mu R_1\cos\theta}{\nu}$	$\mp k\frac{\mu R_1}{\nu}$	$-\left(1+\mu\frac{R_1}{R_2}\right)$	0	$\frac{R_1}{B}$	0	0	0	0	0	0
$\frac{dX_2}{d\theta}=$	$\pm k\frac{R_1}{\nu}$	$\frac{R_1\cos\theta}{\nu}$	$\pm k\frac{h^2}{3\nu^2}\frac{R_1\cos\theta}{R_2}$	$\mp k\frac{h^2}{3R_2\nu}R_1$	0	$\frac{2R_1}{B(1-\mu)}$	0	0	0	0	0
$\frac{dX_3}{d\theta}=$	1	0	0	R_1	0	0	0	0	0	0	0
$\frac{dX_4}{d\theta}=$	0	$\pm k\frac{\mu R_1\sin\theta}{\nu^2}$	$k^2\frac{\mu R_1}{\nu^2}$	$-\frac{\mu R_1\cos\theta}{\nu}$	0	0	0	$-\frac{R_1}{D}$	0	0	0
$\frac{dX_5}{d\theta}=$	$\frac{EhR_1}{\nu^2}\left(\cos^2\theta+\frac{2k^2}{1+\mu}\frac{h^2}{12R_2^2}\right)$	$\pm Ehk\frac{R_1\cos\theta}{\nu^3}$	$Eh\frac{R_1\cos\theta\sin\theta}{\nu^2}\times\left(1-\frac{2k^2}{1+\mu}\frac{h^2}{12\nu^2}\right)$	$2k^2D(1-\mu)\frac{R_1}{\nu^2R_2}$	$-(1-\mu)\times\frac{R_1\cos\theta}{\nu}$	$\mp k\frac{R_1}{\nu}$	-1	0	$-R_1$	0	0
$\frac{dX_6}{d\theta}=$	$\pm Ehk\frac{R_1\cos\theta}{\nu^2}$	$Ehk^2\frac{R_1}{\nu^2}$	$\pm\frac{EhkR_1}{\nu^2}\times\left(\sin\theta+\frac{h^2}{12}\frac{k^2}{R_2\nu}\right)$	$\mp\frac{Eh^3}{12}\frac{kR_1\cos\theta}{R_2\nu^2}$	$\pm k\frac{\mu R_1}{\nu}$	$-\frac{2R_1\cos\theta}{\nu}$	0	$\pm k\frac{\mu R_1}{R_2\nu}$	0	$-R_1$	0
$\frac{dX_7}{d\theta}=$	$\frac{EhR_1\cos\theta}{\nu R_2}\times\left[1-\frac{h^2}{12(1+\mu)}\frac{2k^2}{\nu^2}\right]$	$\pm Ehk\frac{R_1}{R_2\nu}\left(1+k^2\frac{h^2}{12\nu^2}\right)$	$Eh\frac{R_1}{R_2^2}\left[1+\frac{h^2k^4}{12\nu^2\sin^2\theta}+\frac{h^2}{12(1+\mu)}\frac{2k^2\cos^2\theta}{\nu^2}\right]$	$-\frac{Eh^3}{12}\frac{(3+\mu)}{(1+\mu)}\times k^2\frac{R_1\cos\theta}{\nu^3}$	$1+\mu\frac{R_1}{R_2}$	$\mp 4k\times\frac{R_1\cos\theta h^2}{R_2\cdot 12\nu^2}$	$-\frac{R_1\cos\theta}{\nu}$	$\mu\frac{k^2R_1}{\nu^2}$	0	0	$-R_1$
$\frac{dX_8}{d\theta}$	$-\frac{Eh^3}{12(1+\mu)}\frac{2k^2R_1}{\nu^2R_2}$	$\pm\frac{Eh^3}{12}\frac{kR_1\sin\theta\cos\theta}{\nu^3}$	$\frac{Eh^3}{12}k^2\frac{(3+\mu)}{1+\mu}\times\frac{R_1\cos\theta}{\nu^3}$	$-\frac{Eh^3}{12}\frac{R_1}{\nu^2}\times\left(\cos^2\theta-\frac{2k^2}{1+\mu}\right)$	0	$\mp 4k\times\frac{h^2}{12}\frac{R_1}{\nu R_2}$	R_1	$-\frac{(1-\mu)R_1\cos\theta}{\nu}$	0	0	0

or, taking into account (1.35),

$$\frac{d}{d\theta}\left[\nu^2\left(S+\frac{H}{R_2}\right)\right]+\frac{d}{d\theta}(\nu\sin\theta H)+q_2\nu^2R_1=0. \tag{10.2}$$

In this way, it is easily located the first integral of equations (8.8), which is the condition of equilibrium of moments of all forces twisting a section of the shell (θ_0, θ),

$$\nu^2\left(S+\frac{2H}{R_2}\right)+\int_{\theta_0}^{\theta} q_2\nu^2R_1\,d\theta=\nu_0^2\left(S+\frac{2H}{R_2}\right)_{\theta=\theta_0}. \tag{10.3}$$

In the right side will stand the constant of integration, while ν_0 is the radius of the parallel circle in section $\theta = \theta_0$ $\left(S+\frac{2H}{R_2}\right)_{\theta=\theta_0}$ — value of the combination from S and H, written in parentheses, at $\theta = \theta_0$.

If we attempt to express the combination $S+\frac{2H}{R_2}$ with (8.10) and (8.12) through ν displacement and its derivative, then it is simple to see that those terms of this expression which proceed from the applied twisting moment H will have the order h^2/R_2^2 in comparison with the basic components which proceed from S if the amount of the latter is taken as unity. Since in the creation of the general theory of thin shells an error in h/R_2 (h/R_1) in comparison with unity was allowed, then terms of order h^2/R_2^2 in the presence of terms of the array of unity all the more should be rejected. This is equivalent to the fact that in order to set

$$H\approx 0 \text{ and } S\approx S_{12}.$$

The second approximation equality follows from the first and formulas determining the magnitude of S (4.7). For the basic system of equilibrium of equations (8.8) this means the possibility of replacing it by one equation

$$\frac{d}{d\theta}(\nu S_{12})+S_{12}R_1\cos\theta+q_2\nu R_1=0. \tag{10.4}$$

Thus, with sufficient correctness, the condition of equilibrium of a finite part of the shell can be rewritten in the form

$$S\approx S_{12}=\frac{\nu_0^2}{\nu^2}(S_{12})_{\theta=\theta_0}-\frac{1}{\nu^2}\int_{\theta_0}^{\theta} q_2\nu^2R_1\,d\theta. \tag{10.5}$$

After attribute of amount of tangential force $S_{12} = S$ displacement v is governed from equation

$$\frac{1}{R_1}\frac{dv}{d\vartheta} - \frac{v\cos\theta}{\nu} = \frac{2(1+\mu)}{Eh}S, \qquad (10.6)$$

solving which we will have

$$v = C\nu + \frac{2(1+\mu)}{Eh}\nu\int_{\theta_0}^{\theta} S\frac{R_1}{\nu}\,d\theta, \qquad (10.7)$$

where C – new constant of integration. Moreover, if at $\theta = \theta_0$ $v = 0$, then $C = 0$.

§ 11. Axisymmetric Deformation

The solution to the problem about axisymmetric bending of a shell is considerably more intricate. To estimate the first integral of system of equilibrium equations (8.7) we rewrite them in projections onto the vertical and radial directions. Multiplying the first and second equations of (8.7) once by cos θ and sin θ respectively and combining, and the second time by -sin θ and cos θ and again combining, after regrouping terms we obtain

$$\left.\begin{aligned}&\frac{d}{d\theta}[\nu(-T_1\sin\theta + N_1\cos\theta)] + q_zR_1\nu = 0,\\&\frac{d}{d\theta}[\nu(T_1\cos\theta + N_1\sin\theta)] - T_2R_1 + q_eR_1\nu = 0,\\&\nu R_1N_1 = \frac{d}{d\theta}(\nu M_1) - M_2R_1\cos\theta,\end{aligned}\right\} \qquad (11.1)$$

where

$$\left.\begin{aligned}q_z &= q_n\cos\theta - q_1\sin\theta,\\ q_e &= q_n\sin\theta + q_1\cos\theta.\end{aligned}\right\} \qquad (11.2)$$

Integrating the first equation (11.1), we have

$$2\pi\nu(T_1\sin\theta - N_1\cos\theta) = 2\pi\int_{\theta_0}^{\theta} q_zR_1\nu\,d\theta + P_z^0. \qquad (11.3)$$

The left side of (11.3) is the resultant of system of internal forces in the current shell section θ, directed along the axis of revolution, P_z^0 – constant of integration, equal to axial force in edge section θ_0. Using the statics-geometric analogy, from (11.3) it is simple to obtain the first integral of continuity equations

(8.15). For this in (11.3) one ought to set $q_z = 0$, designate by another character the constant of integration and substitute

$$\left.\begin{array}{l} T_1 \text{ by } \varkappa_2, \\ \nu N_1 \text{ by } -\frac{1}{R_1}\left[\frac{d}{d\theta}(\nu\varepsilon_2) - \varepsilon_1 R_1 \cos\theta\right]. \end{array}\right\} \quad (11.4)$$

As a result of the indicated action we obtain

$$\nu\varkappa_2 \sin\theta + \frac{\cos\theta}{R_1}\left[\frac{d}{d\theta}(\nu\varepsilon_2) - \varepsilon_1 R_1 \cos\theta\right] = C_1. \quad (11.5)$$

Substituting into (11.5) expressions for deformation through displacements (8.11), we verify that the constant

$$C_1 = 0. \quad (11.6)$$

One more equation of continuity, corresponding (in the sense of the statics-geometric analogy) to the second equation of statics (11.1), has the form

$$\frac{d}{d\theta}\left\{\nu\cos\theta\varkappa_2 - \frac{\sin\theta}{R_1}\left[\frac{d(\nu\varepsilon_2)}{d\theta} - \varepsilon_1 R_1 \cos\theta\right]\right\} - \varkappa_1 R_1 = 0. \quad (11.7)$$

Rewriting (11.5) allowing for (11.6) and expressions (8.11), (8.13), we derive the equation

$$\frac{d}{d\theta}(\nu\varepsilon_2) - \varepsilon_1 R_1 \cos\theta - \vartheta_1 R_1 \sin\theta = 0. \quad (11.8)$$

Substituting into (11.7) expressions for curvatures $\varkappa_1$ and $\varkappa_2$ through angle of rotation ϑ_1

$$\varkappa_1 = -\frac{1}{R_1}\frac{d\vartheta_1}{d\theta}, \quad \varkappa_2 = -\frac{\cos\theta}{\nu}\vartheta_1, \quad (11.9)$$

it is simple to verify that it is fulfilled identically in force of (11.8). In this way (11.8) is the only condition which connects the quantities ε_1, ε_2, ϑ_1 during axisymmetric deformation.

Returning to the equations of statics – to the second equation of (11.1) and to equation (11.3) – it is easy to see that they will be identically satisfied if we express forces T_1, T_2, N_1 through stress function V in the following manner:

$$\left.\begin{array}{l} \nu T_1 = V\cos\theta + \Phi_1(\theta), \quad T_2 = \frac{1}{R_1}\frac{dV}{d\theta}, \\ \nu N_1 = V\sin\theta + \Phi_2(\theta). \end{array}\right\} \quad (11.10)$$

where

$$\left.\begin{aligned}\Phi_1(\theta) &= -\cos\theta\int_{\theta_0}^{\theta} q_z\nu R_1\,d\theta + \sin\theta\left(\frac{P_z^0}{2\pi} + \int_{\theta_0}^{\theta} q_z\nu R_1\,d\theta\right),\\ \Phi_2(\theta) &= -\sin\theta\int_{\theta_0}^{\theta} q_z\nu R_1\,d\theta - \cos\theta\left(\frac{P_z^0}{2\pi} + \int_{\theta_0}^{\theta} q_z\nu R_1\,d\theta\right).\end{aligned}\right\} \qquad (11.11)$$

Using (11.9), (11.10) and elasticity relationships (8.9) we can write the expressions for ε_1, ε_2 through stress function V, and bending moments M_1, M_2 express through function ϑ_1:

$$\left.\begin{aligned}\varepsilon_1 &= \frac{1}{Eh}\left[\frac{V\cos\theta}{\nu} - \frac{\mu}{R_1}\frac{dV}{d\theta} + \frac{\Phi_1(\theta)}{\nu}\right],\\ \varepsilon_2 &= \frac{1}{Eh}\left[\frac{1}{R_1}\frac{dV}{d\theta} - \mu\left(\frac{V\cos\theta}{\nu} + \frac{\Phi_1}{\nu}\right)\right].\end{aligned}\right\} \qquad (11.12)$$

$$\left.\begin{aligned}M_1 &= -\frac{Eh^3}{12(1-\mu^2)}\left(\frac{1}{R_1}\frac{d\vartheta_1}{d\theta} + \mu\frac{\cos\theta}{\nu}\vartheta_1\right),\\ M_2 &= -\frac{Eh^3}{12(1-\mu^2)}\left(\frac{\cos\theta}{\nu}\vartheta_1 + \frac{\mu}{R_1}\frac{d\vartheta_1}{d\theta}\right).\end{aligned}\right\} \qquad (11.13)$$

In this way (see equations (11.9)-(11.13)), all deformations of force and bending moments in the shell proved to be expressed through two basic unknown functions V and ϑ_1. Such a way of solving the problem was first shown by Meissner. He obtained two resolvent equations connecting the basic functions. Let us note that the force of function introducted by Meissner in the case of the heterogeneous problem (i.e., in the presence of distributed external loads and tensile force) differs from the function introduced in [12]. Here and subsequently we will use the Meissner-Lur'e function, since in this instance the resolvent equation is obtained with a simpler right side than in [49].

§ 12 Meissner Equations

Let us write out the Meissner equations for a shell of varying thickness, supposing that the thickness of the shell is an even function of coordinate θ, differentiable the necessary number of times. For this let us substitute expressions for M_1, M_2, N_1 and

ε_1, ε_2 through Meissner-Lur'e functions into the third equation of statics (11.1) as well as into equation (11.8); as a result we will obtain two equations in V and ϑ_1:

$$\left.\begin{aligned}&\frac{\nu}{R_1}\frac{d^2V}{d\theta^2}+\frac{dV}{d\theta}\left[\alpha\frac{d}{d\theta}\left(\frac{1}{\alpha}\frac{\nu}{R_1}\right)\right]+\left[-\mu\alpha\frac{d}{d\theta}\left(\frac{\cos\theta}{\alpha}\right)-\frac{R_1\cos^2\theta}{\nu}\right]V-\\&\qquad-Eh_0\vartheta_1\alpha R_1\sin\theta=\mu\alpha\frac{d}{d\theta}\left(\frac{\Phi_1}{\alpha}\right)+\frac{R_1\cos\theta}{\nu}\Phi_1.\\&\frac{\nu}{R_1}\frac{d^2\vartheta_1}{d\theta^2}+\frac{d\vartheta_1}{d\theta}\left[\frac{1}{\alpha^3}\frac{d}{d\theta}\left(\alpha^3\frac{\nu}{R_1}\right)\right]+\left[\frac{\mu}{\alpha^3}\frac{d}{d\theta}(\alpha^3\cos\theta)-\frac{R_1\cos^2\theta}{\nu}\right]\vartheta_1+\\&\qquad+\frac{R_1V\sin\theta}{D_0}\frac{1}{\alpha^3}=-\frac{R_1\Phi_2(\theta)}{D_0}\frac{1}{\alpha^3}.\end{aligned}\right\}\tag{12.1}$$

where the following designations have been introduced:

$$D_0=\frac{Eh_0^3}{12(1-\mu^2)},\quad \alpha=\frac{h(\theta)}{h_0}.\tag{12.2}$$

h_0 – thickness of wall in a certain characteristic section of the shell, for example in secion $\theta = \theta_0$

By changing variables

$$V=V_0\alpha^2b,\quad \vartheta_1=\frac{\Psi_0}{Eh_0},\tag{12.3}$$

where b is a certain characteristic linear of shell dimension, equations (12.1) are brought to the form

$$\left.\begin{aligned}&\frac{d^2V_0}{d\theta^2}+\left(\frac{3\alpha'}{\alpha}+\frac{R_1\cos\theta}{\nu}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)\frac{dV_0}{d\theta}+\\&+V_0\left[\frac{2\alpha''}{\alpha}+\frac{\alpha'}{\alpha}(2+\mu)\frac{R_1\cos\theta}{\nu}-\frac{\alpha'}{\alpha}\frac{2}{R_1}\frac{dR_1}{d\theta}+\frac{\mu R_1\sin\theta}{\nu}-\right.\\&\left.-\frac{R_1^2\cos^2\theta}{\nu^2}\right]-\Psi_0\frac{R_1^2}{b\nu}\frac{\sin\theta}{\alpha}=\frac{1}{b\alpha}\frac{R_1}{\nu}\left[\mu\frac{d}{d\theta}\left(\frac{\Phi_1}{\alpha}\right)+\frac{R_1\cos\theta}{\nu\alpha}\Phi_1\right],\\&\frac{d^2\Psi_0}{d\theta^2}+\left(\frac{3\alpha'}{\alpha}+\frac{R_1\cos\theta}{\nu}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)\frac{d\Psi_0}{d\theta}+\Psi_0\left[\frac{3\alpha'}{\alpha}\frac{R_1\cos\theta}{\nu}-\right.\\&\left.-\frac{\mu R_1\sin\theta}{\nu}-\frac{R_1^2\cos^2\theta}{\nu^2}\right]+4\gamma_0^4\frac{R_1^2\sin\theta}{b\alpha}\frac{V_0}{\nu}=-4\gamma_0^4\frac{R_1^2}{\nu b^2}\frac{\Phi_2}{\alpha^3}.\end{aligned}\right\}\tag{12.4}$$

where

$$4\gamma_0^4=\frac{12(1-\mu^2)b^2}{h_0^2}.\tag{12.5}$$

From equations (12.4) it is evident that the thickness of the shell should be at least a twice differentiable functions θ. For shell of constant thickness $h(\theta)=h_0=h,\ \alpha=1,\ \alpha'=\alpha''=0,\ 4\gamma_0^4=4\gamma^4$ and equations (12.4) are simplified. The assume the form

$$
\left.\begin{aligned}
&\frac{d^2V_0}{d\theta^2}+\left(\frac{R_1\cos\theta}{v}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)\frac{dV_0}{d\theta}+V_0\left(\frac{\mu R_1\sin\theta}{v}-\frac{R_1^2\cos^2\theta}{v^2}\right)-\\
&\qquad-\Psi_0\frac{R_1^2\sin\theta}{bv}=\frac{R_1}{vb}\left[\mu\frac{d\Phi_1}{d\theta}+\frac{R_1\cos\theta}{v}\Phi_1\right].\\
&\frac{d^2\Psi_0}{d\theta^2}+\left(\frac{R_1\cos\theta}{v}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)\frac{d\Psi_0}{d\theta}+\Psi_0\left(-\frac{\mu R_1\sin\theta}{v}-\frac{R_1^2\cos^2\theta}{v^2}\right)+\\
&\qquad+4\gamma^4\frac{R_1^2\sin\theta}{bv}V_0=-4\gamma^4\frac{R_1^2}{vb^2}\Phi_2.
\end{aligned}\right\}\tag{12.6}
$$

In this way, the calculation of an axisymmetrically loaded shell of revolution is reduced to the solution of a system of two differential equations in two variables V and Φ_1, through which are expressed all forces, moments and components of deformation in the shell.

Methods of solving the obtained systems of equations (12.4) or (12.6) will be considered below. Here we will be limited to the remark that for thin shells parameter $4\gamma^4$ (or $4\gamma_0^4$) is great in comparison with unity, and this fact essentially facilitates the creation of an approximation solution. Let us note that equations (12.6), as was to be expected, as a result of statics-geometric analogy have a symmetric form: operations which are accomplished in the left side of these equations on V_0 and Ψ_0 differ only by the sign on the Poisson coefficient μ. This allows replacing two equations in V_0 and Ψ_0 by one equation of the second degree in the imaginary function

$$
\sigma=\Psi_0-2i\gamma^2V_0. \tag{12.7}
$$

$$
\begin{aligned}
&\frac{d^2\sigma}{d\theta^2}+\left(\frac{R_1\cos\theta}{v}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)\frac{d\sigma}{d\theta}-\frac{\mu R_1\sin\theta}{v}\bar{\sigma}+\sigma\left(2i\gamma^2\frac{R_1^2\sin\theta}{bv}-\frac{R_1^2\cos^2\theta}{v^2}\right)=\\
&\quad=2\gamma^2\left[-\frac{R_1^2}{vb^2}2\gamma^2\Phi_2(0)-i\frac{R_1}{vb}\left(\mu\frac{d\Phi_1}{d\theta}+\frac{R_1\cos\theta}{v}\Phi_1\right)\right].
\end{aligned}\tag{12.8}
$$

where

$$
\bar{\sigma}=\Psi_0+2i\gamma^2V_0,
$$

i – imaginary unit.

Equations (12.4) for a shell of alternating thickness by the replacement

$$
\sigma=\Psi_0-2i\gamma_0^2V_0 \tag{12.9}
$$

also can be reduced to one equation:

$$\begin{aligned}
&\frac{d^2\sigma}{d\theta^2}+\left(\frac{3\alpha'}{\alpha}+\frac{R_1\cos\theta}{\nu}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)\frac{d\sigma}{d\theta}-\frac{\mu R_1\sin\theta}{\nu}\bar{\sigma}+\\
&\quad+i\left[\frac{2\alpha''}{\alpha}-\frac{\alpha'(1-\mu)}{\alpha}\frac{R_1\cos\theta}{\nu}-\frac{\alpha'}{\alpha}\frac{2}{R_1}\frac{dR_1}{d\theta}\right]\operatorname{Im}\sigma+\\
&\quad+\left(2i\gamma_0^2\frac{R_1^2\sin\theta}{b\nu\alpha}+\frac{3\alpha'}{\alpha}\frac{R_1\cos\theta}{\nu}-\frac{R_1^2\cos^2\theta}{\nu^2}\right)\sigma=\\
&=2\gamma_0^2\left\{-2\gamma_0^2\frac{R_1^2}{\nu b^2}\frac{\Phi_2(\theta)}{\alpha^3}-i\frac{R_1}{\nu b\alpha}\left[\mu\frac{d}{d\theta}\left(\frac{\Phi_1}{\alpha}\right)+\frac{R_1\cos\theta}{\nu\alpha}\Phi_1(\theta)\right]\right\}. \qquad (12.10)
\end{aligned}$$

We introduce expressions for forces and bending moments through the introduced function σ:

$$\left.\begin{aligned}
\nu T_1&=-\frac{\alpha^2 b}{2\gamma_0^2}\operatorname{Im}\sigma\cos\theta+\Phi_1(\theta),\\
\nu N_1&=-\frac{\alpha^2 b}{2\gamma_0^2}\operatorname{Im}\sigma\sin\theta+\Phi_2(\theta),\\
\nu H_e&=-\frac{\alpha^2 b}{2\gamma_0^2}\operatorname{Im}\sigma+\Phi_1\cos\theta+\Phi_2\sin\theta,\\
T_2&=-\frac{b}{R_1 2\gamma_0^2}\frac{d}{d\theta}(\alpha^2\operatorname{Im}\sigma),\\
M_1&=-\frac{1}{4\gamma_0^4}\alpha^3 b^2\left(\frac{1}{R_1}\operatorname{Re}\frac{d\sigma}{d\theta}+\frac{\mu\cos\theta}{\nu}\operatorname{Re}\sigma\right),\\
M_2&=-\frac{1}{4\gamma_0^4}\alpha^3 b^2\left(\frac{\cos\theta}{\nu}\operatorname{Re}\sigma+\frac{\mu}{R_1}\operatorname{Re}\frac{d\sigma}{d\theta}\right).
\end{aligned}\right\} \qquad (12.11)$$

The corresponding equations for a shell of constant thickness are derived if in (12.11) we set $\alpha = 1$. By itself, replacing a system of equations in two functions Ψ_0 and V_0 by one equation (12.8) or (12.10) bears a formal character and does not facilitate the solution of the system, since along with unknown σ in these equations $\bar{\sigma}$ and Im σ are present. However, if one takes into account that for a thin shell parameter $2\gamma_0^2$ is large and consider shells with a smoothly changing thickness such that

$$\alpha'\ll 2\gamma_0^2,\quad \alpha''\ll 2\gamma_0^2,$$

then equation (12.10) can be simplified by dropping the $1/2\gamma_0^2$ terms (or, which is the same, terms of order h_0/b) in comparison with unity. Then instead of (12.10) we will have

$$\begin{aligned}
&\frac{d^2\sigma}{d\theta^2}+\left(\frac{3\alpha'}{\alpha}+\frac{R_1\cos\theta}{\nu}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)\frac{d\sigma}{d\theta}+\left(2i\gamma_0^2\frac{R_1^2\sin\theta}{b\nu\alpha}-\frac{R_1^2\cos^2\theta}{\nu^2}\right)\sigma=\\
&=-4\gamma_0^4\left\{\frac{R_1^2}{\nu b^2}\frac{1}{\alpha^3}\Phi_2(\theta)+i\frac{1}{2\gamma_0^2}\frac{R_1}{\nu b\alpha}\left[\mu\frac{d}{d\theta}\left(\frac{\Phi_1}{\alpha}\right)+\frac{R_1\cos\theta}{\nu\alpha}\Phi_1\right]\right\}. \qquad (12.12)
\end{aligned}$$

In this case it was assumed that α nowhere turns into zero. Let us note that in the coefficient of σ in (12.12) the term $\frac{R_1^2 \cos^2\theta}{\nu^2}$ is preserved, having at $\nu >> 0$ a magnitude of order $1/2\gamma_0^2$ from the first term. For large ν it can be dropped. However, keeping this term for small ν is necessary, since at $\nu \to 0$ it increases as $1/\nu^2$. An analogous situation can arise even in the right side of (12.12), therefore until clarification of the concrete form of load functions $\Phi_1(\theta)$, $\Phi_2(\theta)$ one ought to keep both terms.

The basic resolvent equation of axisymmetric deformation for a shell of constant thickness is derived from (12.12) at $\alpha = 1$, $\alpha' = 0$ and has the form

$$\frac{d^2\sigma}{d\theta^2} + \frac{d\sigma}{d\theta}\left(\frac{R_1 \cos\theta}{\nu} - \frac{1}{R_1}\frac{dR_1}{d\theta}\right) + 2i\gamma^2\sigma\left(\frac{R_1^2 \sin\theta}{b\nu} + i\frac{R_1^2\cos^2\theta}{2\gamma^2\nu^2}\right) = \\ = -4\gamma^4\left\{\frac{R_1^2}{\nu b^2}\Phi_2 + i\frac{1}{2\gamma^2}\frac{R_1}{\nu b}\left[\mu\frac{d\Phi_1}{d\theta} + \frac{R_1\cos\theta}{\nu}\Phi_1\right]\right\}. \quad (12.13)$$

§ 13. Boundary Conditions. Determination of Displacements

Before passing to the formulation of boundary conditions for the introduced functions, let us note that the quantity N_1, determined by equation (4.20), in this case is equal to shearing force Q_1, since with a solution to the problem of bending it is assumed that there is no twisting and also $H_{21} = H_{12} = 0$. For the same reason $S_{12} = S_{21} = 0$, $v = 0$, and from four boundary conditions of form (6.15), (6.16) for every edge there remains only three conditions:

$$T_1 = T_1^i, \quad N_1 = Q_1^i, \quad M_1 = M_1^i \text{ for } \Gamma_i \quad (13.1)$$

or

$$u = u^i, \quad w = w^i, \quad \vartheta_1 = \vartheta_1^i \text{ for } \Gamma_i, \quad (13.2)$$

where Γ_i – parallel circle $\theta = \theta_i$ $(i = 0.1)$. One ought to note however, that the quantities T_1^i, Q_1^i cannot be designated at both edges of the shell arbitrarily. They should be such that the equilibrium of the shell in the whole would be fulfilled. In order to formulate boundary conditions, taking into consideration the equation of equilibrium of the shell as a whole body, we turn to consideration of contour integral (6.14). Setting in this case $v = 0$ and expressing δu and δw through variations of axial and radial displacements $\delta\Delta_z$ and $\delta\Delta_e$

$$\left.\begin{aligned}\delta u &= \cos\theta\,\delta\Delta_z - \sin\theta\,\delta\Delta_r,\\ \delta w &= \sin\theta\,\delta\Delta_z + \cos\theta\,\delta\Delta_r,\end{aligned}\right\} \tag{13.3}$$

we will write contour integral (6.14) in the form

$$\begin{aligned}\int_{\Gamma_i} \{[(-T_1\cos\theta - N_1\sin\theta) + (T_1^i\cos\theta_i + N_1^i\sin\theta_i)]\,\delta\Delta_r + \\ + [(T_1\sin\theta - N_1\cos\theta) + (-T_1^i\sin\theta_i + N_1^i\cos\theta_i)]\,\delta\Delta_z + \\ + (M_1 - M_1^i)\,\delta\theta_1\}\,v\,d\varphi = 0.\end{aligned} \tag{13.4}$$

Designating axial and radial force through T_z and H_e respectively

$$\left.\begin{aligned}T_z &= T_1\sin\theta - N_1\cos\theta,\\ H_r &= T_1\cos\theta + N_1\sin\theta\end{aligned}\right\} \tag{13.5}$$

and integrating over ϕ, instead of (13.4) we obtain

$$(-H_r + H_r^i)\,\delta\Delta_r + (M_1 - M_1^i)\,\delta\theta_1 = 0 \quad (i = 0.1). \tag{13.6}$$

at $i = 0.1$

Radial forces H_e^i and bending moments M_1^i form on each of the edges of the shell a self-balancing system of forces, i.e., a system with principal vector and principal moment equal to zero. Systems of external forces T_z^i and forces T_z in the shell acting on the edge ($i = 0.1$) should have the same principal vector, directed along the axis of the shell. Therefore $T_z = T_z^i$ on each of the edges and instead of (13.6) we have the condition

$$(-H_r + H_r^i)\,\delta\Delta_r + (T_z - T_z^i)\,\delta\Delta_z + (M_1 - M_1^i)\,\delta\theta_1 = 0 \tag{13.7}$$

Since variations $\delta\Delta_r$, $\delta\theta_1$ are independent and arbitrary, then conditions (13.7) can be fulfilled only in the case when

$$H_r = H_r^i, \quad M_1 = M_1^i \qquad (i = 0.1). \tag{13.8}$$

If on the edges are assigned radial displacement Δ_e^i and rotation of the normal when deformation ϑ_1^i, then $\delta\Delta_e=0$, $\delta\vartheta_1^i=0$ and (13.7) also is fulfilled. Boundary conditions in this instance have the form

$$\Delta_e=\Delta_e^i, \quad \vartheta_1=\vartheta_1^i. \tag{13.9}$$

Conditions (13.8) are purely static, and (13.9) purely geometric. Combinations are possible also, for example:

$$H_e=H_e^i, \quad \vartheta_1=\vartheta_1^i$$

or (13.10)

$$M_1=M_1^i, \quad \Delta_e=\Delta_e^i.$$

Let us give several illustrations of possible boundary conditions:

1) $H_e=0$, $M_1=0$ – free edge,
2) $\Delta_e=0$, $\vartheta_1=0$ – sealed edge,
3) $H_e=0$, $\vartheta_1=0$ – sliding seal,
4) $M_1=0$, $\Delta_e=0$ – edge has no radial displacement, but can be turned, etc.

The above variants of the boundary conditions should be written through Meissner functions V and ϑ_1. The expression for meridian bending moment M_1 through ϑ_1 has been already obtained and is given by equation (11.13). It remained to express radial force H_e and radial displacement Δ_e through these functions. On the basis of (13.5) and (11.10) we have

$$H_e=\frac{V}{\nu}-\frac{1}{\nu}\int_{\theta_e}^{\theta} q_e\nu R_1 d\theta. \tag{13.11}$$

Furthermore, from the second relationship of (8.11) follows

$$\Delta_e=u\cos\theta+w\sin\theta=\nu e_2. \tag{13.12}$$

Taking into account (11.12), from (13.12) we obtain the expression for radial displacement through function V:

$$\Delta_e=\frac{1}{Eh}\left[\frac{\nu}{R_1}\frac{dV}{d\theta}-\mu V\cos\theta-\mu\Phi_1(\theta)\right]. \tag{13.13}$$

Thus, to solve a system of differential equations of the fourth degree (12.4) or (12.6) there are four boundary conditions (two on each edge), expressed in functions V and ϑ_1. After solving this system under preassigned boundary conditions by the above-mentioned formulas all forces and bending moments in any section of shell can be found.

Let us stop now on the determination of displacements. Instead of direct determination of tangential and normal displacements (u and w) it is more simple to find their combinations – radial displacement Δ_e and axial Δ_z, where

$$\Delta_z = -u\sin\theta + w\cos\theta. \tag{13.14}$$

Radial displacement Δ_e already has been determined by formula (13.13). To estimate Δ_z we differentiate (13.14) with respect to θ we obtain

$$\frac{d\Delta_z}{d\theta} = -\left(\frac{du}{d\theta} + w\right)\sin\theta + \left(\frac{dw}{d\theta} - u\right)\cos\theta.$$

Remembering (8.11), (8.13), we will have

$$\frac{d\Delta_z}{d\theta} = -R_1\varepsilon_1\sin\theta + R_1\vartheta_1\cos\theta. \tag{13.15}$$

From (13.15) Δ_z is found with the aid of one quadrature

$$\Delta_z = \int_{\theta_0}^{\theta}(-R_1\varepsilon_1\sin\theta + R_1\vartheta_1\cos\theta)\,d\theta + \Delta_z(\theta_0) =$$

$$= \int_{\theta_0}^{\theta}\left\{-\frac{\sin\theta}{Eh}\left[\frac{V\cos\theta}{\nu} - \frac{\mu}{R_1}\frac{dV}{d\theta} + \frac{\Phi_1(\theta)}{\nu}\right] + \vartheta_1\cos\theta\right\}R_1\,d\theta + \Delta_z(\theta_0). \tag{13.16}$$

In this way, axial displacement in any section of shell is found to within the displacement of a solid $\Delta_z(\theta_0)$, which in section θ_0 can be designated arbitrarily. It is natural to set $\Delta_z(\theta_0) = 0$. Thus, as a result of integration of the basic system, consisting of equations (8.7), (8.9), (8.11), six arbitrary constants appear. Two of them, P_z^0 and $\Delta_z(\theta_0)$, have the sense of axial force acting in the extreme section of the shell, and displacement of the shell as a solid; four enter the solution of the system of Meissner equations and should be determined according to preassigned boundary conditions of form (13.8)-(13.10).

§ 14. Deformation of a Shell Under the Action of a Bending Load

Let us examine the deformation of a shell under the action of a bending load of form

$$q_1 = q_{1(1)} \cos\varphi, \quad q_2 = q_{2(1)} \sin\varphi, \quad q_n = q_{n(1)} \cos\varphi. \tag{14.1}$$

We call this bending load "symmetric," since it causes in the shell a stressed state symmetric relative to plane $\phi = 0$. Correspondingly a load of the form

$$q_1 = q_1^{(1)} \sin\varphi, \quad q_2 = q_2^{(1)} \cos\varphi, \quad q_n = q_n^{(1)} \sin\varphi \tag{14.2}$$

we call "antisymmetric," since the stressed state caused by it is antisymmetric (odd) relative to plane $\phi = 0$. Let us note that in literature to designate load (14.1) and (14.2) the term "wind load" is used also.

Let us write the system of basic equations for a symmetric bending load. Assuming in (8.17), (8.18), (8.19) $k = 1$, we obtain equations of equilibrium connecting the amplitudes of forces and moments:

$$\left.\begin{aligned}
&\frac{d}{d\theta}(\nu T_{1(1)}) - T_{2(1)} R_1 \cos\theta + R_1\left(S_{(1)} + \frac{H_{(1)}}{R_1}\right) + \\
&\qquad + \nu N_{1(1)} + q_{1(1)} R_1 \nu = 0, \\
&\frac{d}{d\theta}\left[\nu\left(S_{(1)} + \frac{H_{(1)}}{R_1}\right)\right] + R_1 \cos\theta\left(S_{(1)} + \frac{H_{(1)}}{R_1}\right) - R_1 T_{2(1)} + \\
&\qquad + N_{2(1)} R_1 \sin\theta + q_{2(1)} \nu R_1 = 0, \\
&\frac{d}{d\theta}(\nu N_{1(1)}) + R_1 N_{2(1)} - T_{1(1)} \nu - T_{2(1)} R_1 \sin\theta + \\
&\qquad + q_{n(1)} \nu R_1 = 0;
\end{aligned}\right\} \tag{14.3}$$

$$\left.\begin{aligned}
R_1 \nu N_{1(1)} &= \frac{d}{d\theta}(\nu M_{1(1)}) - M_{2(1)} R_1 \cos\theta + R_1 H_{(1)}, \\
R_1 \nu N_{2(1)} &= \nu \frac{dH_{(1)}}{d\theta} + 2H_{(1)} R_1 \cos\theta - R_1 M_{2(1)},
\end{aligned}\right\} \tag{14.4}$$

and three equations of compatibility, which should be satisfied by the amplitudes of the component of deformation:

$$\left.\begin{aligned}
&\frac{d}{d\theta}(\nu\varkappa_{2(1)}) - R_1\varkappa_{1(1)}\cos\theta - R_1\tau_{(1)} + \gamma_{(1)} - \\
&\qquad - \frac{1}{R_1}\frac{d(\nu\varepsilon_{2(1)})}{d\theta} + \varepsilon_{1(1)}\cos\theta = 0, \\
&-R_1\varkappa_{1(1)} - \frac{d(\nu\tau_{(1)})}{d\theta} - R_1\cos\theta\,\tau_{(1)} + \gamma_{(1)}\cos\theta + \frac{d\gamma_{(1)}}{d\theta}\sin\theta + \\
&\qquad + \gamma_{(1)}\frac{R_1}{\nu}\cos\theta\sin\theta + \frac{R_1\sin\theta}{\nu}\varepsilon_{1(1)} = 0, \\
&\nu\varkappa_{2(1)} + \varkappa_{1(1)}R_1\sin\theta + \frac{d}{d\theta}\left[-\frac{\gamma_{(1)}}{2} - \varepsilon_{1(1)}\cos\theta + \frac{1}{R_1}\frac{d(\nu\varepsilon_{2(1)})}{d\theta}\right] - \\
&\qquad - \frac{\gamma_{(1)}}{2} - \gamma_{(1)}\frac{R_1\cos\theta}{\nu} - \frac{R_1}{\nu}\varepsilon_{1(1)} = 0.
\end{aligned}\right\} \qquad (14.5)$$

To equations (14.3), (14.4), (14.5) it is necessary to associate elasticity relationships (8.21) and boundary conditions (8.25) or (8.26), setting in them k = 1 and selecting in formulas the primary sign of k.

After obtaining the solution of the written system for a symmetric bending load the solution for an antisymmetric load can be obtained from it by introducing and auxiliary variable instead of angular coordinate ϕ. Really, for load (14.1) let us assume that a solution has been constructed. It is necessary to find the solution of system (8.20), (8.21), (8.24), at k = 1 with an upper index describing a shell deformation under antisymmetric load (14.2) with either power (formulas (8.6))

$$\left.\begin{aligned}
&T_1^l = T_1^{(1)l}\sin\varphi, \quad S^l + \frac{2H^l}{R_2} = \left(S^{(1)l} + \frac{2H^{(1)l}}{R_2}\right)\cos\varphi, \\
&M_1^l = M_1^{(1)l}\sin\varphi, \quad N_1^l + \frac{1}{\nu}\frac{\partial H^l}{\partial\varphi} = \left(N_1^{(1)l} - \frac{H^{(1)l}}{\nu}\right)\sin\varphi
\end{aligned}\right\} \qquad (14.6)$$

or geometric boundary conditions

$$\left.\begin{aligned}
&u^l = u^{(1)l}\sin\varphi, \quad v^l = v^{(1)l}\cos\varphi, \\
&w^l = w^{(1)l}\sin\varphi, \quad \vartheta_1^l = \vartheta_1^{(1)l}\sin\varphi,
\end{aligned}\right\} \qquad (14.7)$$

where $T_1^{(1)l}$, $\left(S^{(1)l} + \frac{2H^{(1)l}}{R_2}\right)$, $M_1^{(1)l}$, $\left(N_1^{(1)l} - \frac{H^{(1)l}}{\nu}\right)$ or $u^{(1)l}$, $v^{(1)l}$, $w^{(1)l}$, $\vartheta_1^{(1)l}$ – preassigned numbers.

Substituting the variable

$$\varphi = \varphi^* - \frac{\pi}{2}, \qquad (14.8)$$

instead of load (14.2) we obtain a symmetric edge load (relative to plane $\phi^* = 0$) with components

$$q^*_{1(1)} = -q_1^{(1)}, \quad q^*_{2(1)} = q_2^{(1)}, \quad q^*_{n(1)} = -q_n^{(1)}. \tag{14.9}$$

Boundary conditions (14.6), (14.7) take the form

$$\left.\begin{aligned} &T_1^l = T^{*l}_{1(1)}\cos\varphi^*, \quad S^l + \frac{2H^l}{R_2} = \left(S^l_{(1)} + \frac{2H^l_{(1)}}{R_2}\right)^* \sin\varphi^*, \\ &M_1^l = M^{*l}_{1(1)}\cos\varphi^*, \quad N_1^l + \frac{1}{\nu}\frac{\partial H^l}{\partial\varphi^*} = \left(N^l_{1(1)} - \frac{H^l_{(1)}}{\nu}\right)^* \cos\varphi^*, \end{aligned}\right\} \tag{14.10}$$

$$\left.\begin{aligned} &u^l = u^{l*}_{(1)}\cos\varphi^*, \quad v^l = v^{l*}_{(1)}\sin\varphi^*, \\ &w^l = w^{l*}_{(1)}\cos\varphi^*, \quad \vartheta_1^l = \vartheta^{l*}_{1(1)}\cos\varphi^*, \end{aligned}\right\} \tag{14.11}$$

where

$$T^{*l}_{1(1)} = -T_1^{(1)l}, \quad \left(S^l_{(1)} + \frac{2H^l_{(1)}}{R_2}\right)^* = \left(S^{(1)l} + \frac{2H^{(1)l}}{R^2}\right),$$

$$M^{*l}_{1(1)} = -M_1^{(1)l}, \quad \left(N^l_{1(1)} - \frac{H^l_{(1)}}{\nu}\right) = -\left(N_1^{(1)l} - \frac{H^{(1)l}}{\nu}\right),$$

$$u^{l*}_{(1)} = -u^{(1)l}, \quad v^{l*}_{(1)} = v^{(1)l},$$

$$w^{l*}_{(1)} = -w^{(1)l}, \quad \vartheta^{l*}_{1(1)} = -\vartheta_1^{(1)l}.$$

Constructing for a load (14.9) under boundary conditions (14.10) or (14.11) the solution

$$\left.\begin{aligned} &T_1 = +T^*_{1(1)}\cos\varphi^*, \quad S = S^*_{(1)}\sin\varphi^*, \ldots, \\ &u = u^*_{(1)}\cos\varphi^*, \quad v = v^*_{(1)}\sin\varphi^*, \ldots \end{aligned}\right\} \tag{14.12}$$

and returning to the old variable ϕ, we obtain

$$\left.\begin{aligned} &T_1 = -T^*_{1(1)}\sin\varphi, \quad S = S^*_{(1)}\cos\varphi, \ldots, \\ &u = -u^*_{(1)}\sin\varphi, \quad v = v^*_{(1)}\cos\varphi, \ldots \end{aligned}\right\} \tag{14.13}$$

In this way

$$\left.\begin{aligned} &T_1^{(1)} = -T^*_{1(1)}, \quad S^{(1)} = S^*_{(1)}, \ldots, \\ &u^{(1)} = -u^*_{(1)}, \quad v^{(1)} = v^*_{(1)}, \ldots \end{aligned}\right\} \tag{14.14}$$

This allows subsequently being limited to analysis of only a symmetric bending load (14.1).

Let us return to system of equations (14.3)-(14.5). If from system (14.3), (14.4) we exclude the quantities $N_{1(1)}$ and $N_{2(1)}$ and write equations (14.5) with the aid of (8.21) through forces and moments, then we obtain a system of differential equations of the eighth degree, consisting of six equations and containing six unknowns $T_{1(1)}$, $T_{2(1)}$, $S_{(1)}$, $M_{1(1)}$, $M_{2(1)}$, $H_{(1)}$. As already was mentioned above, the degree of this system can be lowered to four units. To do this there is no need to write out the system in explicit form, since the decrease in degree is generated because of the search for first integrals of systems of equilibrium equations and equations of continuity separately.

§ 15. First Integrals of Equations of Equilibrium and Equations of Continuity

Let us turn to equations of equilibrium. Multiplying the first and third equations of (14.3) by (-cos θ) and (-sin θ) respectively and combining with the second, we obtain

$$\frac{d}{d\theta}\left[-\nu T_{1(1)}\cos\theta-\nu N_{1(1)}\sin\theta+\nu\left(S_{(1)}+\frac{H_{(1)}}{R_2}\right)\right]+$$
$$+(-q_{1(1)}\cos\theta+q_{2(1)}-q_{n(1)}\sin\theta)R_1\nu=0. \qquad (15.1)$$

whence it follows that the relationship

$$-\nu T_{1(1)}\cos\theta-\nu N_{1(1)}\sin\theta+\nu\left(S_{(1)}+\frac{H_{(1)}}{R_2}\right)+$$
$$+\int_{\theta_0}^{\theta}(-q_{1(1)}\cos\theta+q_{2(1)}-q_{n(1)}\sin\theta)R_1\nu\,d\theta=C_1 \qquad (15.2)$$

is the first integral of system (14.3), (14.4) (C_1 - constant of integration).

Equations of statics (14.3), (14.4) have one first integral. We exclude $M_{2(1)}$ and $N_{2(1)}$ from the third equation of (14.3) and equations (14.4). As a result we will have

$$\nu\cos\theta\frac{d}{d\theta}(\nu N_{1(1)})+R_1\nu N_{1(1)}-\frac{d}{d\theta}(\nu M_{1(1)})-R_1\sin^2\theta H_{(1)}+$$
$$+\cos\theta\frac{d(\nu H_{(1)})}{d\theta}-T_{1(1)}\nu^2\cos\theta-T_{2(1)}\nu R_1\sin\theta\cos\theta+q_{n(1)}\nu^2R_1\cos\theta=0. \qquad (15.3)$$

Excluding from (15.3) and the first equation of (14.3) force $T_{2(1)}$, we obtain

$$-\nu^2\cos\theta T_{1(1)}-\nu\sin\theta\frac{d}{d\theta}(\nu T_{1(1)})+R_1\nu N_{1(1)}-\nu^2 N_{1(1)}\sin\theta+$$
$$+\nu\cos\theta\frac{d}{d\theta}(\nu N_{1(1)})-\frac{d}{d\theta}(\nu M_{1(1)})-R_1\sin^2\theta H_{(1)}+\cos\theta\frac{d(\nu H_{(1)})}{d\theta}-$$
$$-\nu R_1\sin\theta\left(S_{(1)}+\frac{H_{(1)}}{R_1}\right)+\nu^2 R_1(q_{n(1)}\cos\theta-q_{1(1)}\sin\theta)=0. \quad (15.4)$$

In (15.4) regrouping terms with the aid of identities

$$\left.\begin{aligned}&-\nu^2\cos\theta T_{1(1)}-\nu\sin\theta\frac{d}{d\theta}(\nu T_{1(1)})=\\&\qquad=-\frac{d}{d\theta}(\nu^2\sin\theta T_{1(1)})+T_{1(1)}\nu R_1\sin\theta\cos\theta,\\&\nu\cos\theta\frac{d}{d\theta}(\nu N_{1(1)})-\nu^2 N_{1(1)}\sin\theta+R_1\nu N_1=\\&\qquad=\frac{d}{d\theta}(\nu^2\cos\theta N_{1(1)})+R_1\nu N_1\sin^2\theta,\end{aligned}\right\} \quad (15.5)$$

we have

$$\frac{d}{d\theta}(-\nu^2\sin\theta T_{1(1)}+\nu^2\cos\theta N_{1(1)}-\nu M_{1(1)}+\nu H_{(1)}\cos\theta)+$$
$$+R_1\sin\theta\left[\nu T_{1(1)}\cos\theta+\nu N_{1(1)}\sin\theta-\nu\left(S_{(1)}+\frac{H_{(1)}}{R_2}\right)\right]+$$
$$+\nu^2 R_1(q_{n(1)}\cos\theta-q_{1(1)}\sin\theta)=0. \quad (15.6)$$

From (15.6), taking into account (15.2), we obtain the desired relationship:

$$-\nu^2\sin\theta T_{1(1)}+\nu^2 N_{1(1)}\cos\theta-\nu M_{1(1)}+\nu H_{(1)}\cos\theta+$$
$$+\int_{\theta_0}^{\theta}R_1\sin\theta\left[\int_{\theta_0}^{\theta}(-q_{1(1)}\cos\theta+q_{2(1)}-q_{n(1)}\sin\theta)R_1\nu\,d\theta\right]d\theta+$$
$$+\int_{\theta_0}^{\theta}(q_{n(1)}\cos\theta-q_{1(1)}\sin\theta)R_1\nu^2\,d\theta=C_2+C_1\int_{\theta_0}^{\theta}R_1\sin\theta\,d\theta. \quad (15.7)$$

In order to establish the meaning of relationships (15.2) and (15.7) and determine the constants of integration C_1, C_2, let us compose the condition of equilibrium of a finite element of the shell contained between section θ_0 and flow section θ (Fig. 9). Let section θ_0 be applied to an external load, statically equivalent to force P_x and to moment M_y. In section θ = const on an element of arc $\nu d\varphi$ acts forces $K_1\nu d\varphi$ and moment $M_1\nu d\varphi$, equal to

$$\begin{aligned}K_1\nu d\varphi&=(T_1\tau_1+S_{12}\tau_2+Q_1 n)\nu d\varphi=\\&=(T_{1(1)}\cos\varphi\tau_1+S_{12(1)}\sin\varphi\tau_2+Q_{1(1)}\cos\varphi n)\nu d\varphi.\end{aligned} \quad (15.8)$$

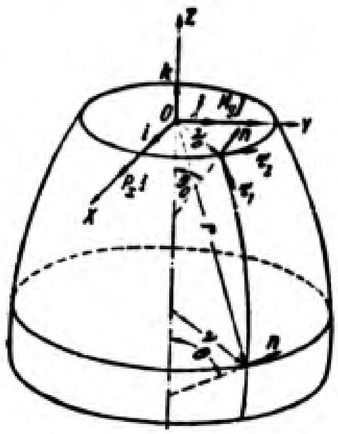


Fig. 9. System of external bending loads applied to edge θ_0.

$$M_1 \nu\, d\varphi = (M_1 \tau_2 - H_{12} \tau_1) = (M_{1(1)} \cos\varphi\, \tau_2 - H_{12(1)} \sin\varphi\, \tau_1)\, \nu\, d\varphi. \qquad (15.9)$$

Onto an area element of the middle surface of the shell $R_1 \nu d\phi d\theta$ is applied force

$$\begin{aligned} qR_1 \nu\, d\theta\, d\varphi &= \\ &= (q_1\tau_1 + q_2\tau_2 + q_n n)\, R_1 \nu\, d\theta\, d\varphi = \\ &= (q_{1(1)} \cos\varphi\, \tau_1 + q_{2(1)} \sin\varphi\, \tau_2 + \\ &+ q_{n(1)} \cos\varphi\, n)\, R_1 \nu\, d\theta\, d\varphi. \end{aligned} \qquad (15.10)$$

Using Table 2 of the cosines of angles between axes τ_1, τ_2, n and i, j, k vectors K_1, M_1, q are easily written in projections on to the X, Y, Z

$$\left.\begin{aligned} K_1 = K_{1x} i + K_{1y} j + K_{1z} k, \quad M_1 = M_{1x} i + M_{1y} j + M_{1z} k, \\ q = q_x i + q_y j + q_z k, \end{aligned}\right\} \qquad (15.11)$$

where

$$\begin{aligned} K_{1x} &= T_{1(1)} \cos\theta \cos^2\varphi - S_{12(1)} \sin^2\varphi + Q_{1(1)} \sin\theta \cos^2\varphi, \\ K_{1y} &= (T_{1(1)} \cos\theta + S_{12(1)} + Q_{1(1)} \sin\theta) \sin\varphi \cos\varphi, \\ K_{1z} &= (-T_{1(1)} \sin\theta + Q_{1(1)} \cos\theta) \cos\varphi, \\ M_{1x} &= (-M_{1(1)} - H_{12(1)} \cos\theta) \sin\varphi \cos\varphi, \\ M_{1y} &= M_{1(1)} \cos^2\varphi - H_{12(1)} \cos\theta \sin^2\varphi, \\ M_{1z} &= H_{12(1)} \sin\theta \sin\varphi, \\ q_x &= q_{1(1)} \cos\theta \cos^2\varphi - q_{2(1)} \sin^2\varphi + q_{n(1)} \sin\theta \cos^2\varphi, \\ q_y &= (q_{1(1)} \cos\theta + q_{2(1)} + q_{n(1)} \sin\theta) \sin\varphi \cos\varphi, \\ q_z &= (-q_{1(1)} \sin\theta + q_{n(1)} \cos\theta) \cos\varphi. \end{aligned}$$

Table 2.

	i	j	k
τ_1	$\cos\theta \cos\varphi$	$\cos\theta \sin\varphi$	$-\sin\theta$
τ_2	$-\sin\varphi$	$\cos\varphi$	0
n	$\sin\theta \cos\varphi$	$\sin\theta \sin\varphi$	$\cos\theta$

The radius vectors of a certain point M in section θ = const is equal to

$$r = OM = \nu\cos\varphi i + \nu\sin\varphi j + Zk, \tag{15.12}$$

while on the basis of (1.36)

$$Z = -\int_{\theta_0}^{\theta} R_1 \sin\theta \, d\theta. \tag{15.13}$$

In vectorial form the conditions of equilibrium of the chosen part of the shell has the form

$$\int_0^{2\pi} K_1 \nu \, d\varphi + P_x i + \int_{\theta_0}^{\theta}\int_0^{2\pi} q R_1 \nu \, d\theta \, d\varphi = 0. \tag{15.14}$$

$$\int_0^{2\pi} [M + (r \times K_1)] \nu \, d\varphi + M_y j + \int_{\theta_0}^{\theta}\int_0^{2\pi} (r \times q) R_1 \nu \, d\theta \, d\varphi = 0. \tag{15.15}$$

writing (15.14), (15.15) in projections onto the X, Y, Z axes with the aid of relationships (15.8)-(15.13) and integrating over ϕ, we obtain two equations, one of which is the equation of forces, and the other the equation of moments in projections onto the OX and OY axes respectively:

$$\nu(T_{1(1)}\cos\theta - S_{12(1)} + Q_{1(1)}\sin\theta) + \\ + \int_{\theta_0}^{\theta} (q_{1(1)}\cos\theta - q_{2(1)} + q_{n(1)}\sin\theta)\nu R_1 \, d\theta = -\frac{P_x}{\pi}. \tag{15.16}$$

$$\nu\pi(M_{1(1)} - H_{12(1)}\cos\theta + \nu T_{1(1)}\sin\theta - \nu Q_{1(1)}\cos\theta) + \\ + \pi\int_{\theta_0}^{\theta} (q_{1(1)}\cos\theta - q_{2(1)} + q_{n(1)}\sin\theta) Z\nu R_1 \, d\theta - \\ - \pi\int_{\theta_0}^{\theta} (-q_{1(1)}\sin\theta + q_{n(1)}\cos\theta)\nu^2 R_1 \, d\theta - \\ - \pi Z\int_{\theta_0}^{\theta} (q_{1(1)}\cos\theta - q_{2(1)} + q_{n(1)}\sin\theta)\nu R_1 \, d\theta + M_y - ZP_x = 0. \tag{15.17}$$

It is simple to verify that the four remaining equations are trivial equalities of the form 0 = 0 because every term individually vanishes. For example, projecting (15.14) onto axis OY, we have

$$\int_0^{2\pi} (T_{1(1)}\cos\theta + S_{12(1)} + Q_{1(1)}\sin\theta)\nu\sin\varphi\cos\varphi\, d\varphi +$$

$$+ \int_{\theta_0}^{\theta} \left[\int_0^{2\pi} (q_{1(1)}\cos\theta + q_{2(1)} + q_{n(1)}\sin\theta)\sin\varphi\cos\varphi\, d\varphi \right] R_1 \nu\, d\theta = 0.$$

In the second term of the left side of (15.17) integrating in parts and taking into account (15.13), we rewrite (15.17) in the form

$$\nu(M_{1(1)} - H_{12(1)}\cos\theta + \nu T_{1(1)}\sin\theta - \nu Q_{1(1)}\cos\theta) +$$
$$+ \int_{\theta_0}^{\theta} R_1 \sin\theta \left[\int_{\theta_0}^{\theta} (q_{1(1)}\cos\theta - q_{2(1)} + q_{n(1)}\sin\theta)\nu R_1\, d\theta \right] d\theta -$$
$$- \int_{\theta_0}^{\theta} (q_{n(1)}\cos\theta - q_{1(1)}\sin\theta)\nu^2 R_1\, d\theta + \frac{M_y}{\pi} + \frac{P_x}{\pi} \int_{\theta_0}^{\theta} R_1 \sin\theta\, d\theta = 0. \qquad (15.18)$$

With the aid of formulas (4.7), (4.8), (4.20) and relationships (8.1), we express the connection between amplitudes of forces $S_{12(1)}$, $Q_{1(1)}$, moment $H_{12(1)}$ and amplitudes of the corrected quantities $S_{(1)}$, $N_{1(1)}$ and $H_{(1)}$, then we obtain

$$\left.\begin{aligned} S_{12(1)} &= S_{(1)} + \frac{H_{21(1)}}{R_2}, \\ \nu Q_{1(1)} &= \nu N_{1(1)} + \frac{1}{2}(H_{21(1)} - H_{12(1)}), \\ H_{(1)} &= \frac{1}{2}(H_{21(1)} + H_{12(1)}) \end{aligned}\right\} \qquad (15.19)$$

consequently,

$$\left.\begin{aligned} -S_{12(1)} + Q_{1(1)}\sin\theta &= -S_{(1)} + N_{1(1)}\sin\theta - \frac{H_{(1)}}{R_2}, \\ H_{12(1)} + \nu Q_{1(1)} &= \nu N_{1(1)} + H_{(1)}. \end{aligned}\right\} \qquad (15.20)$$

Using (15.20), we write conditions (15.16), (15.18) in final form:

$$-\nu\left(T_{1(1)}\cos\theta + N_{1(1)}\sin\theta - S_{(1)} - \frac{H_{(1)}}{R_2}\right) +$$
$$+ \int_{\theta_0}^{\theta} (-q_{1(1)}\cos\theta + q_{2(1)} - q_{n(1)}\sin\theta)\nu R_1\, d\theta = \frac{P_x}{\pi}. \qquad (15.21)$$

$$-\nu M_{1(1)} - \nu^2 T_{1(1)}\sin\theta + \nu^2 N_{1(1)}\cos\theta + \nu H_{(1)}\cos\theta +$$
$$+ \int_{\theta_0}^{\theta} R_1 \sin\theta \left[\int_{\theta_0}^{\theta} (-q_{1(1)}\cos\theta + q_{2(1)} - q_{n(1)}\sin\theta) R_1 \nu\, d\theta \right] d\theta +$$
$$+ \int_{\theta_0}^{\theta} (q_{n(1)}\cos\theta - q_{1(1)}\sin\theta)\nu^2 R_1\, d\theta = \frac{M_y}{\pi} + \frac{P_x}{\pi} \int_{\theta_0}^{\theta} R_1 \sin\theta\, d\theta. \qquad (15.22)$$

Comparing (15.21), (15.22) and (15.2), (15.7), we observe that the latter are nothing else but the equilibrium conditions of a finite section of the shell, and constants of integration C_1, C_2 are equal to

$$C_1=\frac{P_x}{\pi},\quad C_2=\frac{M_y}{\pi}. \tag{15.23}$$

On the basis of the statics-geometric analogy it is now simple to write out two integrals of equations of continuity. For this in equations (15.2), (15.7) we set $q_{1(1)} = q_{2(1)} = q_{n(1)} = 0$, make the substitution

$$T_{1(1)}\to \varkappa_{2(1)},$$
$$S_{(1)}\to -\tau_{(1)},$$
$$H_{(1)}\to \frac{\gamma_{(1)}}{2},$$
$$M_{(1)}\to -\varepsilon_{2(1)},$$
$$N_{1(1)}\to \frac{1}{\nu R_1}\left[-\frac{d}{d\theta}(\nu\varepsilon_{2(1)})+\varepsilon_{1(1)}R_1\cos\theta+R_1\frac{\gamma_{(1)}}{2}\right]$$

and introduce new designations for the constants of integration; then we obtain

$$\nu(\varkappa_{2(1)}\cos\theta+\tau_{(1)})-\frac{\nu\sin\theta}{R_1}\frac{d\varepsilon_{2(1)}}{d\theta}-(\varepsilon_{2(1)}-\varepsilon_{1(1)})\sin\theta\cos\theta=-C_3. \tag{15.24}$$

$$-\frac{\nu\cos\theta}{R_1}\frac{d\varepsilon_{2(1)}}{d\theta}+\gamma_{(1)}\cos\theta+\varepsilon_{2(1)}\sin^2\theta+\varepsilon_{1(1)}\cos^2\theta-\varkappa_{2(1)}\nu\sin\theta=$$
$$=\frac{1}{\nu}\left(C_4+C_3\int_{\theta_0}^{\theta}R_1\sin\theta\,d\theta\right) \tag{15.25}$$

Substituting into (15.24), (15.25) expressions for deformations through displacements

$$\left.\begin{aligned}
\varepsilon_{1(1)}&=\frac{1}{R_1}\frac{du_{(1)}}{d\theta}+\frac{w_{(1)}}{R_1},\\
\varepsilon_{2(1)}&=\frac{1}{\nu}(v_{(1)}+u_{(1)}\cos\theta+w_{(1)}\sin\theta),\\
\gamma_{(1)}&=\frac{1}{R_1}\frac{dv_{(1)}}{d\theta}-\frac{1}{\nu}(u_{(1)}+v_{(1)}\cos\theta),\\
\varkappa_{1(1)}&=-\frac{1}{R_1}\frac{d}{d\theta}\left(\frac{1}{R_1}\frac{dw_{(1)}}{d\theta}-\frac{u_{(1)}}{R_1}\right),\\
\varkappa_{2(1)}&=-\frac{\cos\theta}{\nu R_1}\left(\frac{dw_{(1)}}{d\theta}-u_{(1)}\right)+\frac{1}{\nu^2}(w_{(1)}+v_{(1)}\sin\theta),\\
\tau_{(1)}&=\frac{1}{\nu}\left[\frac{1}{R_1}\frac{dw_{(1)}}{d\theta}-\frac{u_{(1)}}{R_1}-\frac{\cos\theta}{\nu}(w_{(1)}+v_{(1)}\sin\theta)+\frac{\sin\theta}{R_1}\frac{dv_{(1)}}{d\theta}\right]
\end{aligned}\right\} \tag{15.26}$$

taking into account the fact that equations of continuity should in this case be identically satisfied, we satisfy ourselves that

$$C_3 = C_4 = 0. \tag{15.27}$$

As a result of the search for four integrals of the basic system of differential equations its degree has dropped twice. Now instead of the original system of equations (14.3), (14.4), (14.5) it is possible to take a system from the obtained four first integrals and two equations from the original system, for example: the second equation of statics and second equation of continuity. In the second equation of statics it is useful to exclude the quantity $N_{2(1)}$ with the aid of the second equation of (14.4). Equation of (15.21), (15.22) we replace by two equivalent equations, the first of which is obtained after excluding from (15.21), (15.22) the quantity $N_{1(1)}$, and the second by excluding $T_{1(1)}$. Analogously we rewrite also the integrals of the continuity equations. As a result we obtain the system:

$$\begin{aligned} &\nu T_{1(1)} + M_{1(1)} \sin\theta - \nu S_{(1)} \cos\theta - 2H_{(1)} \sin\theta\cos\theta = \\ &= \nu f_1(P_x, M_y) + \nu f_0(q_{1(1)}, q_{2(1)}, q_{n(1)}) + \cos\theta \int_0^\theta q_{2(1)} \nu R_1 \, d\theta. \end{aligned} \tag{15.28}$$

$$\begin{aligned} &\frac{1}{R_1 \nu}\left[\frac{d}{d\theta}(\nu M_{1(1)}) - M_{2(1)} R_1 \cos\theta + R_1 H_{(1)}\right] - S_{(1)} \sin\theta - \\ &\qquad - \frac{M_{1(1)} \cos\theta}{\nu} + \frac{H_{(1)}}{\nu}(\cos^2\theta - \sin^2\theta) = \\ &= F_0(q_{1(1)}, q_{2(1)}, q_{n(1)}) + F_1(P_x, M_y) + \frac{\sin\theta}{\nu}\int_0^\theta q_{2(1)} \nu R_1 \, d\theta. \end{aligned} \tag{15.29}$$

$$\nu \varkappa_{2(1)} + \nu \tau_{(1)} \cos\theta - \varepsilon_{2(1)} \sin\theta - \gamma_{(1)} \sin\theta\cos\theta = 0. \tag{15.30}$$

$$\frac{1}{R_1}\frac{d\varepsilon_{2(1)}}{d\theta} - \tau_{(1)} \sin\theta - \frac{\varepsilon_{1(1)} \cos\theta}{\nu} - \frac{\gamma_{(1)} \cos^2\theta}{\nu} = 0. \tag{15.31}$$

$$\begin{aligned} &\nu \frac{dS_{(1)}}{d\theta} + 2S_{(1)} R_1 \cos\theta + 2\sin\theta \frac{dH_{(1)}}{d\theta} + 2H_{(1)} \cos\theta - R_1 T_{2(1)} + \\ &\quad + 2H_{(1)} \frac{R_1}{\nu} \sin\theta\cos\theta - \frac{R_1}{\nu} \sin\theta M_{2(1)} + q_{2(1)} \nu R_1 = 0. \end{aligned} \tag{15.32}$$

$$\begin{aligned} &-R_1 \varkappa_{1(1)} - \nu \frac{d\tau_{(1)}}{d\theta} - 2R_1 \cos\theta \tau_{(1)} + \gamma_{(1)} \cos\theta + \frac{d\gamma_{(1)}}{d\theta} \sin\theta + \\ &\qquad + \gamma_{(1)} \frac{R_1}{\nu} \sin\theta\cos\theta + \frac{R_1}{\nu} \sin\theta \varepsilon_{1(1)} = 0. \end{aligned} \tag{15.33}$$

$$\begin{aligned}
\nu f_4(P_x, M_y) &= -\frac{P_x}{\pi}\cos\theta - \frac{\sin\theta}{\nu}\left(\frac{M_y}{\pi} + \frac{P_x}{\pi}\int_{\theta_0}^{\theta} R_1 \sin\theta \, d\theta\right), \\
\nu f_0(q_{1(1)}, q_{2(1)}, q_{n(1)}) &= -\cos\theta \int_{\theta_0}^{\theta} (q_{1(1)}\cos\theta + q_{n(1)}\sin\theta)\nu R_1 \, d\theta + \\
&+ \frac{\sin\theta}{\nu}\int_{\theta_0}^{\theta} (q_{n(1)}\cos\theta - q_{1(1)}\sin\theta)\nu^2 R_1 \, d\theta - \\
&- \frac{\sin\theta}{\nu}\int_{\theta_0}^{\theta} R_1 \sin\theta \left[\int_{\theta_0}^{\theta} (q_{1(1)}\cos\theta - q_{2(1)} + q_{n(1)}\sin\theta) R_1 \nu \, d\theta\right] d\theta.
\end{aligned} \tag{15.34}$$

$$\begin{aligned}
F_1(P_x, M_y) &= -\frac{P_x}{\pi\nu}\sin\theta + \frac{\cos\theta}{\nu^2}\left(\frac{M_y}{\pi} + \frac{P_x}{\pi}\int_{\theta_0}^{\theta} R_1 \sin\theta \, d\theta\right), \\
F_0(q_{1(1)}, q_{2(1)}, q_{n(1)}) &= -\frac{\sin\theta}{\nu}\int_{\theta_0}^{\theta} (q_{1(1)}\cos\theta + q_{n(1)}\sin\theta)\nu R_1 \, d\theta - \\
&- \frac{\cos\theta}{\nu^2}\int_{\theta_0}^{\theta} (q_{n(1)}\cos\theta - q_{1(1)}\sin\theta)\nu^2 R_1 \, d\theta + \\
&+ \frac{\cos\theta}{\nu^2}\int_{\theta_0}^{\theta} R_1 \sin\theta \left[\int_{\theta_0}^{\theta} (q_{1(1)}\cos\theta - q_{2(1)} + q_{n(1)}\sin\theta) R_1 \nu \, d\theta\right] d\theta.
\end{aligned} \tag{15.35}$$

§ 16. Derivation of Meissner Type Equations

Conditions of compatibility (15.30), (15.31) and (15.33) with the aid of elasticity relationships can be written in forces and moments. The instead of (15.28)-(15.33) a system of six equations (two finite and four differential of the first degree) in six unknown forces and moments. However, despite the decrease in degree, this system nevertheless will be very complex. To obtain a simpler system of equations we use a method analogous to the Meissner method in transforming equations of the axisymmetric problem.

We introduce the function of variables

$$\Psi = -\frac{1}{R_1}\left(\frac{dw_{(1)}}{d\theta} - u_{(1)}\right) + \frac{1}{\nu}(w_{(1)}\cos\theta - u_{(1)}\sin\theta). \tag{16.1}$$

Then, in accordance with (15.26), we will have

$$\begin{aligned}
\varkappa_{1(1)} &= \frac{1}{R_1}\frac{d\Psi}{d\theta} + \frac{1}{\nu}(\Psi\cos\theta + \varepsilon_{1(1)}\sin\theta), \\
\tau_{(1)} &= -\frac{\Psi}{\nu} + \frac{\gamma_{(1)}\sin\theta}{\nu}, \\
\varkappa_{2(1)} &= \frac{\Psi\cos\theta}{\nu} + \frac{\varepsilon_{2(1)}\sin\theta}{\nu}.
\end{aligned} \tag{16.2}$$

It is simple to verify that after the elimination of Ψ from (16.2) we obtain two equations which agree with (15.30) and (15.33). In this way one of the integrals and the second equation of compatibility with the aid of representation (16.2) are satisfied identically.

Let us analogously represent also forces $T_{2(1)}$, $T_{1(1)}$, $S_{(1)}$ through the stress function of plus load terms select so as to satisfy heterogeneous equations (15.28), (15.32):

$$
\left.
\begin{aligned}
T_{2(1)} &= \frac{1}{R_1}\frac{dV}{d\theta} + \frac{V\cos\theta}{\nu} - \frac{M_{2(1)}\sin\theta}{\nu} - \frac{\cos\theta}{\nu}\int_{\theta_0}^{\theta} q_{2(1)} R_1 \nu\, d\theta, \\
T_{1(1)} &= \frac{V\cos\theta}{\nu} - \frac{M_{1(1)}\sin\theta}{\nu} + f_0(q_{1(1)}, q_{2(1)}, q_{n(1)}) + f_1(P_x, M_y), \\
S_{(1)} &= \frac{V}{\nu} - \frac{2H_{(1)}\sin\theta}{\nu} - \frac{1}{\nu}\int_{\theta_0}^{\theta} q_{2(1)} R_1 \nu\, d\theta.
\end{aligned}
\right\} \tag{16.3}
$$

Relationships (16.2), (16.3), (8.21) form a system of twelve equations in twelve unknowns - six strain components and six static quantities. Solving it, we find

$$
\left.
\begin{aligned}
Eh\varepsilon_{1(1)}\left(1 + \frac{h^2}{12}\frac{\sin^2\theta}{\nu^2}\right) &= -\frac{\mu}{R_1}\frac{dV}{d\theta} + (1-\mu)\frac{V\cos\theta}{\nu} + \\
&+ f_0(q_{1(1)}, q_{2(1)}, q_{n(1)}) + f_1(P_x, M_y) + \\
&+ \frac{\mu\cos\theta}{\nu}\int_{\theta_0}^{\theta} q_{2(1)} R_1 \nu\, d\theta - \frac{\sin\theta}{\nu}\frac{Eh^3}{12}\left(\frac{1}{R_1}\frac{d\Psi}{d\theta} + \frac{\Psi\cos\theta}{\nu}\right), \\
Eh\varepsilon_{2(1)}\left(1 + \frac{h^2}{12}\frac{\sin^2\theta}{\nu^2}\right) &= \frac{1}{R_1}\frac{dV}{d\theta} + (1-\mu)\frac{V\cos\theta}{\nu} - \\
&- \Psi\frac{\sin\theta\cos\theta}{\nu^2}\frac{Eh^3}{12} - \mu f_0(q_{1(1)}, q_{2(1)}, q_{n(1)}) - \\
&- \mu f_1(P_x, M_y) - \frac{\cos\theta}{\nu}\int_{\theta_0}^{\theta} q_{2(1)} R_1 \nu\, d\theta, \\
Eh\gamma_{(1)}\left(1 + \frac{h^2}{3}\frac{\sin^2\theta}{\nu^2}\right) &= \\
&= 2(1+\mu)\frac{V}{\nu} + \Psi\frac{\sin\theta}{\nu^2}\frac{Eh^3}{3} - \frac{2(1+\mu)}{\nu}\int_{\theta_0}^{\theta} q_{2(1)} R_1 \nu\, d\theta.
\end{aligned}
\right\} \tag{16.4}
$$

$$
\left.
\begin{aligned}
&\varkappa_{1(1)}\left(1+\frac{h^2}{12}\,\frac{\sin^2\theta}{\nu^2}\right)=\frac{1}{R_1}\,\frac{d\Psi}{d\theta}+\frac{\Psi\cos\theta}{\nu}+\\
&\qquad+\frac{\sin\theta}{\nu Eh}\left[(1-\mu)\frac{V\cos\theta}{\nu}-\frac{\mu}{R_1}\,\frac{dV}{d\theta}+\right.\\
&\left.+f_0(q_{1(1)},\ q_{2(1)},\ q_{n(1)})+f_1(P_x,\ M_y)+\frac{\mu\cos\theta}{\nu}\int_{\theta_0}^{\theta}q_{2(1)}R_1\nu\,d\theta\right],\\
&\varkappa_{2(1)}\left(1+\frac{h^2}{12}\,\frac{\sin^2\theta}{\nu^2}\right)=\frac{\Psi\cos\theta}{\nu}+\frac{\sin\theta}{\nu Eh}\left[\frac{1}{R_1}\,\frac{dV}{d\theta}+\right.\\
&+(1-\mu)\frac{V\cos\theta}{\nu}-\mu f_0(q_{1(1)},\ q_{2(1)},\ q_{n(1)})-f_1(P_x,\ M_y)\cdot\mu-\\
&\left.\qquad-\frac{\cos\theta}{\nu}\int_{\theta_0}^{\theta}q_{2(1)}R_1\nu\,d\theta\right],\\
&\tau_{(1)}\left(1+\frac{h^2}{3}\,\frac{\sin^2\theta}{\nu^2}\right)=\\
&\qquad=-\frac{\Psi}{\nu}+\frac{2(1+\mu)\sin\theta}{\nu Eh}\left(\frac{V}{\nu}-\frac{1}{\nu}\int_{\theta_0}^{\theta}q_{2(1)}R_1\nu\,d\theta\right).
\end{aligned}
\right\}\qquad(16.5)
$$

$$
\left.
\begin{aligned}
&T_{1(1)}\left(1+\frac{h^2}{12}\,\frac{\sin^2\theta}{\nu^2}\right)=\\
&\qquad=\frac{V\cos\theta}{\nu}-\frac{\sin\theta}{\nu}D\left[\frac{1}{R_1}\,\frac{d\Psi}{d\theta}+\frac{(1+\mu)\Psi\cos\theta}{\nu}\right]+\\
&\qquad\qquad+f_0(q_{1(1)},\ q_{2(1)},\ q_{n(1)})+f_1(P_x,\ M_y),\\
&T_{2(1)}\left(1+\frac{h^2}{12}\,\frac{\sin^2\theta}{\nu^2}\right)=\\
&=\frac{1}{R_1}\,\frac{dV}{d\theta}+\frac{V\cos\theta}{\nu}-\frac{\sin\theta}{\nu}D\left[\frac{\mu}{R_1}\,\frac{d\Psi}{d\theta}+\frac{(1+\mu)\Psi\cos\theta}{\nu}\right]-\\
&\qquad\qquad-\frac{\cos\theta}{\nu}\int_{\theta_0}^{\theta}q_{2(1)}R_1\nu\,d\theta,\\
&S_{(1)}\left(1+\frac{h^2}{3}\,\frac{\sin^2\theta}{\nu^2}\right)=\frac{V}{\nu}+\frac{2(1-\mu)\,D\Psi\sin\theta}{\nu^2}-\frac{1}{\nu}\int_{\theta_0}^{\theta}q_{2(1)}R_1\nu\,d\theta,
\end{aligned}
\right\}\qquad(16.6)
$$

$$
\left.
\begin{aligned}
&\frac{M_{1(1)}}{D}\left(1+\frac{h^2}{12}\,\frac{\sin^2\theta}{\nu^2}\right)=\\
&\qquad=\frac{1}{R_1}\,\frac{d\Psi}{d\theta}+\frac{(1+\mu)\Psi\cos\theta}{\nu}+\frac{(1-\mu^2)}{Eh}\,\frac{V\cos\theta\sin\theta}{\nu^2}+\\
&\qquad+\frac{\sin\theta}{\nu}\,\frac{(1-\mu^2)}{Eh}\left[f_0(q_{1(1)},\ q_{2(1)},\ q_{n(1)})+f_1(P_x,\ M_y)\right],\\
&\frac{M_{2(1)}}{D}\left(1+\frac{h^2}{12}\,\frac{\sin^2\theta}{\nu^2}\right)=\frac{\mu}{R_1}\,\frac{d\Psi}{d\theta}+\frac{(1+\mu)\Psi\cos\theta}{\nu}+\\
&\qquad+\frac{\sin\theta\,(1-\mu^2)}{Eh\nu}\left[\frac{1}{R_1}\,\frac{dV}{d\theta}+\frac{V\cos\theta}{\nu}\right]-\\
&\qquad\qquad-\frac{(1-\mu^2)}{Eh\nu^2}\sin\theta\cos\theta\int_{\theta_0}^{\theta}q_{2(1)}R_1\nu\,d\theta,\\
&\frac{H_{(1)}}{D}\left(1+\frac{h^2}{3}\,\frac{\sin^2\theta}{\nu^2}\right)=\\
&=-\frac{(1-\mu)\Psi}{\nu}+\frac{2(1-\mu^2)V\sin\theta}{Eh\nu^2}-2(1-\mu^2)\frac{\sin\theta}{Eh\nu^2}\int_{\theta_0}^{\theta}q_{2(1)}R_1\nu\,d\theta.
\end{aligned}
\right\}\qquad(16.7)
$$

Taking into consideration that $\nu = R_2 \sin\theta$, we assume that R_2 nowhere turns into zero, in the obtained expressions we reject the factor $\frac{h^2 \sin^2\theta}{\nu^2} = \frac{h^2}{R_2^2}$ in comparison with unity. Then, substituting them into equations (15.29), (15.31), we will have two equations for determining the two introduced functions Ψ and V:

$$\begin{aligned} &D_0 \frac{\alpha^3}{R_1^2}\frac{d^2\Psi}{d\theta^2} + D_0 \frac{d\Psi}{d\theta}\left[\frac{1}{R_1^2}\frac{d(\alpha^3)}{d\theta} + \frac{\alpha^3\cos\theta}{R_1\nu} - \frac{\alpha^3}{R_1^3}\frac{dR_1}{d\theta}\right] + \\ &+ D_0\Psi\left[\frac{(1+\mu)\cos\theta}{\nu R_1}\frac{d(\alpha^3)}{d\theta} - \frac{(1+\mu)\alpha^3\sin\theta}{\nu R_1} - \frac{2(1+\mu)\alpha^3\cos^2\theta}{\nu^2} - 2(1-\mu)\frac{\alpha^3}{\nu^2}\right] + \\ &+ V\left\{-\frac{\sin\theta}{\nu} + \frac{h_0^2}{12}\left[\frac{\sin\theta\cos\theta}{R_1\nu^2}\frac{d(\alpha^2)}{d\theta} + \right.\right. \\ &\left.\left. + \frac{\alpha^2(\cos^2\theta - \sin^2\theta)}{\nu^2 R_1} + \frac{\alpha^2\sin\theta\cos^2\theta}{\nu^3}\right]\right\} = \Phi_3, \end{aligned} \qquad (16.8)$$

$$\begin{aligned} &\frac{1}{R_1^2\alpha}\frac{d^2V}{d\theta^2} + \frac{dV}{d\theta}\left[\frac{1}{R_1^2}\frac{d}{d\theta}\left(\frac{1}{\alpha}\right) + \frac{\cos\theta}{\nu\alpha R_1} - \frac{1}{\alpha R_1^3}\frac{dR_1}{d\theta}\right] + \\ &+ V\left[\frac{(1-\mu)\cos\theta}{R_1\nu}\frac{d}{d\theta}\left(\frac{1}{\alpha}\right) - \frac{(1-\mu)\sin\theta}{R_1\nu\alpha} - \frac{2(1-\mu)\cos^2\theta}{\nu^2\alpha} - \frac{2(1+\mu)}{\nu^2\alpha}\right] + \\ &+ Eh_0\Psi\left\{\frac{\sin\theta}{\nu} + \frac{h_0^2}{12}\left[-\frac{\sin\theta\cos\theta}{R_1\nu^2}\frac{d(\alpha^2)}{d\theta} - \frac{\alpha^2(\cos^2\theta - \sin^2\theta)}{\nu^2 R_1} - \right.\right. \\ &\left.\left. - \frac{\alpha^2}{\nu^3}\sin\theta\cos^2\theta\right]\right\} = \Phi_4, \end{aligned} \qquad (16.9)$$

where

$$\left.\begin{aligned} &\Phi_3 = F_0 + F_1 + \frac{h_0^2}{12\nu^2}\left\{-\frac{\nu}{R_1}\frac{d}{d\theta}[\alpha^2\sin\theta(f_0+f_1)] + \right. \\ &+ \alpha^2\sin\theta\cos\theta(f_0+f_1) + [2\sin\theta(\cos^2\theta - \sin^2\theta) + \\ &\left. + \sin\theta(2-\cos^2\theta)]\frac{\alpha^2}{\nu}f_2\right\}, \\ &f_2 = \int_{\theta_0}^{\theta} q_{2(1)}\nu R_1\,d\theta. \end{aligned}\right\} \qquad (16.10)$$

$$\begin{aligned} \Phi_4 = &\frac{\mu}{R_1}\frac{d}{d\theta}\left(\frac{f_0+f_1}{\alpha}\right) + \frac{1}{R_1}\frac{d}{d\theta}\left(\frac{f_2\cos\theta}{\nu\alpha}\right) + \\ &+ \frac{\cos\theta}{\nu\alpha}(f_0+f_1) - \frac{2(1+\mu)}{\nu^2\alpha}f_2 + \frac{\mu\cos^2\theta}{\nu^2\alpha}f_2. \end{aligned} \qquad (16.11)$$

D_0 and α are determined by equations (12.2).

It is easy to see that terms contained in (16.10) in brackets remain finite even at $\nu = 0$ (if $R_2 \neq 0$) and have the order qb, where q is the order of the load, b is a certain linear dimension of the shell. The primary terms F_0, F_1 have the order $\frac{qb^3}{\nu^2}$.

Rejecting in the right side of (16.10) quantities of order $\frac{h^2}{12b^2}$ in comparison with unity, we obtain

$$\Phi_3 \approx F_0 + F_1.$$

Making the substitution

$$V = V_1 a^2 b, \quad \Psi = \frac{\Psi_1}{Eh_0}, \tag{16.12}$$

we have

$$\frac{d^2\Psi_1}{d\theta^2} + \frac{d\Psi_1}{d\theta}\left(\frac{3a'}{a} + \frac{R_1\cos\theta}{\nu} - \frac{1}{R_1}\frac{dR_1}{d\theta}\right) + \\ + \Psi_1\left[\frac{3a'}{a}\frac{(1+\mu)R_1\cos\theta}{\nu} - \frac{(1+\mu)R_1\sin\theta}{\nu} - \right. \\ \left. - \frac{2(1+\mu)R_1^2\cos^2\theta}{\nu^2} - 2(1-\mu)\frac{R_1^2}{\nu^2}\right] + \\ + V_1\cdot 4\gamma_0^4\left\{-\frac{1}{a}\frac{R_1^2\sin\theta}{\nu b} + \frac{1}{4\gamma_0^4}(1-\mu^2)\left[\frac{2a'R_1b}{\nu^2}\sin\theta\cos\theta + \right.\right. \\ \left.\left. + \frac{aR_1b}{\nu^2}(\cos^2\theta - \sin^2\theta) + \frac{aR_1^2b}{\nu^3}\sin\theta\cos^2\theta\right]\right\} = 4\gamma_0^4\frac{\Phi_3}{a^2}\frac{R_1^2}{b^2}. \tag{16.13}$$

$$\frac{d^2V_1}{d\theta^2} + \frac{dV_1}{d\theta}\left(\frac{3a'}{a} + \frac{R_1\cos\theta}{\nu} - \frac{1}{R_1}\frac{dR_1}{d\theta}\right) + V_1\left[\frac{2a''}{a} - \frac{2a'}{aR_1}\frac{dR_1}{d\theta} + \right. \\ \left. + (1+\mu)\frac{a'}{a}\frac{R_1\cos\theta}{\nu} - (1-\mu)\frac{R_1\sin\theta}{\nu} - 2(1-\mu)\cos^2\theta\frac{R_1^2}{\nu^2} - 2(1+\mu)\frac{R_1^2}{\nu^2}\right] + \\ + \Psi_1\left\{\frac{R_1^2\sin\theta}{a\nu b} + \frac{1}{4\gamma_0^4}(1-\mu^2)\left[-\frac{2a'R_1b}{\nu^2}\sin\theta\cos\theta - \frac{aR_1b}{\nu^2}(\cos^2\theta - \sin^2\theta) - \right.\right. \\ \left.\left. - \frac{aR_1^2b}{\nu^3}\sin\theta\cos^2\theta\right]\right\} = \frac{R_1^2}{ab}\Phi_4. \tag{16.14}$$

For a shell of constant thickness equations (16.13), (16.14) are written thus:

$$\frac{d^2\Psi_1}{d\theta^2} + \frac{d\Psi_1}{d\theta}\left(\frac{R_1\cos\theta}{\nu} - \frac{1}{R_1}\frac{dR_1}{d\theta}\right) + \\ + \Psi_1\left[-(1+\mu)\frac{R_1\sin\theta}{\nu} - 2(1+\mu)\frac{R_1^2\cos^2\theta}{\nu^2} - 2(1-\mu)\frac{R_1^2}{\nu^2}\right] + \\ + V_1 4\gamma_0^4\left\{-\frac{R_1^2\sin\theta}{\nu b} + \frac{(1-\mu^2)}{4\gamma_0^4}\left[\frac{R_1b}{\nu^2}(\cos^2\theta - \sin^2\theta) + \right.\right. \\ \left.\left. + \frac{R_1^2b}{\nu^3}\sin\theta\cos^2\theta\right]\right\} = 4\gamma_0^4\frac{R_1^2}{b^2}\Phi_3. \tag{16.15}$$

$$\frac{d^2V_1}{d\theta^2}+\frac{dV_1}{d\theta}\left(\frac{R_1\cos\theta}{\nu}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)+$$
$$+V_1\left[-(1-\mu)\frac{R_1\sin\theta}{\nu}-2(1-\mu)\frac{R_1^2\cos^2\theta}{\nu^2}-2(1+\mu)\frac{R_1^2}{\nu^2}\right]+$$
$$+\Psi_1\left\{\frac{R_1^2\sin\theta}{\nu b}+\frac{(1-\mu^2)}{4\gamma_0^4}\left[-\frac{R_1 b}{\nu^2}(\cos^2\theta-\sin^2\theta)-\right.\right.$$
$$\left.\left.-\frac{R_1^2 b}{\nu^3}\sin\theta\cos^2\theta\right]\right\}=\frac{R_1^2}{b}\Phi_4. \qquad (16.16)$$

where $h_0=h$, $4\gamma_0^4=4\gamma^4$.

Using the complex unknown

$$\sigma=\Psi_1+2i\gamma_0^2V_1, \qquad (16.17)$$

where i is an imaginary unit, equations (16.15), (16.16) can be reduced to one equation

$$\frac{d^2\sigma}{d\theta^2}+\frac{d\sigma}{d\theta}\left(\frac{R_1\cos\theta}{\nu}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)+\sigma\left(-\frac{R_1\sin\theta}{\nu}-\frac{4R_1^2}{\nu^2}+\frac{2R_1^2\sin^2\theta}{\nu^2}\right)-$$
$$-\bar{\sigma}\mu\left(\frac{R_1\sin\theta}{\nu}-\frac{2R_1^2\sin^2\theta}{\nu^2}\right)+2i\gamma_0^2\sigma\frac{R_1^2\sin\theta}{\nu b}-$$
$$-\frac{(1-\mu^2)}{2\gamma_0^2}i\sigma\left[\frac{R_1 b}{\nu^2}(\cos^2\theta-\sin^2\theta)+\frac{R_1^2 b}{\nu^3}\sin\theta\cos^2\theta\right]=$$
$$=4\gamma_0^4\frac{R_1^2}{b^2}\left(\Phi_3+\frac{i}{2\gamma_0^2}\Phi_4 b\right). \qquad (16.18)$$

Similarly (16.13) and (16.14) can be written in the form of one equation

$$\frac{d^2\sigma}{d\theta^2}+\frac{d\sigma}{d\theta}\left(\frac{3\alpha'}{\alpha}+\frac{R_1\cos\theta}{\nu}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)+\left(-\frac{R_1\sin\theta}{\nu}+\frac{2R_1^2\sin^2\theta}{\nu^2}-\frac{4R_1^2}{\nu^2}\right)\sigma-$$
$$-\bar{\sigma}\mu\left(\frac{R_1\sin\theta}{\nu}-\frac{2R_1^2\sin^2\theta}{\nu^2}\right)+\sigma\frac{3\alpha'}{\alpha}(1+\mu)\frac{R_1\cos\theta}{\nu}+$$
$$+i\,\mathrm{Im}\,\sigma\left[\frac{2\alpha''}{\alpha}-\frac{2\alpha'}{\alpha}\frac{1}{R_1}\frac{dR_1}{d\theta}-\frac{2\alpha'}{\alpha}(1+\mu)\frac{R_1\cos\theta}{\nu}\right]+$$
$$+2i\gamma_0^2\sigma\frac{R_1^2\sin\theta}{\alpha\nu b}-i\frac{(1-\mu^2)}{2\gamma_0^2}\sigma\left[2\alpha'\frac{R_1 b}{\nu^2}\sin\theta\cos\theta+\right.$$
$$\left.+\alpha\frac{R_1 b}{\nu^2}(\cos^2\theta-\sin^2\theta)+\alpha\frac{R_1^2 b}{\nu^3}\sin\theta\cos^2\theta\right]=$$
$$=4\gamma_0^4\frac{R_1^2}{b^2}\left(\frac{\Phi_3}{\alpha^3}+\frac{i}{2\gamma_0^2}b\frac{\Phi_4}{\alpha}\right). \qquad (16.19)$$

From (16.12), (16.17) it follows that

$$V=\frac{\alpha^3 b}{2\gamma^3}\operatorname{Im}\sigma, \quad \Psi=\frac{\operatorname{Re}\sigma}{Eh_0}$$

and expressions for forces and bending moments through introduced function σ has the form

$$\left.\begin{aligned}
T_{1(1)}&=\frac{b\alpha^2}{2\gamma_0^2}\operatorname{Im}\sigma\frac{\cos\theta}{\nu}-\frac{b^2\alpha^3}{4\gamma_0^4}\frac{\sin\theta}{\nu}\left[\frac{1}{R_1}\operatorname{Re}\frac{d\sigma}{d\theta}+\right.\\
&\left.+(1+\mu)\operatorname{Re}\sigma\frac{\cos\theta}{\nu}\right]+f_0(q_{1(1)}, q_{2(1)}, q_{n(1)})+f_1(P_x, M_y),\\
T_{2(1)}&=\frac{b}{2\gamma_0^2}\left[\frac{1}{R_1}\operatorname{Im}\frac{d}{d\theta}(\alpha^2\sigma)+\alpha^2\operatorname{Im}\sigma\frac{\cos\theta}{\nu}\right]-\\
&-\frac{b^2\alpha^3}{4\gamma_0^4}\frac{\sin\theta}{\nu}\left[\frac{\mu}{R_1}\operatorname{Re}\frac{d\sigma}{d\theta}+(1+\mu)\operatorname{Re}\sigma\frac{\cos\theta}{\nu}\right]-\frac{\cos\theta}{\nu}f_2,\\
S_{(1)}&=\frac{b\alpha^2}{2\gamma_0^2}\frac{\operatorname{Im}\sigma}{\nu}+2(1-\mu)\frac{b^2\alpha^3}{4\gamma_0^4}\operatorname{Re}\sigma\frac{\sin\theta}{\nu^2}-\frac{1}{\nu}f_2,\\
M_{1(1)}&=\frac{b\alpha^3}{4\gamma_0^4}\left\{\frac{b}{R_1}\operatorname{Re}\frac{d\sigma}{d\theta}+(1+\mu)\frac{b\cos\theta}{\nu}\operatorname{Re}\sigma+\right.\\
&+\frac{(1-\mu^2)}{2\gamma_0^2}\frac{\alpha b^2}{\nu^2}\cos\theta\sin\theta\operatorname{Im}\sigma+\\
&\left.+(1-\mu^2)\frac{b}{\alpha\nu}\sin\theta\left[f_0(q_{1(1)}, q_{2(1)}, q_{n(1)})+f_1(P_x, M_y)\right]\right\},\\
M_{2(1)}&=\frac{b\alpha^3}{4\gamma_0^4}\left\{\mu\frac{b}{R_1}\operatorname{Re}\frac{d\sigma}{d\theta}+(1+\mu)\frac{b\cos\theta}{\nu}\operatorname{Re}\sigma+\right.\\
&+(1-\mu^2)\frac{\sin\theta}{\alpha 2\gamma_0^2}\left[\frac{b^2}{R_1\nu}\operatorname{Im}\frac{d}{d\theta}(\alpha^2\sigma)+\frac{b^2}{\nu^2}\alpha^2\operatorname{Im}\sigma\cos\theta\right]-\\
&\left.-(1-\mu^2)\frac{b}{\alpha\nu^2}\sin\theta\cos\theta f_2\right\},\\
H_{(1)}&=\frac{\alpha^3 b}{4\gamma_0^4}\left\{-(1-\mu)\frac{b}{\nu}\operatorname{Re}\sigma+\frac{2(1-\mu^2)}{2\gamma_0^2}\frac{\alpha b^2}{\nu^2}\sin\theta\operatorname{Im}\sigma-\right.\\
&\left.-\frac{2(1-\mu^2)b\sin\theta}{\nu^2\alpha}f_2\right\}.
\end{aligned}\right| \qquad (16.20)$$

Setting in (16.20) α = 1, we can obtain equations for a shell of constant thickness. Supposing as before that $\alpha'\ll 2\gamma_0^2$, $\alpha''\ll 2\gamma_0^2$ and α nowhere turns into zero, and parameter $2\gamma_0^2\gg 1$, we simplify equation (16.19), rejecting in the coefficients of unknown functions σ term of the order $\frac{1}{2\gamma_0^2}$ in comparison with unity. We obtain

$$\begin{aligned}
&\frac{d^2\sigma}{d\theta^2}+\frac{d\sigma}{d\theta}\left(\frac{3\alpha'}{\alpha}+\frac{R_1\cos\theta}{\nu}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)+\\
&\quad+\sigma\left(2i\gamma_0^2\frac{R_1^2\sin\theta}{\alpha\nu b}-\frac{4R_1^2}{\nu^2}\right)=4\gamma_0^4\frac{R_1^2}{b^2}\left(\frac{\Phi_3}{\alpha^3}+\frac{i}{2\gamma_0^2}\frac{\Phi_4}{\alpha}b\right).
\end{aligned} \qquad (16.21)$$

For a shell of constant thickness the corresponding equations has the form

$$\frac{d^2\sigma}{d\theta^2}+\frac{d\sigma}{d\theta}\left(\frac{R_1\cos\theta}{\nu}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)+\sigma\left(2i\gamma_0^2\frac{R_1^2\sin\theta}{\nu b}-\frac{4R_1^2}{\nu^2}\right)=$$

$$=4\gamma_0^4\frac{R_1^2}{b^2}\left(\Phi_3+\frac{i}{2\gamma_0^2}\Phi_4 b\right). \qquad (16.22)$$

Comparing the left sides of the obtained equations and equations which describe axisymmetric deformation (12.12), (12.13), we note that they differ only in one term in the coefficient of the unknown function, which for large ν is not essential.

In this way, just as at axisymmetric loading, calculation of a shell for a bending load is reduced to the solution of one differential equation of the second degree in unknown imaginary function σ, where at $\nu \neq 0$ the left side of these equations practically coincide [21], [42], [44].

§ 17. Boundary Conditions

Let us formulate boundary conditions for function σ. In the general case of loading all possible boundary conditions on the edges θ_i ($i = 0.1$) are obtained as a result of the requirement that contour integral (6.14) vanish in which through T_1^i, S_{12}^i, H_{12}^i, Q_1^i, M_1^i we designated external forces and moments on the edge. Since in this case

$$\left.\begin{aligned}&(T_1,\ N_1,\ M_1,\ T_1^i,\ Q_1^i,\ M_1^i)=\\&\quad=(T_{1(1)},\ N_{1(1)},\ M_{1(1)},\ T_{1(1)}^i,\ Q_{1(1)}^i,\ M_{1(1)}^i)\cos\varphi,\\&(S,\ H,\ S_{12}^i,\ H_{12}^i)=(S_{(1)},\ H_{(1)},\ S_{12(1)}^i,\ H_{12(1)}^i)\sin\varphi,\\&(u,\ w,\ \vartheta_1)=(u_{(1)},\ w_{(1)},\ \vartheta_{1(1)})\cos\varphi,\\&\qquad v=v_{(1)}\sin\varphi,\end{aligned}\right\} \qquad (17.1)$$

the condition of integral (6.14) vanishing after integration over ϕ can be written in the form

$$(-T_{1(1)}+T_{1(1)}^i)\delta u_{(1)}+\left[-\left(S_{(1)}+\frac{2H_{(1)}}{R_2}\right)+\left(S_{12(1)}^i+\frac{H_{12(1)}^i}{R_2}\right)\right]\delta v_{(1)}+$$

$$+\left[-\left(N_{1(1)}+\frac{1}{\nu_i}H_{(1)}\right)+\left(Q_{1(1)}^i+\frac{1}{\nu_i}H_{12(1)}^i\right)\right]\delta w_{(1)}+$$

$$+(M_{1(1)}-M_{1(1)}^i)\delta\vartheta_{1(1)}=0. \qquad (17.2)$$

Using (8.22) and (16.1), it is simple to obtain the equality

$$\left.\begin{aligned}\Delta_{e(1)} &= u_{(1)}\cos\theta + w_{(1)}\sin\theta = \varepsilon_{2(1)}\nu - v_{(1)},\\ \Delta_{z(1)} &= -u_{(1)}\sin\theta + w_{(1)}\cos\theta = \Psi\nu + \vartheta_{1(1)}\nu.\end{aligned}\right\} \tag{17.3}$$

where through $\Delta_{e(1)}$, $\Delta_{z(1)}$ we designate the amplitudes of radial and axial displacements. Expressing $u_{(1)}$, $w_{(1)}$ through $\varepsilon_{2(1)}$, Ψ, $\vartheta_{1(1)}$ and $v_{(1)}$, we have

$$\left.\begin{aligned}u_{(1)} &= \varepsilon_{2(1)}\nu\cos\theta - v_{(1)}\cos\theta - \Psi\nu\sin\theta - \nu\vartheta_{1(1)}\sin\theta,\\ w_{(1)} &= \varepsilon_{2(1)}\nu\sin\theta - v_{(1)}\sin\theta + \Psi\nu\cos\theta + \nu\vartheta_{1(1)}\cos\theta.\end{aligned}\right\} \tag{17.4}$$

Going from relationships (17.4) to relationships in variations and substituting expressions for $\delta u_{(1)}$, $\delta w_{(1)}$ into (17.2), we obtain

$$\begin{aligned}&\Big\{\left(-T_{1(1)}+T^{i}_{1(1)}\right)\cos\theta_i+\Big[-\Big(N_{1(1)}+\frac{1}{\nu_i}H_{(1)}\Big)+\Big(Q^{i}_{1(1)}+\frac{1}{\nu_i}H^{i}_{12(1)}\Big)\Big]\times\\ &\times\sin\theta_i\Big\}\nu_i\,\delta\varepsilon_{2(1)}-\Big\{\left(-T_{1(1)}+T^{i}_{1(1)}\right)\cos\theta_i+\Big[-\Big(N_{1(1)}+\frac{1}{\nu_i}H_{(1)}\Big)+\\ &\quad+\Big(Q^{i}_{1(1)}+\frac{1}{\nu_i}H^{i}_{12(1)}\Big)\Big]\sin\theta_i-\Big[-\Big(S_{(1)}+\frac{2H_{(1)}}{R^{i}_{2}}\Big)+\\ &\quad+\Big(S^{i}_{12(1)}+\frac{H^{i}_{12(1)}}{R^{i}_{2}}\Big)\Big]\Big\}\delta v_{(1)}+\Big\{-\left(-T_{1(1)}+T^{i}_{1(1)}\right)\nu_i\sin\theta_i+\\ &\quad+\Big[-\Big(N_{1(1)}+\frac{1}{\nu_i}H_{(1)}\Big)+\Big(Q^{i}_{1(1)}+\frac{1}{\nu_i}H^{i}_{12(1)}\Big)\Big]\nu_i\cos\theta_i+\\ &\quad+\left(M_{1(1)}-M^{i}_{1(1)}\right)\Big\}\delta\vartheta_{1(1)}+\Big\{-\left(-T_{1(1)}+T^{i}_{1(1)}\right)\sin\theta_i+\\ &+\Big[-\Big(N_{1(1)}+\frac{1}{\nu_i}H_{(1)}\Big)+\Big(Q^{i}_{1(1)}+\frac{1}{\nu_i}H^{i}_{12(1)}\Big)\Big]\cos\theta_i\Big\}\nu_i\,\delta\Psi=0.\end{aligned} \tag{17.5}$$

System of internal forces $T_{1(1)}$, $S_{(1)}$, $N_{(1)}$, $M_{1(1)}$, $H_{(1)}$, acting in the edge of section θ_i, and external forces and moments $T^{i}_{1(1)}$, $S^{i}_{12(1)}$, $Q^{i}_{1(1)}$, $M^{i}_{1(1)}$, $H^{i}_{12(1)}$ should be balanced. Consequently, in order that the problem is correctly posed, external forces and moments should satisfy relationships of form (15.16), (15.18), written for $\theta = \theta_i$ ($i = 0.1$). Writing the same relationships, but reduced to form (15.21), (15.22) for internal forces and moments acting in sections $\theta = \theta_i$, and deducting the corresponding equations containing identical load terms, one from the other, we find that at $\theta = \theta_i$ we have the equality:

$$\left.\begin{array}{l}(T^{i}_{1(1)}-T_{1(1)})\cos\theta_i+\left[\left(Q^{i}_{1(1)}+\frac{1}{\nu_i}H^{i}_{12(1)}\right)-\right.\\ \left.-\left(N_{1(1)}+\frac{1}{\nu_i}H_{(1)}\right)\right]\sin\theta_i-\left(S^{i}_{12(1)}+\frac{H^{i}_{12(1)}}{R^{i}_{2}}\right)+\left(S_{(1)}+\frac{2H_{(1)}}{R^{i}_{2}}\right)=0.\\ (M^{i}_{1(1)}-M_{1(1)})+(T^{i}_{1(1)}-T_{1(1)})\nu_i\sin\theta_i-\\ \quad-\left[\left(Q^{i}_{1(1)}+\frac{1}{\nu_i}H^{i}_{12(1)}\right)-\left(N_{1(1)}+\frac{1}{\nu_i}H_{(1)}\right)\right]\nu_i\cos\theta_i=0.\end{array}\right\} \qquad (17.6)$$

After this (17.5) assumes the form

$$\left[\left(S^{i}_{12(1)}+\frac{H^{i}_{12(1)}}{R^{i}_{2}}\right)-\left(S_{(1)}+\frac{2H_{(1)}}{R^{i}_{2}}\right)\right]\nu_i\,\delta\varepsilon_{2(1)}+$$
$$+(M^{i}_{1(1)}-M_{1(1)})\delta\Psi=0 \qquad (i=0,\ 1). \qquad (17.7)$$

Condition (17.7) gives the following possible combinations of boundary conditions at the θ_i edge:

$$S_{(1)}+\frac{2H_{(1)}}{R_2}=S^{i}_{12(1)}+\frac{H^{i}_{12(1)}}{R_2},\quad M^{i}_{1(1)}=M_{1(1)}, \qquad (17.8)$$

$$\varepsilon_{2(1)}=\varepsilon^{i}_{2(1)},\quad \Psi=\Psi^{i}, \qquad (17.9)$$

$$S_{(1)}+\frac{2H_{(1)}}{R_2}=S^{i}_{12(1)}+\frac{H^{i}_{12(1)}}{R_2},\quad \Psi=\Psi^{i}, \qquad (1710)$$

$$\varepsilon_{2(1)}=\varepsilon^{i}_{2(1)},\quad M_{1(1)}=M^{i}_{1(1)}, \qquad (17.11)$$

where the i indicates quantities on the edge $\theta = \theta_i$. For example, conditions (17.9) in this case can have the form

$$\varepsilon_{2(1)}=0,\quad \Psi=0. \qquad (17.12)$$

On the basis of (16.1) it is easy to see that at $\cos\theta_i \neq 0$ conditions (17.12) are equivalent to conditions

$$\varepsilon_{2(1)}=0,\quad \varkappa_{2(1)}=0. \qquad (17.13)$$

In this way (17.13) equivalent conditions (17.12) imply a rigid sealing of the edge in the sense that during deformation there is no relative elongation in peripheral direction and change in curvature

κ_2 in extreme section θ_i. It is obvious that the edge section can in this case be moved or turned as a solid body.

The first power condition (17.8), taking into account (17.6), can be replaced by the equivalent condition

$$T_{1(1)} \cos\theta_i + \left(N_{1(1)} + \frac{1}{v_i} H_{(1)}\right) \sin\theta_i = \\ = T^i_{1(1)} \cos\theta_i + \left(Q^i_{1(1)} + \frac{1}{v_i} H^i_{12(1)}\right) \sin\theta_i. \tag{17.14}$$

Conditions which have the form

$$S_{(1)} + \frac{2H_{(1)}}{R_2} = 0, \quad M_{1(1)} = 0. \tag{17.15}$$

man that the edge is free from tangential forces and meridian bending moments. Conditions of the form

$$T_{1(1)} \cos\theta_i + \left(N_{1(1)} + \frac{1}{v_i} H_{(1)}\right) \sin\theta_i = 0, \quad M_{1(1)} = 0 \tag{17.16}$$

correspond to an edge free from bending moments and radial forces. When loads on the edge generally are absent, conditions (17.15) and (17.16) are equivalent. Using equations (16.4), (16.17), (16.20), we compose the expressions for the quantities in boundary conditions (17.8)-(17.11) through basic unknown function σ. Ignoring in this case quantities of order $\frac{1}{2\gamma_0^2}$ in comparison with unity, we obtain

$$\left.\begin{aligned}
&S_{(1)} + \frac{2H_{(1)}}{R_2} = \frac{1}{2\gamma_0^2} \frac{a^2 b}{v} \operatorname{Im}\sigma - \frac{1}{v} f_2, \\
&M_{1(1)} = \frac{a^3 b^2}{4\gamma_0^4} \left\{ \frac{1}{R_1} \operatorname{Re}\frac{d\sigma}{d\theta} + (1+\mu)\frac{\cos\theta}{v} \operatorname{Re}\sigma + \right. \\
&\qquad \left. + (1-\mu^2)\frac{\sin\theta}{av} [f_0(q_{1(1)}, q_{2(1)}, q_{n(1)}) + f_1(P_x, M_y)] \right\}, \\
&Eh_0\varepsilon_{2(1)} = \frac{b}{2\gamma_0^2} \left[\frac{1}{R_1 a} \operatorname{Im}\frac{d(a^2\sigma)}{d\theta} + (1-\mu)\frac{a\cos\theta}{v} \operatorname{Im}\sigma \right] - \\
&\qquad - \frac{\mu}{a} [f_0(q_{1(1)}, q_{2(1)}, q_{n(1)}) + f_1(P_x, M_y)] - \frac{\cos\theta}{va} f_2, \\
&Eh_0\Psi' = \operatorname{Re}\sigma.
\end{aligned}\right\} \tag{17.17}$$

§ 18. Determination of Displacements

Let us study now the determination of displacements through the introduced functions of V and Ψ.

On the basis of the fourth relationship of (15.26) we determine angle of rotation $\vartheta_{1(1)}=\frac{1}{R_1}\left(\frac{dw_{(1)}}{d\theta}-u_{(1)}\right)$ using one quadrature

$$\vartheta_{1(1)}=-\int_{\theta_0}^{\theta} R_1\varkappa_{1(1)}\,d\theta+D_1. \qquad (18.1)$$

where $\kappa_{1(1)}$ – known function of V and Ψ (see (16.5)), D_1 – constant of integration. After defining $\vartheta_{1(1)}$ of the amplitude of axial displacement $\Delta_{z(1)}$ easily is found using the second equation of (17.3)

$$\Delta_{z(1)}=\nu\left(\Psi-\int_{\theta_0}^{\theta} R_1\varkappa_{1(1)}\,d\theta\right)+D_1\nu. \qquad (18.2)$$

Excluding from the second and third relationships of (15.26) displacement $v_{(1)}$, we find the expression for $\frac{dv_{(1)}}{d\theta}$ through $\gamma_{(1)}, \varepsilon_{2(1)}, \Delta_{z(1)}$, which are already known functions of V and Ψ:

$$\frac{1}{R_1}\frac{dv_{(1)}}{d\theta}=\gamma_{(1)}+\varepsilon_{2(1)}\cos\theta-\frac{\Delta_{z(1)}\sin\theta}{\nu}, \qquad (18.3)$$

whence by one integration we determine

$$v_{(1)}=\int_{\theta_0}^{\theta} R_1\left(\gamma_{(1)}+\varepsilon_{2(1)}\cos\theta-\frac{\Delta_{z(1)}\sin\theta}{\nu}\right)d\theta+D_2. \qquad (18.4)$$

on the basis of (17.3) we find the amplitude of radial displacement

$$\Delta_{\rho(1)}=\nu\varepsilon_{2(1)}-\int_{\theta_0}^{\theta}\left(\gamma_{(1)}+\varepsilon_{2(1)}\cos\theta-\frac{\Delta_{z(1)}\sin\theta}{\nu}\right)R_1\,d\theta-D_2. \qquad (18.5)$$

Thus, as a result of integration of a basic system of the eighth degree (8.20), (8.25) at k = 1 the solution contains eight arbitrary constants. Two of them (P_x and M_y) have the sense of total shearing force and bending moment in section θ_0, two (D_1, D_2) appear during determination of displacements with respect to the found functions of V and Ψ. The remaining four enter the solution of equation (16.21) and should be determined according to boundary conditions of form (17.8)-(17.11).

In § 3 it was already mentioned that displacements are determined according to assigned components of deformation accurate to constants. Let us explain the meaning of constants D_1 and D_2. We examine a shell which, without deforming, moved in space with a preassigned displacement vector u_0 and turns relative to point O (origin of coordinates, coinciding with the center of section θ_0) with preassigned vector of rotation ω:

$$u_0 = u_{0x}i + u_{0y}j + u_{0z}k, \quad \omega = \omega_x i + \omega_y j + \omega_z k. \tag{18.6}$$

The displacement of any point of the middle surface of the shell and vector of rotation Ω in this instance are equal to

$$u = u_0 + \omega \times r, \tag{18.7}$$

$$\Omega = \omega, \tag{18.8}$$

where $r = \nu \cos\varphi i + \nu \sin\varphi j + Zk$ — radiusvector of point.

Let us designate the projection of displacement (18.7) onto moving axes τ_1, τ_2, n, calculating the products $u = (u \cdot \tau_1)$, $v = (u \cdot \tau_2)$, $w = (u \cdot n)$ with the aid of (18.7) and the table of cosines between axes τ_1, τ_2, n and OX, OY, OZ, given in § 15. We obtain

$$\left.\begin{aligned} u &= u_{0x}\cos\theta\cos\varphi + u_{0y}\cos\theta\sin\varphi - u_{0z}\sin\theta + \\ &\quad + (\omega_y\cos\varphi - \omega_x\sin\varphi)\nu\sin\theta + (\omega_y\cos\varphi - \omega_x\sin\varphi)Z\cos\theta, \\ v &= -u_{0x}\sin\varphi + u_{0y}\cos\varphi + \omega_z\nu - \omega_y Z\sin\varphi - \omega_x Z\cos\varphi, \\ w &= u_{0x}\sin\theta\cos\varphi + u_{0y}\sin\theta\sin\varphi + u_{0z}\cos\theta + \\ &\quad + (\omega_y\cos\varphi - \omega_x\sin\varphi)(-\nu\cos\theta + Z\sin\theta). \end{aligned}\right\} \tag{18.9}$$

Projections of displacements u onto axes e, k, τ_2 are equal to

$$\left.\begin{aligned} \Delta_e &= u_{0x}\cos\varphi + u_{0y}\sin\varphi + \omega_y Z\cos\varphi - \omega_x Z\sin\varphi, \\ \Delta_z &= u_{0z} - \omega_y\nu\cos\varphi + \omega_x\nu\sin\varphi, \\ v &= -u_{0x}\sin\varphi + u_{0y}\cos\varphi + \omega_z\nu - \omega_y Z\sin\varphi - \omega_x Z\cos\varphi. \end{aligned}\right\} \tag{18.10}$$

The projection of vector ω onto axis τ_2 is equal to

$$(\omega \cdot \tau_2) = -\omega_x\sin\varphi + \omega_y\cos\varphi \tag{18.11}$$

consequently, because of (18.8), (2.14)

$$-\vartheta_1 = -\omega_x \sin\varphi + \omega_y \cos\varphi. \qquad (18.12)$$

We exclude from consideration displacements which are axisymmetric and odd relative to plane $\phi = 0$, i.e., we set $u_{0y} = u_{0z} = 0$, $\omega_x = \omega_z = 0$. Then we will have

$$\left.\begin{aligned} \Delta_r &= (u_{0x} + \omega_y Z)\cos\varphi, & \Delta_z &= -\omega_y \nu \cos\varphi, \\ v &= (-u_{0x} - \omega_y Z)\sin\varphi, & \vartheta_1 &= -\omega_y \cos\varphi. \end{aligned}\right\} \qquad (18.13)$$

Corresponding amplitudes of displacements are equal to

$$\left.\begin{aligned} \Delta_{r(1)} &= u_{0x} + \omega_y Z, & \Delta_{z(1)} &= -\omega_y \nu, \\ v_{(1)} &= -u_{0x} - \omega_y Z, & \vartheta_{1(1)} &= -\omega_y. \end{aligned}\right\} \qquad (18.14)$$

Calculating on the basis of (18.14) components of deformation (8.22) and (8.23) ($k = 1$) and function Ψ (see (16.1)), it is simple to verify that, as one would expect, in this case

$$\varepsilon_{1(1)} = \varepsilon_{2(1)} = \gamma_{(1)} = \varkappa_{1(1)} = \varkappa_{2(1)} = \tau_{(1)} = 0, \quad \Psi = 0. \qquad (18.15)$$

Let us rewrite equations (18.1), (18.2), (18.4), (18.5) allowing for (18.15) and (15.13). We obtain

$$\left.\begin{aligned} \vartheta_{1(1)} &= D_1, \\ \Delta_{z(1)} &= D_1 \nu, \\ v_{(1)} &= D_1 z + D_2, \\ \Delta_{r(1)} &= -D_1 Z - D_2. \end{aligned}\right\} \qquad (18.16)$$

Comparing (18.14) and (18.16), we have

$$D_1 = -\omega_y, \quad D_2 = -u_{0x}. \qquad (18.17)$$

i.e., constants of integration D_1 and D_2, appearing during determination of amplitudes of displacements with respect to assigned amplitudes of the component of deformation (formulas (18.1)-(18.5)), are the rotation of the shell as a whole around axis *OY* and the shift of a solid as a whole in the direction of axis *OX* taken with the opposite sign.

In conclusion we will examine the strain of a shell whose edge θ_0 is joined with a rigid washer. On the washer act force P_x and moment M_y (Fig. 10). The washer can only shift and turn as a solid body, where

$$u_0 = u_{0x} i, \quad \omega = \omega_y j. \tag{18.18}$$

Setting in (18.14) $z = 0$, $\nu = \nu_0$, we find that on the edge of the shell $\theta = \theta_0$ the amplitudes of the displacements are equal to

$$\left.\begin{aligned} \Delta_{\varphi(1)}(\theta_0) &= u_{0x}, & \Delta_{z(1)}(\theta_0) &= -\omega_y \nu_0, \\ v_{(1)}(\theta_0) &= -u_{0x}, & \vartheta_{1(1)}(\theta_0) &= -\omega_y. \end{aligned}\right\} \tag{18.19}$$

Conditions (18.19) are boundary conditions in displacements of type (8.26). The fourth condition of (18.19) means that the angle of inclination of normal n to the plane of the extreme section remains constant with deformation. (In the designations of Fig. 10 $\beta = \beta'$.) Taking into account formula (17.3), we can replace (18.19) by equivalent conditions

$$\Delta_{\varphi(1)}(\theta_0) = u_{0x}, \quad \vartheta_{1(1)}(\theta_0) = -\omega_y, \tag{18.20}$$

$$\varepsilon_{2(1)}(\theta_0) = 0, \qquad \Psi'(\theta_0) = 0. \tag{18.21}$$

Conditions (18.21) serve to determine the constants of integration in the solution of equation (16.21). To them should be combined two conditions on the second edge of the shell, which can have any form from (17.8)-(17.11), depending on the concrete assigned conditions on this edge.

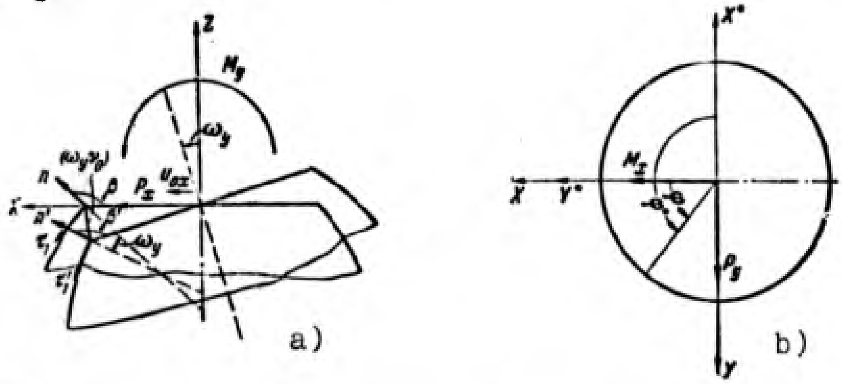


Fig. 10. a) Deformation of edge of shell joined with a rigid washer. b) System of loads which bends shell into plane $\varphi = \frac{\pi}{2}$.

Let the second edge of the shell $\theta = \theta_1$ be sealed into a rigid wall, i.e., it remains motionless and not deformable. For simplicity it can be considered that there are no distributed loads on the shell. The shell is an overhang loaded by an assigned force and moment P_x, M_y. It is necessary to designate shift and rotation u_{0x}, ω_y of edge θ_0. On edge $\theta = \theta_1$ we have conditions in displacements:

$$\left.\begin{aligned} \Delta_{e(1)}(\theta_1)=0, \quad \Delta_{z(1)}(\theta_1)=0, \\ v_{(1)}(\theta_1)=0, \quad \vartheta_{1(1)}(\theta_1)=0. \end{aligned}\right\} \tag{18.22}$$

On the basis of formulas (17.3) they can be replaced by the equivalent conditions:

$$\Delta_{e(1)}(\theta_1)=0, \quad \vartheta_{1(1)}(\theta_1)=0. \tag{18.23}$$

$$\varepsilon_{2(1)}(\theta_1)=0, \quad \Psi(\theta_1)=0. \tag{18.24}$$

Conditions (18.21), (18.24) serve for determining of constants of integration in the general solution of equations of Meissner (type equations (16.13), (16.14) or equation (16.21)). Since these constants have been defined and the solution of equation (16.21) satisfying assigned boundary conditions has been constructed, we can consider as known all components of deformation $\varepsilon_{1(1)}$, $\varepsilon_{2(1)}$, $\gamma_{(1)}$, $\varkappa_{1(1)}$, $\varkappa_{2(1)}$, $\tau_{(1)}$ and the function of Ψ in any section of the shell. Let us remember that they linearly depend on loads P_x, M_y, assigned on the edge, since the right side of (16.21) is a linear function of these quantities. Conditions (18.20) serve for determination of constants of integration D_1, D_2, figuring in equations (18.1), (18.5). Setting $\theta = \theta_0$, we find

$$D_1 = -\omega_y, \quad D_2 = -u_{0x}, \tag{18.25}$$

i.e., constants D_1, D_2 are the desired unknowns. We require now the execution of conditions (18.23). Setting in (18.1), (18.5) $\theta = \theta_1$ and taking into account (18.2), (18.25), we obtain

$$\left.\begin{aligned}&\int_{\theta_0}^{\theta_1} R_1 \varkappa_{1(1)}\, d\theta + \omega_y = 0,\\&\nu\varepsilon_{2(1)}(\theta_1) - \int_{\theta_0}^{\theta_1}\Big[\gamma_{(1)} + \varepsilon_{2(1)}\cos\theta - \Psi\sin\theta +\\&\qquad + \sin\theta\int_{\theta_0}^{\theta} R_1\varkappa_{1(1)}\, d\theta + \omega_y\sin\theta\Big] R_1\, d\theta + u_{0x} = 0.\end{aligned}\right\} \qquad (18.26)$$

From (18.26) we find

$$\left.\begin{aligned}&\omega_y = -\int_{\theta_0}^{\theta_1} R_1\varkappa_{1(1)}\, d\theta,\\&u_{0x} = \omega_y\int_{\theta_0}^{\theta_1} R_1\sin\theta\, d\theta - \nu\varepsilon_{2(1)}(\theta_1) +\\&+ \int_{\theta_0}^{\theta_1}\left[\gamma_{(1)} + \varepsilon_{2(1)}\cos\theta - \Psi\sin\theta + \sin\theta\int_{\theta_0}^{\theta} R_1\varkappa_{1(1)}\, d\theta\right] R_1\, d\theta.\end{aligned}\right\} \qquad (18.27)$$

The case of a rigidly sealed edge loaded by force P_y and with moment M_x is considered analogously. Really, introducing new axes OX^*, OY^* (Fig. 10b), we find

$$P_x^* = -P_y, \quad M_y^* = M_x,$$
$$u_{0x}^* = -u_{0y}, \quad \omega_y^* = \omega_x$$

and all previous arguments are repeated in curvilinear coordinates θ, ϕ^*, whereupon we can return to previous coordinates θ, φ $\left(\varphi^* = \varphi + \frac{\pi}{2}\right)$.

§ 19. Character of the Stressed State of a Shell During Axisymmetric and Bending Loads

In §§ 12 and 16 of this chapter it was shown that the problem about the equilibrium of a shell which undergoes to action of non-self-balancing loads reduces to the solution of the equation

$$L(\sigma) + 2i\gamma_0^2 \frac{R_1^2 \sin\theta}{b\nu\alpha}\sigma - \frac{R_1^2 \cos^2\theta}{\nu^2}\sigma = -2\gamma_0^2\left\{\frac{R_1^2}{\nu b^2} 2\gamma_0^2 \frac{\Phi_2(\theta)}{\alpha^3} + i\left[\mu \frac{d}{d\theta}\left(\frac{\Phi_1}{\alpha}\right) + \frac{R_1\cos\theta}{\nu\alpha}\Phi_1(\theta)\right]\frac{R_1}{\nu b\alpha}\right\} \quad (19.1)$$

during axisymmetric loading and the equation

$$L(\sigma) + 2i\gamma_0^2 \frac{R_1^2 \sin\theta}{b\nu\alpha}\sigma - \frac{4R_1^2}{\nu^2}\sigma = 2\gamma_0^2\left[2\gamma_0^2 \frac{\Phi_3(\theta)}{\alpha^3}\frac{R_1^2}{b^2} + i\frac{R_1^2}{b}\Phi_4(\theta)\right] \quad (19.2)$$

during a bending load, where

$$L(\sigma) = \frac{d^2\sigma}{d\theta^2} + \frac{d\sigma}{d\theta}\left(\frac{3\alpha'}{\alpha} + \frac{R_1\cos\theta}{\nu} - \frac{1}{R_1}\frac{dR_1}{d\theta}\right). \quad (19.3)$$

Φ_1, Φ_2 and Φ_3, Φ_4 are known functions of the load and can be calculated according to formulas (11.11), (16.10), (16.11) $\alpha = \frac{h(\theta)}{h_0}$ – relative thickness of shell, b - certain characteristic dimension of shell (for example, b can be the radius of curvature in extreme section R_1^0 or R_2^0).

$$2\gamma_0^2 = \sqrt{12(1-\mu^2)}\frac{b}{h_0} \quad (19.4)$$

- large parameter.

The left sides of equations (19.1) and (19.2) differ from another in only one term in the coefficient of unknown function σ, which has no essential value if $\nu \neq 0$ and parameter $2\gamma_0^2$ is great in comparison with unity.

The general solution of equation (19.1) or (19.2) is made up from the general solution of the corresponding uniform equation and a certain particular solution of an equation with a non-zero right side. Let us show that in certain cases the particular solution of these equations can be the approximate solution of the initial system of equations of the theory of shells, the so-called solution to the "zero-moment" theory. In calculation from the zero-moment theory it is assumed that amounts M_1/b, M_2/b, H/b, N_1, N_2 are small in comparison with forces T_1, T_2 and S and in accordance with this in equations of statics (4.22) they are dropped. "Zero-moment" forces T_1, T_2, S depend on three simplified equations (4.22). By the found forces with the aid of three elasticity relationships (15.17) deformations ε_1, ε_2, γ, are determined, then integration the first three relationships of (3.19) to the displacements which correspond to the zero-moment state. The zero-moment theory of shells and questions connected with the satisfaction of boundary conditions within the framework of this theory have been comprehensively examined in monograph [5].

We will stop first on constructing a particular solution to equation (19.1). We look for it in the form of a series in degrees $\frac{1}{2\gamma_0^2}$ (or, which is the same, in degrees $\frac{h_0}{b}$ [12]

$$\sigma=\sigma_0+\frac{1}{2\gamma_0^2}\sigma_1+\left(\frac{1}{2\gamma_0^2}\right)\sigma_2+\cdots \tag{19.5}$$

Substituting (19.5) into (19.1) and equating terms with identical quantities $1/2\gamma_0^2$, we obtain

$$\left.\begin{aligned}
&\sigma_0=i\,\frac{2\gamma_0^2}{b\sin\theta}\,\frac{\Phi_2(\theta)}{a^2},\\
&\sigma_1=2\gamma_0^2\left\{\operatorname{ctg}^2\theta\,\frac{\Phi_2(\theta)}{\nu a}-\frac{\nu a}{\hat{R}_1^2\sin\theta}\,L\left[\frac{\Phi_2(\theta)}{\sin\theta a^2}\right]-\right.\\
&\qquad\left.-\frac{1}{R_1\sin\theta}\left[\mu\,\frac{d}{d\theta}\left(\frac{\Phi_1}{a}\right)+\frac{R_1\cos\theta}{\nu a}\,\Phi_1(\theta)\right]\right\}
\end{aligned}\right\} \tag{19.6}$$

etc.

Since in deriving resolvent equations (19.1), (19.2) terms of degree h_0/b in comparison with unity were dropped, then in developing a solution it is natural to be limited to the same correctness. In this way, the particular solution of (19.1) can be taken in the form

$$\sigma = i2\gamma_0^2 \frac{1}{b \sin\theta} \frac{\Phi_1(\theta)}{a^2}. \tag{19.7}$$

The particular solution of equation (19.2), constructed analogously, has the form of

$$\sigma = -i2\gamma_0^2 \frac{\nu}{b \sin\theta} \frac{\Phi_3(\theta)}{a^2}. \tag{19.8}$$

It is necessary, however, to emphasize that in dropping small terms in the course of deriving equations(19.1), (19.2), and also in developing a particular solution, one ought to see that the dropped terms nowhere turn into infinity, as happens at $\nu = 0$ or $\sin\theta = 0$. Because of this the constructed particular solutions are adequate only in the vicinity of the change of θ, rather far from points $\theta = 0$, or $\nu = 0$. By formulas §§ 12 and 16 we will find the forces and moments which correspond to particular solutions (19.7), (19.8).

For and axisymmetrically loaded shell on the basis of (19.7) and (12.11) we have

$$\left.\begin{aligned} \nu T_1 &= \frac{1}{\sin\theta}\left(\frac{P_z^0}{2\pi} + \int_{\theta_0}^{\theta} q_z \nu R_1 \, d\theta\right), \\ T_2 &= q_n R_2 - \frac{1}{R_1 \sin^2\theta}\left(\frac{P_z^0}{2\pi} + \int_{\theta_0}^{\theta} q_z \nu R_1 \, d\theta\right), \\ N_1 &= M_1 = M_2 = 0. \end{aligned}\right\} \tag{19.9}$$

The stressed state corresponding to particular solution (19.7) is zero-moment — shear forces and bending moments in the shell are equal to zero. By a direct check one can ascertain that forces (19.9) satisfy equations of equilibrium of the zero-moment theory, which are derived from formulas (11.1) at $N_1 = M_1 = M_2 = 0$. Therefore subsequently particular solution (19.7) will be called the zero-moment solution, and forces (19.9) are the forces found from zero-moment theory.

Under a bending load by (19.8) and (16.20) we obtain the following forces:

$$\left.\begin{aligned} T_{1(1)} &= -\frac{\cos\theta}{\sin\theta}(F_0+F_1) + (f_0+f_1), \\ S_{(1)} &= -\frac{1}{\sin\theta}(F_0+F_1) - \frac{1}{\nu} f_2, \\ T_{2(1)} &= \frac{\nu\cos\theta}{R_1\sin^2\theta}(F_0+F_1) - \frac{\nu}{R_1\sin\theta}(f_0+f_1) + q_{n(1)}R_2, \end{aligned}\right\} \quad (19.10)$$

where F_0, F_1, f_0, f_1, f_2 are functions of the external load, designated by formulas (15.34), (15.35), (16.10).

If we calculate according to formulas (16.20) and (19.8), (14.4) moments $M_{1(1)}, M_{2(1)}, H_{(1)}$ and forces $N_{1(1)}, N_{2(1)}$, then we obtain, generally speaking, quantities differing from zero, but small: having the order of $\frac{1}{4\gamma_0^4}$ of forces multiplied by dimension b for moments, and the order of $\frac{1}{4\gamma_0^4}$ of forces for shear forces. Therefore at the substitution of the obtained expressions into equilibrium equations the quantities $M_{1(1)}/b, M_{2(1)}/b, H_{(1)}/b, N_{1(1)}, N_{2(1)}$ should be dropped in comparison with the corresponding terms containing $T_{1(1)}, T_{2(1)}, S_{(1)}$. Substituting forces (19.10) into equations (15.28), (15.29) and the third equation of (14.3), setting preliminarily $N_{1(1)} = M_{1(1)} = M_{2(1)} = H_{(1)} = 0$, we verify that they are satisfied. This implies that the basic equilibrium equations of zero-moment theory, which are derived from system (14.3) in the absence of moments and shear forces, are satisfied also.

We go now to the solution fo the uniform equation which corresponds to (19.1). We need to transform this equation to the form

$$\frac{d^2\tau}{dx^2} + f(x)\tau = 0.$$

which makes possible the application of several asymptotic methods of integration [47], [45], [12]. Let us replace the independent variable

$$dx = \frac{R_1}{\sqrt{b}} \sqrt{\frac{\sin\theta}{\nu a}}\, d\theta. \tag{19.11}$$

Having in view that

$$\frac{d\sigma}{d\theta} = \frac{d\sigma}{dx}\frac{dx}{d\theta} = \frac{d\sigma}{dx}\frac{R_1}{\sqrt{b}}\sqrt{\frac{\sin\theta}{\nu a}},$$

$$\frac{d^2\sigma}{d\theta^2} = \frac{d^2\sigma}{dx^2}\left(\frac{dx}{d\theta}\right)^2 + \frac{d\sigma}{dx}\frac{d^2x}{d\theta^2} = \frac{d^2\sigma}{dx^2}\frac{R_1^2}{b}\frac{\sin\theta}{\nu a} + \frac{d\sigma}{dx}\frac{d}{d\theta}\left(\frac{R_1}{\sqrt{b}}\sqrt{\frac{\sin\theta}{\nu a}}\right),$$

we derive the equation

$$\frac{d^2\sigma}{dx^2} + \frac{d\sigma}{dx}\left[\frac{d^2x}{d\theta^2} + \left(\frac{3a'}{a} + \frac{R_1\cos\theta}{\nu} - \frac{1}{R_1}\frac{dR_1}{d\theta}\right)\frac{dx}{d\theta}\right]\frac{b\nu a}{R_1^2\sin\theta} +$$
$$+ 2i\gamma_0^2\sigma - \frac{ba\cos^2\theta}{\nu\sin\theta}\sigma = 0.$$

It is known that in an equation of the second degree by a special replacement of variables it is always possible for the coefficient of the first derivative to turn into zero, for example, setting

$$\sigma = \tau u,$$

where τ – new unknown function. Taking into account that

$$\frac{d\sigma}{dx} = \frac{d\tau}{dx}u + \tau\frac{du}{dx}, \quad \frac{d^2\sigma}{dx^2} = \frac{d^2\tau}{dx^2}u + 2\frac{d\tau}{dx}\frac{du}{dx} + \tau\frac{d^2u}{dx^2},$$

and gathering terms, we require that the sum of equations containing $\frac{d\tau}{dx}$, be zero. we obtain the equation for determining u.

$$2\frac{du}{d\theta}\frac{d\theta}{dx}+u\left[\frac{d^2x}{d\theta^2}+\left(\frac{3\alpha'}{\alpha}+\frac{R_1\cos\theta}{\nu}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)\frac{dx}{d\theta}\right]\frac{b\nu\alpha}{R_1^2\sin\theta}=0.$$

or writing out $\frac{d^2x}{d\theta^2}$, $\frac{dx}{d\theta}$ in detail, and gathering terms,

$$2\frac{du}{u}+\left(\frac{1}{2}\frac{\cos\theta}{\sin\theta}+\frac{1}{2}\frac{R_1\cos\theta}{\nu}+\frac{5}{2}\frac{\alpha'}{\alpha}\right)d\theta=0,$$

whence follows

$$\ln u=\ln\left(\frac{1}{\alpha\sqrt[4]{\nu\alpha\sin\theta}}\right),\quad u=\frac{1}{\alpha\sqrt[4]{\nu\alpha\sin\theta}}.$$

Thus, setting

$$\sigma=\frac{\tau}{\alpha\sqrt[4]{\nu\alpha\sin\theta}}, \tag{19.12}$$

we arrive at an equation for unknown τ

$$\frac{d^2\tau}{dx^2}+\tau\left[2i\gamma_0^2-\frac{b\alpha\cos^2\theta}{\nu\sin\theta}-\psi(\theta)\right]=0, \tag{19.13}$$

where it is noted that

$$\psi(\theta)=\frac{b\nu\alpha}{4R_1^2\sin\theta}\left[-\frac{1}{\sin^2\theta}-\frac{\cos^2\theta}{4\sin^2\theta}+\frac{1}{2}\frac{R_1\cos^2\theta}{\nu\sin\theta}-\frac{1}{4}\frac{R_1^2\cos^2\theta}{\nu^2}-\right.$$
$$-\frac{1}{R_1}\frac{\cos\theta}{\sin\theta}\frac{dR_1}{d\theta}+\frac{\alpha'}{\alpha}\left(\frac{1}{2}\frac{\cos\theta}{\sin\theta}+\frac{11}{2}\frac{R_1\cos\theta}{\nu}\right)-\frac{R_1\sin\theta}{\nu}-$$
$$\left.-\frac{5\alpha'}{\alpha}\frac{1}{R_1}\frac{dR_1}{d\theta}+\frac{15}{4}\frac{(\alpha')^2}{\alpha^2}+\frac{5\alpha''}{\alpha}\right]. \tag{19.14}$$

Transforming likewise homogeneous equation (19.2) for a bending load, we obtain

$$\frac{d^2\tau}{dx^2} + \tau\left[2i\gamma_0^2 - \frac{4ba}{v\sin\theta} - \psi(\theta)\right] = 0. \tag{19.15}$$

where ψ is determined again according to (19.14).

To investigate the character of solutions of equation (19.13), (19.15) at $v \neq 0$, $\sin\theta \neq 0$ we neglect in coefficient of τ variable terms as possessing the order of unity in comparison with the large term $2i\gamma_0^2$. Then we arrive at equation

$$\frac{d^2\tau}{dx^2} + 2i\gamma_0^2\tau = 0. \tag{19.16}$$

the general solution of which has the form of

$$\tau = C_1 e^{-\gamma_0(1-i)x} + C_2 e^{\gamma_0(1-i)x}, \tag{19.17}$$

where C_1, C_2 – constants of integration – generally speaking, imaginary numbers. Variable x on the basis of (19.11) can be expressed through original variable θ

$$x = \frac{1}{\sqrt{b}}\int_{\theta_0}^{\theta}\sqrt{\frac{\sin\theta}{va}}R_1\,d\theta. \tag{19.18}$$

Replacing constant C_2 by a certain other constant

$$C_2 e^{-\gamma_0(1-i)x(\theta_1)},\quad x(\theta_1) = \frac{1}{\sqrt{b}}\int_{\theta_0}^{\theta_1}\sqrt{\frac{\sin\theta}{va}}R_1\,d\theta$$

returning to basic variable σ, we will write the approximate solution of equations (19.1), (19.2) (right side equal to zero) in the form

$$\sigma = \frac{1}{a\sqrt[4]{va\sin\theta}}\left[C_1 e^{-\gamma_0(1-i)x} + C_2 e^{-\gamma_0(1-i)x_1}\right], \tag{19.19}$$

where x is determined according to (19.18), and

$$x_1 = \frac{1}{\sqrt{b}} \int_{\theta}^{\theta_1} \sqrt{\frac{\sin\theta}{va}} R_1 \, d\theta. \tag{19.20}$$

It is easy to see that x increases from θ_0 to θ and, consequently, the solution $e^{-\gamma_0(1-i)x}$ decreases in proportion to the distance from edge θ_0. On the contrary, x_1 increases from θ_1 to θ and, correspondingly, the solution $e^{-\gamma_0(1-i)x}$ decreases in proportion to the distance from edge $\theta = \theta_1$. Variables x and x_1 are dimensicnless. At a sufficiently large value of x and parameter γ_0, proportional to $\sqrt{\frac{b}{h_0}}$, the value of the first solution $e^{-\gamma_0(1-i)x_1}$ in the neighborhood of edge $\theta = \theta_1$ will be negligibly small as compared to the second $e^{-\gamma_0(1-i)x_1}$. The same can be said about the behavior of the second solution in the neighborhood of edge $\theta = \theta_0$. Thus, for a sufficiently large parameter γ_0 and shell length, i.e., when $\gamma_0 x(\theta_1) = \gamma_0 x_1(\theta_0)$ has the order of several units, each of the independent solutions in (19.19) describes the state of the shell in the neighborhood of its edge:

$$\sigma = \frac{1}{a\sqrt[4]{va\sin\theta}} C_1 e^{-\gamma_0(1-i)x}, \qquad \theta \approx \theta_0. \tag{19.21}$$

$$\sigma = \frac{1}{a\sqrt[4]{va\sin\theta}} C_2 e^{-\gamma_0(1-i)x_1}, \qquad \theta \approx \theta_1. \tag{19.22}$$

Frror connected with the separate representation of the solutions for each of the edges for a given shell is simple to evaluate. Thus, if $\gamma_0 x(\theta_1) = 3$, then at edge $\theta = \theta_0$ the first solution assumes the value

$$C_1 \left(\frac{1}{a\sqrt[4]{va\sin\theta}} \right)_{\theta=\theta_0},$$

the second solution at this edge gives

$$C_2 \frac{1}{\left(a\sqrt[4]{va\sin\theta}\right)_{\theta=\theta_0}} \cdot e^{-3(1-i)} = \frac{C_2 e^{3i}}{\left(a\sqrt[4]{va\sin\theta}\right)_{\theta=\theta_0}} \cdot 0.04.$$

In this way, if the quantities $\alpha, v\sin\theta$ during extension of the shell change not too strongly, the error is of the order of 4%. If the shell is sufficiently long, then in the middle part of the shell, far from both edges, both solutions (19.21) and (19.22) will be negligible in comparison with the zero-moment solution. In this way, the stressed state of a thin rather long shell consists of a fundamental slowly changing zero-moment stressed state, onto which in the boundary zones are imposed states which correspond to solutions (19.21), (19.22). This phenomenon of the existence of local perturbation of the stressed state in the area of the edges of a thin shell received the name "edge effect." One ought to note that edge effect, i.e., local elevation in the stress in a shell, can be observed not only in the neighborhood of the edges, but also in places of a pronounced change in load, thickness or angle of inclination, or curvature of the meridian of the shell.

Thus, for instance, if along a certain parallel $\theta = \theta^*$ is applied a distributed normal or tangential load, or a distributed bending moment, then, dividing the shell into two sections (θ_0, θ^*) and (θ^*, θ_1) and replacing the action of section (θ^*, θ_1) on section (θ_0, θ^*) by a certain system of boundary forces applied in section θ^*, we arrive at consideration of section of shell (θ_0, θ^*), for which by the above method a zero-moment solution and a solution of the edge effect type should be constructed. The same should be done even for section (θ^*, θ_1), where in section θ^* should be applied a system of boundary forces and moments giving in sum with forces and moments applied in section θ^* to the first section of shell an assigned external load in θ^*. These boundary forces should be determined from conditions of continuity of several geometric and static curves of the deformed state during transition through section θ^*. In this way, in cases of a pronounced change in load or pronounced change in geometrics quantities (curvature, thickness) it is necessary to divide the shell into individual parts with smoothly changing load, thickness and curvature and then solve the problem of connecting these parts with each other. Subsequently we consider such problems for concrete forms of shells of rotation. Here these considerations were given in order to explain that edge effect can appear not only at the edge as such, but also in places of a pronounced change in load or geometric curves of the shell.

§ 20. Temperature Stresses. Formulation of the Problem

In a shell a stressed state can arise not only from the influence of external forces, but also as a result of nonuniform distribution of temperature. Let us assume that there are no external loads, and edge of the shell can freely move. Because of nonuniform heating individual elements of the shell tend to broaden also unevenly, and since they are interconnected, in the shell appears a stressed state. Forces and moments statically equivalent to this internal stressed state satisfy uniform equilibrium equations. Forces and moments at the edges of the shell are equal to zero. In this case the zero stressed state $(T_1=T_2=S=M_1=M_2=H=0)$ is statically possible, i.e., satisfies equilibrium equations and power boundary conditions. But it can be realized only for definite conditions imposed on temperature distribution. To determine a non-zero internal stressed state it is necessary to add physical and geometric relationships to the equations of equilibrium.

Total relative elongations of elements of an elastic body are made up of temperature elongations and elongations connected with internal stress by Hooke's law [39]. Components of deformation are expressed through displacements by the usual method. Taking these positions, we need to write out the total system of equations describing the deformation of a shell during nonuniform temperature distribution.

We assume that with respect to the thickness of the wall temperature changes linearly, i.e.

$$t(\theta, \varphi)=t^m(\theta, \varphi)+\frac{\zeta}{h}\Delta t(\theta, \varphi). \qquad (20.1)$$

where ζ – distance from middle surface, read along the normal, t^m – average temperature of wall, Δt – drop in temperature with respect to depth. If we designate temperature of the external $(\zeta=+h/2)$ and internal $(\zeta=-h/2)$ surfaces of the shell through t^+ and t^-, then

$$t^m=\frac{t^++t^-}{2}, \quad \Delta t=t^+-t^-. \tag{20.2}$$

The components of deformation of an element of the shell lying on layer $\zeta=\text{const}$, are equal to

$$\begin{aligned} e_1&=\frac{1}{E}(\sigma_1-\mu\sigma_2)+\beta t,\\ e_2&=\frac{1}{E}(\sigma_2-\mu\sigma_1)+\beta t, \quad \omega=\frac{2(1+\mu)}{E}\tau_{12}, \end{aligned} \tag{20.3}$$

here β – coefficient of linear temperature expansion. It is a physical constant of the material from which the shell is made. Solving equation (20.3) in $\sigma_1, \sigma_2, \tau_{12}$, we obtain

$$\left.\begin{aligned} \sigma_1&=\frac{E}{1-\mu^2}[e_1+\mu e_2-(1+\mu)\beta t],\\ \sigma_2&=\frac{E}{1-\mu^2}[e_2+\mu e_1-(1+\mu)\beta t],\\ \tau_{12}&=\frac{E}{2(1+\mu)}\omega. \end{aligned}\right\} \tag{20.4}$$

Ignoring in equations (3.13), (3.18) the quantities $\frac{\zeta}{R_1}$ and $\frac{\zeta}{R_2}$ in comparison with unity, we find that elongations and shear in surface points $\zeta=\text{const}$ are expressed through components of deformation of the middle surface in the following manner:

$$\left.\begin{aligned} e_1&=\varepsilon_1+\zeta\varkappa_1, \quad e_2=\varepsilon_2+\zeta\varkappa_2,\\ \omega&=\gamma+\zeta 2\tau. \end{aligned}\right\} \tag{20.5}$$

Substituting expressions (20.5) into formulas (20.4) and using simplified expressions to calculate forces and bending moments (in formulas (4.5), just as in (3.13), we can neglect quantities of the order of ζ/R_1 and ζ/R_2 in comparison with unity), we obtain

$$
\left.\begin{aligned}
T_1 &= B\,[\varepsilon_1 + \mu\varepsilon_2 - (1+\mu)\,\beta t^m],\\
T_2 &= B\,[\varepsilon_2 + \mu\varepsilon_1 - (1+\mu)\,\beta t^m],\\
S &= B\,\frac{(1-\mu)}{2}\,\gamma,
\end{aligned}\right\} \qquad (20.6)
$$

$$
\left.\begin{aligned}
M_1 &= D\left[\varkappa_1 + \mu\varkappa_2 - (1+\mu)\,\beta\,\frac{\Delta t}{h}\right],\\
M_2 &= D\left[\varkappa_2 + \mu\varkappa_1 - (1+\mu)\,\beta\,\frac{\Delta t}{h}\right],\\
H &= D\,(1-\mu)\,\tau,
\end{aligned}\right\} \qquad (20.7)
$$

where we have introduced the designations

$$
B = \frac{Eh}{(1-\mu^2)}, \quad D = \frac{Eh^3}{12\,(1-\mu^2)}.
$$

Components of deformation of the middle surface are connected with displacements *u, v, w* by the formulas (3.19). Equations of equilibrium, elasticity relationships (20.6), (20.7) and expressions (3.19) form the total system of equations for determining forces, moments and displacements in a shell for an assigned temperature distribution. It is necessary to combine it with the boundary conditions, which in the considered case, when external forces are absent, are uniform static conditions.

In conclusion let us note that relationships (20.6), (20.7) can be rewritten in the form

$$
\left.\begin{aligned}
\varepsilon_1 &= \varepsilon_1^e + \varepsilon_1^t, & \varkappa_1 &= \varkappa_1^e + \varkappa_1^t,\\
\varepsilon_2 &= \varepsilon_2^e + \varepsilon_2^t, & \varkappa_2 &= \varkappa_2^e + \varkappa_2^t,\\
\gamma &= \gamma^e + \gamma^t, & \tau &= \tau^e + \tau^t,
\end{aligned}\right\} \qquad (20.8)
$$

where

$$
\left.\begin{aligned}
\varepsilon_1^e &= \frac{1}{Eh}(T_1 - \mu T_2), & \varkappa_1^e &= \frac{12}{Eh^3}(M_1 - \mu M_2),\\
\varepsilon_2^e &= \frac{1}{Eh}(T_2 - \mu T_1), & \varkappa_2^e &= \frac{12}{Eh^3}(M_2 - \mu M_1),\\
\gamma^e &= \frac{2\,(1+\mu)}{Eh}\,S, & \tau^e &= \frac{12\,(1+\mu)}{Eh}\,H,
\end{aligned}\right\} \qquad (20.9)
$$

$$\left.\begin{array}{c}\varepsilon_1^t=\varepsilon_2^t=\beta t^m, \quad \varkappa_1^t=\varkappa_2^t=\beta\frac{\Delta t}{h}, \\ \gamma^t=\tau^t=0.\end{array}\right\} \tag{20.10}$$

The quantities in (20.9) are given an e to show that they are connected with forces and moments by elasticity relationships of the usual form.

The total components of deformation (20.8) should satisfy equations of continuity (3.30). Since $t^m, \Delta t$ are assigned functions of coordinates θ, φ, and $\varepsilon_1^e, \ldots, \tau^e$ are expressed through forces and moments by (20.9), then the equations of continuity after substituting into them expressions (20.8) will turn into three heterogeneous differential equations in six unknown forces and moments. Together with the equations of equilibrium they form a system of six differential equations of the eighth degree in T_1, T_2, S, M_1, M_2, H with uniform static boundary conditions. This system of equations in heterogeneous because of the equations of continuity. If the distribution of temperature is such that deformation components $\varepsilon_1^t, \varepsilon_2^t, \ldots, \tau^t$ identically satisfy the continuity equations, then in this instance the equations of continuity in forces and moments will have free terms identical equal to zero. To attribute forces and moments $(T_1, T_2, S, M_1, M_2, H)$ we derive a uniform system of six equations with uniform boundary conditions. The solution of this system is identically zero, and a stressed state does not appear in the shell. This case exists for a linear distribution of temperature in the space taken by the shell.

§ 21. Linear Distribution of Temperature. Determination of Displacements

Let us examine a linear distribution of temperature

$$t=K+A_{(0)}Z+A_{(1)}X+A^{(1)}Y, \tag{21.1}$$

where X, Y, Z are the Cartesian coordinates of an arbitrary point of the shell. They can be expressed through curvilinear coordinates θ, φ, ζ in the following manner (see Fig. 1 and formulas § 1):

$$X=(\nu+\zeta\sin\theta)\cos\varphi,\quad Y=(\nu+\zeta\sin\theta)\sin\varphi,$$
$$Z=-\int_{\theta_0}^{\theta}R_1\sin\theta\,d\theta+\zeta\cos\theta. \tag{21.2}$$

where ν, R_1 are known functions of coordinate θ.

Taking into account (21.2), we will rewrite (21.1) in the form

$$t=K+A_{(0)}\left(-\int_{\theta_0}^{\theta}R_1\sin\theta\,d\theta+\zeta\cos\theta\right)+ \\ +A_{(1)}(\nu+\zeta\sin\theta)\cos\varphi+A^{(1)}(\nu+\zeta\sin\theta)\sin\varphi \tag{21.3}$$

Comparing (21.3) and (20.1), we find that in the considered case

$$t^{m}=K-A_{(0)}\int_{\theta_0}^{\theta}R_1\sin\theta\,d\theta+A_{(1)}\nu\cos\varphi+A^{(1)}\nu\sin\varphi,$$
$$\frac{\Delta t}{h}=A_{(0)}\cos\theta+A_{(1)}\sin\theta\cos\varphi+A^{(1)}\sin\theta\sin\varphi. \tag{21.4}$$

It is simple to verify that in accordance with the general law of the theory of elasticity stresses in a free shell during linear temperature distribution (21.1) or, which is the same (21.3), are equal to zero.

By (21.4) and (20.10) we have

$$\varepsilon_1^t=\varepsilon_2^t=\varepsilon_{(0)}^t+\varepsilon_{(1)}^t\cos\varphi+\varepsilon^{(1)t}\sin\varphi,$$
$$\varkappa_1^t=\varkappa_2^t=\varkappa_{(0)}^t+\varkappa_{(1)}^t\cos\varphi+\varkappa^{(1)t}\sin\varphi,$$
$$\gamma^t=\tau^t=0. \tag{21.5}$$

where we introduced the designations

$$\varepsilon_0^t=\beta K-\beta A_{(0)}\int_{\theta_0}^{\theta}R_1\sin\theta\,d\theta,$$
$$\varkappa_{(0)}^t=\beta A_{(0)}\cos\theta. \tag{21.6}$$

$$\varepsilon'_{(1)} = \beta A_{(1)}\nu, \quad \varkappa'_{(1)} = \beta A_{(1)} \sin\theta, \tag{21.7}$$

$$\varepsilon^{(1)\prime} = \beta A^{(1)}\nu, \quad \varkappa^{(1)\prime} = \beta A^{(1)} \sin\theta. \tag{21.8}$$

Let us examine deformation during axisymmetric temperature distribution (first term in formulas (21.5)). To realize the statically possible zero stressed state it is necessary that the components of strain

$$\left.\begin{aligned} \varepsilon_{1(0)} = \varepsilon_{2(0)} = \varepsilon'_{(0)}, \quad \gamma^{(0)} = 0, \\ \varkappa_{1(0)} = \varkappa_{2(0)} = \varkappa'_{(0)}, \quad \tau^{(0)} = 0 \end{aligned}\right\} \tag{21.9}$$

identically satisfy equations of continuity (8.15), (8.16). It is clear that equation (8.16), connecting only $\gamma^{(0)}$, $\tau^{(0)}$, is satisfied in § 11 it was shown that equations of continuity (8.15) allow one first integral (formulas (11.5), (11.6)). Therefore (8.15) can be replaced by the equations:

$$\left.\begin{aligned} \nu\varkappa_{2(0)} \sin\theta + \frac{\cos\theta}{R_1}\left[\frac{d}{d\theta}(\nu\varepsilon_{2(0)}) - \varepsilon_{1(0)} R_1 \cos\theta\right] = 0, \\ \frac{d}{d\theta}(\nu\varkappa_{2(0)}) - R_1\varkappa_{1(0)} \cos\theta - \frac{1}{R_1}\frac{d}{d\theta}(\nu\varepsilon_{2(0)}) + \varepsilon_{1(0)} \cos\theta = 0. \end{aligned}\right\} \tag{21.10}$$

Substitution of expressions (21.9), (21.6) into (21.10) turns (21.10) into identities. We will examine now temperature distribution proportional to $\cos\varphi$. Corresponding amplitudes of deformations are equal to

$$\left.\begin{aligned} \varepsilon_{1(1)} = \varepsilon_{2(1)} = \varepsilon'_{(1)}, \quad \gamma_{(1)} = 0, \\ \varkappa_{1(1)} = \varkappa_{2(1)} = \varkappa'_{(1)}, \quad \tau_{(1)} = 0. \end{aligned}\right\} \tag{21.11}$$

They should satisfy equations of compatibility (8.24), which, as shown in § 15, can be replaced by equations (15.30), (15.31), (15.33). Substituting into these equations components of strain (21.11), we have

$$\left.\begin{aligned} \nu\varkappa'_{(1)} - \varepsilon'_{(1)}\sin\theta = 0, \\ \frac{1}{R_1}\frac{d\varepsilon'_{(1)}}{d\theta} - \varepsilon'_{(1)}\frac{\cos\theta}{\nu} = 0, \\ -R_1\varkappa'_{(1)} + \frac{R_1\sin\theta}{\nu}\varepsilon'_{(1)} = 0. \end{aligned}\right\} \quad (21.12)$$

The three relationships of (21.12) connect two quantities: $\varkappa'_{(1)}, \varepsilon'_{(1)}$. When $\varkappa'_{(1)}, \varepsilon'_{(1)}$ have the form of (21.7), relationships (21.12) are not contradictory, since each of them is satisfied identically. The case of a temperature distribution which is odd relative to plane $\varphi=0$. is checked similarly. Thus, during linear temperature distribution in the space taken up by a shell the components of temperature deformation $\varepsilon'_1, \varepsilon'_2, \varkappa'_1, \varkappa'_2$ satisfy equations of continuity and, consequently, the statically possible zero stressed state is realized.

Let us determine the displacements of points of the middle surface of a free shell during linear temperature distribution. First in (21.4) we set $A_{(1)}=A^{(1)}=0$ and find displacements $u_{(0)}, v^{(0)}, w_{(0)}$ in terms of preassigned deformation components (21.9), (21.6):

$$\left.\begin{aligned} \varepsilon_{1(0)} = \varepsilon_{2(0)} = \beta A_0 Z + \beta K, \\ \varkappa_{1(0)} = \varkappa_{2(0)} = \beta A_0\cos\theta, \quad \gamma^{(0)} = \tau^{(0)} = 0. \end{aligned}\right\} \quad (21.13)$$

From (8.12) it is clear that $v^{(0)}=0$, i.e., there is no twisting using the first relationship of (11.9) we have

$$\frac{d\vartheta_{1(0)}}{d\theta} = -\beta A_{(0)} R_1 \cos\theta,$$

whence, taking into account (1.33) and integrating, we obtain

$$\vartheta_{1(0)} = -\beta A_{(0)}\nu + D.$$

Substituting the obtained expression for $\vartheta_{1(0)}$ and deformation (21.13) into equation (11.8), we see that constant of integration $D=0$. Thus,

$$\vartheta_{1(0)} = -\beta A_{(0)}\nu. \quad (21.14)$$

Now the determination of axial and radial displacements can be conducted according to the system given in § 13. By equations (13.12), (13.15) we derive

$$\left.\begin{aligned}\Delta_r &= \beta\nu(K + A_{(0)}Z),\\ \frac{d\Delta_z}{d\theta} &= -\beta(K + A_{(0)}Z)R_1\sin\theta - \beta A_{(0)}\nu\cos\theta R_1.\end{aligned}\right\} \tag{21.15}$$

Integrating the second relationship of (21.15), we find

$$\Delta_z = D_1 - \beta\int\limits_{\theta_0}^{\theta}(K + A_{(0)}Z)R_1\sin\theta\,d\theta - \beta A_{(0)}\frac{\nu^2 - \nu_0^2}{2}. \tag{21.16}$$

Note that Δ_z can be represented also in the form of [12]

$$\Delta_z = D_2 - \beta\int\limits_{\theta_0}^{\theta}(K + A_{(0)}Z)\frac{R_1 - R_2}{\sin\theta}\,d\theta + \nu\beta(K + A_{(0)}Z)\operatorname{ctg}\theta. \tag{21.17}$$

Really, differentiating (21.17) and comparing the result with the right side of the second equation of (21.15), it is simple to verify the identity of (21.16) and (21.17).

For temperature distribution according to the law

$$t = A_{(1)}(\nu + \zeta\sin\theta)\cos\varphi$$

the amplitudes of deformations in a free shell are equal to

$$\left.\begin{aligned}\varepsilon_{1(1)} &= \varepsilon_{2(1)} = \beta A_{(1)}\nu, \quad \gamma_{(1)} = 0,\\ \varkappa_{1(1)} &= \varkappa_{2(1)} = \beta A_{(1)}\sin\theta, \quad \tau_{(1)} = 0.\end{aligned}\right\} \tag{21.18}$$

The determination of displacements we begin again from angle of rotation $\vartheta_{1(1)}$. According to (8.23), (8.22) $(k=1)$ we have

$$\frac{d\vartheta_{1(1)}}{d\theta} = -R_1\varkappa_{1(1)} = -\beta A_{(1)}R_1\sin\theta. \tag{21.19}$$

Integrating (21.19) and taking into account (15.13), we obtain

$$\vartheta_{1(1)} = \beta A_{(1)} Z + \mathscr{E}. \tag{21.20}$$

Here Z – coordinate of points of the middle surface in section $\theta = \text{const}$. After this $\Delta_{z(1)}$ can be foun. by the formula

$$\Delta_{z(1)} = \mathscr{E}_1 + \int_{\theta_0}^{\theta} (-\varepsilon_{1(1)} \sin\theta + \vartheta_{1(1)} \cos\theta) R_1 \, d\theta =$$
$$= \mathscr{E}_1 + \int_{\theta_0}^{\theta} [(\beta A_{(1)} Z + \mathscr{E}) \cos\theta - \beta A_{(1)} \nu \sin\theta] R_1 \, d\theta. \tag{21.21}$$

and peripheral and radial displacements with (18.4), (18.5)

$$\left. \begin{aligned} v_{(1)} &= \mathscr{E}_2 + \int_{\theta_0}^{\theta} \left(A_{(1)} \beta \nu \cos\theta - \frac{\Delta_{z(1)} \sin\theta}{\nu} \right) R_1 \, d\theta, \\ \Delta_{e(1)} &= \varepsilon_{2(1)} \nu - v_{(1)}. \end{aligned} \right\} \tag{21.22}$$

The obtained expressions contain three constants: $\mathscr{E}$, $\mathscr{E}_1$, $\mathscr{E}_2$. One of them, for example $\mathscr{E}_1$, should be determined from the condition of compatibility of expressions (8.22) at $k=1$. To determine $\mathscr{E}_1$ we use an equation which is a corollary of the fifth and sixth equations of (8.22):

$$\varkappa_{2(1)} \cos\theta + \tau_{(1)} = \frac{\sin\theta}{\nu R_1} \frac{dv_{(1)}}{d\theta} + \frac{1}{\nu} \vartheta_{1(1)} \sin^2\theta. \tag{21.23}$$

Substituting into it the found expressions for $v_{(1)}$ and $\vartheta_{1(1)}$, and also $\varkappa_{2(1)}$ and $\tau_{(1)}$ in accordance with (21.18) and requiring that (21.23) be satisfied identically, after a series of transformations we obtain

$$\mathscr{E}_1 = \mathscr{E} \nu_0.$$

Taking into consideration that

$$\int_{\theta_0}^{\theta} \nu R_1 \cos\theta \, d\theta = \frac{\nu^2 - \nu_0^2}{2}, \quad -\int_{\theta_0}^{\theta} R_1 Z \sin\theta \, d\theta = \frac{Z^2}{2},$$

We derive the final expressions for displacements:

$$\left.\begin{aligned}\vartheta_{1(1)} &= A_{(1)}\beta Z + \mathscr{E},\\ \Delta_{z(1)} &= \mathscr{E}\nu + A_{(1)}\beta Z\nu,\\ v_{(1)} &= \mathscr{E}_2 + \mathscr{E}Z + A_{(1)}\beta\left(\frac{\nu^2-\nu_0^2}{2} + \frac{Z^2}{2}\right),\\ \Delta_{e(1)} &= -\mathscr{E}_2 - \mathscr{E}Z + A_{(1)}\beta\left(\frac{\nu^2+\nu_0^2}{2} - \frac{Z^2}{2}\right).\end{aligned}\right\} \quad (21.24)$$

§ 22. Particular Solution of Meissner Equations Considering an Axisymmetric Temperature Field

In the general case of temperature distribution according to the law

$$\left.\begin{aligned} t^m(\theta,\varphi) &= \sum_{k=0}^{\infty}\left[t^m_{(k)}(\theta)\cos k\varphi + t^{m(k)}(\theta)\sin k\varphi\right],\\ \Delta t(\theta,\varphi) &= \sum_{k=0}^{\infty}\left[\Delta t_{(k)}(\theta)\cos k\varphi + \Delta t^{(k)}(\theta)\sin k\varphi\right],\end{aligned}\right\} \quad (22.1)$$

when $t^m_{(k)}(\theta)$, $t^{m(k)}(\theta)$, $\Delta t_{(k)}(\theta)$, $\Delta t^{(k)}(\theta)$ are several arbitrary functions of coordinate θ, internal forces and moments in the shell are non-zero. Using representations of the desired quantities in the form of trigonometric series (8.1) - (8.3), to determine the k-th harmonic of the static and geometric quantities we will obtain a system of equations consisting of equations of equilibrium (8.20), equations (8.22) and physical relationships of the form

$$\left.\begin{aligned} T_{1(k)} &= B\left[\varepsilon_{1(k)} + \mu\varepsilon_{2(k)} - (1+\mu)\beta t^m_{(k)}\right],\\ T_{2(k)} &= B\left[\varepsilon_{2(k)} + \mu\varepsilon_{1(k)} - (1+\mu)\beta t^m_{(k)}\right],\\ S_{(k)} &= B\frac{1-\mu}{2}\gamma_{(k)},\end{aligned}\right\} \quad (22.2)$$

$$\left.\begin{aligned} M_{1(k)} &= D\left[\varkappa_{1(k)} + \mu\varkappa_{2(k)} - (1+\mu)\beta\frac{\Delta t_{(k)}}{h}\right],\\ M_{2(k)} &= D\left[\varkappa_{2(k)} + \mu\varkappa_{1(k)} - (1+\mu)\beta\frac{\Delta t_{(k)}}{h}\right],\\ H_{(k)} &= D(1-\mu)\tau_{(k)}.\end{aligned}\right\} \quad (22.3)$$

For $k=0, 1$, in the same way as in §§ 12, 16, we can obtain Meissner equations, in the right side of which will now stand specific temperatures functions, which can be considered as a certain fictitious "temperature" load.

Let us examine the stressed state of a shell during preassigned axisymmetric temperature distribution $(k=0)$ [12]. Tangential forces in the shell do not appear, since it experiences in this case only axisymmetric curvature and elongation.

Since the shell is free of external loads, equations of equilibrium (11.1) in this instance will be uniform. The first two of them are satisfied if we set

$$\nu T_{1(0)} = V\cos\theta, \quad \nu N_{1(0)} = V\sin\theta, \quad T_{2(0)} = \frac{1}{R_1}\frac{dV}{d\theta}. \tag{22.4}$$

From (22.4) and (22.2) it's simple to find expressions for $\varepsilon_{1(0)}, \varepsilon_{2(0)}$ through function V:

$$\left.\begin{aligned} \varepsilon_{1(0)} &= \frac{1}{Eh}\left[\frac{V\cos\theta}{\nu} - \frac{\mu}{R_1}\frac{dV}{d\theta}\right] + \beta t^{m}_{(0)}, \\ \varepsilon_{2(0)} &= \frac{1}{Eh}\left[\frac{1}{R_1}\frac{dV}{d\theta} - \mu\frac{V\cos\theta}{\nu}\right] + \beta t^{m}_{(0)}. \end{aligned}\right\} \tag{22.5}$$

Bending moments are expressed now through angle of rotation $\vartheta_{1(0)}$ with the aid of relationships which come from (11.9) and (22.3):

$$\left.\begin{aligned} M_{1(0)} &= -\frac{Eh^3}{12(1-\mu^2)}\left[\frac{1}{R_1}\frac{d\vartheta_{1(0)}}{d\theta} + \mu\frac{\cos\theta}{\nu}\vartheta_{1(0)} + (1+\mu)\beta\frac{\Delta t_{(0)}}{h}\right], \\ M_{2(0)} &= -\frac{Eh^3}{12(1-\mu^2)}\left[\frac{\cos\theta}{\nu}\vartheta_{1(0)} + \frac{\mu}{R_1}\frac{d\vartheta_{1(0)}}{d\theta} + (1+\mu)\beta\frac{\Delta t_{(0)}}{h}\right]. \end{aligned}\right\} \tag{22.6}$$

Substituting (22.4), (22.5), (22.6) into the third equation of equilibrium (11.1) and equation (11.8), we derive the desired equations for V and $\vartheta_{1(0)}$ functions for preassigned temperature distribution $t^{m}_{(0)}(\theta)$, $\Delta t_{(0)}(\theta)$:

$$\frac{\nu}{R_1}\frac{d^2V}{d\theta^2}+\left(\cos\theta-\frac{\nu}{R_1^2}\frac{dR_1}{d\theta}\right)\frac{dV}{d\theta}+V\left(-\frac{R_1\cos^2\theta}{\nu}+\mu\sin\theta\right)-$$
$$-Eh\vartheta_{1(0)}R_1\sin\theta=-Eh\beta\nu\frac{dt^m_{(0)}}{d\theta}. \tag{22.7}$$

$$\frac{\nu}{R_1}\frac{d^2\vartheta_{1(0)}}{d\theta^2}+\left(\cos\theta-\frac{\nu}{R_1^2}\frac{dR_1}{d\theta}\right)\frac{d\vartheta_{1(0)}}{d\theta}+$$
$$+\vartheta_{1(0)}\left(-\frac{R_1\cos^2\theta}{\nu}-\mu\sin\theta\right)+\frac{R_1V\sin\theta}{D}=-(1+\mu)\frac{\beta\nu}{h}\frac{d(\Delta t_{(0)})}{d\theta}. \tag{22.8}$$

By direct check one can be certain that for linear temperature distribution

$$t^m_{(0)}=K-A_{(0)}\int_{\theta_0}^{\theta}R_1\sin\theta\,d\theta,\quad \Delta t_{(0)}=A_{(0)}h\cos\theta,$$

equations (22.7), (22.8) have the solution

$$V=0,\quad \vartheta_{1(0)}=-\beta A_{(0)}\nu. \tag{22.9}$$

using equations (22.4), (22.6) in this instance we obtain

$$T_{1(0)}=T_{2(0)}=M_{1(0)}=M_{2(0)}=N_{1(0)}=0.$$

Replacing variables

$$\sigma_0=\Psi_0-2i\gamma^2V_0, \tag{22.10}$$

where

$$\Psi_0=Eh\vartheta_{1(0)},\quad V_0=\frac{1}{b}V,$$

we go from two equations (22.7), (22.8) to one equation for σ_0. Dropping unessential terms, we obtain

$$\frac{d^2\sigma_0}{d\theta^2}+\left(\frac{R_1\cos\theta}{\nu}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)\frac{d\sigma_0}{d\theta}+2i\gamma^2\sigma_0\frac{R_1^2\sin\theta}{b\nu}-\frac{R_1^2\cos^2\theta}{\nu^2}\sigma_0=$$
$$=Eh\beta\frac{R_1}{b}2\gamma^2\left[-\frac{(1+\mu)}{\sqrt{12(1-\mu^2)}}\frac{d(\Delta t_{(0)})}{d\theta}+i\frac{dt^m_{(0)}}{d\theta}\right]. \tag{22.11}$$

We find the approximate particular solution (22.11) by dividing the right side of the equation by the coefficient of σ_0. Setting $\nu \neq 0$, and dropping quantities of the order of $\frac{1}{2\gamma^2}$ in comparison with unity, we obtain

$$\sigma_0 = Eh\beta \frac{\nu}{R_1 \sin\theta}\left[\frac{dt^m_{(0)}}{d\theta} + i\frac{(1+\mu)}{\sqrt{12(1-\mu^2)}}\cdot\frac{d(\Delta t_{(0)})}{d\theta}\right] \qquad (22.12)$$

consequently,

$$\left.\begin{aligned} \vartheta_{1(0)} &= \beta\frac{R_2}{R_1}\frac{dt^m_{(0)}}{d\theta}, \\ V &= -Eh^2\beta\frac{(1+\mu)}{12(1-\mu^2)}\frac{R_2}{R_1}\frac{d(\Delta t_{(0)})}{d\theta}. \end{aligned}\right\} \qquad (22.13)$$

Forces, moments and displacements corresponding to this solution in the shell will have the form

$$T_{1(0)} = -\frac{Eh^2\beta}{12(1-\mu)}\frac{\cos\theta}{\sin\theta}\frac{1}{R_1}\frac{d(\Delta t_{(0)})}{d\theta}. \qquad (22.14)$$

$$T_{2(0)} = -\frac{Eh^2\beta}{12(1-\mu)}\left\{\frac{R_2}{R_1}\frac{d}{d\theta}\left[\frac{1}{R_1}\frac{d(\Delta t_{(0)})}{d\theta}\right] + \left(1-\frac{R_2}{R_1}\right)\frac{\cos\theta}{\sin\theta}\frac{1}{R_1}\frac{d(\Delta t_{(0)})}{d\theta}\right\}. \qquad (22.15)$$

$$N_{1(0)} = -\frac{Eh^2\beta}{12(1-\mu)}\frac{1}{R_1}\frac{d(\Delta t_{(0)})}{d\theta}, \qquad (22.16)$$

$$M_{1(0)} = -\frac{Eh^3\beta}{12(1-\mu)}\left[\frac{\Delta t_{(0)}}{h} + \frac{R_2}{1+\mu}\frac{1}{R_1}\frac{d}{d\theta}\left(\frac{1}{R_1}\frac{dt^m_{(0)}}{d\theta}\right) + \frac{\cos\theta}{\sin\theta}\frac{1}{R_1}\frac{dt^m_{(0)}}{d\theta} - \frac{R_2}{R_1(1+\mu)}\frac{\cos\theta}{\sin\theta}\frac{1}{R_1}\frac{dt^m_{(0)}}{d\theta}\right]. \qquad (22.17)$$

$$M_{2(0)} = -\frac{Eh^3\beta}{12(1-\mu)}\left[\frac{\Delta t_{(0)}}{h} + \frac{\mu}{1+\mu}\frac{R_2}{R_1}\frac{d}{d\theta}\left(\frac{1}{R_1}\frac{dt^m_{(0)}}{d\theta}\right) - \frac{\mu}{(1+\mu)}\frac{R_2}{R_1}\frac{\cos\theta}{\sin\theta}\frac{1}{R_1}\frac{dt^m_{(0)}}{d\theta} + \frac{\cos\theta}{\sin\theta}\frac{1}{R_1}\frac{dt^m_{(0)}}{d\theta}\right]. \qquad (22.18)$$

$$\vartheta_{2(0)} = \beta\frac{R_2}{R_1}\frac{dt^m_{(0)}}{d\theta}. \qquad (22.19)$$

$$\Delta_{c(0)} = \nu\beta \left\{ t^m_{(0)} - \frac{h}{12} \left[\frac{R_2}{(1-\mu) R_1} \frac{d}{d\theta} \left(\frac{1}{R_1} \frac{d\,\Delta t_{(0)}}{d\theta} \right) + \right.\right.$$

$$\left.\left. + \frac{\cos\theta}{\sin\theta} \left(1 - \frac{R_2}{R_1} \right) \frac{1}{R_1} \frac{d\,\Delta t_{(0)}}{d\theta} \right] \right\}. \quad (22.20)$$

$$\Delta_{z(0)} = D_2 + \beta R_2 \cos\theta\, t^m_{(0)} - \beta \int_{\theta_0}^{\theta} t^m_{(0)} \left(1 - \frac{R_2}{R_1} \right) R_1 \frac{d\theta}{\sin\theta} -$$

$$- \frac{h\beta\nu\mu}{12(1-\mu)} \frac{1}{R_1} \frac{d\,\Delta t_{(0)}}{d\theta} + \frac{h\beta}{12(1-\mu)} \int_{\theta_0}^{\theta} \left(1 + \mu \frac{R_2}{R_1} \right) \cos\theta \frac{d\,(\Delta t_{(0)})}{d\theta}\, d\theta. \quad (22.21)$$

§ 23. Equations of Meissner Type with a Temperature Distribution Varying as $\cos\varphi$ [43], [31]

Let us examine the temperature distribution

$$\left.\begin{aligned} t^m(\theta, \varphi) &= t^m_{(1)}(\theta) \cos\varphi, \\ \Delta t(\theta, \varphi) &= \Delta t_{(1)}(\theta) \cos\varphi. \end{aligned}\right\} \quad (23.1)$$

Our task is to derive Meissner equations analogous to (16.15), (16.16), with the temperature terms in the right side. In deriving these equations one ought to have in mind that relationships (16.2), identically satisfying the two equations of continuity (15.30) and (15.33), stay in force; the same is true of (16.3), if we set the load terms equal to zero. Adding to them physical relationships (22.2), (22.3) $(k=1)$, we express all involved quantities through V and Ψ:

$$\left.\begin{aligned} Eh\varepsilon_{1(1)} &= -\frac{\mu}{R_1} \frac{dV}{d\theta} + V(1-\mu) \frac{\cos\theta}{\nu} - \\ &\quad - \frac{Eh^3}{12} \frac{\sin\theta}{\nu} \left(\frac{1}{R_1} \frac{d\Psi}{d\theta} + \frac{\Psi\cos\theta}{\nu} \right) + \\ &\quad + Eh\beta t^m_{(1)} + \frac{Eh^3}{12} \beta \frac{\sin\theta}{\nu} \left(\frac{\Delta t_{(1)}}{h} - \frac{\sin\theta}{\nu} t^m_{(1)} \right), \\ Eh\varepsilon_{2(1)} &= \frac{1}{R_1} \frac{dV}{d\theta} + V(1-\mu) \frac{\cos\theta}{\nu} - \frac{Eh^3}{12} \Psi \frac{\sin\theta\cos\theta}{\nu^2} + \\ &\quad + Eh\beta t^m_{(1)} + \frac{Eh^3}{12} \beta \frac{\sin\theta}{\nu} \left(\frac{\Delta t_{(1)}}{h} - \frac{\sin\theta}{\nu} t^m_{(1)} \right), \\ Eh\gamma_{(1)} &= 2(1+\mu) \frac{V}{\nu} + \frac{Eh^3}{3} \frac{\Psi \sin\theta}{\nu^2}. \end{aligned}\right\} \quad (23.2)$$

$$
\left.
\begin{aligned}
\varkappa_{1(1)} &= \frac{1}{R_1}\frac{d\Psi}{d\theta} + \frac{\Psi\cos\theta}{\nu} + \\
&\quad + \frac{\sin\theta}{Eh\nu^2}\left[(1-\mu)V\cos\theta - \mu\frac{\nu}{R_1}\frac{dV}{d\theta}\right] + \\
&\quad + \beta t^m_{(1)}\frac{\sin\theta}{\nu} + \frac{\sin^2\theta}{\nu^2}\frac{h^2}{12}\beta\left(\frac{\Delta t_{(1)}}{h} - \frac{\sin\theta}{\nu}t^m_{(1)}\right), \\
\varkappa_{2(1)} &= \frac{\Psi\cos\theta}{\nu} + \frac{\sin\theta}{Eh\nu^2}\left[(1-\mu)V\cos\theta + \frac{\nu}{R_1}\frac{dV}{d\theta}\right] + \\
&\quad + \beta t^m_{(1)}\frac{\sin\theta}{\nu} + \frac{\sin^2\theta}{\nu^2}\frac{h^2}{12}\beta\left(\frac{\Delta t_{(1)}}{h} - \frac{\sin\theta}{\nu}t^m_{(1)}\right), \\
\tau_{(1)} &= -\frac{\Psi}{\nu} + V\frac{2(1+\mu)\sin\theta}{Eh\nu^2}, \\
T_{1(1)} &= \frac{V\cos\theta}{\nu} - \frac{D\sin\theta}{\nu}\left[\frac{1}{R_1}\frac{d\Psi}{d\theta} + (1+\mu)\frac{\Psi\cos\theta}{\nu}\right] + \\
&\quad + \frac{Eh^3}{12(1-\mu)}\frac{\beta\sin\theta}{\nu}\left(\frac{\Delta t_{(1)}}{h} - \frac{\sin\theta}{\nu}t^m_{(1)}\right), \\
T_{2(1)} &= \frac{1}{R_1}\frac{dV}{d\theta} + \frac{V\cos\theta}{\nu} - \frac{D\sin\theta}{\nu}\left[\frac{\mu}{R_1}\frac{d\Psi}{d\theta} + (1+\mu)\frac{\Psi\cos\theta}{\nu}\right] + \\
&\quad + \frac{Eh^3}{12(1-\mu)}\frac{\beta\sin\theta}{\nu}\left(\frac{\Delta t_{(1)}}{h} - \frac{\sin\theta}{\nu}t^m_{(1)}\right), \\
S_{(1)} &= \frac{V}{\nu} + D\cdot 2(1-\mu)\frac{\Psi\sin\theta}{\nu^2},
\end{aligned}
\right\} \qquad (23.3)
$$

$$
\left.
\begin{aligned}
\frac{M_{1(1)}}{D} &= \frac{1}{R_1}\frac{d\Psi}{d\theta} + (1+\mu)\frac{\Psi\cos\theta}{\nu} + \frac{(1-\mu^2)}{Eh}\frac{V\sin\theta\cos\theta}{\nu^2} + \\
&\quad + (1+\mu)\beta\left(\frac{t^m_{(1)}\sin\theta}{\nu} - \frac{\Delta t_{(1)}}{h}\right), \\
\frac{M_{2(1)}}{D} &= \frac{\mu}{R_1}\frac{d\Psi}{d\theta} + (1+\mu)\frac{\Psi\cos\theta}{\nu} + \\
&\quad + \frac{(1-\mu^2)}{Eh}\frac{\sin\theta}{\nu}\left(\frac{1}{R_1}\frac{dV}{d\theta} + \frac{V\cos\theta}{\nu}\right) + (1+\mu)\beta\left(\frac{t^m_{(1)}\sin\theta}{\nu} - \frac{\Delta t_{(1)}}{h}\right), \\
\frac{H_{(1)}}{D} &= -(1-\mu)\frac{\Psi}{\nu} + 2(1-\mu^2)\frac{V\sin\theta}{Eh\nu^2}, \\
N_{1(1)} &= -\frac{T_{1(1)}\cos\theta}{\sin\theta} + \frac{S_{(1)}}{\sin\theta} + \frac{H_{(1)}}{\nu}.
\end{aligned}
\right\} \qquad (23.4)
$$

Equations for the determination of V and Ψ are derived by substituting the expressions for forces, moments and deformations into uniform equation (15.29) and equation of compatibility (15.31).

Writing these equations in variables

$$
\Psi_1 = Eh\Psi, \quad V_1 = \frac{V}{b}. \qquad (23.5)
$$

we will have two equations which differ from (16.15) and (16.16) only in the right sides. The right side of (16.15) is replaced by

$$-(1+\mu)Eh\nu R_1 \frac{d}{d\theta}\left[\frac{1}{\nu}\left(\beta\frac{t_{(1)}^{m}\sin\theta}{\nu}-\beta\frac{\Delta t_{(1)}}{h}\right)\right]. \qquad (23.6)$$

the right side of (16.16) by

$$-\frac{EhR_1\nu}{b}\frac{d}{d\theta}\left(\frac{\beta t_{(1)}^{m}}{\nu}\right)-\frac{R_1}{b}\frac{Eh^3}{12\nu}\frac{d}{d\theta}\left[\sin\theta\left(\frac{\beta\,\Delta t_{(1)}}{h}-\frac{\sin\theta}{\nu}\beta t_{(1)}^{m}\right)\right]. \qquad (23.7)$$

Going to complex functions $\sigma_1=\Psi_1+2i\gamma^2V_1$, instead of two equations in V_1 and Ψ_1 we obtain one

$$\begin{aligned}L(\sigma_1)+\left[2i\gamma^2\frac{R_1^2\sin\theta}{\nu b}-\frac{4R_1^2}{\nu^2}\right]\sigma_1=&\\ =-(1+\mu)Eh\beta\nu R_1\frac{d}{d\theta}\left[\frac{1}{\nu}\left(\frac{t_{(1)}^{m}\sin\theta}{\nu}-\frac{\Delta t_{(1)}}{h}\right)\right]-&\\ -2i\gamma^2\frac{Eh\beta R_1\nu}{b}\frac{d}{d\theta}\left(\frac{t_{(1)}^{m}}{\nu}\right).&\end{aligned} \qquad (23.8)$$

where L is a differential operator

$$L=\frac{d^2}{d\theta^2}+\left(\frac{R_1\cos\theta}{\nu}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)\frac{d}{d\theta}.$$

On making this change, just as in § 16, we dropped quantities of the order of $\frac{1}{2\gamma^2}$ in comparison with unity. The reverse change from (23.8) to two equations gives a result which differs from the original, namely:

$$\begin{aligned}L(\Psi_1)-\frac{4R_1^2}{\nu^2}\Psi_1-4\gamma^4V_1\frac{R_1^2\sin\theta}{\nu b}=&\\ =-(1+\mu)Eh\beta\nu R_1\frac{d}{d\theta}\left[\frac{1}{\nu}\left(\frac{t_{(1)}^{m}\sin\theta}{\nu}-\frac{\Delta t_{(1)}}{h}\right)\right].&\end{aligned} \qquad (23.9)$$

$$L(V_1)-\frac{4R_1^2}{\nu^2}V_1+\Psi_1\frac{R_1^2\sin\theta}{b\nu}=-Eh\beta\frac{R_1\nu}{b}\frac{d}{d\theta}\left(\frac{t_{(1)}^{m}}{\nu}\right). \qquad (23.10)$$

It is easy to see that for linear temperature distribution $t^m_{(1)} = A_{(1)}\nu$, $\Delta t_{(1)} = A_{(1)} h \sin\theta$ the right sides of equations (23.9), (23.10) turn into zero and the equations are satisfied at $\Psi_1 = V_1 = 0$. Calculating forces and moments using equations (23.3), (23.4), we obtain $T_{1(1)} = T_{2(1)} = S_{(1)} = M_{1(1)} = M_{2(1)} = H_{(1)} = 0$.

In general the particular solution of equations (23.9), (23.10) can be approximately represented in the form

$$V_1 = \frac{1}{4\gamma^4} Eh\beta(1+\mu)\frac{\nu^2 b}{R_1 \sin\theta}\frac{d}{d\theta}\left[\frac{1}{\nu}\left(\frac{t^m_{(1)}\sin\theta}{\nu} - \frac{\Delta t_{(1)}}{h}\right)\right]. \tag{23.11}$$

$$\Psi_1 = -Eh\beta\frac{\nu^2}{R_1\sin\theta}\frac{d}{d\theta}\left(\frac{t^m_{(1)}}{\nu}\right). \tag{23.12}$$

To it correspond the following forces, bending moments and displacements:

$$\left.\begin{aligned}
T_{1(1)} = {} & \frac{Eh^3\beta}{12(1-\mu^2)}\left\{(1+\mu)\frac{\nu\cos\theta}{R_1\sin\theta}\frac{d}{d\theta}\left(\frac{t^m_{(1)}\sin\theta}{\nu^2} - \frac{1}{\nu}\frac{\Delta t_{(1)}}{h}\right) + \right.\\
& + (1+\mu)\frac{\cos\theta}{R_1}\frac{d}{d\theta}\left(\frac{t^m_{(1)}}{\nu}\right) + \frac{\sin\theta}{\nu R_1}\frac{d}{d\theta}\left[\frac{\nu^2}{R_1\sin\theta}\frac{d}{d\theta}\left(\frac{t^m_{(1)}}{\nu}\right)\right] - \\
& \left. - (1+\mu)\frac{\sin\theta}{\nu}\left(\frac{t^m_{(1)}\sin\theta}{\nu} - \frac{\Delta t_{(1)}}{h}\right)\right\},\\
T_{2(1)} = {} & \frac{Eh^3\beta}{12(1-\mu^2)}\left\{\frac{(1+\mu)}{\nu R_1}\frac{d}{d\theta}\left[\frac{\nu^3}{R_1\sin\theta}\frac{d}{d\theta}\left(\frac{t^m_{(1)}\sin\theta}{\nu^2} - \frac{1}{\nu}\frac{\Delta t_{(1)}}{h}\right)\right] + \right.\\
& + (1+\mu)\frac{\cos\theta}{R_1}\frac{d}{d\theta}\left(\frac{t^m_{(1)}}{\nu}\right) + \mu\frac{\sin\theta}{\nu R_1}\frac{d}{d\theta}\left[\frac{\nu^2}{R_1\sin\theta}\frac{d}{d\theta}\left(\frac{t^m_{(1)}}{\nu}\right)\right] - \\
& \left. - (1+\mu)\frac{\sin\theta}{\nu}\left(\frac{t^m_{(1)}\sin\theta}{\nu} - \frac{\Delta t_{(1)}}{h}\right)\right\},\\
S_{(1)} = {} & \frac{Eh^3\beta}{12(1-\mu)}\left[\frac{\nu}{R_1\sin\theta}\frac{d}{d\theta}\left(\frac{t^m_{(1)}\sin\theta}{\nu^2} - \frac{1}{\nu}\frac{\Delta t_{(1)}}{h}\right) - \right.\\
& \left. - 2\frac{(1-\mu)}{(1+\mu)}\frac{1}{R_1}\frac{d}{d\theta}\left(\frac{t^m_{(1)}}{\nu}\right)\right],
\end{aligned}\right\} \tag{23.13}$$

$$
\left.
\begin{aligned}
M_{1\,(1)} &= \frac{Eh^3\beta}{12(1-\mu^2)}\Big\{(1+\mu)\Big(\frac{t^m_{(1)}\sin\theta}{\nu}-\frac{\Delta t_{(1)}}{h}\Big)-\\
&-\frac{1}{R_1\nu}\frac{d}{d\theta}\Big[\frac{\nu^3}{R_1\sin\theta}\frac{d}{d\theta}\Big(\frac{t^m_{(1)}}{\nu}\Big)\Big]-\mu\frac{\nu\cos\theta}{R_1\sin\theta}\frac{d}{d\theta}\Big(\frac{t^m_{(1)}}{\nu}\Big)\Big\},\\
M_{2\,(1)} &= \frac{Eh^3\beta}{12(1-\mu^2)}\Big\{(1+\mu)\Big(\frac{t^m_{(1)}\sin\theta}{\nu}-\frac{\Delta t_{(1)}}{h}\Big)-\\
&-\frac{\mu}{R_1\nu}\frac{d}{d\theta}\Big[\frac{\nu^3}{R_1\sin\theta}\frac{d}{d\theta}\Big(\frac{t^m_{(1)}}{\nu}\Big)\Big]-\frac{\nu\cos\theta}{R_1\sin\theta}\frac{d}{d\theta}\Big(\frac{t^m_{(1)}}{\nu}\Big)\Big\},\\
H_{(1)} &= \frac{Eh^3\beta}{12(1+\mu)}\frac{\nu}{R_1\sin\theta}\frac{d}{d\theta}\Big(\frac{t^m_{(1)}}{\nu}\Big).
\end{aligned}
\right\}
\qquad (23.14)
$$

$$
\begin{aligned}
N_{1\,(1)} &= \frac{Eh^3\beta}{12(1-\mu^2)}\Big\{(1+\mu)\frac{\nu\sin\theta}{R_1}\frac{d}{d\theta}\Big(\frac{t^m_{(1)}\sin\theta}{\nu^2}-\frac{1}{\nu}\frac{\Delta t_{(1)}}{h}\Big)-\\
&-\frac{\cos\theta}{\nu R_1}\frac{d}{d\theta}\Big[\frac{\nu^2}{R_1\sin\theta}\frac{d}{d\theta}\Big(\frac{t^m_{(1)}}{\nu}\Big)\Big]+\frac{(1+\mu)\cos\theta}{\nu}\Big(\frac{t^m_{(1)}\sin\theta}{\nu}-\frac{\Delta t_{(1)}}{h}\Big)-\\
&-\frac{(1-\mu)}{R_1\sin\theta}\frac{d}{d\theta}\Big(\frac{t^m_{(1)}}{\nu}\Big)-(1+\mu)\frac{\cos^2\theta}{R_1\sin\theta}\frac{d}{d\theta}\Big(\frac{t^m_{(1)}}{\nu}\Big)\Big\}.
\end{aligned}
\qquad (23.15)
$$

$$
\left.
\begin{aligned}
\vartheta_{1\,(1)} &= \mathscr{E}+\int_{\theta_0}^{\theta}\Big\{-\frac{\beta t^m_{(1)}R_1\sin\theta}{\nu}+\\
&\qquad+\frac{\beta}{\nu}\frac{d}{d\theta}\Big[\frac{\nu^3}{R_1\sin\theta}\frac{d}{d\theta}\Big(\frac{t^m_{(1)}}{\nu}\Big)\Big]\Big\}\,d\theta,\\
\Delta_{z\,(1)} &= \mathscr{E}\nu-\frac{\beta\nu^3}{R_1\sin\theta}\frac{d}{d\theta}\Big(\frac{t^m_{(1)}}{\nu}\Big)+\\
&+\nu\int_{\theta_0}^{\theta}\Big\{-\frac{\beta t^m_{(1)}R_1\sin\theta}{\nu}+\frac{\beta}{\nu}\frac{d}{d\theta}\Big[\frac{\nu^3}{R_1\sin\theta}\frac{d}{d\theta}\Big(\frac{t^m_{(1)}}{\nu}\Big)\Big]\Big\}\,d\theta,\\
v_{(1)} &= \delta^{\circ}_2-\mathscr{E}\int_{\theta_0}^{\theta}R_1\sin\theta\,d\theta+\beta\Big\{\int_{\theta_0}^{\theta}\nu^2\frac{d}{d\theta}\Big(\frac{t^m_{(1)}}{\nu}\Big)d\theta+\\
&+\int_{\theta_0}^{\theta}t^m_{(1)}R_1\cos\theta\,d\theta+\int_{\theta_0}^{\theta}R_1\sin\theta\Big(\int_{\theta_0}^{\theta}t^m_{(1)}\frac{R_1\sin\theta}{\nu}d\theta\Big)d\theta-\\
&-\int_{\theta_0}^{\theta}R_1\sin\theta\int_{\theta_0}^{\theta}\frac{1}{\nu}\frac{d}{d\theta}\Big[\frac{\nu^3}{R_1\sin\theta}\frac{d}{d\theta}\Big(\frac{t^m_{(1)}}{\nu}\Big)\Big]d\theta\,d\theta\Big\}.\\
\Delta_{r\,(1)} &= \varepsilon_{2\,(1)}\nu-v_{(1)}=\beta t^m_{(1)}\nu-v_{(1)}.
\end{aligned}
\right\}
\qquad (23.16)
$$

For linear temperature distribution, the displacements computable using formulas (23.16) agree with (21.24).

CHAPTER III

CYLINDRICAL SHELL

§ 24. The Total System of Equations of a Randomly Loaded Cylindrical Shell

Figures 11, 12 show an element of the middle surface of a cylindrical shell and applied forces and moments, replacing the action of the neglected part of the shell. Let us compose directly equations of equilibrium of the element in the general load case, i.e., without any assumptions about the change in forces and moments with respect to coordinate ϕ. In this case we will suppose that on the chosen element in directions τ_1, τ_2, n act external loads (Fig. 13)

$$\left.\begin{aligned} q_1 R\,d\varphi\,ds &= (p_1^+ + p_1^- + F_1 h)\,R\,d\varphi\,ds, \\ q_2 R\,d\varphi\,ds &= (p_2^+ + p_2^- + F_2 h)\,R\,d\varphi\,ds, \\ q_n R\,d\varphi\,ds &= (p_n^+ + p_n^- + F_n h)\,R\,d\varphi\,ds, \end{aligned}\right\} \qquad (24.1)$$

where $p_1^+, p_2^+, p_n^+, p_1^-, p_2^-, p_n^-$ – components of vectors of surface loads p^+, p^-, on the surfaces $\zeta = \pm h/2$; F_1, F_2, F_n – components of the vector of volume force.

The projections of the primary moment of external loads onto the directions τ_1, τ_2, n are equal to

$$\left.\begin{aligned} R\,d\varphi\,ds L_1 &= \left[(-p_2^+ + p_2^-)\frac{h}{2} - \int_{-h/2}^{+h/2} F_2\zeta\,d\zeta\right] R\,d\varphi\,ds, \\ R\,d\varphi\,ds L_2 &= \left[(p_1^+ - p_1^-)\frac{h}{2} + \int_{-h/2}^{+h/2} F_1\zeta\,d\zeta\right] R\,d\varphi\,ds. \end{aligned}\right\} \qquad (24.2)$$

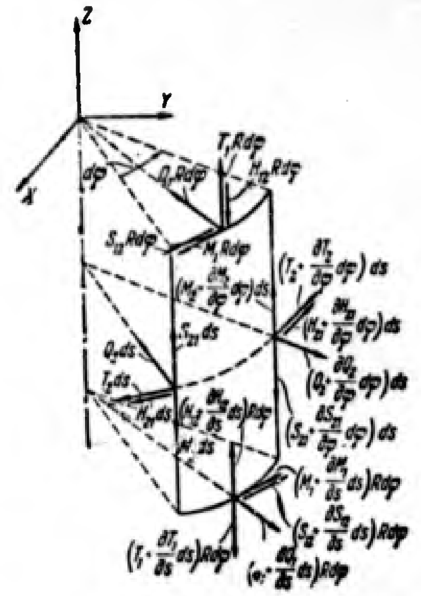


Fig. 11. Element of cylindrical shell and internal forces and moments acting on it.

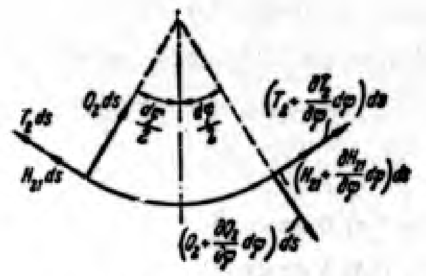


Fig. 12.

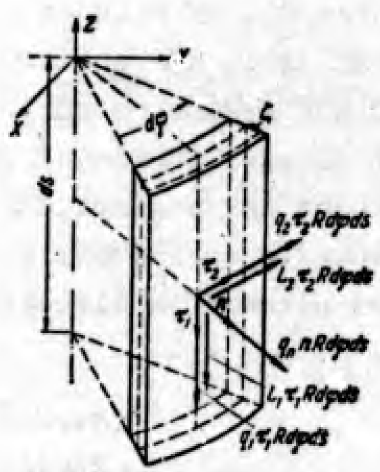


Fig. 13.

Fig. 12. Direction of peripheral tensile forces acting on element of a shell with angle $d\phi$.

Fig. 13. External loads applied to an element from a cylindrical shell.

Equating to zero the projections of the principal vector of the forces on the element onto directions τ_1, τ_2, and n (Fig. 11), we obtain

$$\left.\begin{aligned}&\left(T_1+\frac{\partial T_1}{\partial s}ds\right)R\,d\varphi-T_1R\,d\varphi-S_{21}\,ds+\left(S_{21}+\frac{\partial S_{21}}{\partial\varphi}d\varphi\right)ds+\\&\qquad+q_1R\,d\varphi\,ds=0,\\&-T_2\,ds+\left(T_2+\frac{\partial T_2}{\partial\varphi}d\varphi\right)ds-S_{12}R\,d\varphi+\left(S_{12}+\frac{\partial S_{12}}{\partial s}ds\right)Rd\varphi+\\&\qquad+Q_2\,ds\frac{d\varphi}{2}+\left(Q_2+\frac{\partial Q_2}{\partial\varphi}d\varphi\right)ds\frac{d\varphi}{2}+q_2R\,d\varphi\,ds=0,\end{aligned}\right\}\qquad(24.3)$$

$$\left(Q_1+\frac{\partial Q_1}{\partial s}ds\right)R\,d\varphi-Q_1R\,d\varphi+\left(Q_2+\frac{\partial Q_2}{\partial\varphi}d\varphi\right)ds-Q_2ds-$$
$$-T_2ds\frac{d\varphi}{2}-\left(T_2+\frac{\partial T_2}{\partial\varphi}d\varphi\right)ds\frac{d\varphi}{2}+q_nR\,d\varphi\,ds=0. \qquad (24.3)$$

cont'd.

Of course, this projection assumes

$$\cos\left(\frac{d\varphi}{2}\right)\approx 1.\quad \sin\left(\frac{d\varphi}{2}\right)\approx\frac{d\varphi}{2}.$$

Canceling terms in (24.3) and ignoring quantities which possess a high degree of smallness in comparison with the main terms (in the second equation the term $\frac{\partial Q_2}{\partial\varphi}ds\frac{(d\varphi)^2}{2}$ is rejected, and in the third $\frac{\partial T_2}{\partial\varphi}ds\frac{(d\varphi)^2}{2}$), after removing the common factor $R\,d\varphi\,ds$ we obtain three equations of equilibrium:

$$\begin{aligned}
&\frac{\partial T_1}{\partial s}+\frac{1}{R}\frac{\partial S_{21}}{\partial\varphi}+q_1=0.\\
&\frac{1}{R}\frac{\partial T_2}{\partial\varphi}+\frac{\partial S_{12}}{\partial s}+\frac{Q_2}{R}+q_2=0.\\
&\frac{\partial Q_1}{\partial s}+\frac{1}{R}\frac{\partial Q_2}{\partial\varphi}-\frac{T_2}{R}+q_n=0.
\end{aligned} \qquad (24.4)$$

In the same way composing three equations of moments, we have

$$\begin{aligned}
&M_2ds-\left(M_2+\frac{\partial M_2}{\partial\varphi}d\varphi\right)ds-\left(H_{12}+\frac{\partial H_{12}}{\partial s}ds\right)R\,d\varphi+\\
&\quad+H_{12}R\,d\varphi+Q_2dsR\frac{d\varphi}{2}+\left(Q_2+\frac{\partial Q_2}{\partial\varphi}d\varphi\right)R\frac{d\varphi}{2}+\\
&\quad+L_1R\,d\varphi\,ds=0.\\
&\left(M_1+\frac{\partial M_1}{\partial s}ds\right)R\,d\varphi-M_1R\,d\varphi-H_{21}ds+\\
&\quad+\left(H_{21}+\frac{\partial H_{21}}{\partial\varphi}d\varphi\right)ds-Q_1R\,d\varphi\frac{ds}{2}-\\
&\quad-\left(Q_1+\frac{\partial Q_1}{\partial s}ds\right)R\,d\varphi\frac{ds}{2}+L_2R\,d\varphi\,ds=0.\\
&-S_{21}dsR\frac{d\varphi}{2}-\left(S_{21}+\frac{\partial S_{21}}{\partial\varphi}d\varphi\right)R\frac{d\varphi}{2}+S_{12}R\,d\varphi\frac{ds}{2}+\\
&\quad+\left(S_{12}+\frac{\partial S_{12}}{\partial s}ds\right)R\,d\varphi\frac{ds}{2}-H_{21}ds\frac{d\varphi}{2}-\\
&\quad-\left(H_{21}+\frac{\partial H_{21}}{\partial\varphi}d\varphi\right)ds\frac{d\varphi}{2}=0.
\end{aligned} \qquad (24.5)$$

Canceling terms in (24.5) and dropping terms of a higher order of smallness in comparison with the main terms, we derive an additional three equations of statics:

$$\left.\begin{aligned} -\frac{1}{R}\frac{\partial M_2}{\partial\varphi} - \frac{\partial H_{12}}{\partial s} + Q_2 + L_1 = 0, \\ \frac{\partial M_1}{\partial s} + \frac{1}{R}\frac{\partial H_{21}}{\partial\varphi} - Q_1 + L_2 = 0, \\ S_{21} - S_{12} + \frac{1}{R}H_{21} = 0. \end{aligned}\right\} \tag{24.6}$$

Equations (24.4), (24.6) can be obtained from equations (4.18) if in them we set: $R_1=\infty$, $v=R_2=R$, $\theta=\frac{\pi}{2}$, $R_1 d\theta = ds$. Equations of equilibrium of a cylinder shell, as also (4.18), are simple to transform so that they contain instead of the four quantities S_{12}, S_{21}, H_{12}, H_{21}, only the two amounts S, H,

$$S = S_{12} - \frac{1}{R}H_{21} = S_{21}, \quad H = \frac{1}{2}(H_{12}+H_{21}). \tag{24.7}$$

In this case the third equation of (24.6) is identically satisfied, and the remaining five equations of statics assume the form

$$\left.\begin{aligned} &\frac{\partial T_1}{\partial s} + \frac{1}{R}\frac{\partial S}{\partial\varphi} + q_1 = 0, \\ &\frac{\partial}{\partial s}\left(S + \frac{H}{R}\right) + \frac{1}{R}\frac{\partial T_2}{\partial\varphi} + \frac{N_2}{R} + q_2 = 0, \\ &\frac{\partial N_1}{\partial s} + \frac{1}{R}\frac{\partial N_2}{\partial\varphi} - \frac{T_2}{R} + q_n = 0. \end{aligned}\right\} \tag{24.8}$$

$$\left.\begin{aligned} N_1 = \frac{\partial M_1}{\partial s} + \frac{1}{R}\frac{\partial H}{\partial\varphi} + L_2, \\ N_2 = \frac{\partial H}{\partial s} + \frac{1}{R}\frac{\partial M_2}{\partial\varphi} - L_1, \end{aligned}\right\} \tag{24.9}$$

where N_1, N_2 designate quantities connected with shear forces Q_1, Q_2 by the formulas:

$$\left.\begin{aligned} N_1 = Q_1 - \frac{1}{2}\frac{1}{R}\frac{\partial}{\partial\varphi}(H_{21} - H_{12}), \\ N_2 = Q_2 + \frac{1}{2}\frac{\partial}{\partial s}(H_{21} - H_{12}). \end{aligned}\right\} \tag{24.10}$$

Eliminating from (24.8), (24.9) the quantities N_1, N_2, we can obtain three equations of equilibrium in the quantities T_1, T_2, S, M_1, M_2, H, into which the load terms will have the form

$$q_1, \quad q_2 - \frac{L_1}{R}, \quad q_n + \frac{\partial L_2}{\partial s} - \frac{1}{R}\frac{\partial L_1}{\partial\varphi}.$$

Since L_1, L_2 have the order of load multiplied by shell thickness, it is easy to see that the quantities $\frac{L_1}{R}$, $\frac{\partial L_2}{\partial s}$, $\frac{1}{R}\frac{\partial L_1}{\partial \varphi}$ in these equations can be neglected if the ratio of the thickness of the shell to its other linear dimensions (length or radius) is small, and moments L_1, L_2 have a definite derivative. Therefore subsequently in (24.9) we neglect L_1, L_2.

In view of the simplicity of the geometric shape of the involved shell, expressions for components of deformation of the middle surface through displacements u, v, w are considerably simplified. These expressions can be obtained with formulas (3.19) if in them we set $R_1 d\theta = ds$, $\theta = \frac{\pi}{2}$, $R_1 = \infty$, $R_2 = \nu = R$. Having done this, we will have

$$\left.\begin{aligned} \varepsilon_1 &= \frac{\partial u}{\partial s}, & \varkappa_1 &= -\frac{\partial^2 w}{\partial s^2}, \\ \varepsilon_2 &= \frac{w}{R} + \frac{1}{R}\frac{\partial v}{\partial \varphi}, & \varkappa_2 &= -\frac{1}{R^2}\frac{\partial}{\partial \varphi}\left(\frac{\partial w}{\partial \varphi} - v\right), \\ \gamma &= \frac{\partial v}{\partial s} + \frac{1}{R}\frac{\partial u}{\partial \varphi}, & \tau &= -\frac{1}{R}\frac{\partial^2 w}{\partial \varphi\, \partial s} + \frac{1}{R}\frac{\partial v}{\partial s}. \end{aligned}\right\} \qquad (24.11)$$

Here ε_1, ε_2, γ – relative elongations and shear of the middle surface caused by shifts of the middle surface u, v, w; κ_1, κ_2 – changes in curvatures of rectilinear generatrix and parallel circle during strain; τ – twisting. Angles of rotation of normal n around axes τ_2, τ_1 during strain are equal to

$$\begin{aligned} -\vartheta_1 &= -\frac{\partial w}{\partial s}, \\ \vartheta_2 &= \frac{1}{R}\frac{\partial w}{\partial \varphi} - \frac{v}{R}. \end{aligned} \qquad (24.12)$$

Since because of the Kirchhoff-Love hypothesis the normal during deformation keeps its perpendicularity to the middle surface, then $-\vartheta_1$ is at the same time the angle of rotation of the generatrix around axis τ_2, and angle ϑ_2 is the angle of rotation of the tangential to the parallel circle around axis τ_1. Changes in curvatures κ_1, κ_2 and twisting τ can be expressed through angles of rotation $-\vartheta_1$ and ϑ_2. Really, taking into account (24.12), from (24.11) we obtain

$$\varkappa_1 = -\frac{\partial\theta_1}{\partial s}, \quad \varkappa_2 = -\frac{1}{R}\frac{\partial\theta_2}{\partial\varphi}, \quad \tau = -\frac{\partial\theta_2}{\partial s}. \tag{24.13}$$

Formulas (24.13) and Fig. 14 explain why components of deformation κ_1, κ_2, τ bear the name of "change in curvatures" and "twisting."

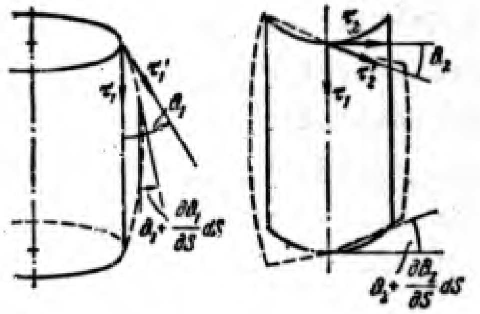


Fig. 14. Change in curvature of meridian and twisting of middle surface during deformation of shell.

The six components of deformation $\varepsilon_1, \varepsilon_2, \gamma, \varkappa_1, \varkappa_2$ are connected with the six static quantities T_1, T_2, S, M_1, M_2, H by relationships expressing Hooke's law for a thin shell:

$$\begin{aligned} \varepsilon_1 &= \frac{1}{Eh}\left(T_1 - \mu T_2\right), & \varkappa_1 &= \frac{12}{Eh^3}\left(M_1 - \mu M_2\right), \\ \varepsilon_2 &= \frac{1}{Eh}\left(T_2 - \mu T_1\right), & \varkappa_2 &= \frac{12}{Eh^3}\left(M_2 - \mu M_1\right), \\ \gamma &= \frac{2(1+\mu)}{Eh} S, & \tau &= \frac{12(1+\mu)}{Eh^3} H. \end{aligned} \tag{24.14}$$

Relationships (24.8), (24.9), (24.11), (24.14) form the total system of equations which describes the equilibrium of a randomly loaded cylindrical shell, to which it is necessary to add only the boundary conditions.

§ 25. Axisymmetric Deformation of a Cylindrical Shell

Let us examine the simplest load case of a shell: the axisymmetric load. Let us assume that the conditions of the fastening of the edges also possess symmetry relative to the axis of revolution of the shell. In this instance the forces, moments, components of deformation and displacements in the shell are not a function of coordinate ϕ and in the trigonometric series of § 8, representing static and geometric quantities, only the first terms marked with a "(0)," are not zero.

To avoid double indexes we will agree that within this chapter and wherever it will not give rise to misunderstanding forces, moments, deformations and displacements during axisymmetric deformation will be designated without the lower and upper zero indexes. In this way, instead of the designations of § 8 $T_{1(0)}, T_{2(0)}, S^{(0)}, M_{1(0)}, M_{2(0)}, H^{(0)}$ we will use the designations T_1, T_2, S, M_1, M_2, H, instead of $\varepsilon_{1(0)}, \varepsilon_{2(0)}, \gamma^{(0)}, \varkappa_{1(0)}, \varkappa_{2(0)}, \tau^{(0)} - \varepsilon_1, \varepsilon_2, \gamma, \varkappa_1, \varkappa_2, \tau$, we will use $u_{(0)}, v^{(0)}, w_{(0)} - u, v, w$, keeping, however, the zero index in designation of a distributed load on the shell $q_{1(0)}, q_{2(0)}, q_{n(0)}$. In the case of axisymmetric deformation of a shell system of equations of equilibrium (24.8), (24.9) is split into two groups of equations:

$$\left.\begin{aligned} &\frac{d}{ds}\left(S+\frac{H}{R}\right)+\frac{N_2}{R}+q_{2(0)}=0,\\ &N_2=\frac{dH}{ds}. \end{aligned}\right\} \tag{25.1}$$

$$\left.\begin{aligned} &\frac{dT_1}{ds}+q_{1(0)}=0,\\ &\frac{dN_1}{ds}-\frac{T_2}{R}+q_{n(0)}=0,\\ &N_1=\frac{dM_1}{ds}. \end{aligned}\right\} \tag{25.2}$$

The first of them describes the twisting of the shell, and the duplicate - the elongation in the axial direction and axisymmetric curvature. Rewriting expressions (24.11) for the case of displacements not depending on ϕ, we obtain an additional two groups of relationships:

$$\gamma=\frac{dv}{ds}, \quad \tau=\frac{1}{R}\frac{dv}{ds}, \tag{25.3}$$

$$\left.\begin{aligned} &\varepsilon_1=\frac{du}{ds}, \quad \varkappa_1=-\frac{d^2w}{ds^2},\\ &\varepsilon_2=\frac{w}{R}, \quad \varkappa_2=0. \end{aligned}\right\} \tag{25.4}$$

In this way, taking into account relationships (24.14), expressing Hooke's law, it is simple to see that the problem of twisting is completely isolated and is solved elementarily; actually, from (25.1) after the elimination of N_2 and integration over s we obtain

$$S + \frac{2H}{R} + \int_0^s q_{2(0)} ds + A = 0. \qquad (25.5)$$

where A is the constant of integration. Expressing with (24.14), (25.3) S and H through $\frac{dv}{ds}$, we obtain

$$S = \frac{Eh}{2(1+\mu)} \frac{dv}{ds}, \quad H = \frac{Eh^3}{12(1+\mu)} \frac{1}{R} \frac{dv}{ds}. \qquad (25.6)$$

Substituting (25.6) into (25.5) and dropping terms of the order of h^2/R^2 in comparison with unity, obtain one differential equation of the first degree for the determination of v

$$\frac{Eh}{2(1+\mu)} \frac{dv}{ds} + \int_0^s q_{2(0)} ds + A = 0, \qquad (25.7)$$

from which it follows that

$$v = -\frac{2(1+\mu)}{Eh} \int_0^s \left(\int_0^s q_{2(0)} ds \right) ds - As + B. \qquad (25.8)$$

Constants of integration A and B are defined according to the boundary conditions, where, as is easy to see from (25.5), on one of the boundaries force can be given, for example:

when $s = 0$ $S = S^0$,

then, dropping $2H/R$ in comparison with S as a small term, from (25.5) when $s = 0$ we obtain $S^0 + A = 0$ and

$$S = -\int_0^s q_{2(0)} ds + S^0. \qquad (25.9)$$

Displacement v is defined from relationship (25.8) accurate to a constant B, characterizing the rotation of the shell as a whole around axis OZ.

The problem of elongation and bend has a more complex solution. The first equation of equilibrium (25.2) is integrated directly:

$$T_1 = \frac{P_0}{2\pi R} - \int_0^s q_{1(0)}\,ds. \tag{25.10}$$

while the constant of integration P_0 has a simple mechanical sense – the axial force acting in section $s = 0$. Eliminating from the second and third equations (25.2) the quantity N_1 we arrive at one equation

$$\frac{d^2M_1}{ds^2} - \frac{T_2}{R} + q_{n(0)} = 0. \tag{25.11}$$

connecting two quantities: M_1 and T_2. The latter with the aid of (24.14), (25.4), (25.10) can be expressed through displacement w and external loads

$$M_1 = -\frac{Eh^3}{12(1-\mu^2)}\frac{d^2w}{ds^2}. \tag{25.12}$$

$$T_2 = Eh\frac{w}{R} + \mu\frac{P_0}{2\pi R} - \mu\int_0^s q_{1(0)}\,ds. \tag{25.13}$$

Rewriting (25.11) allowing for (25.12), (25.13), we obtain one difference equation of the fourth degree relative to displacement w

$$\frac{Eh^3}{12(1-\mu^2)}\frac{d^4w}{ds^4} + \frac{Eh}{R^2}w = q_{n(0)} - \mu\frac{P_0}{2\pi R} + \mu\int_0^s q_{1(0)}\,ds. \tag{25.14}$$

Equation (25.14) coincides with the equation of the elastic line of a beam on a continuous elastic Winkler base [12], [59]. This coincidence is not chance. Really, an element of a shell cut out in the meridian direction (Figs. 11, 12) and possessing in the middle plane dimensions $Rd\varphi$, ds, undergoes during a radial resistance proportional to the amount of peripheral force T_2, which in turn is proportional to normal displacement w.

The resolvant equation for the axisymmetric problem can be written also in variables V, ϑ_1, introduced in § 11. Namely, introducing stress function V, connected fo forces N_1, T_2 and the external load thus, since this follows from equations (11.10), (11.11), (11.12) if in them we set $v=R,\ R_1 d\theta=ds,\ \theta=\frac{\pi}{2}$, we obtain

$$\left.\begin{aligned} N_1 &= \frac{V}{R} - \int_0^s q_{n(0)}\,ds, \\ T_2 &= \frac{dV}{ds}. \end{aligned}\right\} \tag{25.15}$$

Furthermore, bending moment M_1 can be expressed through angle of rotation ϑ_1, if one takes into account (24.13), (24.14) and the last relationship of (25.4),

$$M_1 = -\frac{Eh^3}{12(1-\mu^2)}\frac{d\vartheta_1}{ds}. \tag{25.16}$$

Substituting the expressions for N_1, T_2, M_1 in accordance with (25.15), (25.16) into the second equation in this case is identically satisfied, and the third gives

$$-\frac{Eh^3}{12(1-\mu^2)}\frac{d^2\vartheta_1}{ds^2} = \frac{V}{R} - \int_0^s q_{n(0)}\,ds. \tag{25.17}$$

One more equation of the connection between V and ϑ_1 can be obtained by using the equation

$$R\frac{d\varepsilon_2}{ds} - \vartheta_1 = 0, \tag{25.18}$$

which is an identity relative to displacement w, since in accordance with (25.4), (24.12)

$$\varepsilon_2 = \frac{w}{R}, \quad \vartheta_1 = \frac{dw}{ds}.$$

Using Hooke's law (24.14) the first integral of equations of equilibrium (25.10) and (25.15), from (25.18) we obtain

$$\frac{R}{Eh}\frac{d^2V}{ds^2} - \frac{R\mu}{Eh}\frac{d}{ds}\left(\frac{P_0}{2\pi R} - \int_0^s q_{1(0)}\,ds\right) - \vartheta_1 = 0. \tag{25.19}$$

Relationships (25.17) and (25.19) form a system of equations for the determination of V and ϑ_1. Replacing variables V and ϑ_1 by variables V_0, Ψ_0,

$$V_0 = \frac{V}{R}, \quad \Psi_0 = Eh\vartheta_1, \tag{25.20}$$

and introducing dimensionless coordinate

$$\xi = \frac{s}{R}, \tag{25.21}$$

while

$$\text{when } s = L, \quad \xi = \frac{L}{R} = l, \tag{25.22}$$

the obtained equations reduce to the form

$$\left.\begin{aligned} \frac{d^2\Psi_0}{d\xi^2} + 4\gamma^4 V_0 &= 4\gamma^4 \int_0^\xi Rq_{n\,(0)}\,d\xi, \\ \frac{d^2V_0}{d\xi^2} - \Psi_0 &= -R\mu q_{1\,(0)}, \end{aligned}\right\} \tag{25.23}$$

where

$$4\gamma^4 = 12(1-\mu^2)\frac{R^2}{h^2}. \tag{25.24}$$

By the introduction of the imaginary variable

$$\sigma_0 = \Psi_0 - 2i\gamma^2 V_0 \tag{25.25}$$

system of equations (25.23) is replaced by one equation of the second degree in σ_0

$$\frac{d^2\sigma_0}{d\xi^2} + 2i\gamma^2\sigma_0 = 4\gamma^4 \int_0^\xi Rq_{n\,(0)}\,d\xi + 2i\gamma^2 R\mu q_{1\,(0)}. \tag{25.26}$$

Dropping in the right side of (25.26) the second term, having in the comparison with the first the same order of smallness as $\left(\frac{h}{R}\right)$ (or, which is the same $\frac{1}{2\gamma^2}$) in comparison with unity, finally we obtain

$$\frac{d^2\sigma_0}{d\xi^2} + 2i\gamma^2\sigma_0 = 4\gamma^4 \int_0^\xi Rq_{n\,(0)}\,d\xi. \tag{25.27}$$

The particular solution of this equation can be approximately written in the form

$$\tilde{\sigma}_0 = -i2\gamma^2 \int_0^{\xi} Rq_{n(0)}\, d\xi. \tag{25.28}$$

Separating the real and imaginary parts, we have

$$\tilde{\Psi}_0 = 0, \quad \tilde{V}_0 = \int_0^{\xi} Rq_{n(0)}\, d\xi \tag{25.29}$$

To particular solution (25.29) corresponds the zero-moment state of a cylindrical shell. Really, by formulas (25.20), (25.16), (25.15), (25.10) we obtain

$$\left.\begin{aligned} \tilde{M}_1 &= -\frac{R}{4\gamma^4}\frac{d\tilde{\Psi}_0}{d\xi} = 0, \\ \tilde{N}_1 &= \tilde{V}_0 - \int_0^{\xi} Rq_{n(0)}\, d\xi = 0, \\ \tilde{T}_2 &= \frac{d\tilde{V}_0}{d\xi} = Rq_{n(0)}, \\ \tilde{T}_1 &= \frac{P_0}{2\pi R} - \int_0^{\xi} Rq_{1(0)}\, d\xi. \end{aligned}\right\} \tag{25.30}$$

The obtained particular solution is sufficiently accurate if load $q_{n(0)}$ changes smoothly, i.e., if $\frac{dq_{n(0)}}{d\xi}$ is little. Really, let us substitute

$$\sigma_0 = \sigma - i2\gamma^2 \int_0^{\xi} Rq_{n(0)}\, d\xi,$$

then instead of (25.27) we will have the equation for the determination of σ

$$\frac{d^2\sigma}{d\xi^2} + 2i\gamma^2\sigma = i2\gamma^2 R\frac{dq_{n(0)}}{d\xi}. \tag{25.31}$$

At $\frac{dq_{n(0)}}{d\xi} \approx 0$ the particular solution of this heterogeneous equation is also approximately equal to zero and $\tilde{\sigma}_0$ has the form (25.28).

In this way, with a smooth change in load the particular solution can be solution (25.28), corresponding to the zero-moment state. However, as will be shown subsequently (§ 33), in certain cases such and approach can lead to errors of the order of $\sqrt{\frac{h}{R}}$ in comparison with unit during determination of stress at the edges of the shell.

When $\frac{dq_{n(0)}}{d\xi}$ is not small, the particular solution of (25.31) is approximately equal to

$$\sigma \approx R\frac{dq_{n(0)}}{d\xi}$$

thus

$$\tilde{\sigma}_0 = R\frac{dq_{n(0)}}{d\xi} - 12\gamma^2\int_0^{\xi} Rq_{n(0)}\,d\xi. \qquad (25.32)$$

To this value of $\tilde{\sigma}_0$ correspond

$$\tilde{\Psi}_0 = R\frac{dq_{n(0)}}{d\xi}, \quad \tilde{V}_0 = \int_0^{\xi} Rq_{n(0)}\,d\xi. \qquad (25.33)$$

If we keep also the second term in the righ side of (25.26), then the solution should be taken in the form

$$\tilde{\Psi}_0 = R\frac{dq_{n(0)}}{d\xi} + \mu Rq_{1(0)}, \quad \tilde{V}_0 = \int_0^{\xi} Rq_{n(0)}\,d\xi. \qquad (25.34)$$

§ 26. Deformation of a Shell Under the Action of a Symmetric Bending Load

Considering the deformation of a shell under the action of a bending load of form (8.28), we will use for the designation of amplitudes of forces and moments lower case Latin characters, for example, we will write $T_1 = t_1\cos\varphi$, $M_1 = m_1\cos\varphi$ etc. In this way, within the limits of this chapter and subsequently where no misunderstanding will be caused instead of the designations $T_{1(1)}$, $T_{2(1)}$, $S_{(1)}$, $M_{1(1)}$, $M_{2(1)}$, $H_{(1)}$, $N_{1(1)}$, $N_{2(1)}$, introduced in § 8, we will use the designations t_1, t_2, s_1,

m_1, m_2, h_1, n_1, n_2. The subscripts (1) in the designations for amplitudes of deformations, displacements and components of external loading are kept. The dependences of static and geometric curves on coordinate ϕ in the involved case have the form

$$(T_1, T_2, M_1, M_2, N_1)=(t_1, t_2, m_1, m_2, n_1)\cos\varphi,$$
$$(\varepsilon_1, \varepsilon_2, \varkappa_1, \varkappa_2, \vartheta_1, u, w)=$$
$$=(\varepsilon_{1(1)}, \varepsilon_{2(1)}, \varkappa_{1(1)}, \varkappa_{2(1)}, \vartheta_{1(1)}, u_{(1)}, w_{(1)}) \cdot \cos\varphi,$$
$$(q_1, q_n)=(q_{1(1)}, q_{n(s)})\cos\varphi,$$
$$(S, H, N_2, \gamma, \tau, \vartheta_2, v, q_2)=$$
$$=(s_1, h_1, n_2, \gamma_{(1)}, \tau_{(1)}, \vartheta_{2(1)}, v_{(1)}, q_{2(1)})\sin\varphi. \tag{26.1}$$

After singling out cos ϕ, sin ϕ from (24.8), (24.9) we will obtain the following equations of equilibrium for the amplitudes of forces and moments

$$t_1, t_2, s_1, n_1, n_2, m_1, m_2, h_1:$$
$$\left.\begin{aligned}&\frac{dt_1}{ds}+\frac{s_1}{R}+q_{1(1)}=0,\\&\frac{d}{ds}\left(s_1+\frac{h_1}{R}\right)-\frac{t_2}{R}+\frac{n_2}{R}+q_{2(1)}=0,\\&\frac{dn_1}{ds}+\frac{n_2}{R}-\frac{t_2}{R}+q_{n(1)}=0,\end{aligned}\right\} \tag{26.2}$$

$$\left.\begin{aligned}n_1&=\frac{dm_1}{ds}+\frac{h_1}{R},\\n_2&=\frac{dh_1}{ds}-\frac{m_2}{R}.\end{aligned}\right\} \tag{26.3}$$

In this case, just as in § 25, moments of distributed loads L_1 and L_2 are dropped.

System of equations (26.2), (26.3) allows a decrease in degree by two. Actually, subtracting the second equation of (26.2) from the third and integrating, we obtain

$$n_1-\left(s_1+\frac{h_1}{R}\right)=-\frac{P_1}{\pi R}-\int_0^s (q_{n(1)}-q_{2(1)})ds, \tag{26.4}$$

where P_1 is the constant of integration, equal in magnitude to the primary vector of the edge loads acting in section s = 0 (Fig. 15).

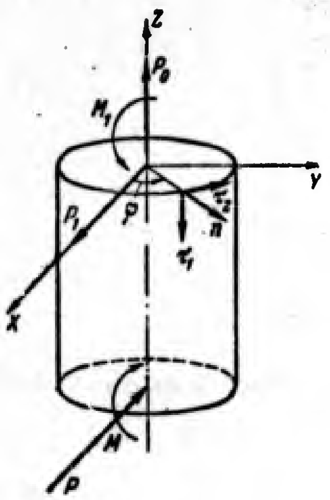


Fig. 15. System of external loads applied to the edges of a cylindrical shell.

Substituting the expression for n_1, in accordance with (26.3), into equation (26.4) and combining the obtained relationship with the first equation of (26.2), one can easily find one more integral of system of equations of equilibrium

$$Rt_1 + m_1 = -\frac{P_1}{\pi R}s - \frac{M_1}{\pi R} - \int_0^s \int_0^s (q_{n(1)} - q_{2(1)})\,ds\,ds - \int_0^s Rq_{1(1)}\,ds. \qquad (26.5)$$

where M_1 is the new constant of integration, equal to the total moment of edge loads in section $s = 0$. In order for the involved stressed state to exist the edge loads should have a principal vector acting along axis OX, and a main moment directed along axis OY. We consider positive the direction of moment M_1 at which looking from the side of positive axis OY the moment acts counter-clockwise. That the introduced constants of integration P_1 and M_1 really have the above mechanical sense one can be certain of by directly composing the conditions of equilibrium of the internal and external forces and moments at edge $s = 0$.

For cylindrical shell the directions τ_1, τ_2, n make with the directions of axes OX, OY, OZ angles whose cosines are given in the table:

	i	j	k
τ_1	0	0	−1
τ_2	$-\sin\varphi$	$\cos\varphi$	0
n	$\cos\varphi$	$\sin\varphi$	0

In section s = const on a certain elementary arc $R\,d\varphi$ act internal force

$$KR\,d\varphi = (T_1\tau_1 + S_{12}\tau_2 + Q_1 n)R\,d\varphi =$$
$$= (t_1\cos\varphi\tau_1 + S_{12\,(1)}\sin\varphi\tau_2 + Q_{1\,(1)}\cos\varphi n)R\,d\varphi$$

and moment

$$MR\,d\varphi = (M_1\tau_2 - H_{12}\tau_1)R\,d\varphi = (m_1\cos\varphi\tau_2 - H_{12\,(1)}\sin\varphi\tau_1)R\,d\varphi.$$

Their projections onto axes OX, OY, OZ are equal to

$$K_xR\,d\varphi = (-S_{12\,(1)}\sin^2\varphi + Q_{1\,(1)}\cos^2\varphi)R\,d\varphi.$$
$$K_yR\,d\varphi = (S_{12\,(1)}\sin\varphi\cos\varphi + Q_{1\,(1)}\cos\varphi\sin\varphi)R\,d\varphi.$$
$$K_zR\,d\varphi = -t_1\cos\varphi R\,d\varphi.$$
$$M_xR\,d\varphi = (-m_1\cos\varphi\sin\varphi - H_{12\,(1)}\sin\varphi\cos\varphi)R\,d\varphi.$$
$$M_yR\,d\varphi = (m_1\cos^2\varphi)R\,d\varphi.$$
$$M_zR\,d\varphi = H_{12\,(1)}\sin\varphi R\,d\varphi.$$

The principal vector and principal moment of external load P_1 and M_1 should be equal in magnitude and opposite in sign to the principal vector and moment of internal forces in section s = 0. It is easy to see that the principal vector and principal moment of the system of internal forces in any section have components equal to zero along axes OY, OZ and OX, OZ respectively. For example, at s = 0 we have

$$\int_0^{2\pi} K_yR\,d\varphi = R\int_0^{2\pi}(S_{12\,(1)} + Q_{1\,(1)})\sin\varphi\cos\varphi\,d\varphi = 0.$$

$$\int_0^{2\pi}(M_z + K_yR\cos\varphi - K_xR\sin\varphi)R\,d\varphi =$$
$$= \int_0^{2\pi}[H_{12\,(1)}\sin\varphi + (S_{12\,(1)} + Q_{1\,(1)})R\sin\varphi\cos^2\varphi +$$
$$+ S_{12\,(1)}R\sin^3\varphi - Q_{1\,(1)}R\cos^2\varphi\sin\varphi]R\,d\varphi = 0$$

etc.

In this way there exist the equalities

$$P_1 = -\int_0^{2\pi} (K_x)_{s=0} R\, d\varphi,$$

$$M_1 = -\int_0^{2\pi} (M_y - K_z R \cos\varphi)_{s=0} R\, d\varphi.$$

After integration we obtain

$$-\frac{P_1}{\pi R} = (-S_{12(1)} + Q_{1(1)})_{s=0},$$

$$-\frac{M_1}{\pi R} = (m_1 + t_1 R)_{s=0}.$$

Taking into consideration that according to (24.7), (24.10)

$$S_{12(1)} = s_1 + \frac{H_{21(1)}}{R},$$

$$n_1 = Q_{1(1)} - \frac{1}{2R}(H_{21(1)} - H_{12(1)}),$$

$$h_1 = \frac{1}{2}(H_{21(1)} + H_{12(1)}),$$

we rewrite the obtained relationships in the form

$$-\frac{P_1}{\pi R} = \left[n_1 - \left(s_1 + \frac{h_1}{R}\right)\right]_{s=0},$$

$$-\frac{M_1}{\pi R} = (m_1 + t_1 R)_{s=0}.$$

Assuming in (26.4), (26.5) $s = 0$, we arrive again at the same relationships.

System of equations (26.2) can be considered as a system of three equations in six unknown forces and moments under the condition that n_1 and n_2 have been determined according to formulas (26.3). Since two first integrals of this have been found, it can be replaced by a system consisting of (26.4), (26.5) and the second equation of (26.2). Eliminating from these three equations forces n_1, n_2 with the aid of (26.3), we obtain

$$\left.\begin{aligned}\frac{ds_1}{ds}+\frac{2}{R}\frac{dh_1}{ds}-\frac{t_2}{R}-\frac{m_2}{R^2}+q_{2(1)}=0,\\ \frac{dm_1}{ds}-s_1=-\frac{P_1}{\pi k}-\int\limits_0^s(q_{n(1)}-q_{2(1)})\,ds,\\ Rt_1+m_1=R(f_0+f_1),\end{aligned}\right\}\qquad(26.6)$$

where the designation

$$R(f_0+f_1)=-\frac{P_1 s}{\pi R}-\frac{M_1}{\pi R}-\int\limits_0^s\int\limits_0^s(q_{n(1)}-q_{2(1)})\,ds\,ds-R\int\limits_0^s q_{1(1)}\,ds\qquad(26.7)$$

has been introduced. Using the statico-geometric analogy, from equations (26.6) we will obtain equations of continuity. We set in (26.6) all load terms equal to zero and go from the quantities t_1, t_2, s_1, $2h_1$, m_1, m_2 to the quantities $\varkappa_{2(1)}$, $\varkappa_{1(1)}$, $-\tau_{(1)}$, $\gamma_{(1)}$, $-\varepsilon_{2(1)}$, $-\varepsilon_{1(1)}$ respectively. Having done this, we will have

$$\left.\begin{aligned}-\frac{d\tau_{(1)}}{ds}+\frac{1}{R}\frac{d\gamma_{(1)}}{ds}-\frac{\varkappa_{1(1)}}{R}+\frac{\varepsilon_{1(1)}}{R^2}=0,\\ -\frac{d\varepsilon_{2(1)}}{ds}+\tau_{(1)}=0,\\ R\varkappa_{2(1)}-\varepsilon_{2(1)}=0.\end{aligned}\right\}\qquad(26.8)$$

The obtained equations are really conditions of continuity of the middle surface of the shell during deformation. Actually from (26.1) and (24.11) it follows that amplitudes of deformations and displacements are interconnected by these relationships:

$$\left.\begin{aligned}&\varepsilon_{1(1)}=\frac{du_{(1)}}{ds}, &&\varkappa_{1(1)}=-\frac{d^2w_{(1)}}{ds^2},\\ &\varepsilon_{2(1)}=\frac{1}{R}(w_{(1)}+v_{(1)}), &&\varkappa_{2(1)}=\frac{1}{R^2}(w_{(1)}+v_{(1)}),\\ &\gamma_{(1)}=\frac{dv_{(1)}}{ds}-\frac{u_{(1)}}{R}, &&\tau_{(1)}=\frac{1}{R}\left(\frac{dw_{(1)}}{ds}+\frac{dv_{(1)}}{ds}\right),\\ &\vartheta_{1(1)}=\frac{dw_{(1)}}{ds}, &&\vartheta_{2(1)}=-\frac{1}{R}(w_{(1)}+v_{(1)}).\end{aligned}\right\}\qquad(26.9)$$

Substituting the expressions for amplitudes of deformations (26.9) into equations (26.8), we see that the latter in this case are satisfied identically.

Let us introduce a certain function of displacements Ψ and connect to it deformations according to the following law:

$$
\left.\begin{aligned}
\varkappa_{1(1)} &= \frac{d\Psi}{ds} + \frac{\varepsilon_{1(1)}}{R}, \\
\tau_{(1)} &= -\frac{\Psi}{R} + \frac{\gamma_{(1)}}{R}, \\
\varkappa_{2(1)} &= \frac{\varepsilon_{2(1)}}{R}.
\end{aligned}\right\} \tag{26.10}
$$

At such a presentation of deformations the first and third equations of (26.8) are satisfied identically. Furthermore, from the second relationship of (26.10) and expressions (26.9) it follows that the introduced function of displacements Ψ is equal to

$$
\Psi = -\left(\vartheta_{1(1)} + \frac{u_{(1)}}{R}\right). \tag{26.11}
$$

The first and third equations of (26.6) are satisfied if we express forces and moments through stress function V_1 by formulas similar to equations (26.10), allowing for specially selected load terms:

$$
\left.\begin{aligned}
t_2 &= R\frac{dV_1}{ds} - \frac{m_2}{R}, \\
t_1 &= -\frac{m_1}{R} + f_0 + f_1, \\
s_1 &= V_1 - \frac{2h_1}{R} - \int_0^s q_{2(1)}\,ds.
\end{aligned}\right\} \tag{26.12}
$$

Equations (26.10), (26.12) after adding to them the six relationships (24.14) form a system of twelve equations for determining twelve unknowns (six components of deformation and six static quantities) through the introduced functions Ψ and V_1. The result of solving this system is the following:

$$
\left.\begin{aligned}
Eh\varepsilon_{1(1)} &= -\mu R\frac{dV_1}{ds} - \frac{Eh^3}{12R}\frac{d\Psi}{ds} + f_0 + f_1, \\
Eh\varepsilon_{2(1)} &= R\frac{dV_1}{ds} - \mu(f_0 + f_1), \\
Eh\gamma_{(1)} &= 2(1+\mu)V_1 + \frac{Eh^3}{3R^2}\Psi, \\
\varkappa_{1(1)} &= \frac{d\Psi}{ds} - \frac{\mu}{Eh}\frac{dV_1}{ds} + \frac{1}{REh}(f_0 + f_1), \\
\varkappa_{2(1)} &= \frac{1}{Eh}\frac{dV_1}{ds} - \frac{\mu}{REh}(f_0 + f_1), \\
\tau_{(1)} &= -\frac{\Psi}{R} + \frac{2(1+\mu)}{REh}\left(V_1 - \int_0^s q_{2(1)}\,ds\right).
\end{aligned}\right\} \tag{26.13}
$$

$$\left.\begin{aligned}
t_1 &= f_0 + f_1 - \frac{D}{R}\frac{d\Psi}{ds},\\
t_2 &= R\frac{dV_1}{ds} - \mu\frac{D}{R}\frac{d\Psi}{ds},\\
s_1 &= V_1 + 2(1-\mu)\frac{D}{R^2}\Psi - \int_0^s q_{2(1)}\,ds,\\
\frac{m_2}{D} &= \mu\frac{d\Psi}{ds} + \frac{(1-\mu^2)}{Eh}\frac{dV_1}{ds},\\
\frac{m_1}{D} &= \frac{d\Psi}{ds} + \frac{(1-\mu^2)}{REh}(f_0+f_1),\\
\frac{h_1}{D} &= -\frac{(1-\mu)}{R}\Psi + \frac{2(1-\mu^2)}{EhR}V_1 - \frac{2(1-\mu^2)}{EhR}\int_0^s q_{2(1)}\,ds,
\end{aligned}\right\} \tag{26.14}$$

where $D = \frac{Eh^3}{12(1-\mu^2)}$.

Substituting the expressions for $\varepsilon_{2(1)}, \tau_{(1)}$ and m_1, s_1 in the second equation of continuity of (26.8) and the second equation of statics (26.6), we obtain two equations for determination of unknown functions Ψ and V_1:

$$\left.\begin{aligned}
&\frac{d^2\Psi}{ds^2} - \frac{2(1-\mu)}{R^2}\Psi - \frac{V_1}{D} =\\
&\qquad = -\frac{1}{D}\left(\frac{P_1}{\pi R} + \int_0^s q_{n(1)}\,ds\right) - \frac{(1-\mu^2)}{REh}\frac{d}{ds}(f_0+f_1),\\
&\frac{d^2V_1}{ds^2} - \frac{2(1+\mu)}{R^2}V_1 + \frac{Eh}{R^2}\Psi =\\
&\qquad = \frac{\mu}{R}\frac{d}{ds}(f_0+f_1) - \frac{2(1+\mu)}{R^2}\int_0^s q_{2(1)}\,ds.
\end{aligned}\right\} \tag{26.15}$$

As in the axisymmetric case, we go to dimensionless coordinate $\xi = \frac{s}{R}$ and introduce new function

$$\Psi_1 = Eh\Psi. \tag{26.16}$$

Then equations (26.15) are brought to the form

$$\left.\begin{aligned}
&\frac{d^2\Psi_1}{d\xi^2} - 2(1-\mu)\Psi_1 - 4\gamma^4 V_1 =\\
&\qquad = -4\gamma^4\left(\frac{P_1}{\pi R} + \int_0^\xi q_{n(1)}R\,d\xi\right) - (1-\mu^2)\frac{d}{d\xi}(f_0+f_1),\\
&\frac{d^2V_1}{d\xi^2} - 2(1+\mu)V_1 + \Psi_1 =\\
&\qquad = -2(1+\mu)R\int_0^\xi q_{2(1)}\,d\xi + \mu\frac{d}{d\xi}(f_0+f_1).
\end{aligned}\right\} \tag{26.17}$$

Multiplying the second equation by $2i\gamma^2$ and adding to the first, we obtain one equation in σ_1:

$$\frac{d^2\sigma_1}{d\xi^2}-2\sigma_1+2\mu\bar{\sigma}_1+2i\gamma^2\sigma_1=$$
$$=-4\gamma^4\left(\frac{P_1}{\pi R}+\int_0^\xi Rq_{n(1)}\,d\xi\right)-(1-\mu^2)\frac{d}{d\xi}(f_0+f_1)+$$
$$+2i\gamma^2\left[-2(1+\mu)R\int_0^\xi q_{2(1)}\,d\xi+\mu\frac{d}{d\xi}(f_0+f_1)\right]. \qquad (26.18)$$

where

$$\left.\begin{aligned}\sigma_1&=\Psi_1+2i\gamma^2V_1,\\ \bar{\sigma}_1&=\Psi_1-2i\gamma^2V_1.\end{aligned}\right\} \qquad (26.19)$$

Ignoring in the right side of this equation and in coefficient of σ_1 terms with an order of smallness $\left(\frac{1}{2\gamma^2}\right)$ and above in comparison with unity, we obtain

$$\frac{d^2\sigma_1}{d\xi^2}+2i\gamma^2\sigma_1=-4\gamma^4\left(\frac{P_1}{\pi R}+\int_0^\xi Rq_{n(1)}\,d\xi\right). \qquad (26.20)$$

In order to clarify how deleting in the left side (26.18) the term containing $-2\sigma_1+2\mu\bar{\sigma}_1$ affects the correctness of the solution, we write out a uniform systems of equations corresponding to (26.18) and (26.20)

$$\left.\begin{aligned}\frac{d^2\Psi_1}{d\xi^2}-2(1-\mu)\Psi_1-4\gamma^4V_1&=0,\\ \frac{d^2V_1}{d\xi^2}-2(1+\mu)V_1+\Psi_1&=0\end{aligned}\right\} \qquad (26.21)$$

and

$$\left.\begin{aligned}\frac{d^2\Psi_1}{d\xi^2}-4\gamma^4V_1&=0,\\ \frac{d^2V_1}{d\xi^2}+\Psi_1&=0.\end{aligned}\right\} \qquad (26.22)$$

The first system is equivalent to the equation of the fourth degree

$$\frac{d^4V_1}{d\xi^4}-4\frac{d^2V_1}{d\xi^2}+4\gamma^4V_1=0, \qquad (26.23)$$

and the second to the equation

$$\frac{d^4V_1}{d\xi^4} + 4\gamma^4 V_1 = 0. \tag{26.24}$$

Roots of the corresponding characteristic equations

$$k^4 - 4k^2 + 4\gamma^4 = 0 \tag{26.25}$$

and

$$k^4 + 4\gamma^4 = 0 \tag{26.26}$$

have a magnitude of order γ and differ from one another only in terms of the order $1/\gamma$ [57].

The detailed analysis of the stressed state of a closed cylindrical shell at different loads of form (8.4) conducted in monograph [5] indicates that in the case of a nonsymmetric load (k = 1) for the approximate creation of an edge effect we can use equation (26.26). In finding the particular solution of heterogeneous equation (26.18) the deletion of the quantity $(-2\sigma_1 + 2\mu\bar{\sigma}_1)$ in comparison with $2l\gamma^2\sigma_1$ is possible only when the right side of (26.18) is a slowly changing function of coordinate ξ.

To the simplifications made during the transition from (26.18) and (26.20) corresponds the following variant of relationships (26.13), (26.14):

$$\left.\begin{aligned} &t_1 = f_0 + f_1, \quad t_2 = \frac{dV_1}{d\xi}, \quad s_1 = V_1 - \int_0^\xi Rq_{2(1)}\,d\xi, \\ &m_1 = \frac{R}{4\gamma^4}\frac{d\Psi_1}{d\xi}, \quad m_2 = \mu m_1, \quad h_1 = -\frac{(1-\mu)}{4\gamma^4}R\Psi_1. \end{aligned}\right\} \tag{26.27}$$

$$\left.\begin{aligned} &Ehe_{1(1)} = f_0 + f_1 - \mu\frac{dV_1}{d\xi}, \quad Eh\varepsilon_{2(1)} = \frac{dV_1}{d\xi} - \mu(f_0 + f_1), \\ &Eh\gamma_{(1)} = 2(1+\mu)V_1, \\ &\varkappa_{1(1)} = \frac{1}{REh}\frac{d\Psi_1}{d\xi}, \quad \varkappa_{2(1)} = 0, \quad \tau_{(1)} = -\frac{1}{EhR}\Psi_1. \end{aligned}\right\} \tag{26.28}$$

Equation (26.20) differs from equation of axisymmetric deformation (25.26) only in the right side. In exactly the same way as was done in the case of axisymmetric deformation, the particular solution of equation (26.20) can be approximately found by dividing the right side by the coefficient of σ_1

$$\tilde{\sigma}_1 = 2\gamma^2 l\left(\frac{P_1}{\pi R} + \int_0^\xi Rq_{n(1)}\,d\xi\right) \tag{26.29}$$

or

$$\tilde{\Psi}_1 = 0, \quad \tilde{V}_1 = \frac{P_1}{\pi R} + \int_0^{\xi} R q_{n(1)} d\xi. \tag{26.30}$$

As follows from equations (26.27) to it corresponds the zero-moment stressed state of a shell

$$\left.\begin{aligned}
&\tilde{t}_1 = f_0 + f_1 = \\
&= -\frac{P_1 \xi}{\pi R} - \frac{M_1}{\pi R^2} - \int_0^{\xi}\int_0^{\xi} R(q_{n(1)} - q_{2(1)})\, d\xi\, d\xi - \int_0^{\xi} R q_{1(1)} d\xi, \\
&\tilde{t}_2 = R q_{n(1)}, \\
&\tilde{s}_1 = \frac{P_1}{\pi R} + \int_0^{\xi} R(q_{n(1)} - q_{2(1)})\, d\xi, \\
&\tilde{m}_1 = \tilde{m}_2 = \tilde{h}_1 = 0, \quad \tilde{n}_1 = 0,
\end{aligned}\right| \tag{26.31}$$

Stressed state (26.31), if we exclude from consideration $\tilde{t}_2$ and set $q_{1(1)} = 0$, coincides with the stressed state in a beam of tubular cross section, loaded on the edge $s = 0$ by force P_1 and moment M_1 and by distributed transverse load of intensity

$$q = (q_{n(1)} - q_{2(1)})\pi R. \tag{26.32}$$

The bending moment in a certain cut of beam s = const is equal to

$$-P_1 s - M_1 - \int_0^{s} q(s_1)(s - s_1)\, ds_1 = -P_1 s - M_1 - \int_0^{s}\int_0^{s} q\, ds\, ds. \tag{26.33}$$

It is balanced by moment of interior stress $\sigma = \frac{T_1}{h}$, equal to

$$2\int_0^{\pi} \left(\frac{T_1}{h}\right) hXR\, d\varphi = \pi R^2 t_1. \tag{26.34}$$

Equating expressions (26.33) and (26.34), we arrive at the first equation of (26.31). Analogously, composing the expression for shear force and equating it to the resultant of tangential forces $\tau = \frac{S}{h}$, we obtain the third equation of (26.31).

§ 27. Stressed State of a Long Axisymmetrically Loaded Cylindrical Shell

In §§ 25, 26 it was shown that the calculation of a cylindrical shell for axisymmetric and bending loads under known conditions is reduced to the solution of the same equation in complex function σ_0 or σ_1, but with different right sides. Let us examine the uniform equation

$$\frac{d^2\sigma}{d\xi^2}+2i\gamma^2\sigma=0. \tag{27.1}$$

Particular solutions of equation (27.1) have the form

$$\left.\begin{array}{ll} e^{-\gamma\xi}\cos\gamma\xi, & e^{-\gamma\xi}\sin\gamma\xi, \\ e^{\gamma\xi}\cos\gamma\xi, & e^{\gamma\xi}\sin\gamma\xi. \end{array}\right\} \tag{27.2}$$

Solutions written in the first line decrease with an increase of ξ, i.e., going from edge $\xi = 0$ to edge $\xi = l$, on the contrary, the solutions in the second line decrease with a decrease of ξ, i.e. going from edge $\xi = l$ to edge $\xi = 0$. In this way each pair of solutions (27.2) describes the stressed state of a shell in the neighborhood of its edge. This feature of the solutions of uniform equation (27.2) is called the edge effect. We introduce along with variable ξ, changing from 0 to l going from edge $s = 0$ to edge $s=L$. another variable

$$\xi_1=l-\xi. \tag{27.3}$$

which changes from 0 to l from edge $s=L$ to edge $s = 0$. Then the general solution of equation (27.1) can be written in the form

$$\sigma=(A_1-iB_1)[\theta(\gamma\xi)+i\zeta(\gamma\xi)]+(A_2-iB_2)[\theta(\gamma\xi_1)+i\zeta(\gamma\xi_1)]. \tag{27.4}$$

where by θ, ζ we designate functions [59], [12]

$$\theta(x)=e^{-x}\cos x,\quad \zeta(x)=e^{-x}\sin x. \tag{27.5}$$

Subsequently more combinations of these functions will be required

$$\varphi(x)=\theta(x)+\zeta(x),\quad \psi(x)=\theta(x)-\zeta(x)$$

and their derivatives

$$\left.\begin{aligned}\theta'(x)=-\varphi(x),\quad \zeta'(x)=\psi(x),\\ \theta''(x)=2\zeta(x)=-\varphi'(x),\quad \zeta''(x)=-2\theta(x)=\psi'(x).\end{aligned}\right\} \tag{27.6}$$

The values of functions θ, ζ, φ and ψ are given in Table 1 of the appendix. Solutions $\theta(\gamma\xi), \zeta(\gamma\xi)$ decrease with distance from edge $s = 0$, and solutions $\theta(\gamma\xi_1), \zeta(\gamma\xi_1)$ decrease with distance from edge $s=L$. As the tables show, $\theta(x), \zeta(x)$ already at $x=\pi$ have the order 0.04. This means that already at $\gamma\xi=\pi$ the influence of edge $s = 0$ on the stressed state in the given section of shell can be neglected. Using (25.22), (25.24) we can calculate the absolute length of the generatrix s which corresponds to the value $\xi=\frac{\pi}{\gamma}$:

$$s=\frac{\pi R}{\gamma}=\frac{\pi R}{\sqrt[4]{3(1-\mu^2)}}\sqrt{\frac{h}{R}}\approx 2.4\sqrt{Rh}.$$

In this way if the total length of a cylindrical shell L exceeds $2.4\sqrt{Rh}$, then the determination of arbitrary constants A_1, B_1 and A_2, B_2 solution (27.4) can be done separately by the conditions on each of the edges. In writing the conditions on edge $s = 0$ it is possible to set $A_2=B_2=0$, and during the determination of constants A_2, B_2 from conditions on edge $s=L$ consider $A_1=B_1=0$.

Let us turn to consideration of different conditions and to the determination of the constants of integration in solution (27.4) for an axisymmetrically loaded shell. In this instance the total solution of equation (25.27) is composed as the sum of solution (27.4) and the particular solution of the heterogeneous equation determined from formula (25.28). Separating the real and imaginary parts, we obtain

$$\left.\begin{aligned}\Psi_0=Eh\vartheta_1=A_1\theta(\gamma\xi)+B_1\zeta(\gamma\xi)+A_2\theta(\gamma\xi_1)+B_2\zeta(\gamma\xi_1),\\ V_0=-\frac{1}{2\gamma^2}[A_1\zeta(\gamma\xi)-B_1\theta(\gamma\xi)+A_2\zeta(\gamma\xi_1)-B_2\theta(\gamma\xi_1)]+\int_0^{\xi}Rq_{n(0)}d\xi.\end{aligned}\right\} \tag{27.7}$$

Further using equations (25.10), (25.15), (25.16) we have

$$T_1=\frac{P_0}{2\pi R}-\int_0^s q_{1(0)}\,ds. \tag{27.8}$$

$$\left.\begin{aligned} M_1 &= \frac{R}{4\gamma^3}[A_1\varphi(\gamma\xi) - B_1\psi(\gamma\xi) - A_2\varphi(\gamma\xi_1) + B_2\psi(\gamma\xi_1)], \\ M_2 &= \mu M_1. \end{aligned}\right\} \tag{27.9}$$

$$T_2 = \frac{1}{2\gamma}[-A_1\psi(\gamma\xi) - B_1\varphi(\gamma\xi) + A_2\psi(\gamma\xi_1) + B_2\varphi(\gamma\xi_1)] + Rq_{n(0)}. \tag{27.10}$$

$$N_1 = \frac{1}{2\gamma^2}[-A_1\zeta(\gamma\xi) + B_1\theta(\gamma\xi) - A_2\zeta(\gamma\xi_1) + B_2\theta(\gamma\xi_1)]. \tag{27.11}$$

Radial and axial displacements are calculated using the formulas

$$w = \frac{R}{Eh}(T_2 - \mu T_1), \tag{27.12}$$

$$u = u^0 + \frac{1}{Eh}\int_0^s (T_1 - \mu T_2)\,ds, \tag{27.13}$$

where u^0 is the constant of integration which is the assigned axial shift of edge $s = 0$. The four constants of integration A_1, B_1, A_2, B_2 are easily calculated by values of shear forces and bending moments assigned on every edge: N_1^0, M_1^0, N_1^L, M_1^L. Note that on the basis of equations (24.10) the quantity N_1 during axisymmetric stress in accuracy is equal to shear force Q_1. The positive directions of shearing force and bending moment are shown in Fig. 16. The bending moment is considered positive if it causes elongation of filaments on the external surface and compression on the internal.

On edge $s = L$ positively directed shearing force acts from the inside to the outside of the cylinder, and on edge $s = 0$, conversely, it acts inside the cylinder.

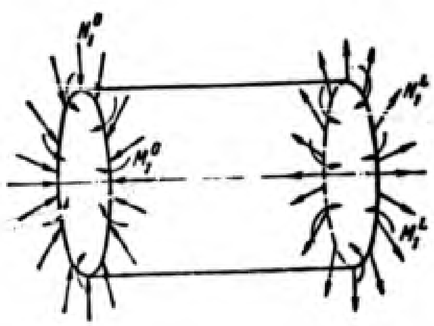


Fig. 16. Shearing forces and bending moments, evenly distributed on the edges of an axisymmetrically loaded cylindrical shell.

After simple calculations, ignoring the mutual influence of the edges, we have

$$N_1 = N_1^0\psi(\gamma\xi) - \frac{2\gamma}{R}M_1^0\zeta(\gamma\xi) + N_1^L\psi(\gamma\xi_1) + \frac{2\gamma}{R}M_1^L\zeta(\gamma\xi_1). \tag{27.14}$$

$$M_1 = M_1^0\varphi(\gamma\xi) + \frac{R}{\gamma}N_1^0\zeta(\gamma\xi) + M_1^L\varphi(\gamma\xi_1) - \frac{R}{\gamma}N_1^L\zeta(\gamma\xi_1). \tag{27.15}$$

$$M_2 = \mu M_1. \tag{27.16}$$

$$T_2 = -2\gamma\left[N_1^0\theta(\gamma\xi) + \frac{\gamma}{R}M_1^0\psi(\gamma\xi) - N_1^L\theta(\gamma\xi_1) + \frac{\gamma}{R}M_1^L\psi(\gamma\xi_1)\right] + Rq_{n(0)}. \tag{27.17}$$

$$\vartheta_1 = \frac{2\gamma^2}{Eh}\left[N_1^0\varphi(\gamma\xi) + \frac{2\gamma}{R}M_1^0\theta(\gamma\xi) + N_1^L\varphi(\gamma\xi_1) - \frac{2\gamma}{R}M_1^L\theta(\gamma\xi_1)\right]. \tag{27.18}$$

$$w = -\frac{2\gamma R}{Eh}\left[N_1^0\theta(\gamma\xi) + \frac{\gamma}{R}M_1^0\psi(\gamma\xi) - N_1^L\theta(\gamma\xi_1) + \right.$$
$$\left. + \frac{\gamma}{R}M_1^L\psi(\gamma\xi_1)\right] + \frac{R^2q_{n(0)}}{Eh} - \frac{\mu R}{Eh}\left(\frac{P_0}{2\pi R} - \int_0^s q_{1(0)}\,ds\right). \tag{27.19}$$

Calculating by these formulas the forces, moments and displacements in the neighborhood of one of the edges, it is necessary to drop the quantities which belong to the other edge. For example, in the neighborhood of $\xi = 0$ calculation of peripheral force T_2 must be done by the formula

$$T_2 = -2\gamma\left[N_1^0\theta(\gamma\xi) + \frac{\gamma}{R}M_1^0\psi(\gamma\xi)\right] + Rq_{n(0)},$$

but in the neighborhood of $\xi = l$ $(\xi_1 = 0)$ by the formula

$$T_2 = -2\gamma\left[-N_1^L\theta(\gamma\xi_1) + \frac{\gamma}{R}M_1^L\psi(\gamma\xi_1)\right] + Rq_{n(0)}$$

Proceeding in this way it is easy to obtain expressions for the angle of rotation and normal displacement of the edge of a shell loaded only by shearing forces and bending moments at this edge,

$$\left.\begin{aligned} \vartheta_1^0 &= \frac{2\gamma^2}{Eh}N_1^0 + \frac{4\gamma^3}{REh}M_1^0, \\ w^0 &= -\frac{2\gamma R}{Eh}N_1^0 - \frac{2\gamma^2}{Eh}M_1^0. \end{aligned}\right\} \tag{27.20}$$

$$\left.\begin{aligned} \vartheta_1^L &= \frac{2\gamma^2}{Eh}N_1^L - \frac{4\gamma^3}{REh}M_1^L. \\ w^L &= \frac{2\gamma R}{Eh}N_1^L - \frac{2\gamma^2}{Eh}M_1^L. \end{aligned}\right\} \tag{27.21}$$

As one would expect, following the principle of reciprocity of displacements, the shift from the action of a unit bending moment proved to be equal to the angle of rotation under the action of rotation under the action of unit shearing force. The inversion of equations (27.20), (27.21) gives

$$\left.\begin{aligned} M_1^0 &= \frac{Eh}{2\gamma^2} w^0 + \frac{EhR}{2\gamma^3} \vartheta_1^0, \\ N_1^0 &= -\frac{Eh}{\gamma R} w^0 - \frac{Eh}{2\gamma^2} \vartheta_1^0, \end{aligned}\right\} \quad (27.22)$$

$$\left.\begin{aligned} M_1^L &= \frac{Eh}{2\gamma^2} w^L - \frac{EhR}{2\gamma^3} \vartheta_1^L, \\ N_1^L &= \frac{Eh}{\gamma R} w^L - \frac{Eh}{2\gamma^2} \vartheta_1^L. \end{aligned}\right\} \quad (27.23)$$

From the previous reasoning and from equations (27.8), (27.14)-(27.16) it follows that if the edges of the shell are free, i.e., at the edges the following conditions hold:

$$N_1^0 = M_1^0 = 0, \quad N_1^L = M_1^L = 0. \quad (27.24)$$

and the distributed loads are smooth functions of coordinate s, then the shell is in the zero-moment stressed state (25.30). If edge conditions differ from (27.24), then on the zero-moment state is imposed an edge effect. We can give different variants of the edge conditions. For example, to the supported edge correspond the conditions

$$M_1^0 = 0, \quad w^0 = 0. \quad (27.25)$$

and to the fixed edge – the conditions

$$\vartheta_1^0 = 0, \quad w^0 = 0. \quad (27.26)$$

Conditions (27.26) can be expressed also through N_1^0, M_1^0, as directly follows from (27.20). For example, the first condition (27.26) can be written in the form

$$N_1^0 + \frac{2\gamma}{R} M_1^0 = 0 \quad (27.27)$$

In the following sections (§§ 28-33) we will examine the simplest examples of calculating a shell for different edge conditions and loads.

§ 28. A Cylinder with Rigid Bottoms Under Internal Pressure

1. Let us examine a cylinder with rigid bottoms which is under

internal pressure p [12]. The bottoms permit axial elongation of the cylinder. In this instance

$$q_{1(0)}=0, \quad q_{n(0)}=p, \quad P_0=p\pi R^2. \qquad (28.1)$$

and on the edges the conditions of zero angle of rotation and radial displacement should be held. Consequently, shearing forces and bending moments on edge $s = 0$ should satisfy condition (27.27) and on edge $s=L$ the analogous relationship

$$N_1^L-\frac{2\gamma}{R}M_1^L=0. \qquad (28.2)$$

Radial displacement in this case is

$$w=\frac{pR^2}{Eh}\left(1-\frac{\mu}{2}\right)-\frac{\gamma R}{Eh}\left[N_1^0\varphi(\gamma\xi)-N_1^L\varphi(\gamma\xi_1)\right]. \qquad (28.3)$$

and from the condition of w turning into zero at $\xi=0,\ \xi_1=0$ is obtained

$$N_1^0=\frac{pR}{\gamma}\left(1-\frac{\mu}{2}\right), \quad N_1^L=-\frac{pR}{\gamma}\left(1-\frac{\mu}{2}\right). \qquad (28.4)$$

Using formulas (27.14)-(27.16) and (28.4), (27.27), (28.2), we will make up the expressions for forces and moments:

$$\left.\begin{aligned} &T_2=pR\left\{1-\frac{(2-\mu)}{2}\left[\varphi(\gamma\xi)+\varphi(\gamma\xi_1)\right]\right\}, \quad T_1=\frac{pR}{2},\\ &M_1=-\frac{(2-\mu)}{2}\frac{pR^2}{2\gamma^2}\left[\psi(\gamma\xi)-\psi(\gamma\xi_1)\right], \quad M_2=\mu M_1,\\ &N_1=\frac{(2-\mu)}{2}\frac{pR}{\gamma}\left[\theta(\gamma\xi)-\theta(\gamma\xi_1)\right]. \end{aligned}\right\} \qquad (28.5)$$

In the calculation of forces using these formulas in the neighborhood of edge $s = 0$ we consider only terms depending on ξ, and in the area of edge $s=L$ only quantities depending on ξ_1. At a sufficiently large distance from the edges exists the zero-moment stressed state. The stresses of this state

$$\sigma_1=\frac{T_1}{h}, \quad \sigma_2=\frac{T_2}{h} \qquad (28.6)$$

we assume as nominal $\sigma_1^N,\ \sigma_2^N$, estimating stresses in the edge zones in comparison with them. In the given problem

$$\sigma_1^N = \frac{pR}{2h}, \quad \sigma_2^N = \frac{pR}{h}. \qquad (28.7)$$

Maximum stresses from the bending moment, computable using formula (4.46), exist on the edges of the shell, for example, at $\xi = 0$

$$\sigma_1^B = \pm \frac{6M_1^0}{h^2} = \mp 3(2-\mu)\frac{pR^2}{2\gamma^2 h^2}$$

or, taking into account (25.24),

$$\sigma_1^B = \mp \frac{(2-\mu)\sqrt{3}}{\sqrt{1-\mu^2}}\sigma_1^N \approx \mp 3.1\sigma_1^N \qquad (28.8)$$

at $\mu^2 = 0.1$.

In these formulas the upper sign is put on to the external, and the lower on the internal surfaces of the shell. In this way, on the external surface in section $s = 0$ we have compressive stresses $\sigma_1 = \sigma_1^N + \sigma_1^B = -2.1\sigma_1^N = -1.05\sigma_2^N$, and on the internal surface tensile stress

$$\sigma_1 = 4.1\sigma_1^N = 2.05\sigma_2^N.$$

The maximum tangential stress in section $s = 0$ exists at $\zeta = 0$ and can be determined according to formula (4.43)

$$\tau_1 = \frac{3}{2}\frac{N_1}{h} = \frac{3}{2}\frac{pR}{\gamma h}\frac{(2-\mu)}{2} \approx 1.75\sqrt{\frac{h}{R}}\sigma_2^N \qquad (\mu^2 = 0.1).$$

Hence it follows that tangential stresses have the order $\sqrt{\frac{h}{R}}$ in comparison with the nominal if the latter hold quantities of the order of unity. In this way tangential stresses are small in comparison with the stresses of the zero-moment state, and even more in comparison with the stresses of edge effect. An increase in stresses connected with the beginning of edge effect, bears a clearly expressed local character. Thus, in the examined example the amount of flexural stress from moment M_1 in section ξ will be

$$\sigma_1^B = \mp 1.55\, \sigma_2^N \psi(\gamma\xi).$$

i.e., already at $\gamma\xi = 0.6$ $|\sigma_1^B| \approx 0.22\ \sigma_2^N$. For a thin shell $\left(\frac{h}{R} = 0.01\right)$ such a drop of flexural stress is achieved at a length of the order of several shell thicknesses $(s \approx 0.0465R \approx 5h)$.

If, besides internal pressure, the cylinder experiences axial tensile forces from forces P applied to the bottoms, then

$$T_1 = \frac{P_0}{2\pi R}, \quad P_0 = p\pi R^2 + P \qquad (28.9)$$

and

$$N_1^0 = -N_1^L = \frac{pR}{\gamma}\left(1 - \frac{\mu}{2}\right) - \frac{\mu}{\gamma}\frac{P}{2\pi R},$$
$$M_1^0 = M_1^L = -\frac{pR^2}{2\gamma^2}\left(1 - \frac{\mu}{2}\right) + \frac{\mu}{2\gamma^2}\frac{P}{2\pi}.$$

When $p = 0$ the increase in stresses on the edge of the shell is comparatively small:

$$\left.\begin{aligned} \sigma_1^N &= \frac{P}{2\pi Rh}, \\ \sigma_1^B &= \mu\frac{6R}{h2\gamma^2}\,\sigma_1^N = 0.580\sigma_1^N \end{aligned}\right\} \qquad (28.10)$$

when $\mu^2 = 0.1$.

2. Assume now rigid diaphragms on the ends of a cylinder which is under the action of internal pressure such that they do not admit relative axial displacement of the ends. Constant P_0 in this instance should be determined from the condition

$$u^L - u^0 = \frac{1}{Eh}\int_0^L (T_1 - \mu T_2)\,ds = 0, \qquad (28.11)$$

hich gives

$$\frac{P_0 L}{2\pi R} = \mu pRL - \mu R\left[-N_1^0\int_0^l \varphi(\gamma\xi)\,\gamma\,d\xi + N_1^L\int_0^l \varphi(\gamma\xi_1)\,\gamma\,d\xi_1\right]. \qquad (28.12)$$

As it is easy to see from the previous example, terms containing N_1^0, N_1^L in the right parts of (28.12) will be quantities of order $1/\gamma$ in comparison with the basic quantities, taken as unity. Dropping them, we get

$$P_0 = \mu 2\pi p R^2. \quad T_1 = \mu p R. \tag{28.13}$$

Radial displacement is equal to

$$w = (1-\mu^2)\frac{pR^2}{Eh} - \frac{\gamma R}{Eh}\left[N_1^0 \varphi(\gamma\xi) - N_1^L \varphi(\gamma\xi_1)\right]. \tag{28.14}$$

Since w should turn into zero on the ends of the cylinder,

$$N_1^0 = -N_1^L = (1-\mu^2)\frac{pR}{\gamma}, \quad M_1^0 = M_1^L = -(1-\mu^2)\frac{pR^2}{2\gamma^2}. \tag{28.15}$$

Hence it is apparent that bending stresses in this instance will be altogether only $\frac{2(1-\mu^2)}{2-\mu}$ times greater than bending stresses in a cylinder with shifting ends.

§ 29. Cylindrical Shell Loaded in the Middle Section by a Normal Load

Let us examine a cylinder of considerable length, loaded on the middle section by forces of constant intensity q (kg/cm) and directed as shown in Fig. 17a [12]. For that part of the cylinder on the right of loaded section $\xi = 0$ we have from conditions of symmetry

$$\vartheta_1^0 = 0, \quad N_1^0 = \frac{q}{2} \tag{29.1}$$

when $\xi = 0$.

On the basis of (27.27) we determine

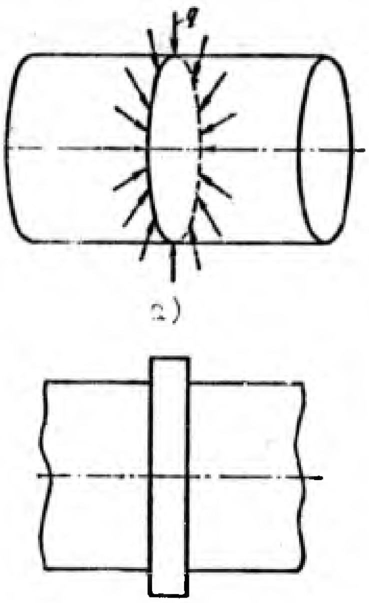
$$M_1^0 = -\frac{R}{2\gamma} N_1^0 = -\frac{qR}{4\gamma} \tag{29.2}$$

using formulas (27.14)-(27.19) we find

$$\left.\begin{array}{l} N_1 = \frac{1}{2} q\theta(\gamma\xi), \quad M_1 = -\frac{qR}{4\gamma}\psi(\gamma\xi), \\ T_2 = -\frac{1}{2} q\gamma\varphi(\gamma\xi), \quad w = -\frac{qR\gamma}{2Eh}\varphi(\gamma\xi), \\ \vartheta_1 = \frac{\gamma^2}{Eh} q\zeta(\gamma\xi). \end{array}\right\} \tag{29.3}$$

Maximum radial displacement exists in a loaded cut and is

$$w = -\frac{qR\gamma}{2Eh}. \qquad (29.4)$$



b)

Fig. 17. Cylindrical shell: a) loaded on the middle section by compressive normal forces, b) shell with ring fit onto it.

Using the solution to this problem, one can determine pressure q, induced by the influence of an elastic ring fit onto the cylinder in the hot state (Fig. 17b). The radius of the cold ring is less than the external radius of the cylinder by an amount δ. Under the action of a normal tensile load q elastic displacement of the ring with transverse section F, made from a material with Young's modulus E_1, is equal to

$$\delta_1 = \frac{qR^2}{E_1F}. \qquad (29.5)$$

and the displacement of the cylinder under a compressive load of the same amount is equal to the right part of (29.4) with the opposite sign. The sum of these displacements is equal to

$$\frac{qR^2}{E_1F} + \frac{qR\gamma}{2Eh} = \delta,$$

whence

$$q = \delta \frac{2Eh}{R\gamma} \frac{1}{1+4\beta_1}, \qquad (29.6)$$

where $\beta_1 = \frac{EhR}{2\gamma E_1F}$.

If a cylinder with closed heads is under the action of internal pressure and has in a certain section far from the ends a circular rib of rigidity, then force of interaction of the cylinder with the rib also can be determined according to previous formula if we set

$$\delta = \frac{2-\mu}{2}\frac{pR^2}{Eh}, \tag{29.7}$$

since namely such would be the radial displacement in the given section of cylinder in the absence of a stiffening rib. In this way, in this instance

$$q = \frac{pR}{\gamma}\frac{2-\mu}{1+4\beta_1}. \tag{29.8}$$

At an increase in the rigidity of the rib E_1F to infinity $(\beta_1 \to 0)$

$$q = (2-\mu)\frac{pR}{\gamma},$$

and the values of shearing force and bending moment in this section

$$N_1^0 = \frac{2-\mu}{2}\frac{pR}{\gamma}, \quad M_1^0 = -\frac{(2-\mu)}{2}\frac{pR}{2\gamma^2}$$

coincide with those which were obtained earlier for a fixed section (§ 28).

§ 30. Shell Equipped with a Ring of Rigidity

In practice frequently we find shells strengthened on the ends by reinforcing rings. Let us examine examine the coupling of a shell with ring (Fig. 18) on which act compressive forces N^* and twisting moments M^*, referred to a unit length of the middle line of the ring. The ring, just as the shell, is considered to be thin, and it is possible to set that the radius of the shell is equal to the radius of the middle line of the ring. The forces of interaction of the shell with the ring we designate N_1^0, M_1^0. Under the action of an applied system of forces the ring will receive radial displacement

$$\delta_1 = \frac{(N_1^0 - N^*)R^2}{E_1F} \tag{30.1}$$

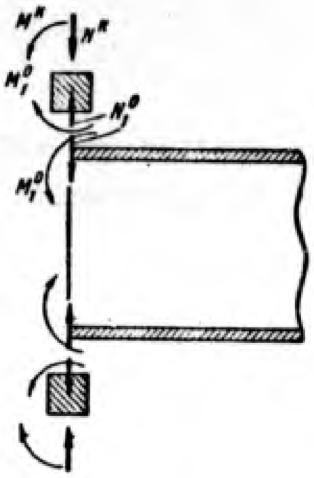


Fig. 18. Coupling of cylindrical shell with ring.

and its cross section turns through an angle

$$\varphi_1 = \frac{(M^k - M_1^0) R^2}{E_1 I}, \tag{30.2}$$

where I – moment of inertia of the cross section of the ring relative to the axis passing through the center of gravity of the section perpendicular to the axis of rotation of the ring. Equating the displacement and turning of the ring to the displacement and turning of the shell under edge loads N_1^0, M_1^0, internal pressure p and axial force $P_0 = p\pi R^2$, we obtain the conditions for determination of N_1^0, M_1^0:

$$\left.\begin{aligned} \frac{(N_1^0 - N^k) R^2}{E_1 F} &= -\frac{2\gamma R}{Eh}\left(N_1^0 + \frac{\gamma}{R} M_1^0\right) + \frac{pR^2}{Eh}\frac{2-\mu}{2}, \\ \frac{(M^k - M_1^0) R^2}{E_1 I} &= \frac{2\gamma^2}{Eh}\left(N_1^0 + \frac{2\gamma}{R} M_1^0\right). \end{aligned}\right\} \tag{30.3}$$

From them we have

$$\left.\begin{aligned} N_1^0 &= \frac{1}{2(1+\beta_1) - \frac{1}{1+\beta_2}}\left[-\frac{2\gamma}{R}\frac{\beta_2}{1+\beta_2} M^k + 2\beta_1 N^k + \right. \\ &\qquad \left. + \frac{pR}{\gamma}\frac{(2-\mu)}{2}\right], \\ M_1^0 &= \frac{1}{2(1+\beta_2) - \frac{1}{1+\beta_1}}\left[M^k\beta_2 - \frac{R}{\gamma}\frac{\beta_1}{1+\beta_1} N^k - \right. \\ &\qquad \left. - \frac{pR^2}{2\gamma^2}\frac{(2-\mu)}{2}\frac{1}{1+\beta_1}\right]. \end{aligned}\right\} \tag{30.4}$$

where β_1 is the designation introduced in § 29, and

$$\beta_2 = \frac{EhR^3}{E_1 I 4\gamma^3}.$$

If on the ring external forces have not been applied ($M^* = N^* = 0$) and its rigidity to unscrewing is infinite, then using formulas (30.4) in this instance we obtain

$$\left.\begin{aligned} N_1^0 &= \frac{pR}{\gamma}\frac{2-\mu}{2}\frac{1}{1+2\beta_1}, \\ M_1^0 &= -\frac{pR^2}{2\gamma^2}\frac{2-\mu}{2}\frac{1}{1+2\beta_1}. \end{aligned}\right\} \quad (30.5)$$

Comparing the obtained result with the one which for an absolutely rigid closing (§ 28), we see that the compliance of the ring on elongation reduces the shearing force and bending moment in the ratio $\frac{1}{1+2\beta_1}$. At $M^* = N^* = 0$ and $\beta_1 = 0$, $\beta_2 \neq 0$ (ring is pliable in the sense of angle of rotation, but is rigid with respect to radial displacement) we obtain

$$\left.\begin{aligned} N_1^0 &= \frac{pR}{\gamma}\frac{2-\mu}{2}\frac{1+\beta_2}{1+2\beta_2}, \\ M_1^0 &= -\frac{pR^2}{2\gamma^2}\frac{2-\mu}{2}\frac{1}{1+2\beta_2}. \end{aligned}\right\} \quad (30.6)$$

Setting in these formulas $1/\beta_2 = 0$ (ring absolutely pliable in the sense of angle of rotation), we obtain the values of shearing force and moment on the support edge

$$N_1^0 = \frac{pR}{\gamma}\cdot\frac{2-\mu}{4}, \quad M_1^0 = 0.$$

§ 31. Shell Loaded in the Middle Section by Distributed Bending Moments

Let us determine the angle of rotation in a section loaded by distributed moments of intensity m. The total solution to the problem about stress of a cylinder under such a load is simple to obtain using the solution to the problem examined in § 29.

In two close sections $\xi = 0$ and $\xi = \Delta$ assume the shell is acted on by forces of intensity q and $-q$ (Fig. 19). Then the angle of rotation in a certain section $\xi > \Delta$ can be calculated in the form of the sum

$$\vartheta_1 = -\frac{2\gamma^2}{Eh}\frac{q}{2}\zeta(\gamma\xi - \gamma\Delta) + \frac{2\gamma^2}{Eh}\frac{q}{2}\zeta(\gamma\xi) =$$
$$= \frac{2\gamma^2}{Eh}\frac{qR\Delta\gamma}{2R}\frac{\zeta(\gamma\xi) - \zeta(\gamma\xi - \gamma\Delta)}{\gamma\Delta}.$$

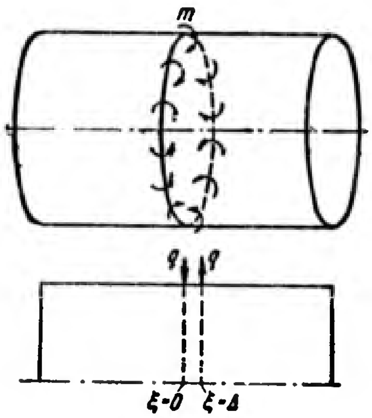


Fig. 19. Cylindrical shell loaded on the middle section by bending moments of constant intensity.

Going to the limit so that at $\Delta s = R \cdot \Delta \to 0 \ q\Delta s \to m,$ we obtain

$$\vartheta_1 = \frac{\gamma^3 m}{EhR}\frac{d\zeta(\gamma\xi)}{d(\gamma\xi)} = \frac{\gamma^3 m}{EhR}\psi(\gamma\xi). \tag{31.1}$$

Expressions for forces, bending moment and radial displacement can be obtained similarly:

$$\left.\begin{aligned} T_2 &= \frac{m\gamma^2}{R}\zeta(\gamma\xi), & N_1 &= -\frac{m\gamma}{2R}\varphi(\gamma\xi), \\ M_1 &= \frac{m}{2}\theta(\gamma\xi), & w &= \frac{m\gamma^2}{Eh}\zeta(\gamma\xi). \end{aligned}\right\} \tag{31.2}$$

§ 32. Deformations of a Welded Heterogeneous Cylindrical Shell

Assume a welded cylindrical shell, composed from heterogeneous parts [60]. The first part has coefficient of linear expansion α_1, and the second part has α_2, where $\alpha_1 > \alpha_2$. At a certain temperature T there are no stresses in the shell. When the shell cools to temperature t the stresses will develop in the neighborhood of the joined heterogenetic parts. Far from the joint radial displacement of the first part is

$$w_1 = -\alpha_1(T - t)R, \tag{32.1}$$

the second part

$$w_2 = -\alpha_2(T - t)R. \tag{32.2}$$

Let us designate the radial displacement and angle of rotation in the butt section through w_0, ϑ_1^0. Then, mentally separating the shell, it can be considered that the first part is bent because of the radial displacement of the edge

$$w_0 - w_1 = w_0 + \alpha_1 (T - t) R \tag{32.3}$$

and the angle of rotation of edge ϑ_1^0 (Fig. 20), and the second part is bent due to the shift and angle, equal to

$$w_0 - w_2 = w_0 + \alpha_2 (T - t) R, \quad \vartheta_1^0. \tag{32.4}$$

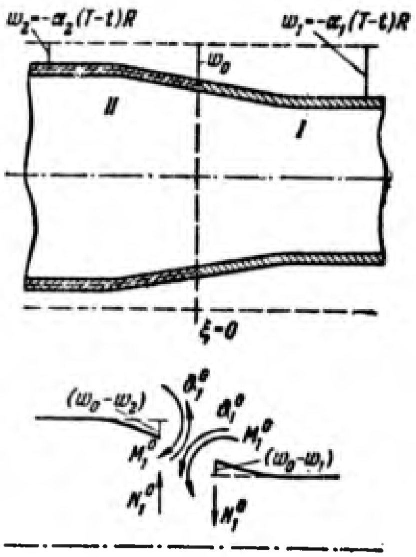


Fig. 20. Temperature deformation of welded heterogeneous cylindrical shell.

By formulas (27.20), (27.21) we obtain for the first part

$$\left.\begin{aligned} w_0 + \alpha_1 (T - t) R &= -\frac{2\gamma R}{E_1 h}\left(N_1^0 + \frac{\gamma}{R} M_1^0\right), \\ \vartheta_1^0 &= \frac{2\gamma^2}{E_1 h}\left(N_1^0 + \frac{2\gamma}{R} M_1^0\right). \end{aligned}\right\} \tag{32.5}$$

for the second part

$$\left.\begin{aligned} w_0 + \alpha_2 (T - t) R &= -\frac{2\gamma R}{E_2 h}\left(-N_1^0 + \frac{\gamma}{R} M_1^0\right), \\ \vartheta_1^0 &= \frac{2\gamma^2}{E_2 h}\left(N_1^0 - \frac{2\gamma}{R} M_1^0\right). \end{aligned}\right\} \tag{32.6}$$

Solving the obtained system in ϑ_1^0, w_0, N_1^0, M_1^0, at $E_1 = E_2 = E$ we have

$$
\left.\begin{array}{l}
N_1^0 = -\frac{Eh}{4\gamma}(\alpha_1 - \alpha_2)(T - t), \quad M_1^0 = 0, \\
w_0 = -\frac{(\alpha_1 + \alpha_2)}{2}(T - t) R, \\
\vartheta_1^0 = -\frac{\gamma}{2}(\alpha_1 - \alpha_2)(T - t).
\end{array}\right\} \qquad (32.7)
$$

Computing T_2, M_1 with the aid of formulas (2.15), (2.16), we find that the peripheral force suffers a discontinuity at the joint, and the bending moment is everywhere continuous:

$$
\left.\begin{array}{l}
T_2^{I} = \frac{Eh}{2}(\alpha_1 - \alpha_2)(T - t)\theta(\gamma\xi), \\
T_2^{II} = -\frac{Eh}{2}(\alpha_1 - \alpha_2)(T - t)\theta(\gamma\xi_1).
\end{array}\right\} \qquad (32.8)
$$

$$
\left.\begin{array}{l}
M_1^{I} = -\frac{EhR}{4\gamma^2}(\alpha_1 - \alpha_2)(T - t)\zeta(\gamma\xi), \\
M_1^{II} = \frac{EhR}{4\gamma^2}(\alpha_1 - \alpha_2)(T - t)\zeta(\gamma\xi_1).
\end{array}\right\} \qquad (32.9)
$$

The bending moment becomes greatest in absolute value in the section $\gamma\xi_* = \frac{\pi}{4}$.

Flexural stresses in this section are equal to

$$
\sigma_1^B = \frac{6E}{4\gamma^2}\frac{R}{h}(\alpha_1 - \alpha_2)(T - t)\,0.322,
$$

at $\mu^2 = 0.1$

$$
\sigma_1^B = 0.294E(\alpha_1 - \alpha_2)(T - t). \qquad (32.10)
$$

Peripheral stresses are the greatest at the joint. They are

$$
\left.\begin{array}{l}
\sigma_2 = \frac{T_2^{I}(0)}{h} = \frac{1}{2}E(\alpha_1 - \alpha_2)(T - t); \\
\sigma_2 = \frac{T_2^{II}(0)}{h} = -\frac{1}{2}E(\alpha_1 - \alpha_2)(T - t).
\end{array}\right\} \qquad (32.11)
$$

§ 33. Cylindrical Shell Under the Action of Hydrostatic Pressure

In all earlier examined examples load $q_{n(0)}$ did not change along the cylinder and the particular solution of the basic resolvant equation (25.27) did not differ from the zero-moment solution.

Let us examine the case of a load changing along the axis [12]. A vertical container is filled to the top with water. The lower edge of the container is rigidly fixed, the upper is free. In this way, at the edges the following conditions should hold:

$$\left.\begin{aligned} \vartheta_1^0 = w^0 = 0, \\ M_1^L = N_1^L = 0. \end{aligned}\right\} \qquad (33.1)$$

The load has the form

$$q_{n(0)} = \rho R (l - \xi), \quad \frac{dq_{n(0)}}{d\xi} = -\rho R, \quad q_{1(0)} = 0. \qquad (33.2)$$

The particular solution of the basic equation is determined now using formulas (25.33)

$$\Psi_0 = -\rho R^2, \quad \tilde{V}_0 = \int_0^\xi \rho R^2 (l - \xi)\, d\xi = \rho R^2 \left(l\xi - \frac{\xi^2}{2}\right). \qquad (33.3)$$

To this solution correspond the following forces, moments and displacements:

$$\left.\begin{aligned} \tilde{T}_1 = 0, \quad \tilde{T}_2 = \rho R^2 (l - \xi), \\ \tilde{M}_1 = \tilde{N}_1 = 0, \end{aligned}\right\}. \qquad (33.4)$$

$$\tilde{\vartheta}_1 = -\frac{\rho R^2}{Eh}, \quad \tilde{w} = \frac{\rho R^3 (l - \xi)}{Eh}. \qquad (33.5)$$

Let us determine shearing force and bending moment in the fixed section. On edge $\xi = 0$ the angle of rotation and displacement corresponding to the above particular solution are equal to

$$\tilde{\vartheta}_1 = -\frac{\rho R^2}{Eh}, \quad \tilde{w} = \frac{\rho R^2 L}{Eh}. \qquad (33.6)$$

Substituting in formulas (27.22)

$$w^0 \text{ by } w^0 - \frac{\rho R^2 L}{Eh},$$

$$\vartheta_1^0 \text{ by } \vartheta_1^0 + \frac{\rho R^2}{Eh}$$

assuming $w^0 = \vartheta_1^0 = 0$, we obtain

$$M_1^0 = -\frac{\rho R^2 L}{2\gamma^2}\left(1 - \frac{1}{\gamma}\frac{R}{L}\right), \quad N_1^0 = \frac{\rho RL}{\gamma}\left(1 - \frac{1}{2\gamma}\frac{R}{L}\right). \qquad (33.7)$$

The calculation of these quantities on the basis of the zero-moment solution

$$\Psi_0 = 0, \quad \tilde{V}_0 = \rho R^2 \left(l\xi - \frac{\xi^2}{2} \right)$$

leads to the values

$$M_1^0 = -\frac{\rho R^2 L}{2\gamma^2}, \quad N_1^0 = \frac{\rho R L}{\gamma}, \qquad (33.8)$$

which differ from those obtained above in terms of order $\frac{1}{\gamma}\frac{R}{L}$ in comparison with unity.

In this way, utilization of the zero-moment solution leads in this case to error of the order $\sqrt{\frac{h}{R}}$ in comparison with unity during the determination of local stresses near the edge.

§ 34. Long Cylindrical Shell Under the Action of a Bending Load

Considering the deformation of a shell under the action of a bending load (Fig. 21), we will make use of simplified resolvant equation (26.20). The solution of the corresponding uniform equation is again taken in form (27.4). Taking into account (26.19), we obtain

$$\left.\begin{aligned} \Psi_1 = Eh\Psi = A_1\theta(\gamma\xi) + B_1\zeta(\gamma\xi) + A_2\theta(\gamma\xi_1) + B_2\zeta(\gamma\xi_1), \\ 2\gamma^2 V_1 = A_1\zeta(\gamma\xi) - B_1\theta(\gamma\xi) + A_2\zeta(\gamma\xi_1) - B_2\theta(\gamma\xi_1). \end{aligned}\right\} \qquad (34.1)$$

To get the general solution to (34.1) it is necessary to add the particular solution of heterogeneous equation (26.20). As such one could use zero-moment solution (26.30)

$$\Psi_1 = 0, \quad \tilde{V}_1 = \frac{P_1}{\pi R} + \int_0^{\xi} R q_{\alpha(1)}\, d\xi. \qquad (34.2)$$

In the example of an axisymmetrically loaded shell it was shown that calculation of the particular solution on the basis of zero-moment theory in certain cases can introduce into the determination of local stress error of the order $\sqrt{\frac{h}{R}}$ in comparison with unity. Solution

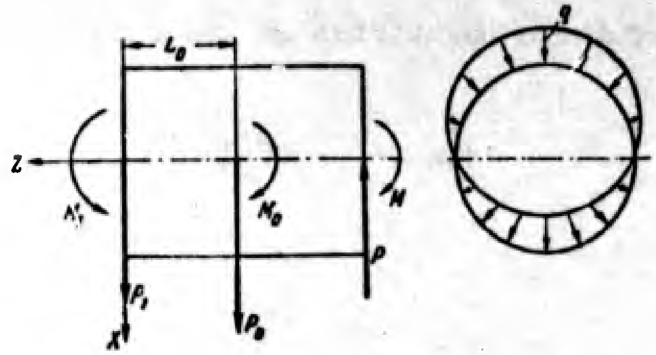


Fig. 21. System of loads bending a cylindrical shell.

(34.1), describing the edge effect, also does not claim greater accuracy, since in the left side of equation (26.18) terms of order $\frac{h}{R}$ in comparison with unity are rejected, which can lead to an error in the solution of the order of $\sqrt{\frac{h}{R}}$ in comparison with unity. Therefore, using (34.1), (34.2) during calculations it is inexpedient to keep terms of the order of $\sqrt{\frac{h}{R}}$ in comparison with unity. Note that developing the solution of the initial equation (26.18) keeping terms of order $\sqrt{\frac{h}{R}}$ in comparison with unity is possible [57]. However, this would lead to a strong complication of the estimated equations, which hardly is justified, since the conditions of fixing the edges of the shell are practically known also with low accuracy.

On the basis of (34.1), (34.2) using formulas (26.27), (26.28) we compute forces, moments and deformations:

$$t_2=\frac{1}{2\gamma}\left[A_1\psi(\gamma\xi)+B_1\varphi(\gamma\xi)-A_2\psi(\gamma\xi_1)-B_2\varphi(\gamma\xi_1)\right]+\tilde{t}_2. \tag{34.3}$$

$$s_1=\frac{1}{2\gamma^2}\left[A_1\zeta(\gamma\xi)-B_1\theta(\gamma\xi)+A_2\zeta(\gamma\xi_1)-B_2\theta(\gamma\xi_1)\right]+\tilde{s}_1. \tag{34.4}$$

$$m_1=\frac{R}{4\gamma^3}\left[-A_1\varphi(\gamma\xi)+B_1\psi(\gamma\xi)+A_2\varphi(\gamma\xi_1)-B_2\psi(\gamma\xi_1)\right]. \tag{34.5}$$

$$m_2=\mu m_1. \tag{34.6}$$

$$h_1=-\frac{(1-\mu)R}{4\gamma^4}\left[A_1\theta(\gamma\xi)+B_1\zeta(\gamma\xi)+A_2\theta(\gamma\xi_1)+B_2\zeta(\gamma\xi_1)\right]. \tag{34.7}$$

$$n_1=s_1+\frac{h_1}{R}-\left[\frac{P_1}{\pi R}+\int_0^{\xi}R\left(q_{x\,(1)}-q_{2\,(1)}\right)d\xi\right]\approx$$

$$\approx\frac{1}{2\gamma^2}\left[A_1\zeta(\gamma\xi)-B_1\theta(\gamma\xi)+A_2\zeta(\gamma\xi_1)-B_2\theta(\gamma\xi_1)\right]. \tag{34.8}$$

$$t_1=-\frac{m_1}{R}+\tilde{t}_1. \tag{34.9}$$

$$Ehe_{2\,(1)}=\frac{1}{2\gamma}\left[A_1\psi(\gamma\xi)+B_1\varphi(\gamma\xi)-A_2\psi(\gamma\xi_1)-B_2\varphi(\gamma\xi_1)\right]+Eh\tilde{e}_{2\,(1)}. \tag{34.10}$$

where (f_0+f_1) is a known function of external loads (26.7), the quantities $\tilde{t}_1, \tilde{t}_2, \tilde{s}_1$ are determined according to formulas (26.31), and

$$\tilde{\varepsilon}_{2(1)}=\frac{1}{Eh}[Rq_{z(1)}-\mu(f_0+f_1)]; \qquad (34.11)$$

is peripheral deformation, which corresponds to the zero-moment state.

Axial displacement $u_{(1)}$ is determined by the first formula of (26.9):

$$u_{(1)}=R\int_0^{\xi}\varepsilon_{1(1)}\,d\xi+u_0=-\frac{\mu R}{Eh}\int_0^{\xi}(t_2-\tilde{t}_2)\,d\xi+\tilde{u}_{(1)}+u_0. \qquad (34.12)$$

Here u_0 – constant of integration, characterizing angular displacement $\left(\frac{u_0}{R}\right)$ of edge $s=0$ as a whole,

$$\tilde{u}_{(1)}=\frac{R}{Eh}\int_0^{\xi}(\tilde{t}_1-\mu\tilde{t}_2)\,d\xi \qquad (34.13)$$

is the displacement corresponding to the zero-moment stressed state. Furthermore, in conformity with (26.27) in (34.12) we accept $t_1=\tilde{t}_1$. On the basis of (17.3), (18.5), assuming in them $R_1d\theta=ds$, $v=R$, $\theta=\frac{\pi}{2}$, we have

$$w_{(1)}=\Delta_{\varphi(1)},\quad \Delta_{z(1)}=-u_{(1)}$$

and

$$w_{(1)}=R\varepsilon_{2(1)}-\int_0^{\xi}\left(\gamma_{(1)}+\frac{u_{(1)}}{R}\right)R\,d\xi+w^0_{(1)},$$

where $w^0_{(1)}$ – displacement of edge $s=0$ in the direction of axis OX. Using this formula we compute the displacement of the zero-moment state

$$\tilde{w}_{(1)}=R\tilde{\varepsilon}_{2(1)}-\int_0^{\xi}\left(\tilde{\gamma}_{(1)}+\frac{\tilde{u}_{(1)}}{R}\right)R\,d\xi. \qquad (34.14)$$

Rewrite formula (26.11) in the following form:

$$\Psi=-\left(\frac{dw_{(1)}}{ds}+\frac{u_{(1)}}{R}\right). \qquad (34.15)$$

Separating the left and right parts, corresponding to the zero-moment solution, we obtain

$$\Psi-\tilde{\Psi}=-\frac{1}{R}\left[\frac{d(w_{(1)}-\tilde{w}_{(1)})}{d\xi}+u_{(1)}-\tilde{u}_{(1)}\right].$$

Assuming in this equality $\Psi=0$, we find

$$w_{(1)}=\tilde{w}_{(1)}-\int_0^{\xi}[R\Psi+u_{(1)}-\tilde{u}_{(1)}]\,d\xi+w^0_{(1)}, \qquad (34.16)$$

where $\tilde{w}_{(1)}$ is determined according to formula (34.14). Note that if we simply used formula (34.15) for determination of zero-moment displacement $\tilde{w}_{(1)}$, assuming in this case $\tilde{\Psi}=0$, then we would obtain the result

$$\tilde{w}_{(1)}=-\int_0^{\xi}\tilde{u}_{(1)}\,d\xi, \qquad (34.17)$$

which differs from (34.14). Such a difference is explained by the fact that the zero-moment solution $\Psi=0$, $V_1=\tilde{V}_1$ does not satisfy accurately the second equation of (26.15), which is the condition of compatibility of deformations.

Peripheral displacement $v_{(1)}$ is easily found without integration. On the basis of (26.9) we have

$$v_{(1)}=Re_{2(1)}-w_{(1)}. \qquad (34.18)$$

At the end of § 26 it was shown that the zero-moment stressed state $\tilde{T}_1$, $\tilde{S}_1$ coincides with the stressed state in a beam. Relative to displacements we can say the following. If in formula (34.13) we set $\mu=0$ and then determine displacement $\tilde{w}_{(1)}$ on the basis of (34.17), then we obtain the expression

$$\tilde{w}_{(1)}=\frac{1}{EI}\left(P_1\frac{s^3}{6}+M_1\frac{s^2}{2}+\iiiint q(s)\,ds\,ds\,ds\,ds\right), \qquad (34.19)$$

also coinciding with the deflection of a beam of tubular cross section ($I=\pi hR^3$ – moment of inertia of cross section of beam, $q(s)$ is transverse load, determined according to (26.32)).

Comparing (34.4) and (34.7), it is easy to see that $\frac{h_1}{R}$ is small in comparison with s_1, and therefore the condition of equilibrium (26.4) can be approximately written in the form

$$n_1 - s_1 = -\frac{P_1}{\pi R} - R\int_0^{\xi} (q_{n\,(1)} - q_{2\,(1)})\, d\xi. \tag{34.20}$$

By the same reason we can set

$$s_1 + \frac{2h_1}{R} \approx s_1 \approx S_{12\,(1)} \tag{34.21}$$

and consider n_1 to be the amplitude of shearing force. Remember that the combination written in the left side of (34.21) appears in the formulation of the power edge conditions (17.8).

Let us assume that at the edges of the shell are assigned amplitudes of tangential forces and bending moments

$$s_1^0,\ m_1^0,\ s_1^L,\ m_1^L.$$

Ignoring the mutual influence of the edges, it is simple to express, using the edge conditions, the constants A_1, B_1, A_2, B_2 through the given quantities $n_1^0, m_1^0, n_1^L, m_1^L$, whereupon for forces, moments and deformations we obtain the expressions

$$n_1 = n_1^0\psi(\gamma\xi) - \frac{2\gamma}{R} m_1^0\zeta(\gamma\xi) + n_1^L\psi(\gamma\xi_1) + \frac{2\gamma}{R} m_1^L\zeta(\gamma\xi_1), \tag{34.22}$$

$$m_1 = \frac{R}{\gamma} n_1^0\zeta(\gamma\xi) + m_1^0\varphi(\gamma\xi) - \frac{R}{\gamma} n_1^L\zeta(\gamma\xi_1) + m_1^L\varphi(\gamma\xi_1), \tag{34.23}$$

$$t_2 = -2\gamma n_1^0\theta(\gamma\xi) - \frac{2\gamma^2}{R} m_1^0\psi(\gamma\xi) + 2\gamma n_1^L\theta(\gamma\xi_1) - \frac{2\gamma^2}{R} m_1^L\psi(\gamma\xi_1) + Rq_{n\,(1)}, \tag{34.24}$$

$$t_1 = f_0 + f_1 - \frac{m_1}{R}, \tag{34.25}$$

$$s_1 = n_1 + \frac{P_1}{\pi R} + R\int_0^{\xi} (q_{n\,(1)} - q_{2\,(1)})\, d\xi, \tag{34.26}$$

$$m_2 = \mu m_1, \tag{34.27}$$

$$\Psi = -\frac{2\gamma^2}{Eh} n_1^0\varphi(\gamma\xi) - \frac{4\gamma^3}{REh} m_1^0\theta(\gamma\xi) - \frac{2\gamma^2}{Eh} n_1^L\varphi(\gamma\xi_1) + \frac{4\gamma^3}{REh} m_1^L\theta(\gamma\xi_1). \tag{34.28}$$

$$Re_{2(1)} = -\frac{2\gamma R}{Eh}\left[n_1^0\theta(\gamma\xi) + \frac{\gamma}{R} m_1^0\psi(\gamma\xi) - n_1^l\theta(\gamma\xi_1) + \right.$$
$$\left. + \frac{\gamma}{R} m_1^l\psi(\gamma\xi_1)\right] + \frac{R^2 q_{n(1)}}{Eh} - \frac{\mu R}{Eh}(f_0 + f_1). \quad (34.29)$$

It is easy to note that formulas (34.22)-(34.29) coincide with formulas (27.14)-(27.19) if we replace in them Ψ by $-\vartheta_1$, $Re_{2(1)}$ by w and change the load terms. In this way, between both events – axisymmetric deformation and deformation with a bending load – is observed a close analogy. Let us observe this in examples.

§ 35. Cylindrical Shell Loaded on the Circumference by a Bending Load

Let us examine a cylinder loaded in a certain section $s = L_0$ rather far from both edges by normal forces of intensity $q_n = q\cos\varphi$ (Fig. 21). The main vector of this load P_0 is equal to πqR. Just as in the axisymmetric case, by considerations of symmetry we assume that the amplitude of shearing force n_1 on going through the loaded section suffers an interruption of continuity in the amount of q:

$$n_1^- = \frac{q}{2} = \frac{P_0}{2\pi R}, \quad n_1^+ = -\frac{q}{2} = -\frac{P_0}{2\pi R}. \quad (35.1)$$

The signs - and + designate the amounts at $s = L_0 - 0$ and $s = L_0 + 0$ respectively.

From conditions of equilibrium it is clear that

$$\left.\begin{aligned} (n_1 - s_1)^- &= -\frac{P_1}{\pi R}, \quad (n_1 - s_1)^+ = -\frac{P_1}{\pi R} - \frac{P_0}{\pi R}, \\ (Rt_1 + m_1)^- &= (Rt_1 + m_1)^+ = -\frac{M_1}{\pi R} - \frac{P_1 L_0}{\pi R}, \end{aligned}\right\} \quad (35.2)$$

consequently,

$$s_1^- = s_1^+ = \frac{P_1}{\pi R} + \frac{P_0}{2\pi R}. \quad (35.3)$$

On the basis of (34.28), (34.29) we can write

$$\left.\begin{aligned} Eh\Psi^- &= -2\gamma^2 n_1^- + \frac{4\gamma^3}{R} m_1^-, \\ Eh\Psi^+ &= -2\gamma^2 n_1^+ - \frac{4\gamma^3}{R} m_1^+. \end{aligned}\right\} \quad (35.4)$$

$$\left.\begin{aligned} EhR\varepsilon^-_{2(1)} &= 2\gamma Rn^-_1 - 2\gamma^2 m^-_1 + \mu\left(\frac{M_1}{\pi R} + \frac{P_1 L_0}{\pi R}\right), \\ EhR\varepsilon^+_{2(1)} &= -2\gamma Rn^+_1 - 2\gamma^2 m^+_1 + \mu\left(\frac{M_1}{\pi R} + \frac{P_1 L_0}{\pi R}\right). \end{aligned}\right\} \qquad (35.5)$$

Note that continuity of quantities Ψ, $\varepsilon_{2(1)}$ during transition through the loaded section should be provided for, inasmuch as it necessary for the continuity of displacements (formulas (26.9), (26.11)). Assuming $\varepsilon^-_{2(1)} = \varepsilon^+_{2(1)}$ and taking into account (35.1), it is simple to see that the bending moment also is continuous. The requirement that $\Psi^- = \Psi^+$ is satisfied if we set

$$\Psi^- = \Psi^+ = 0, \quad m^-_1 = m^+_1 = \frac{R}{2\gamma}\frac{q}{2}. \qquad (35.6)$$

Further, using formulas (34.22)-(34.29) we compute forces and bending moments for that part of the cylinder on the right of the loaded section:

$$\left.\begin{aligned} n_1 &= -\frac{q}{2}\theta(\gamma\xi), \quad t_2 = \frac{\gamma q}{2}\varphi(\gamma\xi), \\ s_1 &= \frac{P_1}{\pi R} + q\left[1 - \frac{1}{2}\theta(\gamma\xi)\right], \quad m_1 = \frac{Rq}{4\gamma}\psi(\gamma\xi), \\ t_1 &= -\frac{M_1}{\pi R^2} - \frac{P_1 s}{\pi R^2} - q\xi - \frac{q}{4\gamma}\psi(\gamma\xi), \end{aligned}\right\} \qquad (35.7)$$

$$\left.\begin{aligned} Eh\Psi &= q\gamma^2\zeta(\gamma\xi), \\ REh\varepsilon_{2(1)} &= \frac{q\gamma R}{2}\varphi(\gamma\xi) + \mu\left(\frac{M_1}{\pi R} + \frac{P_1 L_0}{\pi R}\right). \end{aligned}\right\} \qquad (35.8)$$

In these formulas relative length is lead from the loaded section $\xi = \frac{(s - l_0)}{R}$. The obtained formulas are analogous to formulas (29.3).

Let us examine a shell loaded in section $s = L_0$ moments of intensity $m_0 \cos\varphi$, distributed along the circumference. Total moment M_0 is equal to $m_0 \pi R$. The amplitude of the bending moment during transition through the loaded section suffers a discontinuity:

$$m^-_1 = \frac{m_0}{2}, \quad m^+_1 = -\frac{m_0}{2}. \qquad (35.9)$$

The combination $(n_1 - s_1)$ is continuous, and $(Rt_1 + m_1)$ undergoes discontinuity during transition through the loaded section. This follows from the conditions of statics (26.4), (26.5), written for sections L_0-0 and L_0+0:

$$\left.\begin{aligned} n_1^- - s_1^- &= n_1^+ - s_1^+ = -\frac{P_1}{\pi R},\\ Rt_1^- + m_1^- &= -\frac{M_1}{\pi R} - \frac{P_1L_0}{\pi R},\\ Rt_1^+ + m_1^+ &= -\frac{M_1}{\pi R} - \frac{P_1L_0}{\pi R} - m_0. \end{aligned}\right\} \qquad (35.10)$$

Hence it is easy to see that

$$t_1^- = t_1^+ = -\frac{M_1}{\pi R^2} - \frac{P_1L_0}{\pi R^2} - \frac{m_0}{2R}. \qquad (35.11)$$

Writing the relationships

$$\left.\begin{aligned} EhR\varepsilon_{2(1)}^- &= 2\gamma R n_1^- - 2\gamma^2 m_1^- - \mu t_1^- R,\\ EhR\varepsilon_{2(1)}^+ &= -2\gamma R n_1^+ - 2\gamma^2 m_1^+ - \mu t_1^+ R, \end{aligned}\right\} \qquad (35.12)$$

we are convinced that the condition $\varepsilon_{2(1)}^- = \varepsilon_{2(1)}^+$ is held only when the equalities

$$2\gamma R n_1^- - 2\gamma^2 m_1^- = -2\gamma R n_1^+ - 2\gamma^2 m_1^+ = 0 \qquad (35.13)$$

exist. From these equalities it follows that

$$n_1^- = n_1^+ = \frac{\gamma m_0}{2R}, \qquad (35.14)$$

and consequently,

$$s_1^- = s_1^+ = \frac{P_1}{\pi R} + \frac{m_0\gamma}{2R}. \qquad (35.15)$$

By direct check it is simple to be convinced that in this case $\Psi^- = \Psi^+$; really,

$$Eh\Psi^- = -2\gamma^2 n_1^- + \frac{4\gamma^3}{R} m_1^- = \frac{\gamma^3 m_0}{R},$$

$$Eh\Psi^+ = -2\gamma^2 n_1^+ - \frac{4\gamma^3}{R} m_1^+ = \frac{\gamma^3 m_0}{R}.$$

For forces and moments on the right of the loaded section we obtain the formulas:

$$\left.\begin{aligned} n_1 &= \frac{\gamma m_0}{2R}\varphi(\gamma\xi), \qquad t_2 = -\frac{\gamma^2 m_0}{R}\zeta(\gamma\xi),\\ s_1 &= \frac{P_1}{\pi R} + \frac{\gamma m_0}{2R}\varphi(\gamma\xi), \qquad m_1 = -\frac{m_0}{2}\theta(\gamma\xi),\\ t_1 &= -\frac{M_1}{\pi R^2} - \frac{P_1 s}{\pi R^2} - \frac{m_0}{R} + \frac{m_0}{2R}\theta(\gamma\xi),\\ Eh\Psi &= \frac{\gamma^3 m_0}{R}\psi(\gamma\xi) \qquad \left(\xi = \frac{s - L_0}{R}\right). \end{aligned}\right\} \qquad (35.16)$$

the same formulas could be obtained directly by using (35.7), (35.8) and carrying out the limit transition just as was done in § 31.

§ 36. Stresses and Displacements in a Shell with Rigid Bottoms

Let us determine stresses in a long shell loaded as shown in Fig. 15 at $P_0 = 0$, considering that the extreme sections of a shell are connected with rigid diaphragms and can only be turned and dislocated as a whole. On edges $\xi = 0$ and $\xi_1 = 0$ $(s = L)$ the following conditions should be executed:

$$\varepsilon_{2(1)}^{0} = \Psi^{0} = 0, \quad \varepsilon_{2(1)}^{L} = \Psi^{L} = 0. \tag{36.1}$$

On the basis of (34.28), (34.29) we obtain

$$\left.\begin{aligned} m_1^0 &= -\frac{R}{2\gamma} n_1^0, & n_1^0 &= \frac{1}{\gamma}\left[Rq_{n(1)} - \mu(f_0+f_1)_{s=0}\right], \\ m_1^L &= \frac{R}{2\gamma} n_1^L, & n_1^L &= -\frac{1}{\gamma}\left[Rq_{n(1)} - \mu(f_0+f_1)_{s=L}\right], \end{aligned}\right\} \tag{36.2}$$

where

$$\left.\begin{aligned} (f_0+f_1)_{s=0} &= -\frac{M_1}{\pi R^2}, \\ (f_0+f_1)_{s=L} &= -\frac{M_1}{\pi R^2} - \frac{P_1 L}{\pi R^2} - \\ &\quad -\frac{1}{R}\int_0^L\int_0^s (q_{n(1)} - q_{2(1)})\,ds\,ds - \int_0^L q_{1(1)}\,ds. \end{aligned}\right\} \tag{36.3}$$

Conditions of equilibrium of the shell as a whole is

$$\left.\begin{aligned} -\frac{P_1}{\pi R} - \int_0^L (q_{n(1)} - q_{2(1)})\,ds + \frac{P}{\pi R} &= 0, \\ -\frac{M_1}{\pi R} - \frac{P_1 L}{\pi R} - \int_0^L\int_0^s (q_{n(1)} - q_{2(1)})\,ds\,ds + \frac{M}{\pi R} - \int_0^L Rq_{1(1)}\,ds &= 0, \end{aligned}\right\} \tag{36.4}$$

whence it follows that

$$(f_0+f_1)_{s=L} = -\frac{M}{\pi R^2}.$$

In this way,

$$\left.\begin{aligned} m_1^0 &= -\frac{R}{2\gamma^2}\left[Rq_{n(1)} + \frac{\mu M_1}{\pi R^2}\right], \\ m_1^L &= -\frac{R}{2\gamma^2}\left[Rq_{n(1)} + \frac{\mu M}{\pi R^2}\right]. \end{aligned}\right\} \tag{36.5}$$

Far from ends of the cylinder exists the zero-moment state:

$$\left.\begin{aligned}\tilde{t}_1 &= f_0 + f_1 = -\frac{M_1}{\pi R^2} - \frac{P_1 s}{\pi R^2} - \\ &\quad - \frac{1}{R}\int_0^s\int_0^s \left(q_{n\,(1)} - q_{2\,(1)}\right) ds\, ds - \int_0^s q_{1\,(1)}\, ds, \\ \tilde{t}_2 &= R q_{n\,(1)}, \\ \tilde{s}_1 &= \frac{P_1}{\pi R} + \int_0^s q_{n\,(1)}\, ds - \int_0^s q_{2\,(1)}\, ds.\end{aligned}\right\} \quad (36.6)$$

The stresses calculated according to forces of the zero-moment state we take as the nominal, where

$$\left.\begin{aligned}\sigma_1^{NO} &= \frac{\tilde{t}_1(0)}{h} = -\frac{M_1}{\pi R^2 h}, \quad \sigma_2^N = \frac{R q_{n\,(1)}}{h}, \\ \sigma_1^{NL} &= \frac{\tilde{t}_1(L)}{h} = -\frac{M}{\pi R^2 h}.\end{aligned}\right\} \quad (36.7)$$

One ought to have in mind that amplitudes of stresses or the maximum stresses which exist in points $\phi = 0$ have been here written out. Stresses in points $\phi = \pi$ are equal in magnitude and opposite in sign. In order to find stresses at arbitrary ϕ, it is sufficient to multiply the amplitude values by $\cos\phi$. It is easy to see that the amplitude of stress σ_1^N is equal in magnitude to the maximum stress in a beam of tubular cross section (the moment of mass inertia of the section of the beam relative to axis OY is equal to $I = \pi R^3 h$ and stresses $\sigma_{max} = MR/I = \frac{M}{\pi R^2 h}$). The amplitudes of bending stress from an edge load and from distributed pressure we compute using the formula

$$\sigma_1^B = \pm\frac{6 m_1}{h^2}. \quad (36.8)$$

for example,

$$\sigma_1^{BO} = \mp\frac{6R}{2\gamma^2 h}\left(\frac{R q_{n\,(1)}}{h} + \mu\frac{M_1}{\pi R^2 h}\right). \quad (36.9)$$

where the upper sign is taken during the calculation of stresses in filaments of the external surface of the shell; the lower sign refers to fibers of the internal surface. In this way, flexural stresses from internal pressure on the edge of the shell amount to

$$\frac{6R}{2\gamma^2 h}\sigma_2^N=\sqrt{\frac{3}{1-\mu^2}}\,\sigma_2^N$$

(when $\mu^2=0.1$, $\sqrt{\frac{3}{1-\mu^2}}\approx 1.83$); flexural stresses from the edge load M or M_1 are equal to $\mu\sqrt{\frac{3}{1-\mu^2}}\,\sigma_1^N\approx 0.580\sigma_1^N$. Comparing these results with the estimate of local increase in the stress of a rigidly fixed edge of an axisymmetrically loaded shell (§ 28), it is easy to see that quantitatively they agree.

Let us turn to the determination of displacements. On the basis of formulas (34.12) and (34.24) we find

$$u_{(1)}=\tilde{u}_{(1)}-\frac{\mu R}{Eh}\int_0^{\xi}\left[-2\gamma n_1^0\theta(\gamma\xi)-\frac{2\gamma^2}{R}m_1^0\psi(\gamma\xi)\right]d\xi-$$
$$-\frac{\mu R}{Eh}\int_{\xi_1}^{l}\left[2\gamma n_1^l\theta(\gamma\xi_1)-\frac{2\gamma^2}{R}m_1^l\psi(\gamma\xi_1)\right]d\xi+u_{(1)}^0. \qquad (36.10)$$

Let us integrate taking into consideration that

$$\int\theta(x)\,dx=-\frac{1}{2}\psi(x),\quad \int\psi(x)\,dx=\zeta(x),$$
$$\int\zeta(x)\,dx=-\frac{1}{2}\varphi(x),\quad \int\varphi(x)\,dx=-\theta(x),$$

then we obtain

$$u_{(1)}=\tilde{u}_{(1)}-\frac{\mu R}{Eh}\Big\{n_1^0[\psi(\gamma\xi)-1]-\frac{2\gamma}{R}m_1^0\zeta(\gamma\xi)-$$
$$-n_1^l[\psi(\gamma l)-\psi(\gamma\xi_1)]-\frac{2\gamma}{R}m_1^l[\zeta(\gamma l)-\zeta(\gamma\xi_1)]\Big\}+u_{(1)}^0. \qquad (36.11)$$

Subsequently we will set $u_{(1)}^0=0$. This means that there is no rotation of section $s=0$ around axis OY. Setting in (36.11) $\xi=l$, $\xi_1=0$ and ignoring the quantities $\psi(\gamma l)$, $\zeta(\gamma l)$ in comparison with unity, we find the displacement on the second end

$$u_{(1)}^l=\tilde{u}_{(1)}^l+\frac{\mu R}{Eh}(n_1^0-n_1^l). \qquad (36.12)$$

Calculating the integrals

$$-\int_0^{\xi}(u_{(1)}-\tilde{u}_{(1)})\,d\xi=\frac{\mu R}{Eh}\Big\{-n_1^0\xi-n_1^l\psi(\gamma l)\xi-\frac{2\gamma}{R}m_1^l\zeta(\gamma l)\xi+$$
$$+\frac{1}{\gamma}n_1^0\zeta(\gamma\xi)+\frac{m_1^0}{R}[\varphi(\gamma\xi)-1]+\frac{n_1^l}{\gamma}[\zeta(\gamma l)-\zeta(\gamma\xi_1)]-$$
$$-\frac{m_1^l}{R}[\varphi(\gamma l)-\varphi(\gamma\xi_1)]\Big\}.$$

$$-\int_0^\xi \Psi\, d\xi = -\frac{1}{Eh}\Big\{2\gamma n_1^0[\theta(\gamma\xi)-1]+\frac{2\gamma^2}{R}m_1^0[\psi(\gamma\xi)-1]+$$
$$+2\gamma n_1^l[\theta(\gamma l)-\theta(\gamma\xi_1)]-\frac{2\gamma^2}{R}m_1^l[\psi(\gamma l)-\psi(\gamma\xi_1)]\Big\}$$

and substituting the result into equation (34.16), dropping small terms, we obtain

$$w_{(1)}=\tilde{w}_{(1)}+\frac{R}{Eh}\Big\{2\gamma n_1^0[1-\theta(\gamma\xi)]+\frac{2\gamma^2}{R}[1-\psi(\gamma\xi)]+$$
$$+2\gamma n_1^l\theta(\gamma\xi_1)-\frac{2\gamma^2}{R}m_1^l\psi(\gamma\xi_1)-\mu n_1^0\xi-\mu n_1^l\psi(\gamma l)\,\xi-$$
$$-\mu\frac{2\gamma}{R}m_1^l\zeta(\gamma l)\,\xi-2\gamma n_1^l\theta(\gamma l)+\frac{2\gamma^2}{R}m_1^l\psi(\gamma l)\Big\}+w_{(1)}^0. \qquad (36.13)$$

Let us determine constant $w_{(1)}^0$ from the condition $w_{(1)}(0) = 0$, then we obtain $w_{(1)}^0=-\tilde{w}_{(1)}(0)=-\tilde{w}_{(1)}^0$. On the second end ($\xi=l,\ \xi_1=0$) the shift is

$$w_{(1)}^l=\tilde{w}_{(1)}^l-\tilde{w}_{(1)}^0+\frac{R}{Eh}\Big\{2\gamma n_1^0[1-\theta(\gamma l)]+$$
$$+\frac{2\gamma^2}{R}m_1^0[1-\psi(\gamma l)]+2\gamma n_1^l-\frac{2\gamma^2}{R}m_1^l-\mu n_1^0 l-\mu n_1^l\psi(\gamma l)\cdot l-$$
$$-\mu\frac{2\gamma}{R}m_1^l\zeta(\gamma l)\cdot l-2\gamma n_1^l\theta(\gamma l)+\frac{2\gamma^2}{R}m_1^l\psi(\gamma l)\Big\}. \qquad (36.14)$$

Dropping in (36.14) terms containing $\theta(\gamma l)$, $\zeta(\gamma l)$, $\psi(\gamma l)$, in comparison with unity, derive

$$w_{(1)}^l=\tilde{w}_{(1)}^l-\tilde{w}_{(1)}^0+\frac{R}{Eh}\Big[2\gamma(n_1^0+n_1^l)+\frac{2\gamma^2}{R}(m_1^0-m_1^l)-\mu n_1^0 l\Big]. \qquad (36.15)$$

Let us calculate displacements $\tilde{u}_{(1)}$, $\tilde{w}_{(1)}$, supposing that there is no distributed load. Taking into account (36.3), using formulas (34.13), (36.6) we find

$$Eh\tilde{u}_{(1)}=R\int_0^\xi (f_0+f_1)\,d\xi=-\frac{M_1}{\pi R}\xi-\frac{P_1\xi^2}{2\pi}. \qquad (36.16)$$

On the basis of (34.14) and relationships of elasticity we have

$$\tilde{w}_{(1)}=\frac{R}{Eh}(\tilde{t}_2-\mu\tilde{t}_1)-\frac{2R(1+\mu)}{Eh}\int_0^\xi \tilde{s}_1\,d\xi-\int_0^\xi \tilde{u}_{(1)}\,d\xi. \qquad (36.17)$$

Having in mind that in this case

$$\tilde{t}_2=0,\quad \tilde{t}_1=f_0+f_1,\quad \tilde{s}_1=\frac{P_1}{\pi R},$$

we obtain

$$\left.\begin{aligned}\tilde{w}_{(1)}&=\frac{1}{Eh\pi R}\Big[M_1\frac{\xi^2}{2}+P_1R\frac{\xi^3}{6}+\mu M_1-(2+\mu)P_1R\xi\Big],\\ \tilde{w}_{(1)}^{(0)}&=\frac{1}{Eh\pi R}\mu M_1.\end{aligned}\right\} \qquad (36.18)$$

Shifts on end $s = L$ now are easily calculated:

$$
\left.\begin{aligned}
u_{(1)}^{L} &= -\frac{M_1 l}{\pi R E h} - \frac{P_1}{\pi E h}\frac{l^2}{2} + \frac{\mu R}{E h}\left(n_1^0 - n_1^L\right), \\
w_{(1)}^{L} &= \frac{M_1}{\pi R E h}\frac{l^2}{2} + \frac{P_1}{\pi E h}\frac{l^3}{6} - \frac{(2+\mu)}{E h}\frac{P_1 l}{\pi} + \\
&\quad + \frac{R}{E h}\left[2\gamma\left(n_1^0 + n_1^L\right) + \frac{2\gamma^2}{R}\left(m_1^0 - m_1^L\right) - \mu n_1^0 l\right].
\end{aligned}\right\} \qquad (36.19)
$$

From formula (36.2) follows

$$
\begin{aligned}
n_1^0 + n_1^L &= -\frac{\mu}{\gamma}(f_0 + f_1)_{s=0} + \frac{\mu}{\gamma}(f_0 + f_1)_{s=L}, \\
m_1^0 - m_1^L &= -\frac{R}{2\gamma}\left(n_1^0 + n_1^L\right), \\
n_1^0 - n_1^L &= -\frac{\mu}{\gamma}(f_0 + f_1)_{s=0} - \frac{\mu}{\gamma}(f_0 + f_1)_{s=L}.
\end{aligned}
$$

Substituting these expressions into equations for the calculation of $u_{(1)}^{L}$, $w_{(1)}^{L}$, we find

$$
\left.\begin{aligned}
u_{(1)}^{L} &= -\frac{M_1 l}{\pi R E h} - \frac{P_1}{\pi E h}\frac{l^2}{2} + \frac{\mu^2 R}{E h \gamma}\left(\frac{2M_1}{\pi R^2} + \frac{P_1 l}{\pi R}\right), \\
w_{(1)}^{L} &= \frac{M_1}{\pi R E h}\frac{l^2}{2} + \frac{P_1 l^3}{\pi E h \cdot 6} - \frac{2(1+\mu) P_1 l}{\pi E h} - \frac{\mu^2 l}{\gamma}\frac{M_1}{\pi R E h}.
\end{aligned}\right\} \qquad (36.20)
$$

Taking into account that the relationships

$$P_1 = P, \quad M = M_1 + P_1 L,$$

exist, and introducing the quantities $EI = Eh\pi R^3$, $EF = 2\pi REh$, $G = \frac{E}{2(1+\mu)}$, we write out the formulas for calculation of the angle of rotation and shift of section $s = L$ of the shell in the form

$$
\left.\begin{aligned}
\omega_y^L &= -\frac{u_{(1)}^L}{R} = \frac{1}{EI}\left[ML\left(1 - \frac{2\mu^2 R}{\gamma L}\right) - \frac{P_1 L^2}{2}\left(1 - \frac{2\mu^2 R}{\gamma L}\right)\right], \\
x^L &= w_{(1)}^L = \frac{1}{EI}\left[\frac{ML^2}{2}\left(1 - \frac{2\mu^2 R}{\gamma L}\right) - \frac{PL^3}{3}\left(1 - \frac{3\mu^2 R}{\gamma L}\right)\right] - \frac{2PL}{GF}.
\end{aligned}\right\} \qquad (36.21)
$$

The obtained equations differ from formulas for the calculation of a beam of tubular cross section, conducted allowing for shift, only in terms of order $1/\gamma$ in comparison with unity.

§ 37. Axisymmetric Deformation of a Short Cylindrical Shell

Let us examine a short shell whose length L has order $2.5\sqrt{Rh}$ or less. In this instance the solutions of uniform equation (27.1), taken in the form of (27.2), cannot be significantly diminished from one edge to another. The separate determination of the constants of integration conducted earlier becomes impossible. The general solution of equation (27.1) is now more convenient to write not in the form of (27.4), but through Krylov functions, which are introduced usually during the solution to the problem of curvature of a beam on an elastic base [12], [58]. Below is a table of Krylov functions and their derivatives. As it is easy to see, these functions are linear combinations of solutions (27.2) and that is why they satisfy equation (27.1). The choice of solutions in such a form facilitates the determination of arbitrary constants from edge conditions, since the initial values of the Krylov functions and their derivatives to the third order inclusively form a diagonal unit matrix.

Table 3.

k	$\Omega_k(x)$	$\Omega_k'(x)$	$\Omega_k''(x)$	$\Omega_k'''(x)$	$\Omega_k^{IV}(x)$
1	$\operatorname{ch} x \cos x$	$-4\Omega_4$	$-4\Omega_3$	$-4\Omega_2$	$-4\Omega_1$
2	$\frac{1}{2}(\operatorname{ch} x \sin x + \operatorname{sh} x \cos x)$	Ω_1	$-4\Omega_4$	$-4\Omega_3$	$-4\Omega_2$
3	$\frac{1}{2}\operatorname{sh} x \sin x$	Ω_2	Ω_1	$-4\Omega_4$	$-4\Omega_3$
4	$\frac{1}{4}(\operatorname{ch} x \sin x - \operatorname{sh} x \cos x)$	Ω_3	Ω_2	Ω_1	$-4\Omega_4$

Table 4.

k	$\Omega_k(0)$	$\Omega_k'(0)$	$\Omega_k''(0)$	$\Omega_k'''(0)$
1	1	0	0	0
2	0	1	0	0
3	0	0	1	0
4	0	0	0	1

Let us write out the solution of uniform equation (27.1) in the following form:

$$\sigma = (c_1 - ic_3)[\Omega_1(\gamma\xi) - 2i\Omega_3(\gamma\xi)] + (c_2 - ic_4)[\Omega_2(\gamma\xi) - 2i\Omega_4(\gamma\xi)]. \quad (37.1)$$

For an axisymmetrically loaded shell to (37.1) one ought to add the particular solution of heterogeneous equation (25.26), having the form of (25.28). For a shell under a bending load particular solution which one ought to add to (37.1), has the form of (26.29). We will examine first an axisymmetrically loaded cylindrical shell [12]. separating real and imaginary parts of the solution we obtain

$$\Psi_0 = Eh\vartheta_1 = c_1\Omega_1(\gamma\xi) - 2c_3\Omega_3(\gamma\xi) + c_2\Omega_2(\gamma\xi) - 2c_4\Omega_4(\gamma\xi). \quad (37.2)$$

$$-2\gamma^2 V_0 = -c_3\Omega_1(\gamma\xi) - 2c_1\Omega_3(\gamma\xi) - c_4\Omega_2(\gamma\xi) - 2c_2\Omega_4(\gamma\xi) - 2\gamma^2 \int_0^\xi Rq_{n(0)}\, d\xi. \quad (37.3)$$

Substituting (37.2), (37.3) into formulas (25.20), (25.15), (25.16) and taking into account (25.21) and differentiating with the aid of the table of Krylov derivatives, we have

$$M_1 = -\frac{R}{4\gamma^4}\frac{d\Psi_0}{d\xi} = \frac{R}{4\gamma^3}[4c_1\Omega_4(\gamma\xi) - c_2\Omega_1(\gamma\xi) + 2c_3\Omega_2(\gamma\xi) + 2c_4\Omega_3(\gamma\xi)]. \quad (37.4)$$

$$T_2 = \frac{dV_0}{d\xi} = \frac{1}{2\gamma}[2c_1\Omega_2(\gamma\xi) + 2c_2\Omega_3(\gamma\xi) - 4c_3\Omega_4(\gamma\xi) + c_4\Omega_1(\gamma\xi)] + Rq_{n\,(0)}. \quad (37.5)$$

$$T_1 = \frac{P_0}{2\pi R} - \int_0^{\xi} q_{1\,(0)} R\, d\xi. \quad (37.6)$$

$$N_1 = V_0 - \int_0^{\xi} Rq_{n\,(0)}\, d\xi =$$
$$= \frac{1}{2\gamma^2}[c_3\Omega_1(\gamma\xi) + 2c_1\Omega_3(\gamma\xi) + c_4\Omega_2(\gamma\xi) + 2c_2\Omega_4(\gamma\xi)]. \quad (37.7)$$

$$w = \frac{R}{2Eh\gamma}[2c_1\Omega_2(\gamma\xi) + 2c_2\Omega_3(\gamma\xi) - 4c_3\Omega_4(\gamma\xi) + c_4\Omega_1(\gamma\xi)] +$$
$$+ \frac{R}{Eh}\left[Rq_{n\,(0)} - \frac{\mu P_0}{2\pi R} + \mu\int_0^{\xi} Rq_{1\,(0)}\, d\xi\right]. \quad (37.8)$$

If on edge $s = 0$ the shell is loaded by bending moments M_1^0 and shearing forces N_1^0, and the second edge $s = L$ is free, then arbitrary constants c_1, c_2, c_3, c_4 should be defined from the conditions:

$$s = 0. \quad N_1 = N_1^0. \quad M_1 = M_1^0. \quad (37.9)$$

$$s = L. \quad N_1 = 0. \quad M_1 = 0. \quad (37.10)$$

According to (37.9), (37.4) and (37.7), using the above table of initial values of Krylov functions, immediately we find

$$c_3 = 2\gamma^2 N_1^0. \quad c_2 = -\frac{4\gamma^3}{R}M_1^0. \quad (37.11)$$

Conditions (37.10) allowing for (37.11) give a system of equations for determination of c_1, c_4;

$$\left.\begin{aligned} 2c_1\Omega_3(\gamma l) + c_4\Omega_2(\gamma l) &= -2\gamma^2 N_1^0\Omega_1(\gamma l) + \frac{8\gamma^3}{R}M_1^0\Omega_4(\gamma l). \\ 4c_1\Omega_4(\gamma l) + 2c_4\Omega_3(\gamma l) &= -4\gamma^2 N_1^0\Omega_2(\gamma l) - \frac{4\gamma^3}{R}M_1^0\Omega_1(\gamma l). \end{aligned}\right\} \quad (37.12)$$

Solving (37.12), we obtain

$$\left.\begin{aligned} c_1 &= \frac{4\gamma^3}{R}M_1^0\Phi_3(\gamma l) + 2\gamma^2 N_1^0\Phi_2(\gamma l). \\ c_4 &= -\frac{4\gamma^3}{R}M_1^0\Phi_2(\gamma l) - 4\gamma^2 N_1^0\Phi_1(\gamma l). \end{aligned}\right\} \quad (37.13)$$

where the following designations have been introduced

$$\left.\begin{aligned}&\Phi_1(x)=\frac{1}{\Delta}(\operatorname{sh}2x-\sin 2x), \quad \Phi_2(x)=\frac{1}{\Delta}(\operatorname{ch}2x-\cos 2x),\\ &\Phi_3(x)=\frac{1}{\Delta}(\operatorname{sh}2x+\sin 2x), \qquad \Delta=\operatorname{ch}2x+\cos 2x-2.\end{aligned}\right\} \tag{37.14}$$

Let us give also several relationships between functions Φ_1, Φ_2, Φ_3 and Krylov functions, useful for subsequent calculations:

$$\left.\begin{aligned}&\Phi_1=\frac{8}{\Delta}(\Omega_2\Omega_3-\Omega_1\Omega_4),\\ &\Phi_2=\frac{8}{\Delta}(\Omega_3\Omega_1+4\Omega_4^2)=\frac{8}{\Delta}(\Omega_2^2-\Omega_1\Omega_3),\\ &\Phi_3=\frac{4}{\Delta}(4\Omega_3\Omega_4+\Omega_1\Omega_2), \quad \Delta=16(\Omega_3^2-\Omega_2\Omega_4),\\ &\Phi_2\Omega_1+2\Omega_3-2\Phi_3\Omega_2=-\frac{8\Omega_3}{\Delta},\\ &\Phi_1\Omega_1+2\Omega_4-\Phi_2\Omega_2=-\frac{8}{\Delta}\Omega_4,\\ &\Phi_3\Omega_1-\Omega_2+2\Phi_2\Omega_4=\frac{4}{\Delta}\Omega_2, \quad \Phi_2\Omega_1-2\Omega_3+4\Phi_1\Omega_4=\frac{8}{\Delta}\Omega_3,\\ &-2\Phi_3\Omega_3+2\Omega_4+\Phi_2\Omega_2=0, \quad 2\Phi_2\Omega_3+\Omega_1-2\Phi_1\Omega_2=0,\\ &\Omega_1^2-4\Omega_3^2+8\Omega_2\Omega_4=1.\end{aligned}\right\} \tag{37.15}$$

Constants c_1, c_4 have a simple mechanical sense: the first is proportional to the angle of rotation of section $s=0\,(\vartheta_1^0)$, the second to radial displacement of the edge of the shell at $q_{n(0)}=q_{1(0)}=P_0=0$. Really, from (37.2), (37.8), setting $\xi = 0$, we find

$$c_1=Eh\vartheta_1^0, \quad c_4=\frac{Eh}{R}2\gamma w^0. \tag{37.16}$$

Comparing (37.16) and (37.13), we govern radial displacement w^0 and angle of rotation ϑ_1^0 at edge $s = 0$ depending on forces and moments N_1^0, M_1^0 applied to

$$\left.\begin{aligned}&\vartheta_1^0=\frac{4\gamma^3}{REh}M_1^0\Phi_3(\gamma l)+\frac{2\gamma^2}{Eh}N_1^0\Phi_2(\gamma l),\\ &w^0=-\frac{2\gamma^2}{Eh}M_1^0\Phi_2(\gamma l)-\frac{2\gamma R}{Eh}N_1^0\Phi_1(\gamma l).\end{aligned}\right\} \tag{37.17}$$

At $\gamma l\to\infty$ the values of functions $\Phi_1(\gamma l)$, $\Phi_2(\gamma l)$, $\Phi_3(\gamma l)$ approach toward unity and formulas (37.17) completely coincide with formulas (27.20), obtained earlier for a long shell. Thus, the quantities $\Phi_1(\gamma l)$, $\Phi_2(\gamma l)$, $\Phi_3(\gamma l)$ characterize the influence of a free edge $s = L$ on the displacements of a loaded edge $s = 0$. By analogy with (37.17) and (27.21) it is simple to write the expressions for ϑ_1^L, w^L under the action of forces N_1^L, M_1^L applied to the edge, while edge $s = 0$ now is free ($N_1^0 = M_1^0 = 0$):

$$\left.\begin{aligned} \vartheta_1^L &= \frac{2\gamma^2}{Eh} N_1^L \Phi_2(\gamma l) - \frac{4\gamma^3}{REh} M_1^L \Phi_3(\gamma l), \\ w^L &= \frac{2\gamma R}{Eh} N_1^L \Phi_1(\gamma l) - \frac{2\gamma^2}{Eh} M_1^L \Phi_2(\gamma l). \end{aligned}\right\} \tag{37.18}$$

Substituting the found values of constants c_1, c_2, c_3, c_4 (formulas (37.11), (37.13) into (37.4), (37.8) and taking into account that the case of edge $s = L$ being loaded by forces N_1^L, M_1^L is considered analogously and requires only a change in the signs of the corresponding terms, obtained

$$\begin{aligned} Eh\vartheta_1 = {} & \frac{4\gamma^3}{R} M_1^0 [\Phi_3(\gamma l)\Omega_1(\gamma\xi) - \Omega_2(\gamma\xi) + 2\Phi_2(\gamma l)\Omega_4(\gamma\xi)] + \\ & + 2\gamma^2 N_1^0 [\Phi_2(\gamma l)\Omega_1(\gamma\xi) - 2\Omega_3(\gamma\xi) + 4\Phi_1(\gamma l)\Omega_4(\gamma\xi)] - \\ & - \frac{4\gamma^3}{R} M_1^L [\Phi_3(\gamma l)\Omega_1(\gamma\xi_1) - \Omega_2(\gamma\xi_1) + 2\Phi_2(\gamma l)\Omega_4(\gamma\xi_1)] + \\ & + 2\gamma^2 N_1^L [\Phi_2(\gamma l)\Omega_1(\gamma\xi_1) - 2\Omega_3(\gamma\xi_1) + 4\Phi_1(\gamma l)\Omega_4(\gamma\xi_1)]. \end{aligned} \tag{37.19}$$

$$\begin{aligned} M_1 = {} & M_1^0 [4\Phi_3(\gamma l)\Omega_4(\gamma\xi) + \Omega_1(\gamma\xi) - 2\Phi_2(\gamma l)\Omega_3(\gamma\xi)] + \\ & + \frac{R}{\gamma} N_1^0 [2\Phi_2(\gamma l)\Omega_4(\gamma\xi) + \Omega_2(\gamma\xi) - 2\Phi_1(\gamma l)\Omega_3(\gamma\xi)] + \\ & + M_1^L [4\Phi_3(\gamma l)\Omega_4(\gamma\xi_1) + \Omega_1(\gamma\xi_1) - 2\Phi_2(\gamma l)\Omega_3(\gamma\xi_1)] - \\ & - \frac{R}{\gamma} N_1^L [2\Phi_2(\gamma l)\Omega_4(\gamma\xi_1) + \Omega_2(\gamma\xi_1) - 2\Phi_1(\gamma l)\Omega_3(\gamma\xi_1)]. \end{aligned} \tag{37.20}$$

$$\begin{aligned} N_1 = {} & -\frac{2\gamma}{R} M_1^0 [-2\Phi_3(\gamma l)\Omega_3(\gamma\xi) + 2\Omega_4(\gamma\xi) + \Phi_2(\gamma l)\Omega_2(\gamma\xi)] + \\ & + N_1^0 [2\Phi_2(\gamma l)\Omega_3(\gamma\xi) + \Omega_1(\gamma\xi) - 2\Phi_1(\gamma l)\Omega_2(\gamma\xi)] + \\ & + \frac{2\gamma}{R} M_1^L [-2\Phi_3(\gamma l)\Omega_3(\gamma\xi_1) + 2\Omega_4(\gamma\xi_1) + \Phi_2(\gamma l)\Omega_2(\gamma\xi_1)] + \\ & + N_1^L [2\Phi_2(\gamma l)\Omega_3(\gamma\xi_1) + \Omega_1(\gamma\xi_1) - 2\Phi_1(\gamma l)\Omega_2(\gamma\xi_1)]. \end{aligned} \tag{37.21}$$

$$\begin{aligned} w = {} & -\frac{2\gamma^2}{Eh} M_1^0 [\Phi_2(\gamma l)\Omega_1(\gamma\xi) + 2\Omega_3(\gamma\xi) - 2\Phi_3(\gamma l)\Omega_2(\gamma\xi)] - \\ & - \frac{2\gamma R}{Eh} N_1^0 [\Phi_1(\gamma l)\Omega_1(\gamma\xi) + 2\Omega_4(\gamma\xi) - \Phi_2(\gamma l)\Omega_2(\gamma\xi)] - \\ & - \frac{2\gamma^2}{Eh} M_1^L [\Phi_2(\gamma l)\Omega_1(\gamma\xi_1) + 2\Omega_3(\gamma\xi_1) - 2\Phi_3(\gamma l)\Omega_2(\gamma\xi_1)] + \\ & + \frac{2\gamma R}{Eh} N_1^L [\Phi_1(\gamma l)\Omega_1(\gamma\xi_1) + 2\Omega_4(\gamma\xi_1) - \Phi_2(\gamma l)\Omega_2(\gamma\xi_1)]. \end{aligned} \tag{37.22}$$

In the formulas relative coordinate ξ_1 is read from edge $s = L$ ($\xi_1 = l - \xi$). It is necessary to keep in mind that if besides edge loads, the shell experiences distributed loads $q_{n(0)}$, $q_{1(0)}$ and tension P_0, then in the right side of (37.22) should be terms which correspond to these loads, the same as in formula (37.8). Force T_2 is easily calculated if we multiply the right side of (37.22) by $\frac{Eh}{R}$ and to the result of the multiplication add $Rq_{n(0)}$. Using supplementary relationships (37.15), it is simple to be convinced of the validity of the obtained formulas. For example, setting in (37.21) $\xi = 0$, $\xi_1 = l$, we find that in the right side of (37.21) all terms except the second turn into zero and the value of shearing force on edge $s = 0$ is really equal to N_1^0. With the aid of the same supplementary formulas from (37.19), (37.22) we obtain expressions for ϑ_1^0, w^0 and ϑ_1^L, w^L under the combined action of edge loads N_1^0, M_1^0, N_1^L, M_1^L, tension P_0 and distributed load $q_{n(0)}$, $q_{1(0)}$:

$$\left.\begin{aligned}
\vartheta_1^0 &= \frac{4\gamma^3}{REh} M_1^0 \Phi_3(\gamma l) + \frac{2\gamma^2}{Eh} N_1^0 \Phi_2(\gamma l) - \frac{16\gamma^3}{REh} M_1^L \frac{\Omega_2(\gamma l)}{\Delta(\gamma l)} + \\
&\quad + \frac{16\gamma^2}{Eh} N_1^L \frac{\Omega_3(\gamma l)}{\Delta(\gamma l)}, \\
w^{0*} &= -\frac{2\gamma^2}{Eh} M_1^0 \Phi_2(\gamma l) - \frac{2\gamma R}{Eh} N_1^0 \Phi_1(\gamma l) + \frac{16\gamma^2}{Eh} M_1^L \frac{\Omega_3(\gamma l)}{\Delta(\gamma l)} - \\
&\quad - \frac{16\gamma R}{Eh} N_1^L \frac{\Omega_4(\gamma l)}{\Delta(\gamma l)}.
\end{aligned}\right\} \qquad (37.23)$$

$$\left.\begin{aligned}
w^{L*} &= \frac{16\gamma^2}{Eh} M_1^0 \frac{\Omega_3(\gamma l)}{\Delta(\gamma l)} + \frac{16\gamma R}{Eh} N_1^0 \frac{\Omega_4(\gamma l)}{\Delta(\gamma l)} - \\
&\quad - \frac{2\gamma^2}{Eh} M_1^L \Phi_2(\gamma l) + \frac{2\gamma R}{Eh} N_1^L \Phi_1(\gamma l), \\
\vartheta_1^L &= \frac{16\gamma^3}{REh} M_1^0 \frac{\Omega_2(\gamma l)}{\Delta(\gamma l)} + \frac{16\gamma^2}{Eh} N_1^0 \frac{\Omega_3(\gamma l)}{\Delta(\gamma l)} - \\
&\quad - \frac{4\gamma^3}{REh} M_1^L \Phi_3(\gamma l) + \frac{2\gamma^2}{Eh} N_1^L \Phi_2(\gamma l).
\end{aligned}\right\} \qquad (37.24)$$

Here the following designations have been introduced:

$$\left.\begin{aligned}
w^{0*} &= w^0 - \frac{R}{Eh}\left(Rq_{n(0)} - \frac{\mu P_0}{2\pi R}\right), \\
w^{L*} &= w^L - \frac{R}{Eh}\left(Rq_{n(0)} - \frac{\mu P_0}{2\pi R} + \mu \int_0^l Rq_{1(0)}\, d\xi\right).
\end{aligned}\right\} \qquad (37.25)$$

Considering (37.23), (37.24) as a system of equations for defining M_1^0, N_1^0, M_1^L, N_1^L according to assigned values of the quantities ϑ_1^0, w^{0*}, ϑ_1^L, w^{L*} and solving it, we have

$$
\left.\begin{aligned}
M_1^0 &= \frac{Eh}{2\gamma^2}\left[w^{0*}\Phi_2(\gamma l) - w^{L*}\frac{8\Omega_3(\gamma l)}{\Delta(\gamma l)}\right] + \\
&\quad + \frac{EhR}{2\gamma^3}\left[\vartheta_1^0\Phi_1(\gamma l) + \vartheta_1^L\frac{8\Omega_4(l\gamma)}{\Delta(\gamma l)}\right], \\
N_1^0 &= \frac{Eh}{\gamma R}\left[-w^{0*}\Phi_3(\gamma l) + w^{L*}\frac{4\Omega_2(\gamma l)}{\Delta(\gamma l)}\right] - \\
&\quad - \frac{Eh}{2\gamma^2}\left[\vartheta_1^0\Phi_2(\gamma l) + \vartheta_1^L\frac{8\Omega_3(\gamma l)}{\Delta(\gamma l)}\right].
\end{aligned}\right\} \tag{37.26}
$$

$$
\left.\begin{aligned}
M_1^L &= \frac{Eh}{2\gamma^2}\left[-w^{0*}\frac{8\Omega_3(\gamma l)}{\Delta(\gamma l)} + w^{L*}\Phi_2(\gamma l)\right] - \\
&\quad - \frac{EhR}{2\gamma^3}\left[\vartheta_1^0\frac{8\Omega_4(\gamma l)}{\Delta(\gamma l)} + \vartheta_1^L\Phi_1(\gamma l)\right], \\
N_1^L &= \frac{Eh}{\gamma R}\left[-w^{0*}\frac{4\Omega_2(\gamma l)}{\Delta(\gamma l)} + w^{L*}\Phi_3(\gamma l)\right] - \\
&\quad - \frac{Eh}{2\gamma^2}\left[\vartheta_1^0\frac{8\Omega_3(\gamma l)}{\Delta(\gamma l)} + \vartheta_1^L\Phi_2(\gamma l)\right].
\end{aligned}\right\} \tag{37.27}
$$

Using the above equations, we will consider such a loading of a shell when $M_1^0 = M_1^L$, $N_1^0 = -N_1^L$, $q_{1(0)} = 0$, $q_{n(0)} = \text{const}$. Then on the basis of (37.23), (37.24) in this instance we obtain

$$
\left.\begin{aligned}
w^{0*} = w^{L*} &= -\frac{2\gamma^2}{Eh}M_1^0\chi_2(\gamma l) - \frac{2\gamma R}{Eh}N_1^0\chi_1(\gamma l), \\
\vartheta_1^0 = -\vartheta_1^L &= \frac{4\gamma^3}{EhR}M_1^0\chi_3(\gamma l) + \frac{2\gamma^2}{Eh}N_1^0\chi_2(\gamma l),
\end{aligned}\right\} \tag{37.28}
$$

where

$$
\left.\begin{aligned}
\chi_1(x) &= \frac{\operatorname{ch} x + \cos x}{\operatorname{sh} x + \sin x}, \quad \chi_2(x) = \frac{\operatorname{sh} x - \sin x}{\operatorname{sh} x + \sin x}, \\
\chi_3(x) &= \frac{\operatorname{ch} x - \cos x}{\operatorname{sh} x + \sin x}.
\end{aligned}\right\} \tag{37.29}
$$

The values of functions $\chi_1(x)$, $\chi_2(x)$, $\chi_3(x)$ is given in Table 2 of the appendix.

Comparing formula (37.28) with formula (27.20) for a long shell, it is easy to see that the influence of the second edge is expressed here by the presence of the factors $\chi_1(\gamma l)$, $\chi_2(\gamma l)$, $\chi_3(\gamma l)$ in the corresponding terms.

From the table of values for $\chi_1(x)$, $\chi_2(x)$, $\chi_3(x)$ it follows that already at $\gamma l = 3$ they practically differ little from unity and the shell in this instance can be calculated as long. For very short shells (for example, $\gamma l = 0.4$, which at $h/R = 0.04$, $\mu^2 = 0.1$ corresponds to length $L = 0.48R$) the values of $\chi_2 = 0.0268$, $\chi_3 = 0.200$ are small in comparison with unity and the shell is close to a ring. Angles of rotation of the extreme sections of such a shell are small, and the radial displacements of the edges are determined basically by amount of shearing force N_1^0 ($\chi_1 = 2.502$). Setting γl small, it is easy to show that displacements w^0, w^L coincide in this instance with the displacement of a ring of radius R, area of transverse section $F = hL$, loaded by radial forces of intensity $2N_1^0$. Really, since when x is small $\chi_1(x) \approx \frac{1}{x}$, from (37.28) we obtain

$$w^0 = w^L \approx -\frac{2\gamma R}{Eh} N_1^0 \frac{1}{\gamma l} = -\frac{R^2 (2N_1^0)}{EhL}.$$

§ 38. Examples of the Calculation of a Short Shell

Let us examine several particular problems. We will solve the problem about axisymmetric deformation of a cylinder with rigid bottoms, already examined for a long shell [12]. The cylinder is loaded by internal pressure $q_{n(0)} = p$, the bottoms of the cylinder can freely move in the axial direction. By considerations of symmetry here $M_1^0 = M_1^L$, $N_1^0 = -N_1^L$ and it is possible to use formula (37.28), where

$$P_0 = p\pi R^2, \quad w^{0*} = w^{L*} = w^0 - \frac{pR^2}{Eh}\left(1 - \frac{\mu}{2}\right). \tag{38.1}$$

From the conditions of fixed edges $w^0 = w^L = 0$, $\vartheta_1^0 = \vartheta_1^L = 0$, using (37.28) and (38.1), we obtain equations for determination of forces and moments on the edges:

$$\left.\begin{aligned} -\frac{2\gamma^2}{Eh} M_1^0 \chi_2(\gamma l) - \frac{2\gamma R}{Eh} N_1^0 \chi_1(\gamma l) &= -\frac{pR^2}{Eh}\left(1 - \frac{\mu}{2}\right), \\ \frac{4\gamma^3}{REh} M_1^0 \chi_3(\gamma l) + \frac{2\gamma^2}{Eh} N_1^0 \chi_2(\gamma l) &= 0. \end{aligned}\right\}$$

Solving them, we obtain

$$\left.\begin{aligned} N_1^0 &= \frac{pR}{\gamma}\left(1 - \frac{\mu}{2}\right)\chi_3(\gamma l), \\ M_1^0 &= -\frac{pR^2}{2\gamma^2}\left(1 - \frac{\mu}{2}\right)\chi_2(\gamma l). \end{aligned}\right\} \tag{38.2}$$

Bending moment M_1^0 only in factor $\chi_2(\mu l)$ differs from the bending moment in the fixed section of a long shell which is also under the same load (§ 28). Turning to the table of values of function $\chi_2(\gamma l)$, we note that at $(\gamma l) < 3$ $\chi_2(\gamma l) < 1$, and at $(\gamma l) \approx 3$ $\chi_2(\gamma l)$ insignificantly exceeds unity (at $\gamma l \to \infty$ $\chi_2(\gamma l) \to 1$). This means that in short shells a local increase in stress near the edges, proportional to bending moment M_1, in only a small area of change in γl can insignificantly exceed (in all by 6%) the corresponding increase in the stress taking place in a long shell. For very short shells ($\gamma l \ll 3$) the edge effect is insignificant.

Let us propose now that the load on a shell is the same as in the previous problem, but the edges is hinged. In this way on the edges now it is necessary to set the conditions $M_1^0 = M_1^L = 0$, $w^0 = w^L = 0$. From (37.28) again we find

$$\left.\begin{aligned} N_1^0 &= \frac{pR}{2\gamma}\left(1 - \frac{\mu}{2}\right)\frac{1}{\chi_1(\gamma l)}, \\ \vartheta_1^0 &= -\vartheta_1^L = \frac{pR\gamma}{Eh}\left(1 - \frac{\mu}{2}\right)\frac{\chi_2(\gamma l)}{\chi_1(\gamma l)}. \end{aligned}\right\} \qquad (38.3)$$

As in the first example the edge effect for a short cylinder proves to be very weak. At $\gamma l \approx 5$ the results will not differ from those which for this case can be obtained for a long shell ($\chi_1(5) \approx \chi_2(5) \approx 1$); at $\gamma l \ll 3$ shearing force and the angle of rotation of the edges are small, since they change as $1/\chi_1(\gamma l)$.

§ 39. Calculation of a Short Shell for Bending Load

For a short cylindrical shell which is under the action of a bending load the solution of uniform equation (26.20) is again taken in the form of (37.1). Adding to this solution particular solution (26.29), corresponding to the zero-moment state, we obtain the general solution of equation (26.20):

$$\sigma_1 = \Psi_1 + 2i\gamma^2 V_1 = (c_1 - ic_3)[\Omega_1(\gamma\xi) - 2i\Omega_3(\gamma\xi)] +$$
$$+ (c_2 - ic_4)[\Omega_2(\gamma\xi) - 2i\Omega_4(\gamma\xi)] + 2\gamma^2 i\left[\frac{P_1}{\pi R} + \int_0^{\xi} Rq_{n(1)}\, d\xi\right]. \qquad (39.1)$$

Now it is simple to calculate forces, moments and deformations in the shell using equations (26.27), (26.28). These expressions will contain four arbitrary constants c_1, c_2, c_3, c_4, which, just as in the case of an axisymmetric load, can be determined according to assigned values of the amplitudes of bending moments (m_1^0, m_1^L) and shearing forces (n_1^0, n_1^L) at edges of the shell. Dropping calculations connected to this, since they are similar to the calculations in § 37 for the case of axisymmetric deformation, we obtain formulas analogous to (37.19)-(37.22):

$$\begin{aligned}\Psi_1 = Eh\Psi = &-\frac{4\gamma^3}{R} m_1^0 [\Phi_3(\gamma l)\Omega_1(\gamma\xi) - \Omega_2(\gamma\xi) + 2\Phi_2(\gamma l)\Omega_4(\gamma\xi)] - \\ &- 2\gamma^2 n_1^0 [\Phi_2(\gamma l)\Omega_1(\gamma\xi) - 2\Omega_3(\gamma\xi) + 4\Phi_1(\gamma l)\Omega_4(\gamma\xi)] + \\ &+ \frac{4\gamma^3}{R} m_1^L [\Phi_3(\gamma l)\Omega_1(\gamma\xi_1) - \Omega_2(\gamma\xi_1) + 2\Phi_2(\gamma l)\Omega_4(\gamma\xi_1)] - \\ &- 2\gamma^2 n_1^L [\Phi_2(\gamma l)\Omega_1(\gamma\xi_1) - 2\Omega_3(\gamma\xi_1) + 4\Phi_1(\gamma l)\Omega_4(\gamma\xi_1)].\end{aligned} \quad (39.2)$$

$$\begin{aligned}m_1 = m_1^0 &[4\Phi_3(\gamma l)\Omega_4(\gamma\xi) + \Omega_1(\gamma\xi) - 2\Phi_2(\gamma l)\Omega_3(\gamma\xi)] + \\ &+ \frac{R}{\gamma} n_1^0 [2\Phi_2(\gamma l)\Omega_4(\gamma\xi) + \Omega_2(\gamma\xi) - 2\Phi_1(\gamma l)\Omega_3(\gamma\xi)] + \\ &+ m_1^L [4\Phi_3(\gamma l)\Omega_4(\gamma\xi_1) + \Omega_1(\gamma\xi_1) - 2\Phi_2(\gamma l)\Omega_3(\gamma\xi_1)] - \\ &- \frac{R}{\gamma} n_1^L [2\Phi_2(\gamma l)\Omega_4(\gamma\xi_1) + \Omega_2(\gamma\xi_1) - 2\Phi_1(\gamma l)\Omega_3(\gamma\xi_1)].\end{aligned} \quad (39.3)$$

$$m_2 = \mu m_1.$$

$$\begin{aligned}n_1 = &-\frac{2\gamma}{R} m_1^0 [-2\Phi_3(\gamma l)\Omega_3(\gamma\xi) + 2\Omega_4(\gamma\xi) + \Phi_2(\gamma l)\Omega_2(\gamma\xi)] + \\ &+ n_1^0 [2\Phi_2(\gamma l)\Omega_3(\gamma\xi) + \Omega_1(\gamma\xi) - 2\Phi_1(\gamma l)\Omega_2(\gamma\xi)] + \\ &+ \frac{2\gamma}{R} m_1^L [-2\Phi_3(\gamma l)\Omega_3(\gamma\xi_1) + 2\Omega_4(\gamma\xi_1) + \Phi_2(\gamma l)\Omega_2(\gamma\xi_1)] + \\ &+ n_1^L [2\Phi_2(\gamma l)\Omega_3(\gamma\xi_1) + \Omega_1(\gamma\xi_1) - 2\Phi_1(\gamma l)\Omega_2(\gamma\xi_1)].\end{aligned} \quad (39.4)$$

$$\begin{aligned}Eh\varepsilon_{2(1)} = &Rq_{n(1)} - \mu(f_0 + f_1) - \\ &- \frac{2\gamma^2}{R} m_1^0 [\Phi_2(\gamma l)\Omega_1(\gamma\xi) + 2\Omega_3(\gamma\xi) - 2\Phi_3(\gamma l)\Omega_2(\gamma\xi)] - \\ &- 2\gamma n_1^0 [\Phi_1(\gamma l)\Omega_1(\gamma\xi) + 2\Omega_4(\gamma\xi) - \Phi_2(\gamma l)\Omega_2(\gamma\xi)] - \\ &- \frac{2\gamma^2}{R} m_1^L [\Phi_2(\gamma l)\Omega_1(\gamma\xi_1) + 2\Omega_3(\gamma\xi_1) - 2\Phi_3(\gamma l)\Omega_2(\gamma\xi_1)] + \\ &+ 2\gamma n_1^L [\Phi_1(\gamma l)\Omega_1(\gamma\xi_1) + 2\Omega_4(\gamma\xi_1) - \Phi_2(\gamma l)\Omega_2(\gamma\xi_1)].\end{aligned} \quad (39.5)$$

$$t_2 = Eh\varepsilon_{2(1)} + \mu(f_0 + f_1). \quad (39.6)$$

$$t_1 = -\frac{m_1}{R} + (f_0 + f_1). \quad (39.7)$$

$$s_1 = n_1 + \frac{P_1}{\pi R} + R\int_0^{\xi} (q_{n(1)} - q_{2(1)})\, d\xi. \quad (39.8)$$

$$h_1 = -\frac{(1-\mu)}{4\gamma^4} R\Psi_1. \tag{39.9}$$

where

$$n_1 + \frac{h_1}{R} \approx n_1. \tag{39.10}$$

By analogy with formulas (37.23), (37.24), (37.26), (37.27) replacing in them ϑ_1^0, ϑ_1^L by $-\Psi^0$, $-\Psi^L$ and w^{0*}, w^{L*} by $R\varepsilon_{2(1)}^{0*}$, $R\varepsilon_{2(1)}^{L*}$, we can write out the expressions which connect the values of quantities Ψ and $\varepsilon_{2(1)}$ at the edges of the shell with the given quantities n_1^0, m_1^0, n_1^L, m_1^L, or vice versa; for example, we will write out equations analogous to (37.26):

$$\left.\begin{aligned} m_1^0 &= \frac{EhR}{2\gamma^2}\left[\varepsilon_{2(1)}^{0*}\Phi_2(\gamma l) - \varepsilon_{2(1)}^{L*}\frac{8\Omega_3(\gamma l)}{\Delta(\gamma l)}\right] - \\ &\quad - \frac{EhR}{2\gamma^3}\left[\Psi^0\Phi_1(\gamma l) + \Psi^L\frac{8\Omega_4(\gamma l)}{\Delta(\gamma l)}\right], \\ n_1^0 &= \frac{Eh}{\gamma}\left[-\varepsilon_{2(1)}^{0*}\Phi_3(\gamma l) + \varepsilon_{2(1)}^{L*}\frac{4\Omega_2(\gamma l)}{\Delta(\gamma l)}\right] + \\ &\quad + \frac{Eh}{2\gamma^2}\left[\Psi^0\Phi_2(\gamma l) + \Psi^L\frac{8\Omega_3(\gamma l)}{\Delta(\gamma l)}\right]. \end{aligned}\right\} \tag{39.11}$$

$$\left.\begin{aligned} m_1^L &= \frac{EhR}{2\gamma^2}\left[-\varepsilon_{2(1)}^{0*}\frac{8\Omega_3(\gamma l)}{\Delta(\gamma l)} + \varepsilon_{2(1)}^{L*}\Phi_2(\gamma l)\right] + \\ &\quad + \frac{EhR}{2\gamma^3}\left[\Psi^0\frac{8\Omega_4(\gamma l)}{\Delta(\gamma l)} + \Psi^L\Phi_1(\gamma l)\right], \\ n_1^L &= \frac{Eh}{\gamma}\left[-\varepsilon_{2(1)}^{0*}\frac{4\Omega_2(\gamma l)}{\Delta(\gamma l)} + \varepsilon_{2(1)}^{L*}\Phi_3(\gamma l)\right] + \\ &\quad + \frac{Eh}{2\gamma^2}\left[\Psi^0\frac{8\Omega_3(\gamma l)}{\Delta(\gamma l)} + \Psi^L\Phi_2(\gamma l)\right], \end{aligned}\right\} \tag{39.12}$$

where

$$\varepsilon_{2(1)}^{*} = \varepsilon_{2(1)} - \frac{1}{Eh}\left[Rq_{n(1)} - \mu(f_0 + f_1)\right]. \tag{39.13}$$

§ 40. Shells with Rigid Edges

Using the extracted formulas it is simple to determine shearing forces and bending moments in fixed edge sections of a shell. If both edge sections are fixed then

$$\varepsilon_{2(1)}^0 = \varepsilon_{2(1)}^L = 0, \quad \Psi^0 = \Psi^L = 0 \tag{40.1}$$

consequently,

$$\left.\begin{aligned}\varepsilon^{0*}_{2(1)} &= -\frac{1}{Eh}\left[Rq_{n(1)} - \mu(f_0 + f_1)_{s=0}\right],\\ \varepsilon^{L*}_{2(1)} &= -\frac{1}{Eh}\left[Rq_{n(1)} - \mu(f_0 + f_1)_{s=L}\right].\end{aligned}\right\} \quad (40.2)$$

According to formulas (39.11), (39.12), taking into account the equations of equilibrium of the shell as a unit (36.4), we obtain

$$\left.\begin{aligned}m_1^0 &= \frac{R}{2\gamma^2}\left[-\left(Rq_{n(1)} + \mu\frac{M_1}{\pi R^2}\right)\Phi_2(\gamma l) + \right.\\ &\left. + \left(Rq_{n(1)} + \mu\frac{M}{\pi R^2}\right)\frac{8\Omega_3(\gamma l)}{\Delta(\gamma l)}\right],\\ n_1^0 &= \frac{1}{\gamma}\left[\left(Rq_{n(1)} + \mu\frac{M_1}{\pi R^2}\right)\Phi_3(\gamma l) - \right.\\ &\left. - \left(Rq_{n(1)} + \mu\frac{M}{\pi R^2}\right)\frac{4\Omega_2(\gamma l)}{\Delta(\gamma l)}\right].\end{aligned}\right\} \quad (40.3)$$

$$\left.\begin{aligned}m_1^L &= \frac{R}{2\gamma^2}\left[\left(Rq_{n(1)} + \mu\frac{M_1}{\pi R^2}\right)\frac{8\Omega_3(\gamma l)}{\Delta(\gamma l)} - \left(Rq_{n(1)} + \mu\frac{M}{\pi R^2}\right)\Phi_2(\gamma l)\right],\\ n_1^L &= \frac{1}{\gamma}\left[\left(Rq_{n(1)} + \mu\frac{M_1}{\pi R^2}\right)\frac{4\Omega_2(\gamma l)}{\Delta(\gamma l)} - \left(Rq_{n(1)} + \mu\frac{M}{\pi R^2}\right)\Phi_3(\gamma l)\right].\end{aligned}\right\} \quad (40.4)$$

At $\gamma l \to \infty$ $\Phi_2(\gamma l) \to 1$, $\Phi_3(\gamma l) \to 1$, and $\frac{\Omega_3(\gamma l)}{\Delta(\gamma l)} \to 0$, $\frac{\Omega_2(\gamma l)}{\Delta(\gamma l)} \to 0$ and the above equations agree with those equations for a long shell obtained in § 36.

Let us determine the displacements in a short shell loaded by edge loads P_1, M_1, P, M. The amplitude of axial displacements is

$$\begin{aligned}u_{(1)} &= \frac{R}{Eh}\int_0^{\xi}(t_1 - \mu t_2)\,d\xi + u_1^0 = \tilde{u}_1 -\\ &- \frac{\mu R}{Eh}\int_0^{\xi}\left\{-\frac{2\gamma^2}{R}m_1^0\left[\Phi_2(\gamma l)\Omega_1(\gamma\xi) + 2\Omega_3(\gamma\xi) - 2\Phi_3(\gamma l)\Omega_2(\gamma\xi)\right] - \right.\\ &\left. - 2\gamma n_1^0\left[\Phi_1(\gamma l)\Omega_1(\gamma\xi) + 2\Omega_4(\gamma\xi) - \Phi_2(\gamma l)\Omega_2(\gamma\xi)\right]\right\}d\xi -\\ &- \frac{\mu R}{Eh}\int_{\xi_1}^{l}\left\{-\frac{2\gamma^2}{R}m_1^L\left[\Phi_2(\gamma l)\Omega_1(\gamma\xi_1) + 2\Omega_3(\gamma\xi_1) - 2\Phi_3(\gamma l)\Omega_2(\gamma\xi_1)\right] + \right.\\ &\left. + 2\gamma n_1^L\left[\Phi_1(\gamma l)\Omega_1(\gamma\xi_1) + 2\Omega_4(\gamma\xi_1) - \Phi_2(\gamma l)\Omega_2(\gamma\xi_1)\right]\right\}d\xi_1 + u_1^0.\end{aligned} \quad (40.5)$$

Setting $u_1^0 = 0$ and integrating, we obtain

$$\begin{aligned}u_{(1)} &= \tilde{u}_{(1)} + \frac{\mu 2\gamma}{Eh}m_1^0\left[-2\Phi_3(\gamma l)\Omega_3(\gamma\xi) + 2\Omega_4(\gamma\xi) + \Phi_2(\gamma l)\Omega_2(\gamma\xi)\right] -\\ &- \frac{\mu R}{Eh}n_1^0\left[-1 + 2\Phi_2(\gamma l)\Omega_3(\gamma\xi) + \Omega_1(\gamma\xi) - 2\Phi_1(\gamma l)\Omega_2(\gamma\xi)\right] -\\ &- \frac{\mu 2\gamma}{Eh}m_1^L\left[-2\Phi_3(\gamma l)\Omega_3(\gamma\xi_1) + 2\Omega_4(\gamma\xi_1) + \Phi_2(\gamma l)\Omega_2(\gamma\xi_1)\right] -\\ &- \frac{\mu R}{Eh}n_1^L\left[2\Phi_2(\gamma l)\Omega_3(\gamma\xi_1) + \Omega_1(\gamma\xi_1) - 2\Phi_1(\gamma l)\Omega_2(\gamma\xi_1)\right].\end{aligned} \quad (40.6)$$

where it has been taken into consideration that

$$-2\Phi_3(\gamma l)\Omega_3(\gamma l)+2\Omega_4(\gamma l)+\Phi_2(\gamma l)\Omega_2(\gamma l)=0,$$
$$2\Phi_2(\gamma l)\Omega_3(\gamma l)+\Omega_1(\gamma l)-2\Phi_1(\gamma l)\Omega_2(\gamma l)=0.$$

After determination of $u_{(1)}$ we find the amplitude of normal displacement by formula (34.16)

$$Ehw_{(1)}=Eh\tilde{w}_{(1)}-\int_0^{\xi}[EhR\Psi+Eh(u_{(1)}-\tilde{u}_{(1)})]\,d\xi+Ehw^0_{(1)}=$$
$$=-\int_0^{\xi}(EhR\Psi+\mu Rn_1^0)\,d\xi+Eh(\tilde{w}_{(1)}+w^0_{(1)}). \qquad (40.7)$$

In this case in the subintegral expression we drop those terms in $(u_{(1)}-\tilde{u}_{(1)})$, which have the order $\frac{1}{2\gamma^3}$ in comparison with the corresponding terms in Ψ. Setting $w_{(1)}(0) = 0$ and integrating, we obtain

$$\begin{aligned}Ehw_{(1)}=&\,4\gamma^2m_1^0\Big[\Phi_3(\gamma l)\Omega_2(\gamma\xi)-\Omega_3(\gamma\xi)-\\&-\tfrac{1}{2}\Phi_2(\gamma l)\Omega_1(\gamma\xi)+\tfrac{1}{2}\Phi_2(\gamma l)\Big]+\\&+2\gamma Rn_1^0[\Phi_2(\gamma l)\Omega_2(\gamma\xi)-2\Omega_4(\gamma\xi)-\Phi_1(\gamma l)\Omega_1(\gamma\xi)+\Phi_1(\gamma l)]+\\&+4\gamma^2m_1^L\Big[\Phi_3(\gamma l)\Omega_2(\gamma\xi_1)-\Omega_3(\gamma\xi_1)-\tfrac{1}{2}\Phi_2(\gamma l)\Omega_1(\gamma\xi_1)-\frac{4\Omega_3(\gamma l)}{\Delta(\gamma l)}\Big]-\\&-2\gamma Rn_1^L\Big[\Phi_2(\gamma l)\Omega_2(\gamma\xi_1)-2\Omega_4(\gamma\xi_1)-\Phi_1(\gamma l)\Omega_1(\gamma\xi_1)-\frac{8\Omega_4(\gamma l)}{\Delta(\gamma l)}\Big]+\\&+\tilde{w}_{(1)}-\tilde{w}^0_{(1)}.\end{aligned} \qquad (40.8)$$

Displacements at edge $s = L$ are equal to

$$\left.\begin{aligned}u^L_{(1)}&=\tilde{u}^L_{(1)}+\frac{\mu R}{Eh}(n_1^0-n_1^L),\\w^L_{(1)}&=\frac{2\gamma^2}{Eh}(m_1^0-m_1^L)\zeta_2(\gamma l)+\frac{2\gamma R}{Eh}(n_1^0+n_1^L)\zeta_3(\gamma l)-\\&\quad-\frac{\mu Rl}{Eh}n_1^0+\tilde{w}^L_{(1)}-\tilde{w}^0_{(1)},\end{aligned}\right\} \qquad (40.9)$$

where ζ_2 and ζ_3 designate the functions

$$\left.\begin{aligned}\zeta_2(x)&=\Phi_2(x)+\frac{8\Omega_3(x)}{\Delta(x)}=\frac{\operatorname{sh}x+\sin x}{\operatorname{sh}x-\sin x},\\\zeta_3(x)&=\Phi_1(x)+\frac{8\Omega_4(x)}{\Delta(x)}=\frac{\operatorname{ch}x-\cos x}{\operatorname{sh}x-\sin x}.\end{aligned}\right\} \qquad (40.10)$$

The formula for calculation of $u^L_{(1)}$ in external form coincides with first formula of (36.19). The second formula also will turn into the corresponding formula for a long shell at $\gamma l\to\infty$.

If the edges of the shell are connected with nondeformable diaphragms, then m_1^0, m_1^L, n_1^0, n_1^L are determined according to formulas (40.3), (40.4). Displacements $u_{(1)}^L$, $w_{(1)}^L$ in this instance have the form

$$\left.\begin{aligned} u_{(1)}^L &= \tilde{u}_{(1)}^L + \frac{\mu R}{Eh\gamma}\left[2Rq_{n(1)} + \mu\left(\frac{M_1}{\pi R^2} + \frac{M}{\pi R^2}\right)\chi_3(\gamma l)\right], \\ w_{(1)}^L &= \tilde{w}_{(1)}^L - \tilde{w}_{(1)}^0 - \frac{\mu R l}{Eh\gamma}\left[\left(Rq_{n(1)} + \frac{\mu M_1}{\pi R^2}\right)\Phi_3(\gamma l) - \right. \\ &\left. - \left(Rq_{n(1)} + \frac{\mu M}{\pi R^2}\right)\frac{4\Omega_2(\gamma l)}{\Delta(\gamma l)}\right] + \frac{\mu R}{Eh}\left(\frac{M_1}{\pi R^2} - \frac{M}{\pi R^2}\right). \end{aligned}\right\} \quad (40.11)$$

In this case we have taken into account the identity

$$2\zeta_1(x)\zeta_3(x) - \zeta_2^2(x) = 1,$$

where

$$\zeta_1(x) = \Phi_3(x) + \frac{4\Omega_2(x)}{\Delta(x)} = \frac{\operatorname{ch} x + \cos x}{\operatorname{sh} x - \sin x}.$$

If there are no distributed loads on the shell, then $q_{n(1)} = q_{2(1)} = q_{1(1)} = 0$, $M = M_1 + P_1 lR$ and the formula for displacements assume the form

$$\left.\begin{aligned} u_{(1)}^L &= -\frac{M_1 l}{\pi REh} - \frac{P_1 l^2}{\pi Eh^2} + \frac{\mu^2 R}{Eh\gamma}\left(\frac{2M_1}{\pi R^2} + \frac{P_1 l}{\pi R}\right)\chi_3(\gamma l), \\ w_{(1)}^L &= \frac{1}{\pi REh}\left[M_1\frac{l^2}{2} + P_1 R\frac{l^3}{6} - 2(1+\mu)P_1 R l\right] - \\ &- \frac{\mu^2 l}{\pi REh\gamma}\left[M_1\chi_3(\gamma l) - P_1 R l\frac{4\Omega_2(\gamma l)}{\Delta(\gamma l)}\right]. \end{aligned}\right\} \quad (40.12)$$

These formulas differ from formula (36.20) only in corresponding tersm of order $1/\gamma$ and in the extreme case $\gamma l \to \infty$ they coincide with them.

§ 41. The Deformed State During Axisymmetric Temperature Distribution

At axisymmetric temperature distribution in a shell t^m, Δt are assigned functions of one coordinate s

$$t^m = t_{(0)}^m(s), \quad \Delta t = \Delta t_{(0)}(s).$$

The particular solution of Meissner equations which in the right side have the corresponding temperature terms for a shell of revolution of general form was obtained in § 22. To corresponding forces and shifts expressed by formulas (22.14)-(22.21), which for a cylindrical shell should be rewritten in the form of [12]

$$
\left.\begin{aligned}
&T_1 = 0,\\
&T_2 = -\frac{Eh^2\beta R}{12(1-\mu)}\frac{d^2}{ds^2}(\Delta t_{(0)}),\\
&N_1 = -\frac{Eh^2\beta}{12(1-\mu)}\frac{d}{ds}(\Delta t_{(0)}).
\end{aligned}\right\} \qquad (41.1)
$$

$$
\left.\begin{aligned}
&M_1 = -\frac{Eh^3\beta}{12(1-\mu)}\left(\frac{\Delta t_{(0)}}{h} + \frac{R}{1+\mu}\frac{d^2 t^m_{(0)}}{ds^2}\right),\\
&M_2 = -\frac{Eh^3\beta}{12(1-\mu)}\left(\frac{\Delta t_{(0)}}{h} + \frac{\mu R}{1+\mu}\frac{d^2 t^m_{(0)}}{ds^2}\right).
\end{aligned}\right\} \qquad (41.2)
$$

$$
\left.\begin{aligned}
&w = \Delta_e = R\beta\left[t^m_{(0)} - \frac{hR}{12(1-\mu)}\frac{d^2(\Delta t_{(0)})}{ds^2}\right],\\
&\Delta_z = C - \frac{\mu h\beta R}{12(1-\mu)}\frac{d(\Delta t_{(0)})}{ds} - \beta\int_0^s t^m_{(0)}ds,\\
&\vartheta_1 = \beta R\frac{dt^m_{(0)}}{ds}.
\end{aligned}\right\} \qquad (41.3)
$$

In a rather long shell forces and moments far from the edges of the shell can be determined according to the equations written out above. In the area of the edges appear additional stresses of the edge effect. For calculation of temperature stresses allowing for edge effect we can use formulas obtained in §§ 27, 37 of this chapter, having in mind that now instead of the particular solution, which corresponds to zero-moment state under stress, we take the solution presented by formulas (41.1)-(41.3). In the determination of forces and moments according to displacements assigned on the edge in formulas (27.22), (27.23), (37.26), (37.27) it is necessary to replace

$$
\left.\begin{aligned}
&M_1^0 \to M_1^0 + \frac{Eh^3\beta}{12(1-\mu)}\left(\frac{\Delta t_{(0)}}{h} + \frac{R}{1+\mu}\frac{d^2 t^m_{(0)}}{ds^2}\right)_{s=0},\\
&N_1^0 \to N_1^0 + \frac{Eh^2\beta}{12(1-\mu)}\left(\frac{d\,\Delta t_{(0)}}{ds}\right)_{s=0},\\
&M_1^L \to M_1^L + \frac{Eh^3\beta}{12(1-\mu)}\left(\frac{\Delta t_{(0)}}{h} + \frac{R}{1+\mu}\frac{d^2 t^m_{(0)}}{ds^2}\right)_{s=L},\\
&N_1^L \to N_1^L + \frac{Eh^2\beta}{12(1-\mu)}\left(\frac{d\,\Delta t_{(0)}}{ds}\right)_{s=L}.
\end{aligned}\right\} \qquad (41.4)
$$

$$
\left.\begin{aligned}
&w^0,\ w^{0*} \to w^0 - R\beta\left[t^m_{(0)} - \frac{hR}{12(1-\mu)}\frac{d^2(\Delta t_{(0)})}{ds^2}\right]_{s=0},\\
&\vartheta_1^0 \to \vartheta_1^0 - \beta R\left(\frac{dt^m_{(0)}}{ds}\right)_{s=0},\\
&w^L,\ w^{L*} \to w^L - R\beta\left[t^m_{(0)} - \frac{hR}{12(1-\mu)}\frac{d^2(\Delta t_{(0)})}{ds^2}\right]_{s=L},\\
&\vartheta_1^L \to \vartheta_1^L - \beta R\left(\frac{dt^m_{(0)}}{ds}\right)_{s=L}.
\end{aligned}\right\} \qquad (41.5)
$$

Let us consider several examples.

1. In a free shell during linear distribution of temperature along the axis

$$t_{(0)}^{m}(s)=t^{0}+\frac{s}{L}(t^{L}-t^{0}), \qquad \Delta t_{(0)}=0 \tag{41.6}$$

stress do not appear, and the displacement are equal to

$$\left.\begin{aligned} w&=R\beta t_{(0)}^{m}, \quad \vartheta_1=\frac{R\beta}{L}(t^{L}-t^{0}),\\ \Delta_z^{L}&=-\frac{1}{2}\beta L(t^{0}+t^{L}). \end{aligned}\right\} \tag{41.7}$$

Here Δ_z^L designates the displacement of edge $s = L$ with respect to edge $s = 0$ in the direction of axis OZ. $\Delta_z^L>0$ at a shortening of the shell length.

If the edges of the shell are rigidly fixed, then in the neighborhood of edges appear shearing forces and bending moments, which for a long shell can be calculated using formulas (27.22), (27.23). In this case it is necessary to replace w^0, w^L by $-R\beta t^0$, $-R\beta t^L$ respectively, and ϑ_1^0, ϑ_1^L by $-\frac{R\beta}{L}(t^L-t^0)$. Doing this, we obtain

$$\left.\begin{aligned} M_1^0&=-\frac{EhR\beta}{2\gamma^2}\left[t^0+\frac{R}{\gamma L}(t^L-t^0)\right],\\ M_1^L&=-\frac{EhR\beta}{2\gamma^2}\left[t^L-\frac{R}{\gamma L}(t^L-t^0)\right],\\ N_1^0&=\frac{Eh\beta}{\gamma}\left[t^0+\frac{R}{2\gamma L}(t^L-t^0)\right],\\ N_1^L&=-\frac{Eh\beta}{\gamma}\left[t^L-\frac{R}{2\gamma L}(t^L-t^0)\right]. \end{aligned}\right\} \tag{41.8}$$

If, furthermore, the edges of the cylinder abut against rigid walls, which impede axial elongation, then this is equivalent to the action on the shell of an axial compressive force. The amount of this force is determined from the equation

$$\frac{1}{2}\beta L(t^0+t^L)+\frac{1}{Eh}\int_0^L(T_1-\mu T_2)\,ds=0, \tag{41.9}$$

where $T_1=\frac{P_0}{2\pi R}$, P_0 — axial force.

As was shown in Example 2 § 28, neglecting μT_2 in comparison with T_1 during the calculation of axial displacement gives an error of the order of $\sqrt{\frac{h}{R}}$ in comparison with unity. Limited to this accuracy, we obtain

$$P_0 = -\pi R\beta Eh(t^0 + t^L).$$

The action of axial force P_0 give rise to additional moments and shearing forces in the fixed edge sections:

$$\left.\begin{aligned} M_1^0 = M_1^L = \frac{\mu}{2\gamma^2}\frac{P_0}{2\pi} = -\frac{\mu R}{4\gamma^2}Eh\beta(t^0+t^L), \\ N_1^0 = -N_1^L = -\frac{\mu}{\gamma}\frac{P_0}{2\pi R} = \frac{\mu}{2\gamma}Eh\beta(t^0+t^L). \end{aligned}\right\} \quad (41.10)$$

2. Let us examine a local rise in stress in a shell, one component of which has constant temperature t^b, while the temperature of another component of the shell changes linearly from t^b in section $s = b$ to t^0 in section $s = 0$ [27],

$$\left.\begin{aligned} t = t_{(0)}^m(s) = \begin{cases} t^0 + \frac{s}{b}(t^b - t^0) & \text{at} \quad 0 \leqslant s \leqslant b, \\ t^b & \text{at} \quad s > b. \end{cases} \\ \Delta t_{(0)}(s) = 0. \end{aligned}\right\} \quad (41.11)$$

If we mentally cut the shell along section $s = b$, every component of the shell will be free from stress. The end of the left part of the shell ($s = b$) will have displacement and angle of rotation equal to

$$\vartheta_1^- = \frac{R\beta}{b}(t^b - t^0), \quad w^- = R\beta t^b; \quad (41.12)$$

the adjacent end of the right part has displacements

$$\vartheta_1^+ = 0, \quad w^+ = R\beta t^b. \quad (41.13)$$

In order to remove the discontinuity in the angle of rotation, it is necessary to apply to both shell components bending moment M_1^b, which causes a rotation of the edge of the left part of the shell through an angle $\vartheta_1 - \vartheta_1^-$ and the edge of the second part through an angle $\vartheta_1 - \vartheta_1^+$, where $\vartheta_1 -$ actual angle of rotation in section $s = b$. Using formulas (27.22), (27.23) and setting $N_1^b = 0$, we obtain

$$\begin{aligned} \vartheta_1 - \vartheta_1^- &= -\frac{4\gamma^3}{REh} M_1^b, \\ \vartheta_1 - \vartheta_1^+ &= \frac{4\gamma^3}{REh} M_1^b. \end{aligned} \qquad (41.14)$$

Eliminating from these equalities ϑ_1 and calculating M_1^b, we have

$$M_1^b = \frac{\beta R^2 Eh}{8\gamma^3 b}(t^b - t^0) = D\beta\frac{\gamma}{2b}(t^b - t^0). \qquad (41.15)$$

3. As the next example let us examine a shell with free ends. A drop in temperature with depth of wall is a linear function of coordinate s [27]

$$\begin{aligned} \Delta t_{(0)} &= (\Delta t_{(0)})^0 + \frac{1}{L}[(\Delta t_{(0)})^L - (\Delta t_{(0)})^0]\cdot s, \\ t_{(0)}^m &= 0. \end{aligned} \qquad (41.16)$$

To avoid misunderstandings we note that temperature distribution (41.16) is not a linear function of Cartesian coordinates. Temperature distribution, linearly depending on coordinate z, is represented by formula (41.6) and examined in Example 1.

By formula (41.3) we find that far from the ends the shell is not bent: $w = \vartheta_1 = 0$. Calculating on the basis of (41.1), (41.2) the values of forces and moments, in the edge sections we have

$$\begin{aligned} N_1^0 = N_1^L &= -\frac{Eh^2\beta}{12(1-\mu)}\frac{1}{L}(\Delta t_{(0)}^L - \Delta t_{(0)}^0), \\ M_1^0 = -\frac{Eh^2\beta}{12(1-\mu)}\Delta t_{(0)}^0, &\quad M_1^L = -\frac{Eh^2\beta}{12(1-\mu)}\Delta t_{(0)}^L. \end{aligned} \qquad (41.17)$$

But the leads of the shell should be free from stresses. In order to execute this condition it is necessary to apply to them moments and forces equal in magnitude and opposite in sign to those which were just now obtained. In this case the shell will undergo local bending near the edges. Angle of rotation and radial displacement of the edges are easily computed by (27.20), (27.21). For example, setting in (27.20)

$$N_1^0 = \frac{Eh^2\beta}{12(1-\mu)}\frac{1}{L}(\Delta t_{(0)}^L - \Delta t_{(0)}^0), \quad M_1^0 = \frac{Eh^2\beta}{12(1-\mu)}\Delta t_{(0)}^0,$$

we obtain

$$\begin{aligned} \vartheta_1^0 &= 2\gamma\beta\sqrt{\frac{(1+\mu)}{12(1-\mu)}}\left[\Delta t_{(0)}^0 + \frac{1}{2\gamma}\frac{R}{L}(\Delta t_{(0)}^L - \Delta t_{(0)}^0)\right], \\ w^0 &= -2\gamma^2\frac{h\beta}{12(1-\mu)}\left[\Delta t_{(0)}^0 + \frac{1}{\gamma}\frac{R}{L}(\Delta t_{(0)}^L - \Delta t_{(0)}^0)\right]. \end{aligned} \qquad (41.18)$$

The peripheral bending moment on end $s = 0$ for an assigned temperature distribution is equal to

$$M_2^0 = -\frac{Eh^2\beta}{12(1-\mu)}\Delta t_{(0)}^0 + \mu\frac{Eh^2\beta}{12(1-\mu)}\Delta t_{(0)}^0 = -\frac{Eh^2\beta}{12}\Delta t_{(0)}^0. \tag{41.19}$$

Furthermore, local curvature of the edges is accompanied by initiation of peripheral forces. In section $s = 0$ peripheral force has

$$T_2^0 = -\frac{Eh\beta}{2}\frac{\sqrt{1-\mu^2}}{(1-\mu)\sqrt{3}}\left[\Delta t_{(0)}^0 + \frac{R}{\gamma L}\left(\Delta t_{(0)}^L - \Delta t_{(0)}^0\right)\right]. \tag{41.20}$$

In the particular case when $\Delta t_{(0)}^L = \Delta t_{(0)}^0$ total circumferential stress near the external surface in section $s = 0$ is equal to

$$\sigma_2^0 = \frac{T_2^0}{h} + \frac{6M_2^0}{h^2} = -\frac{E\beta\,\Delta t_{(0)}}{2}\left[1 + \frac{\sqrt{1-\mu^2}}{(1-\mu)\sqrt{3}}\right]. \tag{41.21}$$

It exceeds the maximum flexural stresses taking place far from the edges of the shell by $\left[(1-\mu) + \sqrt{\frac{1-\mu^2}{3}}\right]$ times.

§ 42. Temperature Distribution Proportional to cos ϕ

For temperature distribution according to the law

$$t(s, \varphi, \zeta) = \left[t_{(1)}^m(s) + \frac{\zeta}{h}\Delta t_{(1)}(s)\right]\cos\varphi$$

the amplitudes of forces, moments, deformations and displacements in a cylindrical shell, computable on the basis of particular solutions of Meissner equations (23.11), (23.12), have the form

$$\left.\begin{aligned}
t_1 &= \frac{Eh^3\beta}{12(1-\mu^2)}\frac{d^2t_{(1)}^m}{ds^2} + \frac{Eh^3\beta}{12(1-\mu)R}\left(\frac{\Delta t_{(1)}}{h} - \frac{t_{(1)}^m}{R}\right),\\
t_2 &= \frac{Eh^3\beta}{12(1-\mu^2)}\left\{\left[(1+\mu)R\frac{d^2}{ds^2}\left(\frac{t_{(1)}^m}{R} - \frac{\Delta t_{(1)}}{h}\right) + \mu\frac{d^2t_{(1)}^m}{ds^2}\right] + \right.\\
&\qquad\left. + \frac{1+\mu}{R}\left(\frac{\Delta t_{(1)}}{h} - \frac{t_{(1)}^m}{R}\right)\right\},\\
s_1 &= \frac{Eh^3\beta}{12(1-\mu)}\frac{d}{ds}\left(\frac{t_{(1)}^m}{R} - \frac{\Delta t_{(1)}}{h}\right) - \frac{Eh^3\beta}{6(1+\mu)}\frac{1}{R}\frac{dt_{(1)}^m}{ds},\\
h_1 &= \frac{Eh^3\beta}{12(1+\mu)}\frac{dt_{(1)}^m}{ds},\\
m_1 &= \frac{Eh^3\beta}{12(1-\mu^2)}\left[(1+\mu)\left(\frac{t_{(1)}^m}{R} - \frac{\Delta t_{(1)}}{h}\right) - R\frac{d^2t_{(1)}^m}{ds^2}\right],\\
m_2 &= \frac{Eh^3\beta}{12(1-\mu^2)}\left[(1+\mu)\left(\frac{t_{(1)}^m}{R} - \frac{\Delta t_{(1)}}{h}\right) - \mu R\frac{d^2t_{(1)}^m}{ds^2}\right],\\
n_1 &= \frac{Eh^3\beta}{12(1-\mu^2)}\left[(1+\mu)\frac{d}{ds}\left(\frac{t_{(1)}^m}{R} - \frac{\Delta t_{(1)}}{h}\right) - \frac{(1-\mu)}{R}\frac{dt_{(1)}^m}{ds}\right].
\end{aligned}\right\} \tag{42.1}$$

$$\varepsilon_{2(1)} = \beta t^m_{(1)}, \quad \gamma_{(1)} = \frac{h^2\beta}{6}\frac{(1+\mu)}{(1-\mu)}\frac{d}{ds}\left(\frac{t^m_{(1)}}{R} - \frac{\Delta t_{(1)}}{h}\right),$$
$$\varkappa_{1(1)} = \frac{\beta t^m_{(1)}}{R} - \beta R\frac{d^2 t^m_{(1)}}{ds^2}, \quad \Psi = -\beta R\frac{dt^m_{(1)}}{ds}. \tag{42.2}$$

$$\vartheta_{1(1)} = C_1 + \beta R\frac{dt^m_{(1)}}{ds} - \frac{\beta}{R}\int_0^s t^m_{(1)} ds,$$
$$u_{(1)} = -C_1 R + \beta\int_0^s t^m_{(1)} ds,$$
$$v_{(1)} = C_2 - C_1 s + \frac{\beta}{R}\int_0^s\int_0^s t^m_{(1)} ds\, ds,$$
$$w_{(1)} = \varepsilon_{2(1)}R - v_{(1)} = R\beta t^m_{(1)} + C_1 s - C_2 - \frac{\beta}{R}\int_0^s\int_0^s t^m_{(1)} ds\, ds. \tag{42.3}$$

In a long thin shell far from the edges forces and bending moments (42.1) exist. In calculating the stressed state of a short shell, or local increase in the stress near the ends of a long shell, it is necessary to determine temperature stresses allowing for edge conditions. This is done just as in the case of axisymmetric deformation using the formulas of §§ 34, 39, obtained for a shell which experiences edge bending loads.

As an example let us examine a shell in which the amplitudes of average temperature and temperatures drop with wall thickness change linearly in the axial direction

$$t^m_{(1)} = t^{m0}_{(1)} + (t^{mL}_{(1)} - t^{m0}_{(1)})\frac{s}{L},$$
$$\Delta t_{(1)} = \Delta t^0_{(1)} + (\Delta t^L_{(1)} - \Delta t^0_{(1)})\frac{s}{L}. \tag{42.4}$$

It is possible to show that the temperature distribution is physically possible, i.e., it satisfies with the accepted accuracy, namely, neglecting terms h/R in comparison with unity, the condition of stationary temperature distribution (Laplace equation)

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \varphi^2} + \frac{\partial^2 t}{\partial s^2} = 0.$$

Setting $t = t_{(1)}(r, s)\cos\phi$, where $t_{(1)}$ is a linear function of s, we find that $t_{(1)}$ should satisfy the equation

$$\frac{\partial^2 t_{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial t_{(1)}}{\partial r} - \frac{t_{(1)}}{r^2} = 0.$$

Its general solution has the form

$$t_{(1)} = A(s)r + \frac{B(s)}{r}.$$

Taking into account that in our case $r = R + \zeta$, we have approximately

$$t_{(1)} = A(s)R + \frac{B(s)}{R} + \zeta\left[A(s) - \frac{B(s)}{R^2}\right],$$

where $A(s)$, $B(s)$ are two arbitrary linear functions of coordinate s. In this way $t^m_{(1)}$, $\Delta t_{(1)}$ can be arbitrary linear functions of s.

Calculations made for the given temperature distribution using formulas (42.1), (42.2) on edge $s = 0$ give

$$\left.\begin{aligned} t_1^0 = t_2^0 &= \frac{Eh^3\beta}{12(1-\mu)R}\left(\frac{\Delta t^0_{(1)}}{h} - \frac{t^{m0}_{(1)}}{R}\right), \\ m_1^0 = m_2^0 &= -\frac{Eh^3\beta}{12(1-\mu)}\left(\frac{\Delta t^0_{(1)}}{h} - \frac{t^{m0}_{(1)}}{R}\right), \\ n_1^0 = \left(s_1 + \frac{h_1}{R}\right)^0 &= \frac{Eh^3\beta}{12(1-\mu^2)}\left[(1+\mu)\left(\frac{t^{mL}_{(1)} - t^{m0}_{(1)}}{RL} - \frac{\Delta t^L_{(1)} - \Delta t^0_{(1)}}{hL}\right) - \right. \\ &\left. - (1-\mu)\left(\frac{t^{mL}_{(1)} - t^{m0}_{(1)}}{RL}\right)\right]. \end{aligned}\right\} \quad (42.5)$$

$$\varepsilon^0_{2(1)} = \beta t^{m0}_{(1)}, \quad \psi^0 = -\frac{\beta R}{L}\left(t^{mL}_{(1)} - t^{m0}_{(1)}\right). \quad (42.6)$$

The obtained system of forces is self-balancing, i.e., the following are satisfied

$$m_1^0 + t_1^0 R = 0, \quad n_1^0 - \left(s_1 + \frac{h_1}{R}\right)^0 = 0. \quad (42.7)$$

Adding to edge $s = 0$ also a self-balancing system consisting of forces and moments of the opposite sign, we determine the forces of the edge effect which appears in the free edge. Calculating the forces of the bending moment and peripheral force caused by this system with the aid of (34.23), (34.24), we obtain

$$\left.\begin{aligned} m^0 &= \frac{\mu Eh^3\beta}{12(1-\mu)}\left(\frac{\Delta t^0_{(1)}}{h} - \frac{t^{m0}_{(1)}}{R}\right), \\ t_2^0 &\approx -\frac{2\gamma^2}{R}\frac{Eh^3\beta}{12(1-\mu)}\left(\frac{\Delta t^0_{(1)}}{h} - \frac{t^{m0}_{(1)}}{R}\right); \end{aligned}\right\} \quad (42.8)$$

In calculating t_2^0 quantities of order $1/\gamma$ in comparison with unity were dropped. Adding the forces of edge effect with the main forces, we will have

$$m_1^0 = 0, \quad n_1^0 = 0,$$
$$m_2^0 = \frac{Eh^3\beta}{2}\left(\frac{t_{(1)}^{m0}}{R} - \frac{\Delta t_{(1)}^0}{h}\right),$$
$$t_2^0 = \frac{Eh^2\beta}{2}\sqrt{\frac{(1+\mu)}{(1-\mu)\cdot 3}}\left(\frac{t_{(1)}^{m0}}{R} - \frac{\Delta t_{(1)}^0}{h}\right). \tag{42.9}$$

Analogous expressions could have been written also for the second edge of the shell.

At $t_{(1)}^m = 0$ and $\Delta t_{(1)}$ constant over the entire length of the shell the equation for calculation of maximum peripheral stress in the edge section completely agrees with that obtained earlier for axisymmetric radial drop $\Delta t_{(0)}$; it has the form

$$\sigma_2 = -\frac{E\beta\Delta t_{(1)}}{2}\left[1 + \frac{\sqrt{1-\mu^2}}{(1-\mu)\sqrt{3}}\right]. \tag{42.10}$$

Such stress exists only at $\phi = 0$. To calculate peripheral stress in other sections the amplitude value, written above, must be multiplied by $\cos\phi$.

§ 43. Cylindrical Shell of Varying Thickness

In Chapter II (§§ 12, 16) basic resolvant equations were obtained for a shell of varying thickness, which experiences axisymmetric and bending loads. Setting in them $R_1 = \infty$, $\cos 0 = 0$, $\sin 0 = 1$, $v = R$, $R_1 d\theta = ds$, $b = R$, we obtain the corresponding equations for a cylindrical shell. Namely, equation (12.12) will become equation

$$\frac{d^2\sigma_0}{ds^2} + \frac{3}{\alpha}\frac{d\alpha}{ds}\frac{d\sigma_0}{ds} + \sigma_0 2i\gamma_0^2\frac{1}{R^2\alpha} = -4\gamma_0^4\left[\frac{\Phi_2}{R^3\alpha^3} + \frac{i}{2\gamma_0^2}\frac{1}{\alpha R^2}\mu\frac{d}{ds}\left(\frac{\Phi_1}{\alpha}\right)\right], \tag{43.1}$$

where

$$\alpha = \frac{h(s)}{h_0}, \quad 4\gamma_0^4 = 12(1-\mu^2)\frac{R^2}{h_0^2},$$
$$\Phi_1 = \frac{P_z^0}{2\pi} - \int_0^s Rq_{1(0)}\,ds, \quad \Phi_2 = -\int_0^s Rq_{n(0)}\,ds. \tag{43.2}$$

Equation (16.21) in this case assumes the form

$$\frac{d^2\sigma_1}{ds^2}+\frac{3}{a}\frac{da}{ds}\frac{d\sigma_1}{ds}+\sigma_1\left(2i\gamma_0^2\frac{1}{R^2a}-\frac{4}{R^2}\right)=4\gamma_0^4\left(\frac{\Phi_3}{a^3R^2}+\frac{i}{2\gamma_0^2}\frac{\Phi_4}{R}\right). \quad (43.3)$$

where Φ_3, Φ_4, just as Φ_1, Φ_2, are known functions of load

$$\left.\begin{aligned}
&\Phi_3=-\frac{P_1}{\pi R}-\int_0^s q_{s(1)}\,ds,\\
&\Phi_4=\mu\frac{d}{ds}\left(\frac{f_0+f_1}{a}\right)-\frac{2(1+\mu)}{Ra}\int_0^s q_{2(1)}\,ds,\\
&f_0+f_1=-\frac{P_1 s}{\pi R^2}-\frac{M_1}{\pi R^2}-\\
&\qquad-\frac{1}{R}\int_0^s\int_0^s\left(q_{s(1)}-q_{1(1)}\right)ds\,ds-\int_0^s q_{1(1)}\,ds.
\end{aligned}\right\} \quad (43.4)$$

Equations (43.1) and (43.3) differ in the righ part and in the unessential term in the coefficient of the unknown function. Dropping this term, we find that to construct solutions of the edge effect type in both cases it is necessary to consider the equation

$$\frac{d^2\sigma}{ds^2}+\frac{3}{a}\frac{da}{ds}\frac{d\sigma}{ds}+\sigma 2i\gamma_0^2\frac{1}{R^2a}=0, \quad (43.5)$$

or, going to dimensionless variable $\xi=\frac{s}{R}$, the equation

$$\frac{d^2\sigma}{d\xi^2}+\frac{3}{a}\frac{da}{d\xi}\frac{d\sigma}{d\xi}+\frac{2i\gamma_0^2}{a}\sigma=0. \quad (43.6)$$

Making the change of variables indicated in general in § 19, namely, setting

$$dx=\frac{d\xi}{\sqrt{a}}, \quad \sigma=\frac{\tau}{a\sqrt[4]{a}}, \quad (43.7)$$

equation (43.6) will go to the form

$$\frac{d^2\tau}{dx^2}+\tau\left[2i\gamma_0^2-\psi(s)\right]=0. \quad (43.8)$$

where

$$\psi(s)=\frac{R^2a}{4}\left[\frac{15}{4}\frac{1}{a^2}\left(\frac{da}{ds}\right)^2+\frac{5}{a}\frac{d^2a}{ds^2}\right]=\frac{15}{16}\frac{1}{a}\left(\frac{da}{d\xi}\right)^2+\frac{5}{4}\frac{d^2a}{d\xi^2}.$$

If the thickness of the shell is a so slowly changing function of coordinate ξ, that the inequalities

$$\frac{1}{a}\left(\frac{da}{d\xi}\right)^2 \ll 2\gamma_0^2, \quad \left|\frac{d^2a}{d\xi^2}\right| \ll 2\gamma_0^2. \qquad (43.9)$$

hold, then the term $\psi(s)$ can be neglected in comparison with $2i\gamma_0^2$ and instead of (43.8) thus we obtain the equation

$$\frac{d^2\tau}{dx^2} + 2i\gamma_0^2\tau = 0. \qquad (43.10)$$

As already it was indicated earlier, particular solutions of this equation can be taken in the form

$$e^{\pm\gamma_0 x}\cos\gamma_0 x, \quad e^{\pm\gamma_0 x}\sin\gamma_0 x. \qquad (43.11)$$

respectively particular solutions (43.5) are functions

$$\frac{1}{a\sqrt[4]{a}}e^{\pm\gamma_0 x}\cos\gamma_0 x, \quad \frac{1}{a\sqrt[4]{a}}e^{\pm\gamma_0 x}\sin\gamma_0 x. \qquad (43.12)$$

where

$$x = \int_0^\xi \frac{d\xi}{\sqrt{a}}. \qquad (43.13)$$

In the particular case when thickness is a linear function of coordinate s or, which is the same, a linear function of coordinate ξ,

$$a(\xi) = 1 + k\xi. \qquad (43.14)$$

equation (43.5) is integrated accurately. Really, introducing new variable

$$y = 1 + k\xi. \qquad (43.15)$$

instead of (43.5) we obtain the equation

$$\frac{d^2\sigma}{dy^2} + \frac{3}{y}\frac{d\sigma}{dy} + \frac{2i\gamma_0^2}{k^2}\frac{\sigma}{y} = 0. \qquad (43.16)$$

which with the aid of transformation of dependent and independent variables

$$t=\sqrt{2ly}\,\frac{2\gamma_0}{k},\quad \eta=\sigma y \tag{43.17}$$

becomes Bessel equation

$$\frac{d^2\eta}{dt^2}+\frac{1}{t}\frac{d\eta}{dt}+\eta\left(1-\frac{4}{t^2}\right)=0. \tag{43.18}$$

As the particular linearly independent solutions of this equation we take the Bessel and Hankel functions of the first kind, second order. In this way, the general solution of (43.18) is written in the form

$$\eta=C_1I_2(t)+C_2H_2^{(1)}(t). \tag{43.19}$$

where C_1, C_2 – arbitrary constants, generally speaking, imaginary numbers.

For large values of the argument, which takes place when

$$\frac{2\gamma_0}{k}\gg 1. \tag{43.20}$$

these functions can be represented by an asymptotic decomposition, the first terms of which have the form

$$\left.\begin{aligned} I_2\left(\frac{2\gamma_0}{k}\sqrt{2ly}\right)&\approx-\frac{1}{\sqrt{2\pi\sqrt{2y}\,\frac{2\gamma_0}{k}}}\,e^{\frac{2\gamma_0}{k}\sqrt{y}}\times\\ &\times\left[\cos\left(\frac{2\gamma_0}{k}\sqrt{y}-\frac{\pi}{8}\right)-i\sin\left(\frac{2\gamma_0}{k}\sqrt{y}-\frac{\pi}{8}\right)\right].\\ H_2^{(1)}\left(\frac{2\gamma_0}{k}\sqrt{2ly}\right)&\approx-\frac{1}{\sqrt{\frac{\pi}{2}\sqrt{2y}\,\frac{2\gamma_0}{k}}}\,e^{-\frac{2\gamma_0}{k}\sqrt{y}}\times\\ &\times\left[\sin\left(\frac{2\gamma_0}{k}\sqrt{y}+\frac{\pi}{8}\right)-i\cos\left(\frac{2\gamma_0}{k}\sqrt{y}+\frac{\pi}{8}\right)\right]. \end{aligned}\right\} \tag{43.21}$$

Taking into account these representations, and also that in this case

$$y=\alpha,\quad \sigma=\frac{\eta}{\alpha} \tag{43.22}$$

and

$$x=\int_0^{\xi}\frac{d\xi}{\sqrt{1+k\xi}}=\frac{2}{k}\sqrt{\alpha}. \tag{43.23}$$

it is easy to see that the real and imaginary parts of the expressions

$$\frac{1}{\alpha} J_2\left(\frac{2\gamma_0}{k}\sqrt{2i\alpha}\right), \quad \frac{1}{\alpha} H_2^{(1)}\left(\frac{2\gamma_0}{k}\sqrt{2i\alpha}\right) \qquad (43.24)$$

for a large value of the modulus of the argument are linear combinations of functions (43.12). In this way the correctness of approximate solution (43.12) agrees with that which we have on replacing the accurate values of functions $J_2(t)$, $H_2^{(1)}(t)$ by the first terms of their asymptotic representations. With a more complex law of change of α depending on ξ accurate integration of equation (43.6) is difficult. However, on the basis of the comparison we will assume that also in this instance the correctness of solution (43.12) is practically satisfactory if only conditions (43.9) hold.

Let us make use of the obtained approximate solution for a description of edge effect in a long axisymmetrically loaded cylindrical shell. Repeating the reasoning conducted in § 27, we derive criteria for the determination of shell length. Namely, we will consider the shell to be long if

$$\gamma_0 \int_0^l \frac{d\xi}{\sqrt{\alpha}} > \pi, \qquad (43.25)$$

or

$$\int_0^L \frac{ds}{\sqrt{\alpha(s)}} > 2.4\sqrt{Rh_0}.$$

In carrying out this condition it is convenient to represent the solution of uniform equation (43.6) in the form

$$\sigma_0 = (A_1 - iB_1)\frac{1}{\alpha\sqrt[4]{\alpha}}[\Theta(\gamma_0 x) + i\zeta(\gamma_0 x)] + \\ + (A_2 - iB_2)\frac{1}{\alpha\sqrt[4]{\alpha}}[\Theta(\gamma_0 x_1) + i\zeta(\gamma_0 x_1)]. \qquad (43.26)$$

where

$$x_1 = \int_\xi^l \frac{d\xi}{\sqrt{\alpha}}. \qquad (43.27)$$

At $\alpha = 1$, $x = \xi$, $x_1 = \xi_1$; solution (43.26) agrees with the earlier solution for a long shell of constant thickness.

Separating the real and imaginary parts of σ_0, we have

$$\Psi_0 = \operatorname{Re}\sigma_0 = \frac{1}{\alpha\sqrt[4]{\alpha}}\left[A_1\theta(\gamma_0 x) + B_1\zeta(\gamma_0 x) + A_2\theta(\gamma_0 x_1) + B_2\zeta(\gamma_0 x_1)\right],$$
$$-2\gamma_0^2 V_0 = \operatorname{Im}\sigma_0 =$$
$$= \frac{1}{\alpha\sqrt[4]{\alpha}}\left[A_1\zeta(\gamma_0 x) - B_1\theta(\gamma_0 x) + A_2\zeta(\gamma_0 x_1) - B_2\theta(\gamma_0 x_1)\right]. \qquad (43.28)$$

Forces, moments and displacements are determined through function σ_0 using equations which can be obtained from (12.11) by the corresponding writing for a cylindrical shell:

$$\left.\begin{aligned} &RT_1 = \Phi_1(s), \quad T_2 = -\frac{R}{2\gamma_0^2}\frac{d}{ds}(\alpha^2 \operatorname{Im}\sigma_0), \\ &RN_1 = -\frac{\alpha^2 R}{2\gamma_0^2}\operatorname{Im}\sigma_0 + \Phi_2(s), \\ &M_1 = -\frac{\alpha^3 R^2}{4\gamma_0^4}\operatorname{Re}\frac{d\sigma_0}{ds}, \quad M_2 = \mu M_1. \end{aligned}\right\} \qquad (43.29)$$

$$\left.\begin{aligned} &\vartheta_1 = \frac{1}{Eh_0}\operatorname{Re}\sigma_0, \quad w = \frac{R}{Eh_0}\frac{1}{\alpha}(T_2 - \mu T_1), \\ &\Delta_s = -\frac{1}{Eh_0}\int_0^s \frac{1}{\alpha(s)}(T_1 - \mu T_2)\, ds. \end{aligned}\right\} \qquad (43.30)$$

The particular solution of equation (43.1) is obtained by dividing the right part of the equation by the coefficient of σ. Then after eliminating quantities of the order of $1/2\gamma_0^2$ in comparison with unity we will have

$$\sigma_0 = -i\frac{2\gamma_0^2}{\alpha^2}\int_0^s q_{n(0)}\, ds. \qquad (43.31)$$

To it corresponds the zero-moment stressed state

$$\left.\begin{aligned} &T_1 = \frac{P_z^0}{2\pi R} - \int_0^s q_{1(0)}\, ds, \quad T_2 = Rq_{n(0)}, \\ &N_1 = 0, \quad M_1 = M_2 = 0. \end{aligned}\right\} \qquad (43.32)$$

Displacements of the zero-moment state are calculated using formula (43.30) for values of forces (43.32).

In direct calculation of forces and moments by formula (43.29) it is necessary to differentiate expressions (43.28). In this case one should bear in mind that in accordance with the correctness of the solution itself (43.28) variable coefficient $\frac{1}{\alpha\sqrt[4]{\alpha}}$ during differentiation can be considered as constant. For example,

$$\frac{d}{ds}\left[\frac{1}{\alpha\sqrt[4]{\alpha}}\theta(\gamma_0 x)\right]=$$

$$=\frac{1}{\alpha\sqrt[4]{\alpha}}\theta'(\gamma_0 x)\gamma_0\frac{1}{R\sqrt{\alpha}}+\frac{1}{R}\frac{d}{d\xi}\left(\frac{1}{\alpha\sqrt[4]{\alpha}}\right)\theta(\gamma_0 x)\approx\frac{1}{\alpha^{7/4}}\frac{\gamma_0}{R}\theta'(\gamma_0 x).$$

$$\frac{d}{ds}\left[\frac{1}{\alpha\sqrt[4]{\alpha}}\theta(\gamma_0 x_1)\right]\approx-\frac{1}{\alpha^{7/4}}\frac{\gamma_0}{R}\theta'(\gamma_0 x_1).$$

where, as earlier, the prime indicates differentiation with respect to the argument indicated in brackets.

Taking into account the above and ignoring mutual influence of the edges during determination of constants of integration A_1, B_1, A_2, B_2, it is simple to write the expressions for forces and displacements in a shell of varying thickness, analogous to (27.14) to (27.19):

$$T_2=-2\gamma_0\alpha^{1/4}\Big[N_1^0\theta(\gamma_0 x)+\frac{\gamma_0}{R}M_1^0\psi(\gamma_0 x)-$$
$$-(\alpha^L)^{-1/4}N_1^L\theta(\gamma_0 x_1)+\frac{\gamma_0}{R}(\alpha^L)^{-1/4}M_1^L\psi(\gamma_0 x_1)\Big]+Rq_{n(0)}, \qquad (43.33)$$

$$N_1=\alpha^{3/4}\Big[N_1^0\psi(\gamma_0 x)-\frac{2\gamma_0}{R}M_1^0\zeta(\gamma_0 x)+$$
$$+(\alpha^L)^{-3/4}N_1^L\psi(\gamma_0 x_1)+(\alpha^L)^{-3/4}\frac{2\gamma_0}{R}M_1^L\zeta(\gamma_0 x_1)\Big]. \qquad (43.34)$$

$$M_1=\alpha^{5/4}\Big[M_1^0\varphi(\gamma_0 x)+\frac{R}{\gamma_0}N_1^0\zeta(\gamma_0 x)+$$
$$+(\alpha^L)^{-5/4}M_1^L\varphi(\gamma_0 x_1)-\frac{R}{\gamma_0}(\alpha^L)^{-5/4}N_1^L\zeta(\gamma_0 x_1)\Big]. \qquad (43.35)$$

$$\vartheta_1=\frac{2\gamma_0^2}{Eh_0\alpha^{5/4}}\Big[N_1^0\varphi(\gamma_0 x)+\frac{2\gamma_0}{R}M_1^0\theta(\gamma_0 x)+$$
$$+(\alpha^L)^{-1/4}N_1^L\varphi(\gamma_0 x_1)-\frac{2\gamma_0}{R}(\alpha^L)^{-1/4}M_1^L\theta(\gamma_0 x_1)\Big]. \qquad (43.36)$$

$$w=\frac{R}{Eh_0\alpha}(T_2-\mu T_1)=-\frac{R2\gamma_0}{Eh_0}\alpha^{-3/4}\Big[N_1^0\theta(\gamma_0 x)+\frac{\gamma_0}{R}M_1^0\psi(\gamma_0 x)-$$
$$-(\alpha^L)^{-3/4}N_1^L\theta(\gamma_0 x_1)+\frac{\gamma_0}{R}(\alpha^L)^{-3/4}M_1^L\psi(\gamma_0 x_1)\Big]+$$
$$+\frac{R^2 q_{n(0)}}{Eh_0\alpha}-\frac{\mu R}{Eh_0\alpha}\left(\frac{P_z^0}{2\pi R}-\int_0^s q_{1(0)}\,ds\right). \qquad (43.37)$$

On the basis of (43.33)-(43.37), setting that distributed load and axial force are absent, we easily obtain equations for figuring ϑ_1, w on the edges of the shell. Namely, at $x=0$, ignoring the influence of edge $x_1=0$, we have

$$\left.\begin{aligned}\vartheta_1^0&=\frac{2\gamma_0^2}{Eh_0}\left(N_1^0+\frac{2\gamma_0}{R}M_1^0\right),\\ w^0&=-\frac{2\gamma_0 R}{Eh_0}\left(N_1^0+\frac{\gamma_0}{R}M_1^0\right).\end{aligned}\right\} \qquad (43.38)$$

Correspondingly for edge $x_1=0$ $(s=L)$ we obtain

$$\left.\begin{aligned}\vartheta_1^L&=\frac{2\gamma_0^2}{Eh_0}\frac{1}{(\alpha^L)^2\sqrt{\alpha^L}}\left(\sqrt{\alpha^L}\,N_1^L-\frac{2\gamma_0}{R}M_1^L\right),\\ w^L&=\frac{2\gamma_0 R}{Eh_0}\frac{1}{(\alpha^L)^2}\left(\sqrt{\alpha^L}\,N_1^L-\frac{\gamma_0}{R}M_1^L\right),\end{aligned}\right\} \qquad (43.39)$$

where $\alpha^L=h^L/h^0$.

The case of deformation of a long shell of varying thickness under the action of a bending load can be examined in exactly the same way. In this case for calculation of forces, moments and deformations the following equations are derived:

$$\left.\begin{aligned}t_2&=-\alpha^{1/4}\Big\{2\beta_0 n_1^0\theta(\gamma_0 x)+\frac{2\beta_0^2}{R}m_1^0\psi(\gamma_0 x)+\\&\quad+(\alpha^L)^{-1/4}\Big[-2\beta_L n_1^L\theta(\gamma_0 x_1)+\frac{2\beta_L^2}{R}m_1^L\psi(\gamma_0 x_1)\Big]\Big\},\\ m_1&=\alpha^{3/4}\Big\{m_1^0\varphi(\gamma_0 x)+\frac{R}{\beta_0}n_1^0\zeta(\gamma_0 x)+\\&\quad+(\alpha^L)^{-3/4}\Big[m_1^L\varphi(\gamma_0 x_1)-\frac{R}{\beta_L}n_1^L\zeta(\gamma_0 x_1)\Big]\Big\},\\ n_1&=\alpha^{3/4}\Big\{n_1^0\psi(\gamma_0 x)-\frac{2\beta_0}{R}m_1^0\zeta(\gamma_0 x)+\\&\quad+(\alpha_L)^{-3/4}\Big[n_1^L\psi(\gamma_0 x_1)+\frac{2\beta_L}{R}m_1^L\zeta(\gamma_0 x_1)\Big]\Big\}.\end{aligned}\right| \qquad (43.40)$$

$$
\left.
\begin{aligned}
&t_1 = (f_0 + f_1) - \frac{m_1}{R}, \quad s_1 = n_1 + \frac{P_1}{\pi R} + \int_0^s (q_{n(1)} - q_{2(1)})\, ds, \\
&Eh_0 \varepsilon_{2(1)} = -\alpha^{-1/4} \Big\{ 2\beta_0 n_1^0 \theta(\gamma_0 x) + \frac{2\beta_0^2}{R} m_1^0 \psi(\gamma_0 x) + \\
&\qquad + (\alpha_L)^{-1/4} \Big[-2\beta_L n_1^L \theta(\gamma_0 x_1) + \frac{2\beta_L^2}{R} m_1^L \psi(\gamma_0 x_1) \Big] \Big\} + \\
&\qquad\qquad + \frac{R q_{n(1)}}{\alpha} - \frac{\mu (f_0 + f_1)}{\alpha}, \\
&Eh_0 \Psi = -\alpha^{-3/4} \Big\{ 2\beta_0^2 n_1^0 \varphi(\gamma_0 x) + \frac{4\beta_0^3}{R} m_1^0 \theta(\gamma_0 x) + \\
&\qquad + (\alpha_L)^{3/4} \Big[2\beta_L^2 n_1^L \varphi(\gamma_0 x_1) - \frac{4\beta_L^3}{R} m_1^L \theta(\gamma_0 x_1) \Big] \Big\}.
\end{aligned}
\right]
\quad \text{(43.40) (cont.)}
$$

In (43.40) we accept

$$
2\beta_0^2 = 2\gamma_0^2, \quad 2\beta_L^2 = \sqrt{12(1-\mu^2)}\,\frac{R}{h_L}, \quad \gamma_0 = \beta_L \alpha_L^{1/4}.
$$

CHAPTER IV

THE CONICAL SHELL

§ 44. Axisymmetric Deformation of Conical Shell of Constant Thickness

In the previous chapter, dedicated to the cylindrical shell, the derivation of Meissner equations for axisymmetric and bending loads from considerations of clarity was conducted directly for cylindrical shell without references to Chapter II, where this derivation was given for an arbitrary shell of revolution. Leaving this method, we will consider a conical shell as a particular case of a shell of revolution and will use the equations of Chapter II, setting in them

$$R_1 = \infty, \quad R_1\, d\theta = ds,$$
$$\theta = \frac{\pi}{2} - \beta,$$
$$R_2 = \frac{\nu}{\cos\beta}.$$

For the case of axisymmetric deformation of a conical shell of constant thickness (Fig. 22), rewriting equation (12.6), we obtain

$$\left.\begin{aligned} &\frac{d^2V_0}{ds^2} + \frac{\sin\beta}{\nu}\frac{dV_0}{ds} - \frac{\sin^2\beta}{\nu^2}V_0 - \Psi_0\frac{\cos\beta}{b\nu} = \\ &\qquad = \frac{1}{b\nu}\left[\mu\frac{d\Phi_1}{ds} + \frac{\sin\beta}{\nu}\Phi_1\right], \\ &\frac{d^2\Psi_0}{ds^2} + \frac{\sin\beta}{\nu}\frac{d\Psi_0}{ds} - \frac{\sin^2\beta}{\nu^2}\Psi_0 + 4\gamma^4\frac{\cos\beta}{b\nu}V_0 = -\frac{1}{\nu b^2}4\gamma^4\Phi_2, \end{aligned}\right\} \qquad (44.1)$$

where

$$\left.\begin{aligned}\Phi_1 &= -\sin\beta\int_0^s \nu q_e\,ds + \cos\beta\left(\frac{P_z^0}{2\pi} + \int_0^s \nu q_z\,ds\right),\\ \Phi_2 &= -\cos\beta\int_0^s \nu q_e\,ds - \sin\beta\left(\frac{P_z^0}{2\pi} + \int_0^s \nu q_z\,ds\right).\end{aligned}\right\} \qquad (44.2)$$

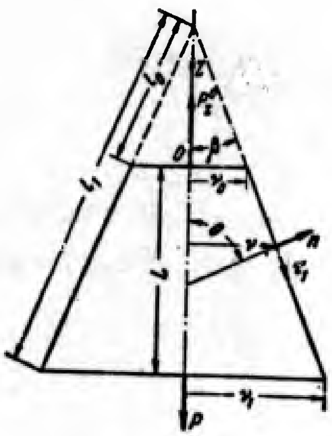


Fig. 22. Conical shell.

Taking into consideration that $ds = \frac{d\nu}{\sin\beta}$, equations (44.1), (44.2) can be written in the form

$$\left.\begin{aligned}&\frac{d^2V_0}{d\nu^2} + \frac{1}{\nu}\frac{dV_0}{d\nu} - \frac{1}{\nu^2}V_0 - \frac{\cos\beta}{b\nu\sin^2\beta}\Psi_0 =\\ &\qquad = \frac{1}{b\nu\sin\beta}\left[\mu\frac{d\Phi_1}{d\nu} + \frac{\Phi_1}{\nu}\right],\\ &\frac{d^2\Psi_0}{d\nu^2} + \frac{1}{\nu}\frac{d\Psi_0}{d\nu} - \frac{1}{\nu^2}\Psi_0 + 4\gamma^4\frac{\cos\beta}{b\nu\sin^2\beta}V_0 = -\frac{4\gamma^4}{\nu b^2\sin^2\beta}\Phi_2.\end{aligned}\right\} \qquad (44.3)$$

$$\left.\begin{aligned}\Phi_1 &= -\int_{\nu_0}^{\nu} \nu q_e\,d\nu + \cos\beta\frac{P_z^0}{2\pi} + \operatorname{ctg}\beta\int_{\nu_0}^{\nu} \nu q_z\,d\nu,\\ \Phi_2 &= -\operatorname{ctg}\beta\int_{\nu_0}^{\nu} \nu q_e\,d\nu - \sin\beta\frac{P_z^0}{2\pi} - \int_{\nu_0}^{\nu} \nu q_z\,d\nu.\end{aligned}\right\} \qquad (44.4)$$

As parameter b we can select the radius of the edge section ν_1, so that

$$4\gamma^4 = 12(1-\mu^2)\frac{\nu_1^2}{h^2}, \quad b = \nu_1. \qquad (44.5)$$

Forces and bending moments are expressed through functions V_0, Ψ_0 in the following manner:

$$\left.\begin{aligned} \nu T_1 = V_0\nu_1 \sin\beta + \Phi_1(\nu), \quad \nu N_1 = V_0\nu_1 \cos\beta + \Phi_2(\nu), \\ T_2 = \nu_1 \sin\beta \frac{dV_0}{d\nu}, \end{aligned}\right\} \tag{44.6}$$

$$\left.\begin{aligned} M_1 = -\frac{h^2}{12(1-\mu^2)} \sin\beta \left(\frac{d\Psi_0}{d\nu} + \mu \frac{\Psi_0}{\nu}\right), \\ M_2 = -\frac{h^2}{12(1-\mu^2)} \sin\beta \left(\frac{\Psi_0}{\nu} + \mu \frac{d\Psi_0}{d\nu}\right) \end{aligned}\right\} \tag{44.7}$$

§ 45. Particular Solution of Meissner Equations for Different Forms of Distributed Load

Let us construct the particular solution of system (44.3) for different forms of distributed load. Almost all practically important cases of loading can be examined if we set [12]

$$q_e = A_0 + A_1\nu, \quad q_z = B_0 + B_1\nu. \tag{45.1}$$

Really, gravity has components $q_e = 0$, $q_z = \rho h$ (ρ — specific weight of material of shell), the force of inertia of revolution — $q_e = \frac{\rho\omega^2}{g} h\nu$, $q_z = 0$ (ω — angular velocity of revolution of shell around axis OZ), uniform interior pressure — $q_e = p\cos\beta$, $q_z = p\sin\beta$, etc. Substituting (45.1) into (44.4), we obtain

$$\left.\begin{aligned} \Phi_1 = a_0 + \frac{1}{2} a_2\nu^2 + \frac{1}{3} a_3\nu^3, \\ \Phi_2 = b_0 + \frac{1}{2} b_2\nu^2 + \frac{1}{3} b_3\nu^3, \end{aligned}\right\} \tag{45.2}$$

where

$$\left.\begin{aligned} a_0 &= \frac{P_z^0}{2\pi}\cos\beta + A_0\frac{\nu_0^2}{2} + A_1\frac{\nu_0^3}{3} - B_0 \operatorname{ctg}\beta \frac{\nu_0^2}{2} - B_1 \operatorname{ctg}\beta \frac{\nu_0^3}{3}, \\ a_2 &= -A_0 + B_0 \operatorname{ctg}\beta, \quad a_3 = -A_1 + B_1 \operatorname{ctg}\beta, \\ b_0 &= -\frac{P_z^0}{2\pi}\sin\beta + A_0 \operatorname{ctg}\beta \frac{\nu_0^2}{2} + A_1 \operatorname{ctg}\beta \frac{\nu_0^3}{3} + B_0\frac{\nu_0^2}{2} + B_1\frac{\nu_0^3}{3}, \\ b_2 &= -A_0 \operatorname{ctg}\beta - B_0, \quad b_3 = -A_1 \operatorname{ctg}\beta - B_1. \end{aligned}\right\} \tag{45.3}$$

Equations (44.3) assume the form

$$\left.\begin{aligned}
&\frac{d^2V_0}{d\nu^2}+\frac{1}{\nu}\frac{dV_0}{d\nu}-\frac{1}{\nu^2}V_0-\frac{1}{\nu\nu_1}\frac{\cos\beta}{\sin^2\beta}\Psi_0=\\
&\qquad=\frac{1}{\nu_1\sin\beta}\left[\frac{a_0}{\nu^2}+\left(\mu+\frac{1}{2}\right)a_2+\left(\mu+\frac{1}{3}\right)a_3\nu\right].\\
&\frac{d^2\Psi_0}{d\nu^2}+\frac{1}{\nu}\frac{d\Psi_0}{d\nu}-\frac{1}{\nu^2}\Psi_0+\frac{4\gamma^4\cos\beta}{\nu\nu_1\sin^2\beta}V_0=\\
&\qquad=-\frac{4\gamma^4}{\nu_1^2\sin^2\beta}\left[\frac{b_0}{\nu}+\frac{1}{2}b_2\nu+\frac{1}{3}b_3\nu^2\right].
\end{aligned}\right\}\qquad(45.4)$$

The particular solution of equations (45.4) we look for in the form

$$\left.\begin{aligned}
&\tilde{\Psi}_0=\frac{c_{-1}}{\nu}+c_1\nu+c_2\nu^2,\\
&\tilde{V}_0=-\frac{1}{\nu_1\cos\beta}\left(b_0+\frac{1}{2}b_2\nu^2+\frac{1}{3}b_3\nu^3\right)+\nu d_1.
\end{aligned}\right\}\qquad(45.5)$$

Unknowns c_{-1}, c_1, c_2, d_1 are determined directly by means of substitution (45.5) into equations (45.4) and equating the coefficients of identical degrees of variable ν. In this case we obtain

$$\left.\begin{aligned}
&c_{-1}=b_0\,\mathrm{tg}^2\beta-a_0\,\mathrm{tg}\,\beta,\\
&c_1=-\frac{3}{2}b_2\,\mathrm{tg}^2\beta-\left(\mu+\frac{1}{2}\right)a_2\,\mathrm{tg}\,\beta,\\
&c_2=-\frac{8}{3}b_3\,\mathrm{tg}^2\beta-\left(\mu+\frac{1}{3}\right)a_3\,\mathrm{tg}\,\beta,\\
&d_1=\frac{b}{4\gamma^4}\left[8b_3\,\mathrm{tg}\,\beta+(3\mu+1)a_3\right]\mathrm{tg}^2\beta\sin\beta.
\end{aligned}\right\}\qquad(45.6)$$

It is easy to see that the last term in the second equation (45.5) can be dropped since it is small in comparison with the first (as a quantity of order $\frac{h^2}{\nu_1^2}$ is small in comparison with unity). The determination of forces and bending moments on the basis of solution (45.5) indicates that bending stresses corresponding to $\tilde{\Psi}_0$, have the order of $\frac{h}{\nu_1}$ in comparison with tensile stresses, computable according to the zero-moment solution

$$\tilde{V}_0=-\frac{1}{\nu_1\cos\beta}\left(b_0+\frac{1}{2}b_2\nu^2+\frac{1}{3}b_3\nu^3\right)=-\frac{\Phi_2}{\nu_1\cos\beta}.\qquad(45.7)$$

Therefore during the calculation of stresses practically, instead of the particular solution, we can use the solution of the zero-moment theory, which in general has the form

$$\tilde{T}_1 = \frac{1}{\cos\beta}\left[\frac{P_z^0}{2\pi\nu} + \frac{1}{\nu\sin\beta}\int_{\nu_1}^{\nu} q_z \nu\, d\nu\right],$$
$$\tilde{T}_2 = \nu q_e + \nu q_z \operatorname{tg}\beta, \quad \tilde{M}_1 = \tilde{M}_2 = 0, \quad \tilde{\Psi}_0 = 0. \qquad (45.8)$$

However, neglecting the quantity $\tilde{\Psi}_0$ in making up boundary conditions can lead to errors of the order of $\sqrt{\frac{h}{\nu_1}}$ or $1/\gamma$ in comparison with unity. To avoid this it is necessary to hold $\tilde{\Psi}_0 \neq 0$. Let us write out the forces and bending moments which correspond to particular solution (45.5) for the examined forms of distributed load:

1) uniform internal pressure

$$q_e = p\cos\beta, \quad q_z = p\sin\beta,$$
$$\tilde{T}_1 = \frac{1}{\nu\cos\beta}\left[\frac{P_z^0}{2\pi} + \frac{p(\nu^2-\nu_0^2)}{2}\right], \quad \tilde{T}_2 = \frac{p\nu}{\cos\beta},$$
$$\tilde{\Psi}_0 = \frac{\sin\beta}{\cos^2\beta}\left[\left(-\frac{P_z^0}{2\pi} + \frac{p\nu_0^2}{2}\right)\frac{1}{\nu} + \frac{3}{2}p\nu\right],$$
$$\tilde{M}_1 = -\frac{h^2}{12(1-\mu^2)}\operatorname{tg}^2\beta\left[-\frac{(1-\mu)}{\nu^2}\left(-\frac{P_z^0}{2\pi} + \frac{p\nu_0^2}{2}\right) + \frac{3}{2}(1+\mu)p\right],$$
$$\tilde{M}_2 = -\frac{h^2}{12(1-\mu^2)}\operatorname{tg}^2\beta\left[\frac{(1-\mu)}{\nu^2}\left(-\frac{P_z^0}{2\pi} + \frac{p\nu_0^2}{2}\right) + \frac{3}{2}(1+\mu)p\right]. \qquad (45.9)$$

For a cone closed in vertex $\nu_0 = 0$. If, moreover, in the vertex there is no concentrated force ($P_z^0 = 0$), then the derived formulas assume the form

$$\tilde{T}_1 = \frac{p\nu}{2\cos\beta}, \quad \tilde{T}_2 = \frac{p\nu}{\cos\beta},$$
$$\tilde{\Psi}_0 = \frac{3}{2}p\nu\frac{\sin\beta}{\cos^2\beta}, \quad \tilde{M}_1 = \tilde{M}_2 = -\frac{1}{4\gamma^4}\frac{3}{2}(1+\mu)p\nu_1^2\operatorname{tg}^2\beta. \qquad (45.10)$$

Angle of rotation, radial and axial displacements in this instance are equal to

$$\tilde{\vartheta}_1 = \frac{3}{2}\frac{p\nu}{Eh}\frac{\sin\beta}{\cos^2\beta}, \quad \tilde{\Delta}_e = \left(1 - \frac{\mu}{2}\right)\frac{p\nu^2}{Eh\cos\beta},$$
$$\tilde{\Delta}_z = \frac{p\nu^2}{2Eh\sin\beta}\left[-\frac{(1-2\mu)}{2} + \frac{3}{2}\operatorname{tg}^2\beta\right] + \Delta_z^0, \qquad (45.11)$$

where Δ_z^0 — displacement of shell as a whole along axis OZ;

2) rotating shell

$$\left.\begin{aligned}
&\tilde{T}_1=0, \quad \tilde{T}_2=\frac{\rho\omega^2hv^2}{g}, \quad \tilde{\vartheta}_1=(3+\mu)\operatorname{tg}\beta\frac{\rho\omega^2v^2}{Eg},\\
&\tilde{M}_1=-\frac{(3+\mu)(2+\mu)}{4\gamma^4}\frac{\sin^2\beta}{\cos\beta}\frac{\rho\omega^2hv_1^2v}{g},\\
&\tilde{M}_2=-\frac{(3+\mu)(1+2\mu)}{4\gamma^4}\frac{\sin^2\beta}{\cos\beta}\frac{\rho\omega^2hv_1^2v}{g},\\
&\tilde{\Delta}_e=\frac{\rho\omega^2v^3}{Eg}; \quad \tilde{\Delta}_z=\frac{\rho\omega^2}{Eg}\operatorname{tg}\beta\left(1+\frac{\mu}{3\sin^2\beta}\right)(v^3-v_0^3)+\Delta_z^0;
\end{aligned}\right\} \qquad (45.12)$$

3) gravity

$$\left.\begin{aligned}
&\tilde{T}_1=-\frac{\rho h}{v\sin\beta}\frac{(v^2-v_0^2)}{2\cos\beta}=-\frac{\rho h}{\sin 2\beta}\frac{(v^2-v_0^2)}{v},\\
&\tilde{T}_2=-\rho hv\operatorname{tg}\beta,\\
&\tilde{\vartheta}_1=\frac{\rho}{E}\left(\mu+\frac{1}{2}-\frac{3}{2}\operatorname{tg}^2\beta\right)v-\frac{\rho}{E}\frac{1}{2\cos^2\beta}\frac{v_0^2}{v}.
\end{aligned}\right\} \qquad (45.13)$$

For a shell closed in vertex $(v_0=0)$, there exist

$$\left.\begin{aligned}
&\tilde{T}_1=-\frac{\rho hv}{\sin 2\beta}, \quad \tilde{T}_2=-\rho hv\operatorname{tg}\beta,\\
&\tilde{\vartheta}_1=\frac{\rho}{E}\left(\mu+\frac{1}{2}-\frac{3}{2}\operatorname{tg}^2\beta\right)v,\\
&\tilde{M}_1=\tilde{M}_2=-\frac{1}{4\gamma^4}\frac{\rho v_1^2}{E}(1+\mu)\left(\mu+\frac{1}{2}-\frac{3}{2}\operatorname{tg}^2\beta\right)\sin\beta,\\
&\tilde{\Delta}_e=-\frac{\rho v^2}{E}\left(\operatorname{tg}\beta-\frac{\mu}{\sin 2\beta}\right),\\
&\tilde{\Delta}_z=\frac{\rho v^2}{2E}\left(\frac{\operatorname{ctg}\beta}{\sin 2\beta}+\frac{1}{2}-\frac{3}{2}\operatorname{tg}^2\beta\right)+\Delta_z^0.
\end{aligned}\right\} \qquad (45.14)$$

§ 46. Solution of Uniform Meissner Equations

Turn to the solution of the system of uniform equations which correspond to (44.3). In § 19 it was shown that the solution of uniform equations which describe axisymmetric deformation and deformation under bending load have the character of an edge effect. At the same time the transformation of these equations given in § 19, and their replacement by approximations resembling the equations gives a method of creating of an approximate solution of boundary effect type. Considering the axisymmetric deformation of a conical shell, there is no need to go to such a method since the solution of a uniform system of equations corresponding to (44.3) can be expressed accurately in Bessel functions. By introducing complex function

$$\sigma = \Psi_0 - 2i\gamma^2 V_0 \tag{46.1}$$

this system is brought to one equation:

$$\frac{d^2\sigma}{d\nu^2} + \frac{1}{\nu}\frac{d\sigma}{d\nu} - \frac{\sigma}{\nu^2} + \frac{2i\gamma^2}{\sin^2\beta}\frac{\cos\beta}{\nu\nu_1}\sigma = 0. \tag{46.2}$$

By replacing variables [12]

$$dx = \gamma\frac{1}{\sqrt{\nu_1}}\frac{\sqrt{\cos\beta}}{\sin\beta}\frac{d\nu}{\sqrt{\nu}}, \quad \sigma = \frac{\tau}{\sqrt[4]{\nu\cos\beta}} \tag{46.3}$$

equation (46.2) is brought to the form

$$\frac{d^2\tau}{dx^2} + \tau\left(2i - \frac{15}{16\gamma^2}\frac{\nu_1}{\nu}\frac{\sin^2\beta}{\cos\beta}\right) = 0. \tag{46.4}$$

Let us note that the first relationship (46.3) is easily integrated

$$x - x_0 = \gamma\frac{2}{\sqrt{\nu_1}}\frac{\sqrt{\cos\beta}}{\sin\beta}(\sqrt{\nu} - \sqrt{\nu_0})$$

and, thus

$$\left.\begin{aligned} x &= \gamma\frac{2}{\sqrt{\nu_1}}\frac{\sqrt{\cos\beta}}{\sin\beta}\sqrt{\nu}, \\ x_0 &= \gamma\frac{2}{\sqrt{\nu_1}}\frac{\sqrt{\cos\beta}}{\sin\beta}\sqrt{\nu_0}. \end{aligned}\right\} \tag{46.5}$$

Taking into account (46.5), we rewrite (46.4) in the following manner:

$$\frac{d^2\tau}{dx^2}+\tau\left(2i-\frac{15}{4}\cdot\frac{1}{x^2}\right)=0. \qquad (46.6)$$

It is known that the obtained equation is integrated accurately. Its solution has the form

$$\tau=\sqrt{x}\left[c_1 I_2(x\sqrt{2i})+c_2 H_2^{(1)}(x\sqrt{2i})\right]. \qquad (46.7)$$

where I_2, $H_2^{(1)}$ — Bessel function and Hankel function of the first kind of second order.

Taking into account equations which connect these functions with zero-order functions

$$\left.\begin{aligned} I_2(q\sqrt{i})&=-\left[\frac{2}{q\sqrt{i}}I_0'(q\sqrt{i})+I_0(q\sqrt{i})\right],\\ H_2^{(1)}(q\sqrt{i})&=-\left[\frac{2}{q\sqrt{i}}H_0^{(1)'}(q\sqrt{i})+H_0^{(1)}(q\sqrt{i})\right],\end{aligned}\right\} \qquad (46.8)$$

while the prime designates the derivative of the argument shown in brackets, and introducing the designations

$$\left.\begin{aligned} I_0(q\sqrt{i})&=\psi_1(q)+i\psi_2(q),\\ H_0^{(1)}(q\sqrt{i})&=\psi_3(q)+i\psi_4(q),\\ \sqrt{i}\,I_0'(q\sqrt{i})&=\psi_1'(q)+i\psi_2'(q),\\ \sqrt{i}\,H_0^{(1)'}(q\sqrt{i})&=\psi_3'(q)+i\psi_4'(q),\end{aligned}\right\} \qquad (46.9)$$

we obtain

$$\left.\begin{aligned} \operatorname{Re} I_2(x\sqrt{2i})&=-\left[\psi_1(x\sqrt{2})+\frac{\sqrt{2}}{x}\psi_2'(x\sqrt{2})\right],\\ \operatorname{Im} I_2(x\sqrt{2i})&=-\left[\psi_2(x\sqrt{2})-\frac{\sqrt{2}}{x}\psi_1'(x\sqrt{2})\right],\\ \operatorname{Re} H_2^{(1)}(x\sqrt{2i})&=-\left[\psi_3(x\sqrt{2})+\frac{\sqrt{2}}{x}\psi_4'(x\sqrt{2})\right],\\ \operatorname{Im} H_2^{(1)}(x\sqrt{2i})&=-\left[\psi_4(x\sqrt{2})-\frac{\sqrt{2}}{x}\psi_3'(x\sqrt{2})\right],\end{aligned}\right\} \qquad (46.10)$$

Since the introduced functions $\psi_1(q)$, $\psi_2(q)$ and $\psi_3(q)$, $\psi_4(q)$ are the real and imaginary parts of solutions of the zero-order Bessel equation, between them exist the relationships

$$\left.\begin{array}{l}\psi_1''(q)=\psi_2(q)-\frac{1}{q}\psi_1'(q), \quad \psi_3''(q)=\psi_4(q)-\frac{1}{q}\psi_3'(q),\\ \psi_2''(q)=-\psi_1(q)-\frac{1}{q}\psi_2'(q), \quad \psi_4''(q)=-\psi_3(q)-\frac{1}{q}\psi_4'(q).\end{array}\right\} \qquad (46.11)$$

Functions $\psi_i(q)$ $(i=1, 2, 3, 4)$ are representable in the form of the following expansion in powers of q:

$$\left.\begin{array}{l}\psi_1(q)=1-\frac{q^4}{(2\cdot 4)^2}+\frac{q^8}{(2\cdot 4\cdot 6\cdot 8)^2}-\dots,\\ \psi_2(q)=-\frac{q^2}{2^2}+\frac{q^6}{(2\cdot 4\cdot 6)^2}-\dots,\\ \psi_3(q)=\frac{1}{2}\psi_1(q)-\frac{2}{\pi}\left[R_1(q)+\psi_2(q)\ln\frac{\gamma_1 q}{2}\right],\\ \psi_4(q)=\frac{1}{2}\psi_2(q)+\frac{2}{\pi}\left[R_2(q)+\psi_1(q)\ln\frac{\gamma_1 q}{2}\right],\end{array}\right| \qquad (46.12$$

where

$$R_1(q)=\left(\frac{q}{2}\right)^2-\frac{S(3)}{(3!)^2}\left(\frac{q}{2}\right)^6+\dots,$$

$$R_2(q)=\frac{S(2)}{(2!)^2}\left(\frac{q}{2}\right)^4-\frac{S(4)}{(4!)^2}\left(\frac{q}{2}\right)^8+\dots,$$

$$S(n)=1+\frac{1}{2}+\dots+\frac{1}{n}, \quad \ln\gamma_1=0.57722.$$

From (46.12) and (46.10) follows

$$\left.\begin{array}{l}\mathrm{Re}\,I_2(x\sqrt{2i})=\frac{x^4}{24}-\frac{x^8}{6^2\cdot 8^2\cdot 5}+\dots,\\ \mathrm{Im}\,I_2(x\sqrt{2i})=\frac{x^2}{4}-\frac{x^6}{2^2\cdot 4^2\cdot 6}+\dots,\\ \mathrm{Re}\,H_2^{(1)}(x\sqrt{2i})=-\frac{2}{\pi x^2}-\frac{x^2}{2\pi}\ln\frac{\gamma_1 x}{\sqrt{2}}+\dots,\\ \mathrm{Im}\,H_2^{(1)}(x\sqrt{2i})=-\frac{1}{\pi}+\dots\end{array}\right| \qquad (46.13)$$

Expansion (46.12) is conveniently used for calculation of $\psi_i(q)$ functions and their derivatives $\psi_i'(q)$ at small values of the argument. At large values of q we have the handy asymptotic presentations

$$\left.\begin{array}{l}\psi_1(x\sqrt{2})\approx\frac{e^x}{\sqrt{2\pi x\sqrt{2}}}\left[\left(1+\frac{1}{16x}\right)\cos\left(x-\frac{\pi}{8}\right)+\right.\\ \qquad\left.+\frac{1}{16x}\sin\left(x-\frac{\pi}{8}\right)\right]=\frac{e^x}{\sqrt{2\pi x\sqrt{2}}}f_1(x),\\ \psi_2(x\sqrt{2})\approx\frac{-e^x}{\sqrt{2\pi x\sqrt{2}}}\left[-\frac{1}{16x}\cos\left(x-\frac{\pi}{8}\right)+\right.\\ \qquad\left.+\left(1+\frac{1}{16x}\right)\sin\left(x-\frac{\pi}{8}\right)\right]=\frac{-e^x}{\sqrt{2\pi x\sqrt{2}}}f_2(x),\end{array}\right| \qquad (46.14)$$

$$\psi_1'(x\sqrt{2}) \approx \frac{e^x}{\sqrt{2}\sqrt{2\pi x\sqrt{2}}}\left[\left(1-\frac{3}{8x}\right)\cos\left(x-\frac{\pi}{8}\right)-\right.$$
$$\left.-\sin\left(x-\frac{\pi}{8}\right)\right]=\frac{e^x}{\sqrt{2}\sqrt{2\pi x\sqrt{2}}}\tilde{f}_1(x),$$
$$\psi_2'(x\sqrt{2}) \approx \frac{-e^x}{\sqrt{2}\sqrt{2\pi x\sqrt{2}}}\left[\cos\left(x-\frac{\pi}{8}\right)+\right.$$
$$\left.+\left(1-\frac{3}{8x}\right)\sin\left(x-\frac{\pi}{8}\right)\right]=\frac{-e^x}{\sqrt{2}\sqrt{2\pi x\sqrt{2}}}\tilde{f}_2(x),$$
$$\psi_3(x\sqrt{2}) \approx \frac{e^{-x}}{\sqrt{\frac{\pi}{2}x\sqrt{2}}}\left[\left(1-\frac{1}{16x}\right)\sin\left(x+\frac{\pi}{8}\right)-\right.$$
$$\left.-\frac{1}{16x}\cos\left(x+\frac{\pi}{8}\right)\right]=\frac{e^{-x}}{\sqrt{\frac{\pi}{2}x\sqrt{2}}}f_3(x),$$
$$\psi_4(x\sqrt{2}) \approx \frac{-e^{-x}}{\sqrt{\frac{\pi}{2}x\sqrt{2}}}\left[\left(1-\frac{1}{16x}\right)\cos\left(x+\frac{\pi}{8}\right)+\right.$$
$$\left.+\frac{1}{16x}\sin\left(x+\frac{\pi}{8}\right)\right]=\frac{-e^{-x}}{\sqrt{\frac{\pi}{2}x\sqrt{2}}}f_4(x),$$
$$\psi_3'(x\sqrt{2}) \approx \frac{e^{-x}}{\sqrt{\pi x\sqrt{2}}}\left[\cos\left(x+\frac{\pi}{8}\right)-\right.$$
$$\left.-\left(1+\frac{3}{8x}\right)\sin\left(x+\frac{\pi}{8}\right)\right]=\frac{e^{-x}}{\sqrt{\pi x\sqrt{2}}}\tilde{f}_3(x),$$
$$\psi_4'(x\sqrt{2}) \approx \frac{e^{-x}}{\sqrt{\pi x\sqrt{2}}}\left[\left(1+\frac{3}{8x}\right)\cos\left(x+\frac{\pi}{8}\right)+\right.$$
$$\left.+\sin\left(x+\frac{\pi}{8}\right)\right]=\frac{e^{-x}}{\sqrt{\pi x\sqrt{2}}}\tilde{f}_4(x).$$

(46.14)
(Cont'd)

$$\psi_1(x\sqrt{2})+\frac{\sqrt{2}}{x}\psi_2'(x\sqrt{2}) \approx$$
$$\approx \frac{e^x}{\sqrt{2\pi x\sqrt{2}}}\left[\left(1-\frac{15}{16x}\right)\cos\left(x-\frac{\pi}{8}\right)-\frac{15}{16x}\sin\left(x-\frac{\pi}{8}\right)\right]=$$
$$=\frac{e^x}{\sqrt{2\pi x\sqrt{2}}}a_1(x),$$
$$\psi_2(x\sqrt{2})-\frac{\sqrt{2}}{x}\psi_1'(x\sqrt{2}) \approx$$
$$\approx \frac{-e^x}{\sqrt{2\pi x\sqrt{2}}}\left[\left(1-\frac{15}{16x}\right)\sin\left(x-\frac{\pi}{8}\right)+\frac{15}{16x}\cos\left(x-\frac{\pi}{8}\right)\right]=$$
$$=\frac{-e^x}{\sqrt{2\pi x\sqrt{2}}}b_1(x),$$
$$\psi_3(x\sqrt{2})+\frac{\sqrt{2}}{x}\psi_4'(x\sqrt{2}) \approx$$
$$\approx \frac{e^{-x}}{\sqrt{\frac{\pi}{2}x\sqrt{2}}}\left[\left(1+\frac{15}{16x}\right)\sin\left(x+\frac{\pi}{8}\right)+\frac{15}{16x}\cos\left(x+\frac{\pi}{8}\right)\right]=$$
$$=\frac{e^{-x}}{\sqrt{\frac{\pi}{2}x\sqrt{2}}}a_2(x),$$

(46.15)

$$
\left.\begin{aligned}
&\psi_4(x\sqrt{2}) - \frac{\sqrt{2}}{x}\psi_3'(x\sqrt{2}) \approx \\
&\approx \frac{-e^{-x}}{\sqrt{\frac{\pi}{2}x\sqrt{2}}}\left[\left(1+\frac{15}{16x}\right)\cos\left(x-\frac{\pi}{8}\right)-\frac{15}{16x}\sin\left(x+\frac{\pi}{8}\right)\right]\approx \\
&\qquad\qquad\approx \frac{-e^{-x}}{\sqrt{\frac{\pi}{2}x\sqrt{2}}}\, b_2(x).
\end{aligned}\right| \qquad \text{(46.15) (Cont'd)}
$$

Finally, taking into account (46.3), (46.7), the solution of uniform equation (46.2) can be written in the following manner:

$$
\sigma = (A_1 - iB_1)\, I_2(x\sqrt{2i}) + (A_2 - iB_2)\, H_2^{(1)}(x\sqrt{2i}). \qquad (46.16)
$$

where $A_1 - iB_1$, $A_2 - iB_2$ — several new imaginary constants introduced instead of c_1, c_2. Using formulas (46.10), (46.15), (46.3), (46.7), it is easy to see that solutions of the uniform system of equations of a conical shell (just as solutions of the uniform equation for a cylindrical shell examined in Chapter III) are solutions of the edge effect type. At large values of x, when asymptotic representations (46.14)-(46.15) are valid, $I_2(x\sqrt{2i})$ decreases with a decrease of x, while $H_2^{(1)}(x\sqrt{2i})$ decreases with an increase of x. Since in accordance with (46.5) $x_1 > x_0 (\nu_1 > \nu_0)$, this means that $I_2(x\sqrt{2i})$ describes the stressed state in the neighborhood of edge $\nu = \nu_1 \left(x_1 = 2\gamma\frac{\sqrt{\cos\beta}}{\sin\beta}\right)$ and decreases in proportion to the distance from it; and solution $H_2^{(1)}(x\sqrt{2i})$ describes the stressed state of edge $\nu = \nu_0 \left(x_0 = 2\gamma\frac{\sqrt{\cos\beta}}{\sin\beta}\sqrt{\frac{\nu_0}{\nu_1}}\right)$ and decreases in proportion to the advancement from this edge.

The general solution of system of equations (44.3) is made up of solution (46.16) and particular solution (45.5) and has the form

$$
\sigma = (A_1 - iB_1)\, I_2(x\sqrt{2i}) + (A_2 - iB_2)\, H_2^{(1)}(x\sqrt{2i}) + \tilde{\sigma}. \qquad (46.17)
$$

where

$$
\tilde{\sigma} = \Psi_0 - 2i\gamma^2 V_0.
$$

§ 47. Conical Shell with Concentrated Force in the Vertex

Let us examine the case when the inner edge degenerates into a point ($v_0=0$). The middle surface of such a shell is a cone with an angle. Let us propose from the beginning that there is no concentrated force in the angle of the cone ($P_z^0=0$). It is natural that all forces and bending moments in the angle of the cone should have limited values. Returning to formulas for the particular solution derived in § 45, we conclude that $\tilde{\sigma}$ in this instance is limited everywhere, including the angle of the cone. In this way it remains to require only boundedness of the solution of the uniform equation. Since $H_2^{(1)}(x\sqrt{2i})$ has a singularity in the angle, this requirement can be completed only by setting $A_2-iB_2=0$. The general solution of equations (44.3) in this instance should have the form

$$\sigma=(A_1-iB_1)I_2(x\sqrt{2i})+\tilde{\sigma}. \qquad (47.1)$$

The values of constants A_1, B_1 are determined according to the edge conditions at edge $x=x_1$. If x_1 is large enough that the values of $I_2(x\sqrt{2i})$ in the neighborhood of the edge can be determined according to asymptotic formulas (46.15), then edge effect fades from the edge of the shell to the angle and the stressed state in the neighborhood of the angle is practically undistinguishable from zero-moment. If, however, x_1 is small, then $I_2(x\sqrt{2i})$ must be calculated, using the representations of (46.12) (or the corresponding tables). The fading effect here no longer takes place, and the stressed state in such a conical shell in this sense resembles the stressed state in a flat plate.

Let us propose now that $P_z^0\neq 0$. In the absence of distributed loads the particular solution is written in the following manner:

$$\tilde{\sigma}=-\frac{P_z^0}{2\pi}\frac{\sin\beta}{\cos^2\beta}\frac{1}{v}-2i\gamma^2\frac{P_z^0}{2\pi}\frac{\sin\beta}{\cos\beta}\frac{1}{v_1}. \qquad (47.2)$$

Using general solution (46.17), we pick up constants A_2, B_2 so that σ remains finite in the angle. This can be attained by the choice of constants, since $H_2^{(1)}(x\sqrt{2i})$ has in the angle a singularity of the same form as $\tilde{\sigma}$, namely:

$$-\frac{2}{\pi x^2} = -\frac{2}{\pi}\frac{\sin^2\beta}{\cos\beta}\frac{\nu_1}{4\gamma^2}\frac{1}{\nu}.$$

Requiring that in (46.17) the sum of terms containing $\frac{1}{\nu}$, be zero, we find

$$\left.\begin{aligned} A_2 &= -P_z^0\frac{\gamma^2}{\nu_1}\frac{1}{\sin\beta\cos\beta} = -P_z^0\frac{\sqrt{3(1-\mu^2)}}{h}\frac{1}{\sin\beta\cos\beta}, \\ B_2 &= 0 \end{aligned}\right\} \tag{47.3}$$

and

$$\begin{aligned} \sigma = (A_1 - iB_1)I_2(x\sqrt{2i}) - P_z^0\frac{\gamma^2}{\nu_1\sin\beta\cos\beta}H_2^{(1)}(x\sqrt{2i}) - \\ -\frac{P_z^0}{2\pi}\frac{\sin\beta}{\cos^2\beta}\frac{1}{\nu} - 2i\gamma^2\frac{P_z^0}{2\pi}\frac{\sin\beta}{\cos\beta}\frac{1}{\nu_1}. \end{aligned} \tag{47.4}$$

At $x_1 \gg 1$ the influence of the edge can be neglected and the stressed state in the neighborhood of the angle can be determined by setting in (47.4) $A_1 = B_1 = 0$. Then we will have

$$\left.\begin{aligned} \Psi_0 &= P_z^0\frac{\gamma^2}{\nu_1\sin\beta\cos\beta}\left[\psi_3(x\sqrt{2}) + \frac{\sqrt{2}}{x}\psi_4'(x\sqrt{2})\right] - \frac{P_z^0}{2\pi}\frac{\sin\beta}{\cos^2\beta}\frac{1}{\nu}, \\ V_0 &= -P_z^0\frac{1}{2\nu_1\sin\beta\cos\beta}\left[\psi_4(x\sqrt{2}) - \frac{\sqrt{2}}{x}\psi_3'(x\sqrt{2})\right] + \frac{P_z^0}{2\pi}\frac{\sin\beta}{\nu_1\cos\beta}. \end{aligned}\right\} \tag{47.5}$$

On the basis of (44.6), (44.7), (46.3), (46.11) we obtain

$$\left.\begin{aligned} \nu T_1 &= -\frac{P_z^0}{2\cos\beta}\left[\psi_4(x\sqrt{2}) - \frac{\sqrt{2}}{x}\psi_3'(x\sqrt{2})\right] + \frac{P_z^0}{2\pi}\frac{1}{\cos\beta}, \\ \nu N_1 &= -\frac{P_z^0}{2\sin\beta}\left[\psi_4(x\sqrt{2}) - \frac{\sqrt{2}}{x}\psi_3'(x\sqrt{2})\right], \\ T_2 &= -P_z^0\frac{\gamma^2\sqrt{2}}{\nu_1\sin^2\beta}\frac{1}{x}\left[\psi_4'(x\sqrt{2}) + \right. \\ &\qquad \left. + \frac{2}{x^2}\psi_3'(x\sqrt{2}) - \frac{\sqrt{2}}{x}\psi_4(x\sqrt{2})\right]. \end{aligned}\right\} \tag{47.6}$$

$$M_1 = -\frac{P_z^0}{\sin^2\beta x\sqrt{2}}\left\{\psi_3'(x\sqrt{2}) - (1-\mu)\frac{\sqrt{2}}{x}\left[\psi_3(x\sqrt{2}) + \frac{\sqrt{2}}{x}\psi_4'(x\sqrt{2})\right]\right\} - (1-\mu)\frac{P_z^0}{2\pi}\frac{\sin\beta}{\cos^2\beta}\frac{1}{4\gamma^4}\frac{\nu_1^2}{\nu^2}. \quad (47.6) \text{ (Cont'd)}$$

Taking into account representations (46.12), it is simple to be convinced that T_1, T_2 have in the angle of the shell finite values, and the behavior of N_1 and M_1 in the immediate vicinity of point $\nu=0$ is described by the formulas

$$\left.\begin{aligned} N_1 &= -\frac{P_z^0}{2\pi\nu\sin\beta} + \dots, \\ M_1 &= -\frac{P_z^0}{\sin^2\beta}\frac{(1+\mu)}{2\pi}\ln q + \dots = -\frac{P_z^0}{\sin^2\beta}\frac{(1+\mu)}{4\pi}\ln\nu + \dots, \end{aligned}\right\} \quad (47.7)$$

where the points designate terms bounded at $\nu=0$. At $\beta=\frac{\pi}{2}$ the cone turns into a flat plate and expressions (47.7) coincide with analogous expressions for an infinite flat plate loaded by a concentrated force.

§ 48. Truncated Conical Shell

Let us examine now a shell whose middle surface is a frustum of a cone (Fig. 23). On the basis of general solution (46.17) and formulas (44.6), (44.7) we will derive the expressions for forces, bending moments and displacements Δ_e:

$$\left.\begin{aligned} \Psi_0 &= -A_1\left(\psi_1 + \frac{\sqrt{2}}{x}\psi_2'\right) - B_1\left(\psi_2 - \frac{\sqrt{2}}{x}\psi_1'\right) - \\ &\quad - A_2\left(\psi_3 + \frac{\sqrt{2}}{x}\psi_4'\right) - B_2\left(\psi_4 - \frac{\sqrt{2}}{x}\psi_3'\right) + \tilde{\Psi}_0, \\ 2\gamma^2 V_0 &= A_1\left(\psi_2 - \frac{\sqrt{2}}{x}\psi_1'\right) - B_1\left(\psi_1 + \frac{\sqrt{2}}{x}\psi_2'\right) + \\ &\quad + A_2\left(\psi_4 - \frac{\sqrt{2}}{x}\psi_3'\right) - B_2\left(\psi_3 + \frac{\sqrt{2}}{x}\psi_4'\right) + 2\gamma^2\tilde{V}_0. \end{aligned}\right\} \quad (48.1)$$

$$\left.\begin{aligned} T_1 &= \frac{2\operatorname{ctg}\beta}{x^2}\left[A_1\left(\psi_2 - \frac{\sqrt{2}}{x}\psi_1'\right) - B_1\left(\psi_1 + \frac{\sqrt{2}}{x}\psi_2'\right) + \right. \\ &\quad \left. + A_2\left(\psi_4 - \frac{\sqrt{2}}{x}\psi_3'\right) - B_2\left(\psi_3 + \frac{\sqrt{2}}{x}\psi_4'\right)\right] + \tilde{T}_1. \\ N_1 &= \frac{2\operatorname{ctg}^2\beta}{x^2}\left[A_1\left(\psi_2 - \frac{\sqrt{2}}{x}\psi_1'\right) - B_1\left(\psi_1 + \frac{\sqrt{2}}{x}\psi_2'\right) + \right. \\ &\quad \left. + A_2\left(\psi_4 - \frac{\sqrt{2}}{x}\psi_3'\right) - B_2\left(\psi_3 + \frac{\sqrt{2}}{x}\psi_4'\right)\right]. \\ H_e &= T_1\sin\beta + N_1\cos\beta. \end{aligned}\right\} \quad (48.2)$$

$$T_2 = \frac{2\,\mathrm{ctg}\,\beta}{x\sqrt{2}}\Big[A_1\Big(\psi_2' + \frac{2}{x^2}\psi_1' - \frac{\sqrt{2}}{x}\psi_2\Big) -$$

$$- B_1\Big(\psi_1' - \frac{2}{x^2}\psi_2' - \frac{\sqrt{2}}{x}\psi_1\Big) + A_2\Big(\psi_4' + \frac{2}{x^2}\psi_3' - \frac{\sqrt{2}}{x}\psi_4\Big) -$$

$$- B_2\Big(\psi_3' - \frac{2}{x^2}\psi_4' - \frac{\sqrt{2}}{x}\psi_3\Big)\Big] + \bar{T}_2,$$

$$M_1 = \frac{h}{\sqrt{3(1-\mu^2)}}\frac{\mathrm{ctg}\,\beta}{x\sqrt{2}}\Big\{A_1\Big[\psi_1' - \frac{\sqrt{2}}{x}(1-\mu)\Big(\psi_1 + \frac{\sqrt{2}}{x}\psi_2'\Big)\Big] +$$

$$+ B_1\Big[\psi_2' - \frac{\sqrt{2}}{x}(1-\mu)\Big(\psi_2 - \frac{\sqrt{2}}{x}\psi_1'\Big)\Big] +$$

$$+ A_2\Big[\psi_3' - \frac{\sqrt{2}}{x}(1-\mu)\Big(\psi_3 + \frac{\sqrt{2}}{x}\psi_4'\Big)\Big] +$$

$$+ B_2\Big[\psi_4' - \frac{\sqrt{2}}{x}(1-\mu)\Big(\psi_4 - \frac{\sqrt{2}}{x}\psi_3'\Big)\Big]\Big\} + \bar{M}_1. \qquad \text{(48.2) (Cont'd)}$$

$$\Delta_e = \frac{x\sqrt{2}\sin\beta}{4E\sqrt{3(1-\mu^2)}}\Big\{A_1\Big[\psi_2' - (1+\mu)\frac{\sqrt{2}}{x}\Big(\psi_2 - \frac{\sqrt{2}}{x}\psi_1'\Big)\Big] -$$

$$- B_1\Big[\psi_1' - (1+\mu)\frac{\sqrt{2}}{x}\Big(\psi_1 + \frac{\sqrt{2}}{x}\psi_2'\Big)\Big] +$$

$$+ A_2\Big[\psi_4' - (1+\mu)\frac{\sqrt{2}}{x}\Big(\psi_4 - \frac{\sqrt{2}}{x}\psi_3'\Big)\Big] -$$

$$- B_2\Big[\psi_3' - \frac{\sqrt{2}}{x}(1+\mu)\Big(\psi_3 + \frac{\sqrt{2}}{x}\psi_4'\Big)\Big]\Big\} + \bar{\Delta}_e. \qquad (48.3)$$

In these equations ψ_i, ψ_i' designate functions from the argument $x\sqrt{2}$.

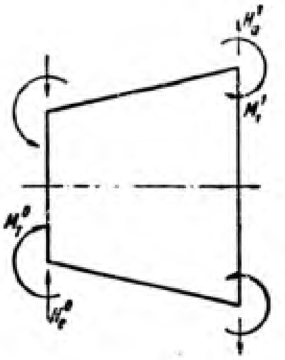


Fig. 23. Frustum of a cone.

The calculation of the shell becomes comparatively simple if the form and dimensions are such that in formulas (48.2) functions ψ_i, ψ_i' can be calculated using asymptotic representations (46.14), (46.15). We will show that even for a flat cone, close in shape to a plate (β is near $\frac{\pi}{2}$), this is admissible if the ratio $\frac{v}{h}$ is large enough. Transform (46.5) to the form

$$\frac{x\sqrt{2}}{2\sqrt[4]{12(1-\mu^2)}} = \frac{\sqrt{\cos\beta}}{\sin\beta}\sqrt{\frac{v}{h}}.$$

The application of formulas (46.14), (46.15) is admissible beginning from the value of the argument equal to six. Setting $x_0\sqrt{2}=6$, $1-\mu^2=0.9$, we find the necessary condition which should be satisfied by $\frac{\nu_0}{h}$ at different β in order that the application of approximation formulas (46.14), (46.15) is justified:

$$\sqrt{\frac{\nu_0}{h}} \geqslant 1.65\frac{\sin\beta}{\sqrt{\cos\beta}}. \tag{48.4}$$

Here follows a table of values $\sqrt{\frac{\nu_0}{h}}$, corresponding to the sign of the equality in (48.4) [12].

Table 5.

β	15°	30°	45°	60°	75°	80°	85°
$\sqrt{\frac{\nu_0}{h}}$	0,44	0,89	1,38	2,03	3,14	3,91	5,57

If condition (48.4) holds for the internal edge of a shell, then it will hold for the outer, since $\nu_1>\nu_0$. Furthermore, if the shell is such that $x_1-x_0\geqslant 3$, then constants A_1, B_1 and A_2, B_2, in solution (46.17) can be determined by ignoring the mutual influence of the edges.

Assuming that all these conditions have been made, we determine the stressed state of edge $x=x_1$, if thrust forces H_e^1 and moments M_1^1 are applied. Distributed loads and axial forces are absent.

Using formulas (48.2) and (46.14), (46.15), we make up the expressions for thrust $H_e=N_1\cos\beta+T_1\sin\beta$ and bending moment M_1 at edge x_1 setting in this case $A_2=B_2=0$. We will have

$$\left.\begin{aligned} &-\frac{2}{x_1^2}\frac{\cos\beta}{\sin^2\beta}\frac{e^{x_1}}{\sqrt{2\pi x_1\sqrt{2}}}[A_1b_1(x_1)+B_1a_1(x_1)]=H_e^1,\\ &\frac{\nu_1}{\gamma^2}\operatorname{ctg}\beta\frac{e^{x_1}}{2x_1\sqrt{2\pi x_1\sqrt{2}}}\left\{A_1\left[\tilde{f}_1(x_1)-\frac{2(1-\mu)}{x_1}a_1(x_1)\right]-\right.\\ &\qquad\left.-B_1\left[\tilde{f}_2(x_1)-\frac{2(1-\mu)}{x_1}b_1(x_1)\right]\right\}=M_1^1.\end{aligned}\right\} \tag{48.5}$$

From these equations we determine constants A_1, B_1:

$$\left.\begin{aligned} A_1 &= -\frac{x_1\sqrt{2\pi x_1\sqrt{2}}\,\mathrm{tg}\,\beta e^{-x_1}}{\Delta(x_1)}\left\{\frac{x_1\sin\beta}{2}H_e^1\left[\tilde{f}_2(x_1)-\frac{2(1-\mu)}{x_1}b_1(x_1)\right]-\right.\\ &\qquad\left.-\frac{2\gamma^2}{\nu_1}M_1^1 a_1(x_1)\right\},\\ B_1 &= -\frac{x_1\sqrt{2\pi x_1\sqrt{2}}\,\mathrm{tg}\,\beta e^{-x_1}}{\Delta(x_1)}\left\{\frac{x_1\sin\beta}{2}H_e^1\left[\tilde{f}_1(x_1)-\frac{2(1-\mu)}{x_1}a_1(x_1)\right]+\right.\\ &\qquad\left.+\frac{2\gamma^2}{\nu_1}M_1^1 b_1(x_1)\right\}. \end{aligned}\right\} \tag{48.6}$$

where

$$\Delta(x_1)=1-\frac{3}{8x_1}-\frac{2(1-\mu)}{x_1}.$$

Knowing these constants, using formulas (48.2), (48.3), one can determine forces, moments and displacements for any value of x in the neighborhood of the involved edge. We will find the value of Ψ_0 and Δ_e at edge ν_1.

$$\left.\begin{aligned} Eh\vartheta_1^1 = \Psi_0^1 &= \frac{x_1^2\sin^2\beta}{2\cos\beta}H_e^1 g_1(x_1)-\frac{2x_1\sqrt{3(1-\mu^2)}}{h}\,\mathrm{tg}\,\beta M_1^1 g_1(x_1),\\ \Delta_e^1 &= \frac{x_1^3\sin^3\beta}{4E\sqrt{3(1-\mu^2)}\cos\beta}H_e^1 g_2(x_1)-\frac{x_1^2\sin^2\beta}{2Eh\cos\beta}M_1^1 g_1(x_1). \end{aligned}\right\} \tag{48.7}$$

where

$$g_1(x_1)=\frac{1-\dfrac{15}{8x_1}}{1-\dfrac{3}{8x_1}-\dfrac{2(1-\mu)}{x_1}}\approx 1+\left(\frac{1}{2}-2\mu\right)\frac{1}{x_1},$$

$$g_2(x_1)=\frac{1-\dfrac{19}{8x_1}}{1-\dfrac{3}{8x_1}-\dfrac{2(1-\mu)}{x_1}}\approx 1-\frac{2\mu}{x_1}.$$

Taking into consideration that in accordance with (46.5)

$$x_1=2\gamma\frac{\sqrt{\cos\beta}}{\sin\beta}.$$

rewrite equations (48.7) again. We obtain

$$\left.\begin{aligned}\vartheta_1^1 &= \frac{2\gamma^2}{Eh} H_e^1 g_1(x_1) - \frac{4\gamma^3}{Eh\nu_1\sqrt{\cos\beta}} M_1^1 g_1(x_1),\\ \Delta_e^1 &= \frac{2\gamma\nu_1\sqrt{\cos\beta}}{Eh} H_e^1 g_2(x_1) - \frac{2\gamma^2}{Eh} M_1^1 g_1(x_1).\end{aligned}\right\} \quad (48.8)$$

Formulas (48.8) are similar to formulas (27.21) for a cylindrical shell. At $\beta=0$ (48.8) and (27.21) they coincide. In the presence of distributed loads or axial tensile force one ougth to take into account also the particular solution and write the obtained equations in the form

$$\left.\begin{aligned}\vartheta_1^1-\tilde{\vartheta}_1^1 &= \frac{2\gamma^2}{Eh}(H_e^1-\tilde{H}_e^1)\, g_1(x_1) - \frac{4\gamma^3}{Eh\nu_1\sqrt{\cos\beta}}(M_1^1-\tilde{M}_1^1)\, g_1(x_1),\\ \Delta_e^1-\tilde{\Delta}_e^1 &= \frac{2\gamma\nu_1\sqrt{\cos\beta}}{Eh}(H_e^1-\tilde{H}_e^1)\, g_2(x_1) - \frac{2\gamma^2}{Eh}(M_1^1-\tilde{M}_1^1)\, g_1(x_1).\end{aligned}\right\} \quad (48.9)$$

where $\tilde{H}_e=\tilde{T}_1\sin\beta$.

It is simple to make analogous calculations for edge $x=x_0$. Namely, dropping in formulas (48.2) terms containing A_1, B_1, and making up expressions for thrust H_e^0 and bending moment M_1^0, we obtain the equations for determination of constants A_2, B_2:

$$\left.\begin{aligned}&A_2 b_2(x_0) + B_2 a_2(x_0) = -H_e^0 \frac{x_0^2}{2}\frac{\sin^2\beta}{\cos\beta} e^{x_0}\sqrt{\frac{\pi}{2}x_0\sqrt{2}},\\ &A_2\left[\tilde{f}_3(x_0) - \frac{2(1-\mu)}{x_0}a_2(x_0)\right] + B_2\left[\tilde{f}_4(x_0) + \frac{2(1-\mu)}{x_0}b_2(x_0)\right] =\\ &\qquad = \frac{\gamma^2}{\nu_1}x_0\sqrt{2}\,\mathrm{tg}\,\beta\, e^{x_0}\sqrt{\pi x_0\sqrt{2}}\, M_1^0.\end{aligned}\right\} \quad (48.10)$$

Hence we find

$$\left.\begin{aligned}A_2 &= \frac{e^{x_0}\sqrt{\pi x_0\sqrt{2}}}{\Delta(x_0)}\left\{-H_e^0\frac{x_0^2\sin^2\beta}{2\sqrt{2}\cos\beta}\left[\tilde{f}_4(x_0) + \frac{2(1-\mu)}{x_0}b_2(x_0)\right] - \right.\\ &\qquad\left. - M_1^0\frac{\gamma^2}{\nu_1}x_0\sqrt{2}\,\mathrm{tg}\,\beta\, a_2(x_0)\right\},\\ B_2 &= \frac{e^{x_0}\sqrt{\pi x_0\sqrt{2}}}{\Delta(x_0)}\left\{H_e^0\frac{x_0^2\sin^2\beta}{2\sqrt{2}\cos\beta}\left[\tilde{f}_3(x_0) - \frac{2(1-\mu)}{x_0}a_2(x_0)\right] + \right.\\ &\qquad\left. + M_1^0\frac{\gamma^2}{\nu_1}x_0\sqrt{2}\,\mathrm{tg}\,\beta\, b_2(x_0)\right\}.\end{aligned}\right\} \quad (48.11)$$

where

$$\Delta(x_0)=1+\frac{3}{8x_0}+\frac{2(1-\mu)}{x_0}.$$

Formulas for calculation of the angle of rotation and radial displacement of edge x_0 have the form

$$\left.\begin{aligned}\theta_1^0&=\frac{2\gamma^2}{Eh}\frac{\nu_0}{\nu_1}H_c^0 g_3(x_0)+\frac{4\gamma^3}{Eh\nu_1\sqrt{\cos\beta}}\sqrt{\frac{\nu_0}{\nu_1}}g_3(x_0)M_1^0,\\ \Delta_c^0&=-\frac{2\gamma\nu_0\sqrt{\cos\beta}}{Eh}\sqrt{\frac{\nu_0}{\nu_1}}g_4(x_0)H_c^0-\frac{2\gamma^2}{Eh}\frac{\nu_0}{\nu_1}g_3(x_0)M_1^0,\end{aligned}\right\} \tag{48.12}$$

where

$$g_3(x_0)=\frac{1+\frac{15}{8x_0}}{1+\frac{3}{8x_0}+\frac{2(1-\mu)}{x_0}},\quad g_4(x_0)=\frac{1+\frac{19}{8x_0}}{1+\frac{3}{8x_0}+\frac{2(1-\mu)}{x_0}}.$$

§ 49. Combining a Cone and Cylinder

As an example of the application of formulas obtained in the previous section we will determine the radial force and bonding moment acting at the junction of a long cylindrical shell with a conical bottom (Fig. 24). The shell is loaded by internal pressure of intensity p. Since $\nu_1=R$, then

$$2\gamma_n^2=2\gamma_k^2=2\gamma^2=\sqrt{12(1-\mu^2)}\frac{R}{h}.$$

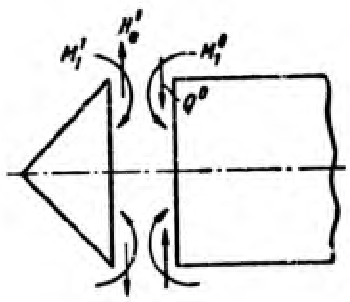


Fig. 24. Junction of a cone and cylinder.

For a cylindrical shell at the place of coupling we have

$$\left.\begin{aligned}\vartheta_1^0 &= \frac{2\gamma^2}{Eh} Q^0 + \frac{4\gamma^3}{EhR} M_1^0,\\ \Delta_e^0 - \frac{pR^2}{Eh}\left(1 - \frac{\mu}{2}\right) &= \\ = -\frac{2\gamma R}{Eh} Q^0 &- \frac{2\gamma^2}{Eh} M_1^0.\end{aligned}\right\} \qquad (49.1)$$

The connection between ϑ_1^1, Δ_e^1 and H_e^1, M_1^1 at rather large values of $x_1\sqrt{2}$ is determined by formulas (48.9), which for a given concrete load (uniform internal pressure) assume the form

$$\left.\begin{aligned}\vartheta_1^1 - \frac{3}{2}\frac{p\nu_1}{Eh}\frac{\sin\beta}{\cos^2\beta} &= \frac{2\gamma^2}{Eh}\left(H_e^1 - \frac{p\nu_1}{2}\operatorname{tg}\beta\right) g_1(x_1) - \\ &- \frac{4\gamma^3}{Eh\nu_1\sqrt{\cos\beta}} g_1(x_1)\left[M_1^1 + \frac{1}{4\gamma^4}\frac{3}{2}(1+\mu)p\nu_1^2\operatorname{tg}^2\beta\right],\\ \Delta_e^1 - \left(1 - \frac{\mu}{2}\right)\frac{p\nu_1^2}{Eh\cos\beta} &= \frac{2\gamma\nu_1\sqrt{\cos\beta}}{Eh}\left(H_e^1 - \frac{p\nu_1}{2}\operatorname{tg}\beta\right) g_2(x_1) - \\ &- \frac{2\gamma^2}{Eh} g_1(x_1)\left[M_1^1 + \frac{1}{4\gamma^4}\frac{3}{2}(1+\mu)p\nu_1^2\operatorname{tg}^2\beta\right].\end{aligned}\right\} \qquad (49.2)$$

At the junction of the shells displacements and forces should be continuous

$$\vartheta_1^1 = \vartheta_1^0, \quad \Delta_e^1 = \Delta_e^0, \quad H_e^1 = Q^0, \quad M_1^1 = M_1^0. \qquad (49.3)$$

Using these conditions and (49.1), (49.2), we derive two equations for determination of Q^0, M_1^0:

$$\left.\begin{aligned}&\frac{2\gamma^2}{Eh}[1 - g_1(x_1)]Q^0 + \frac{4\gamma^3}{EhR}\left[1 + \frac{g_1(x_1)}{\sqrt{\cos\beta}}\right]M_1^0 = \\ &= -\frac{2\gamma^2}{Eh}\frac{pR}{2}\operatorname{tg}\beta\, g_1(x_1) + \frac{3}{2}\frac{pR}{Eh}\frac{\sin\beta}{\cos^2\beta} - \frac{pR}{Eh}\frac{3(1+\mu)}{2\gamma\sqrt{\cos\beta}}\operatorname{tg}^2\beta\, g_1(x_1),\\ &-\frac{2\gamma R}{Eh}\left[1 + \sqrt{\cos\beta}\, g_2(x_1)\right]Q^0 - \frac{2\gamma^2}{Eh}[1 - g_1(x_1)]M_1^0 = \\ &= \frac{pR^2}{Eh}\left(1 - \frac{\mu}{2}\right)\left(\frac{1}{\cos\beta} - 1\right) - \frac{\gamma pR^2}{Eh}\sqrt{\cos\beta}\operatorname{tg}\beta\, g_2(x_1) - \\ &\qquad - \frac{pR^2}{Eh}\frac{3(1+\mu)}{4\gamma^2}\operatorname{tg}^2\beta\, g_1(x_1).\end{aligned}\right\} \qquad (49.4)$$

Dropping in the right side of the first equation the second and third terms and in the right side of the second equation the last term, as terms small in comparison with the remaining (as $\frac{1}{2\gamma^2}$ or $\frac{1}{\gamma^3}$ in comparison with unity), and setting, for simplicity of calculations,

$$\mu = 0{,}25, \quad g_1(x_1) = 1,$$

we find

$$\left.\begin{aligned} M_1^0 &= -\frac{pR^2}{4\gamma}\,\mathrm{tg}\,\beta\,\frac{\sqrt{\cos\beta}}{1+\sqrt{\cos\beta}}, \\ Q^0 &= -\frac{7}{8}\,\frac{pR}{2\gamma}\,\frac{1-\cos\beta}{[1+\sqrt{\cos\beta}\,g_2(x_1)]\cos\beta} + \frac{pR}{2}\,\frac{\mathrm{tg}\,\beta\sqrt{\cos\beta}\,g_2(x_1)}{1+\sqrt{\cos\beta}\,g_2(x_1)}. \end{aligned}\right\} \quad (49.5)$$

Substituting the obtained expressions in the second formula of (49.1), we calculate radial displacement and peripheral force in the abutting section of cylindrical shell

$$\left.\begin{aligned} \Delta_e^0 &\approx -\frac{pR^2}{2Eh}\,\gamma\,\frac{\mathrm{tg}\,\beta\sqrt{\cos\beta}}{1+\sqrt{\cos\beta}}, \\ T_2^0 &\approx -\frac{pR}{2}\,\gamma\,\frac{\mathrm{tg}\,\beta\sqrt{\cos\beta}}{1+\sqrt{\cos\beta}}. \end{aligned}\right\} \quad (49.6)$$

In the calculations for simplicity we dropped terms of order $\sqrt{\frac{h}{R}}$ in comparison with unity. Calculating on the basis of (49.5) and (49.6) flexural and peripheral stresses and referring them to the nominal (where $\sigma_1^N = \frac{pR}{2h}$, $\sigma_2^N = \frac{pR}{h}$), we obtain

$$\left.\begin{aligned} \frac{\sigma_1^0}{\sigma_1^N} &= \mp 2{,}3\sqrt{\frac{R}{h}}\,\mathrm{tg}\,\beta\,\frac{\sqrt{\cos\beta}}{1+\sqrt{\cos\beta}}, \\ \frac{\sigma_2^0}{\sigma_2^N} &= -0{,}65\sqrt{\frac{R}{h}}\,\mathrm{tg}\,\beta\,\frac{\sqrt{\cos\beta}}{1+\sqrt{\cos\beta}}. \end{aligned}\right\} \quad (49.7)$$

Stresses at the junction depend on the angle of conicity of the bottom: at $\beta = 0$ the cone degenerates into a cylinder and $\sigma_1^0 = \sigma_2^0 = 0$. At $\beta \neq 0$ and large $\frac{R}{h}$ stresses can become large and considerably exceed, for example, flexural stresses in a rigidly fixed section of cylindrical shell (§ 28).

With the aid of formulas (49.2) it is simple to find the values of M_1^1 and H_e^1 in the edge section of a complete conical shell loaded by pressure p, with the following attachments of the edge:

a) fixed edge $\vartheta_1^1 = \Delta_e^1 = 0.$

b) supported edge $M_1^1 = \Delta_e^1 = 0.$

Dropping small terms and setting as before $\mu = 0.25$, $g_1(x_1) = 1$, $g_2(x_1) = 1 - \frac{1}{2x_1}$, we have for a fixed edge

$$\left.\begin{aligned} H_e^1 &= \frac{pv_1}{2}\operatorname{tg}\beta - \frac{7}{8}\frac{pv_1}{\gamma(\cos\beta)^{3/2}}\left(1+\frac{1}{x_1}\right), \\ M_1^1 &= -\frac{7}{16\gamma^2}\frac{pv_1^2}{\cos\beta}\left(1+\frac{1}{x_1}\right). \end{aligned}\right\} \tag{49.8}$$

for a supported edge

$$\left.\begin{aligned} H_e^1 &= \frac{pv_1}{2}\operatorname{tg}\beta - \frac{7}{16}\frac{pv_1}{\gamma(\cos\beta)^{3/2} g_2(x_1)}, \\ M_1^1 &= 0. \end{aligned}\right\} \tag{49.9}$$

Bending stresses at the edge of a fixed conical shell, computable according to bending moment (49.8), are equal to

$$\sigma_1 = \mp\frac{pv_1}{h}\frac{1.56}{\cos\beta}\left(1+\frac{1}{x_1}\right). \tag{49.10}$$

§ 50. Truncated Cone Compressed by Axial Forces

As a second illustration let us examine a truncated cone contained between two even plane surfaces, compressed by forces P [12]. The surfaces do not prevent radial displacement and turning in the edge sections, i.e.,

when $x = x_0$

$$H_e^0 = M_1^0 = 0. \tag{50.1}$$

when $x = x_1$

$$H_e^1 = M_1^1 = 0.$$

Determine displacements Δ_e^0, Δ_e^1 and turning angles ϑ_1^0, ϑ_1^1. the particular solution of the basic equations (44.3) for a given load we obtain using formulas (45.8), setting in them $P_e^0 = -P$, $q_e = q_z = 0$:

$$\left.\begin{aligned} \tilde{T}_1 &= -\frac{P}{2\pi\nu\cos\beta}, \quad \tilde{H} = -\frac{P}{2\pi\nu}\operatorname{tg}\beta, \\ \tilde{\vartheta}_1 &= \tilde{M}_1 = \tilde{T}_2 = 0, \quad \tilde{\Delta}_e = \frac{\mu}{Eh}\frac{P}{2\pi\cos\beta}. \end{aligned}\right\} \tag{50.2}$$

In formulas (48.8), (48.12) we replace

$$\left.\begin{aligned} H_e^1 \quad &\text{by} \quad H_e^1 + \frac{P\operatorname{tg}\beta}{2\pi\nu_1}, \\ \Delta_e^1 \quad &\text{by} \quad \Delta_e^1 - \frac{\mu}{Eh}\frac{P}{2\pi\cos\beta}, \\ \Delta_e^0 \quad &\text{by} \quad \Delta_e^0 - \frac{\mu}{Eh}\frac{P}{2\pi\cos\beta}, \\ H_e^0 \quad &\text{by} \quad H_e^0 + \frac{P\operatorname{tg}\beta}{2\pi\nu_0}, \end{aligned}\right\} \tag{50.3}$$

whereupon, taking into account boundary conditions (50.1), we obtain

$$\left.\begin{aligned} \vartheta_1^1 &= \frac{2\gamma^2}{Eh}\frac{P\operatorname{tg}\beta}{2\pi\nu_1} g_1(x_1), \\ \Delta_e^1 &= \frac{\mu}{Eh}\frac{P}{2\pi\cos\beta} + \frac{\gamma}{Eh}\frac{P\operatorname{tg}\beta\sqrt{\cos\beta}}{\pi} g_2(x_1), \end{aligned}\right\} \tag{50.4}$$

$$\left.\begin{aligned} \vartheta_1^0 &= \frac{2\gamma^2}{Eh}\frac{P\operatorname{tg}\beta}{2\pi\nu_0} g_3(x_0), \\ \Delta_e^0 &= \frac{\mu}{Eh}\frac{P}{2\pi\cos\beta} - \frac{\gamma}{Eh}\sqrt{\frac{\nu_0}{\nu_1}}\frac{P\operatorname{tg}\beta\sqrt{\cos\beta}}{\pi} g_4(x_0). \end{aligned}\right\} \tag{50.5}$$

Subsequently for calculation of displacements the quantities A_1, B_1, A_2, B_2 will be needed also. By formulas (48.6), (48.11), using in them also substitution (50.3) and taking into account (50.1), we obtain

$$\left.\begin{aligned} A_1 &= -\frac{\sqrt{2\pi x_1\sqrt{2}}}{\Delta(x_1)} e^{-x_1}\frac{\gamma^2 P\operatorname{tg}\beta}{\pi\nu_1}\left[\tilde{f}_2(x_1) - \frac{2(1-\mu)}{x_1} b_1(x_1)\right], \\ B_1 &= -\frac{\sqrt{2\pi x_1\sqrt{2}}}{\Delta(x_1)} e^{-x_1}\frac{\gamma^2 P\operatorname{tg}\beta}{\pi\nu_1}\left[\tilde{f}_1(x_1) - \frac{2(1-\mu)}{x_1} a_1(x_1)\right], \\ A_2 &= -\frac{\sqrt{\pi x_0\sqrt{2}}}{\Delta(x_0)} e^{x_0}\frac{\gamma^2 P\operatorname{tg}\beta}{\pi\nu_1\sqrt{2}}\left[\tilde{f}_4(x_0) + \frac{2(1-\mu)}{x_0} b_2(x_0)\right], \\ B_2 &= \frac{\sqrt{\pi x_0\sqrt{2}}}{\Delta(x_0)} e^{x_0}\frac{\gamma^2 P\operatorname{tg}\beta}{\pi\nu_1\sqrt{2}}\left[\tilde{f}_3(x_0) - \frac{2(1-\mu)}{x_0} a_2(x_0)\right]. \end{aligned}\right\} \tag{50.6}$$

Let us determine the values of forces T_1, T_2 and moments M_2 at the edges of the shell. Solving these equations

$$\left.\begin{aligned} T_1^0 \sin\beta + N_1^0 \cos\beta &= 0, \\ T_1^0 \cos\beta - N_1^0 \sin\beta &= -\frac{P}{2\pi v_0}, \end{aligned}\right\} \tag{50.7}$$

we find

$$T_1^0 = -\frac{P}{2\pi v_0}\cos\beta. \tag{50.8}$$

Analogously

$$T_1^1 = -\frac{P}{2\pi v_1}\cos\beta.$$

Forces T_2^0, T_2^1 are simple to find knowing the quantities T_1^0, T_1^1 and Δ_ϵ^0, Δ_ϵ^1. For example, using the formula

$$T_2^0 = \mu T_1^0 + \frac{Eh}{v_0}\Delta_\epsilon^0,$$

we obtain

$$T_2^0 = -\frac{P \operatorname{tg}\beta \sqrt{\cos\beta}}{\pi} \frac{\gamma}{\sqrt{v_0 v_1}} g_4(x_0). \tag{50.9}$$

In exactly the same manner we find that

$$T_2^1 = \frac{P \operatorname{tg}\beta \sqrt{\cos\beta}}{\pi} \frac{\gamma}{v_1} g_2(x_1). \tag{50.10}$$

We determine M_2^0, M_2^1 according to (44.7), taking into consideration that because of edge conditions (50.1) between the derivative of the angle of rotation $\frac{d\vartheta_1}{dv}$ and the value of ϑ_1 on the edges exists the relationship

$$\frac{d\vartheta_1}{dv} = -\mu\frac{\vartheta_1}{v}. \tag{50.11}$$

In this way the values of M_2^0, M_2^1 can be calculated by the formula

$$M_2 = -\frac{Eh^3}{12}\sin\beta\,\frac{\vartheta_1}{\nu} \qquad (\nu = \nu_0,\ \nu_1). \tag{50.12}$$

Substituting into (50.12) the values of ϑ_1 in accordance with (50.4), (50.5), we have

$$\left.\begin{aligned} M_2^0 &= -\frac{h^2}{6}\,\frac{\gamma^2}{\nu_0\nu_1}\,\frac{P\sin^2\beta}{2\pi\cos\beta}\,g_3(x_0),\\ M_2^1 &= -\frac{h^2}{6}\,\frac{\gamma^2}{\nu_1^2}\,\frac{P\sin^2\beta}{2\pi\cos\beta}\,g_1(x_1). \end{aligned}\right\} \tag{50.13}$$

Remembering that $\gamma^2 = \sqrt{3(1-\mu^2)}\,\frac{\nu_1}{h}$, we determine circumferential stresses from tension

$$\left.\begin{aligned} \sigma_2^0 &= \frac{T_2^0}{h} = -\frac{P\,\mathrm{tg}\,\beta\sqrt{\cos\beta}}{\pi\nu_0 h}\sqrt{\frac{\nu_0}{h}}\,\sqrt[4]{3(1-\mu^2)}\,g_4(x_0),\\ \sigma_2^1 &= \frac{T_2^1}{h} = \frac{P\,\mathrm{tg}\,\beta\sqrt{\cos\beta}}{\pi\nu_1 h}\sqrt{\frac{\nu_1}{h}}\,\sqrt[4]{3(1-\mu^2)}\,g_2(x_1) \end{aligned}\right\} \tag{50.14}$$

from the bending moment

$$\left.\begin{aligned} \sigma_2^0 &= \pm\frac{6M_2^0}{h^2} = \mp\sqrt{3(1-\mu^2)}\,\frac{P}{2\pi\nu_0 h}\,\frac{\sin^2\beta}{\cos\beta}\,g_3(x_0),\\ \sigma_2^1 &= \pm\frac{6M_1^1}{h^2} = \mp\sqrt{3(1-\mu^2)}\,\frac{P}{2\pi\nu_1 h}\,\frac{\sin^2\beta}{\cos\beta}\,g_1(x_1). \end{aligned}\right\} \tag{50.15}$$

It is easy to see that the stresses of tension have an order larger by an order of $\sqrt{\frac{\nu}{h}}$ bending stresses and they are in this case estimated.

Let us calculate the relative axial displacement of the edges of the truncated cone or, in other words, the sag of the truncated cone under the action of preassigned compressive forces P. In accordance with (13.16), if we set in them $R_1 d\theta = ds$, $\theta = \frac{\pi}{2} - \beta$, axial displacement in a certain instant section is equal to

$$\Delta_z = \int_{l_0}^{s} (-\varepsilon_1 \cos\beta + \vartheta_1 \sin\beta)\, ds + C. \quad (50.16)$$

Constant of integration $C=0$, assuming that edge section $s=l_0$ $(x=x_0)$ remains stationary. Then sag of the truncated cone δ in absolute value will be equal to the displacement of edge $s=l_1$ $(x=x_1)$. Or, taking into consideration that $ds=\frac{dv}{\sin\beta}$,

$$\delta = \Delta_z(x_1) = \int_{v_0}^{v_1} (-\varepsilon_1 \operatorname{ctg}\beta + \vartheta_1)\, dv. \quad (50.17)$$

Since on the basis of (44.4), (44.6)

$$\left.\begin{aligned} \Phi_1 &= -\frac{P}{2\pi}\cos\beta, \quad \Phi_2 = \frac{P}{2\pi}\sin\beta, \\ V_0\frac{v_1\sin\beta}{v} &= N_1 \operatorname{tg}\beta - \frac{P}{2\pi v}\frac{\sin^2\beta}{\cos\beta}, \end{aligned}\right\} \quad (50.18)$$

which can be written

$$\begin{aligned} \varepsilon_1 &= \frac{1}{Eh}(T_1 - \mu T_2) = \\ &= \frac{1}{Eh}\left[-\mu v_1 \sin\beta\,\frac{dV_0}{dv} - \frac{P}{2\pi v}\left(\frac{\sin^2\beta}{\cos\beta} + \cos\beta\right) + N_1 \operatorname{tg}\beta\right]. \end{aligned} \quad (50.19)$$

Angle of rotation

$$\vartheta_1 = \vartheta_1^* + \tilde{\vartheta}_1,$$

where ϑ_1^* corresponds to solution of uniform equation in the form of (46.16), and

$$\tilde{\vartheta}_1 = \frac{P}{2\pi Eh}\,\frac{\sin\beta}{\cos^2\beta}\,\frac{1}{v}.$$

Let us note that during the calculation of stress we used the zero-moment solution $\tilde{\vartheta}_1=0$. We can verify that use of the accurate solution would give in this instance only insignificatn corrections of the order of $\frac{h}{v}$ in comparison with the basic terms. However, while figuring the axial displacement, as will be explained below, such a change of the particular solution to zero-moment would entail errors in terms of order $\sqrt{\frac{h}{v}}$ in comparison with the basic, i.e., already in those terms, which thus far during calculations we have kept.

Thus, taking into account the above expressions for e_1 and ϑ_1, we obtain

$$\delta=\int_{v_0}^{v_1}\left[\frac{\mu v_1 \cos\beta}{Eh}\frac{dV_0}{dv}+\frac{P}{2\pi Eh}\frac{1}{v\sin\beta\cos^2\beta}+\vartheta_1^{*}-\frac{N_1}{Eh}\right]dv=$$
$$=\frac{\mu v_1\cos\beta}{Eh}\left[V_0(v_1)-V_0(v_0)\right]+\frac{P}{2\pi Eh\sin\beta\cos^2\beta}\ln\frac{v_1}{v_0}+$$
$$+\int_{v_0}^{v_1}\left(\vartheta_1^{*}-\frac{N_1}{Eh}\right)dv. \qquad (50.20)$$

Note that from (44.6) and (44.4) at $q_e=0$ it follows that $vH_e= vN_1\cos\beta+ + vT_1\sin\beta=V$. In this way, under the assigned edge conditions (50.1) there exist the equalities

$$V_0(v_1)=V_0(v_0)=0. \qquad (50.21)$$

The quantities ϑ_1^{*} and N_1 in accordance with (46.17) and (44.6) have the form

$$\left.\begin{aligned}\vartheta_1^{*}&=\frac{1}{Eh}\,\mathrm{Re}\left[(A_1-iB_1)I_2(x\sqrt{2i})+(A_2-iB_2)H_2^{(1)}(x\sqrt{2i})\right].\\ N_1&=-\frac{v_1}{v}\frac{\cos\beta}{2\gamma^2}\,\mathrm{Im}\left[(A_1-iB_1)I_2(x\sqrt{2i})+(A_2-iB_2)H_2^{(1)}(x\sqrt{2i})\right].\end{aligned}\right\} \qquad (50.22)$$

From the comparison of these two expressions it is easy to see that the term $\frac{N_1}{Eh}$ in subintegral expression (50.20) is negligible in comparison with ϑ_1^{*}. Therefore

$$\delta=\frac{P}{2\pi Eh}\frac{1}{\sin\beta\cos^2\beta}\ln\frac{v_1}{v_0}+\int_{v_0}^{v_1}\vartheta_1^{*}\,dv. \qquad (50.23)$$

Since

$$dv=\frac{v_1}{2\gamma^2}\frac{\sin^2\beta}{\cos\beta}\,x\,dx.$$

then for calculation of σ it is necessary to find the values of the definite integrals

$$\int_{x_0}^{x_1} J_2(x\sqrt{2i})\,x\,dx, \quad \int_{x_0}^{x_1} H_2^{(1)}(x\sqrt{2i})\,x\,dx.$$

Using formulas (46.8), (46.9) and noticing that

$$\int J_0 z\,dz = zJ_1(z) = -zJ_0'(z),$$
$$\int H_0^{(1)} z\,dz = -zH_0^{(1)\prime}(z),$$

we find

$$\int_{x_0}^{x_1} J_2(x\sqrt{2i})\,x\,dx = \Big\{\frac{x\sqrt{2}}{2}\Big[\psi_2'(x\sqrt{2}) - \frac{\sqrt{2}}{x}\psi_2(x\sqrt{2})\Big] - \frac{x\sqrt{2}}{2}\,i\Big[\psi_1'(x\sqrt{2}) - \frac{\sqrt{2}}{x}\psi_1(x\sqrt{2})\Big]\Big\}_{x_0}^{x_1},$$

$$\int_{x_0}^{x_1} H_2^{(1)}(x\sqrt{2i})\,x\,dx = \Big\{\frac{x\sqrt{2}}{2}\Big[\psi_4'(x\sqrt{2}) - \frac{\sqrt{2}}{x}\psi_4(x\sqrt{2})\Big] - \frac{x\sqrt{2}}{2}\,i\Big[\psi_3'(x\sqrt{2}) - \frac{\sqrt{2}}{x}\psi_3(x\sqrt{2})\Big]\Big\}_{x_0}^{x_1},$$

or, ignoring the mutual influence of the edges,

$$\left.\begin{aligned}\int_{x_0}^{x_1} J_2(x\sqrt{2i})\,x\,dx &= \frac{x_1\sqrt{2}}{2}\Big[\psi_2'(x_1\sqrt{2}) - \frac{\sqrt{2}}{x_1}\psi_2(x_1\sqrt{2})\Big] - \\ &\quad - i\,\frac{x_1\sqrt{2}}{2}\Big[\psi_1'(x_1\sqrt{2}) - \frac{\sqrt{2}}{x_1}\psi_1(x_1\sqrt{2})\Big],\\ \int_{x_0}^{x_1} H_2^{(1)}(x\sqrt{2i})\,x\,dx &= -\frac{x_0\sqrt{2}}{2}\Big[\psi_4'(x_0\sqrt{2}) - \frac{\sqrt{2}}{x_0}\psi_4(x_0\sqrt{2})\Big] + \\ &\quad + i\,\frac{x_0\sqrt{2}}{2}\Big[\psi_3'(x_0\sqrt{2}) - \frac{\sqrt{2}}{x_0}\psi_3(x_0\sqrt{2})\Big].\end{aligned}\right\} \qquad (50.24)$$

In this way, on the basis of (50.22), (50.24) we obtain

$$\int_{v_0}^{v_1} \vartheta_1^* dv = \frac{1}{Eh} \frac{v_1}{2\gamma^2} \frac{\sin^2\beta}{\cos\beta} \Big\{ A_1 \frac{x_1\sqrt{2}}{2} \left[\psi_2'(x_1\sqrt{2}) - \frac{\sqrt{2}}{x_1}\psi_2(x_1\sqrt{2})\right] -$$
$$- B_1 \frac{x_1\sqrt{2}}{2} \left[\psi_1'(x_1\sqrt{2}) - \frac{\sqrt{2}}{x_1}\psi_1(x_1\sqrt{2})\right] -$$
$$- A_2 \left[\psi_4'(x_0\sqrt{2}) - \frac{\sqrt{2}}{x_0}\psi_4(x_0\sqrt{2})\right] \frac{x_0\sqrt{2}}{2} +$$
$$+ B_2 \frac{x_0\sqrt{2}}{2} \left[\psi_3'(x_0\sqrt{2}) - \frac{\sqrt{2}}{x_0}\psi_3(x_0\sqrt{2})\right]\Big\}. \qquad (50.25)$$

Substituting into this expression the above found values of the constants A_1, B_1, A_2, B_2 (formulas (50.6)) and making the necessary calculations, we find

$$\int_{v_0}^{v_1} \vartheta_1^* dv = \frac{P\,\mathrm{tg}\,\beta}{2\pi Eh} \frac{\sin^2\beta}{\cos\beta}(x_0 + x_1). \qquad (50.26)$$

Finally by (50.23), (50.26) we have

$$\delta = \frac{P}{2\pi Eh \sin\beta\cos^2\beta}\Big[\ln\frac{v_1}{v_0} +$$
$$+ 2\sin^3\beta\sqrt{\cos\beta}\sqrt[4]{3(1-\mu^2)} \times$$
$$\times\left(\sqrt{\frac{v_1}{h}} + \sqrt{\frac{v_0}{h}}\right)\Big]. \qquad (50.27)$$

It is easy to see that the second term has an order greater by $\sqrt{\frac{v}{h}}$ than the first.

The obtained formula can be used to determine the total bend of a construction consisting of a set of cones placed on one another (Fig. 25), called a Bellville spring.

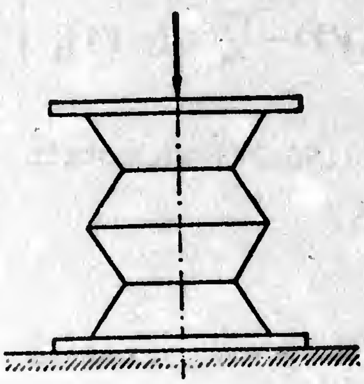


Fig. 25. Set of cones schematically representing a Bellville spring.

§ 51. Conical Compensator Gear

Let us examine a third illustration: the conical compensator (Fig. 26). It is two conic shells joined along the outer edges. It is necessary to determine the stresses and sag of the compensator under the action of compressive forces P and internal pressure.

Mentally separating one shell from the other, we replace the action of the lower truncated cone on the upper by radial forces H_e^1 and bending moments M_1^1 applied on the contour $\nu=\nu_1$. In this case, from considerations of symmetry at this edge we must set

$$H_e^1=0, \quad \vartheta_1^1=0. \qquad (51.1)$$

For simplicity we assume that on the internal contour $(\nu=\nu_0)$ are carried out the same conditions

$$H_e^0=0, \quad \vartheta_1^0=0. \qquad (51.2)$$

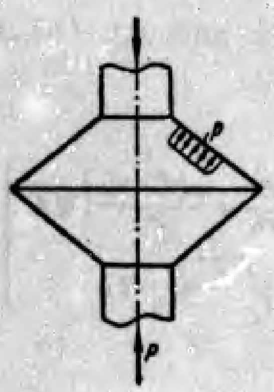


Fig. 26. Conic compensator.

Let us note that in actuality the conic plates of a compensator are coupled with a tube, which, moreover, can have a wall thickness not equal to the thickness of the plate; conditions (51.2) should be exchanged for coupling conditions with a cylindrical shell. Digressing from this fact, let us examine a conical shell loaded by forces $P_z=-P$ and by internal pressure of intensity p under edge conditions (51.1), (51.2).

The particular solution corresponding to this load has the form

$$\left.\begin{aligned}
\tilde{T}_1 &= -\frac{P}{2\pi\nu\cos\beta} + \frac{p(\nu^2-\nu_0^2)}{2\nu\cos\beta},\\
\tilde{H}_e &= \tilde{T}_1\sin\beta = \left[-\frac{P}{2\pi\nu} + \frac{p(\nu^2-\nu_0^2)}{2\nu}\right]\mathrm{tg}\,\beta,\\
\tilde{T}_2 &= \frac{p\nu}{\cos\beta}.
\end{aligned}\right\} \tag{51.3}$$

$$\left.\begin{aligned}
\tilde{\vartheta}_1 &= \frac{1}{Eh}\frac{\sin\beta}{\cos^2\beta}\left[\frac{P}{2\pi\nu} + \frac{3}{2}p\nu + \frac{p}{2}\frac{\nu_0^2}{\nu}\right],\\
\tilde{M}_1 &= -\frac{h^2}{12(1-\mu^2)}\mathrm{tg}^2\beta\left[-\frac{P(1-\mu)}{2\pi\nu^2} + \frac{3}{2}(1+\mu)p - \frac{(1-\mu)}{2}p\frac{\nu_0^2}{\nu^2}\right],\\
\tilde{\Delta}_e &= \frac{\mu}{Eh}\frac{P}{2\pi\cos\beta} + \frac{p}{2Eh\cos\beta}\left[(2-\mu)\nu^2 + \mu\nu_0^2\right].
\end{aligned}\right\} \tag{51.4}$$

In formulas (48.8), (48.12), (48.6), (48.11) we replace

$$\left.\begin{aligned}
&H_e^1 \text{ by } H_e^1 - \tilde{H}_e^1,\\
&\Delta_e^1 \text{ by } \Delta_e^1 - \tilde{\Delta}_e^1,\\
&H_e^0 \text{ by } H_e^0 - \tilde{H}_e^0,\\
&\Delta_e^0 \text{ by } \Delta_e^0 - \tilde{\Delta}_e^0
\end{aligned}\right\} \tag{51.5}$$

and set $H_e^1 = H_e^0 = 0$. In this case, taking into account (51.4) from (48.8), (48.12) we obtain

$$\left.\begin{aligned}
M_1^1 &= -\frac{\nu_1\sqrt{\cos\beta}}{2\gamma}\tilde{H}_e^1 = -\frac{\nu_1\sqrt{\cos\beta}}{2\gamma}\left[-\frac{P}{2\pi\nu_1} + \frac{p(\nu_1^2-\nu_0^2)}{2\nu_1}\right]\mathrm{tg}\,\beta,\\
M_1^0 &= \frac{\nu_1\sqrt{\cos\beta}}{2\gamma}\sqrt{\frac{\nu_0}{\nu}}\tilde{H}_e^0 = -\frac{\nu_1\sqrt{\cos\beta}}{2\gamma}\sqrt{\frac{\nu_0}{\nu_1}}\frac{P}{2\pi\nu_0}\mathrm{tg}\,\beta.
\end{aligned}\right\} \tag{51.6}$$

In calculation (51.6) the small terms are dropped, obliged for their origin to the quantities $\tilde{\vartheta}_1$, $\tilde{M}_1$.

Stress caused by moments (51.6) are equal to

$$\left.\begin{aligned}
\sigma_1^0 &= \pm\frac{6M_1^0}{h^2} = \mp\frac{3}{h^2}\frac{\sqrt{\cos\beta}}{\sqrt[4]{3(1-\mu^2)}}\sqrt{\frac{h}{\nu_0}}\frac{P}{2\pi}\mathrm{tg}\,\beta,\\
\sigma_1^1 &= \pm\frac{6M_1^1}{h^2} = \mp\frac{3}{h^2}\frac{\mathrm{tg}\,\beta\sqrt{\cos\beta}}{\sqrt[4]{3(1-\mu^2)}}\sqrt{\frac{h}{\nu_1}}\left[\frac{P}{2\pi} - \frac{p(\nu_1^2-\nu_0^2)}{2}\right].
\end{aligned}\right\} \tag{51.7}$$

Let us show that in the involved problem these stresses must be considered as the working stresses. To check let us calculate the stresses from peripheral force on the edge $x = x_0$. From the second equation of (48.12) we have

$$\Delta_e'' = \tilde{\Delta}_e'' + \frac{2\gamma\nu_0\sqrt{\cos\beta}}{Eh}\sqrt{\frac{\nu_0}{\nu_1}}\,\tilde{H}_e^0\left[g_4(x_0) - \frac{g_3(x_0)}{2}\right],$$

or, dropping quantities of order $1/\gamma$ in comparison with unity,

$$\Delta_e^0 = \frac{\gamma\nu_0\sqrt{\cos\beta}}{Eh}\sqrt{\frac{\nu_0}{\nu_1}}\,\tilde{H}_e^0. \tag{51.8}$$

Taking into consideration that

$$T_1^0 = -\frac{P}{2\pi\nu_0}\cos\beta,$$

on the basis of the relationship

$$\Delta_e^0 = \frac{\nu_0}{Eh}(T_2^0 - \mu T_1^0),$$

with the same correctness we obtain

$$T_2^0 = \gamma\sqrt{\cos\beta}\sqrt{\frac{\nu_0}{\nu_1}}\,\tilde{H}_e^0 \tag{51.9}$$

and

$$\sigma_2^0 = \frac{T_2^0}{h} = -\frac{\sqrt[4]{3(1-\mu^2)}}{h^2}\sqrt{\cos\beta}\,\mathrm{tg}\,\beta\sqrt{\frac{h}{\nu_0}}\,\frac{P}{2\pi}. \tag{51.10}$$

Comparing (51.7) and (51.10), we find maximum bending stresses are more than the peripheral compressive stresses by $\sqrt{\frac{3}{1-\mu^2}}$ times.

Let us go to the calculation of the sag of one conic plate. Since in this case

$$\left.\begin{aligned}\Phi_1 &= -\frac{P}{2\pi}\cos\beta,\\ \Phi_2 &= \frac{P}{2\pi}\sin\beta - \frac{p}{\sin\beta}\frac{v^2-v_0^2}{2},\end{aligned}\right\} \tag{51.11}$$

then using (44.6) in the same way as in the previous illustration we can write

$$\left.\begin{aligned}T_1 &= N_1\,\mathrm{tg}\,\beta - \frac{P}{2\pi v}\frac{1}{\cos\beta} + p\cos\beta\,\frac{v^2-v_0^2}{2v},\\ T_2 &= v_1\sin\beta\,\frac{dV_0}{dv},\\ -\varepsilon_1\,\mathrm{ctg}\,\beta &= -\frac{1}{Eh}N_1 + \frac{P}{2\pi v}\frac{1}{\sin\beta} - \frac{p\cos^2\beta}{\sin\beta}\frac{v^2-v_0^2}{2v} +\\ &\quad + \mu v_1\cos\beta\,\frac{dV_0}{dv}\end{aligned}\right\} \tag{51.12}$$

and

$$\begin{aligned}\delta &= \int_{v_0}^{v_1}\left[-\varepsilon_1\,\mathrm{ctg}\,\beta + (\vartheta_1^{\circ} + \tilde{\vartheta}_1)\right]dv \approx\\ &\approx \int_{v_0}^{v_1}\vartheta_1^{\circ}\,dv + \frac{P}{2\pi Eh\sin\beta\cos^2\beta}\ln\frac{v_1}{v_0} + \frac{p}{4Eh}\frac{3\sin^2\beta-\cos^2\beta}{\sin\beta\cos^2\beta}(v_1^2-v_0^2) +\\ &\quad + \frac{pv_0^2}{2Eh\sin\beta\cos^2\beta}\ln\frac{v_1}{v_0} + \frac{\mu v_1\cos\beta}{Eh}\left[V_0(v_1) - V_0(v_0)\right].\end{aligned} \tag{51.13}$$

Taking into consideration that

$$vH_e = v_1V_0 - p\,\mathrm{ctg}\,\beta\,\frac{v^2-v_0^2}{2},$$

because of edge conditions (51.1), (51.2) we have

$$v_1V_0(v_1) = p\,\mathrm{ctg}\,\beta\,\frac{v_1^2-v_0^2}{2}, \quad v_1V_0(v_0) = 0.$$

In this way

$$\begin{aligned}\delta &= \int_{v_0}^{v_1}\vartheta_1^{\circ}\,dv + \frac{1}{2Eh\sin\beta\cos^2\beta}\left(\frac{P}{\pi} + pv_0^2\right)\ln\frac{v_1}{v_0} +\\ &\quad + \frac{p(v_1^2-v_0^2)}{2Eh\sin\beta}\left(\mu\cos^2\beta + \frac{3\sin^2\beta-\cos^2\beta}{2\cos^2\beta}\right).\end{aligned} \tag{51.14}$$

The constants of integration also are easily determined:

$$
\left.
\begin{aligned}
A_1 &= \frac{x_1\sqrt{2\pi x_1\sqrt{2}}}{\Delta(x_1)}\, e^{-x_1\gamma\,\mathrm{tg}\,\beta}\sqrt{\cos\beta}\times \\
&\qquad\times\left[\tilde{f}_2(x_1)-\frac{2(1-\mu)}{x_1}\,b_1(x_1)-a_1(x_1)\right]\tilde{H}_e^1. \\
B_1 &= \frac{x_1\sqrt{2\pi x_1\sqrt{2}}}{\Delta(x_1)}\, e^{-x_1\gamma\,\mathrm{tg}\,\beta}\sqrt{\cos\beta}\times \\
&\qquad\times\left[\tilde{f}_1(x_1)-\frac{2(1-\mu)}{x_1}\,a_1(x_1)+b_1(x_1)\right]\tilde{H}_e^1. \\
A_2 &= \frac{x_0\sqrt{\pi x_0\sqrt{2}}}{\sqrt{2}\,\Delta(x_0)}\, e^{x_0\gamma\,\mathrm{tg}\,\beta}\sqrt{\cos\beta}\times \\
&\qquad\times\left[\tilde{f}_4(x_0)+\frac{2(1-\mu)}{x_0}\,b_2(x_0)-a_2(x_0)\right]\sqrt{\frac{\nu_0}{\nu_1}}\,\tilde{H}_e^2. \\
B_2 &= \frac{x_0\sqrt{\pi x_0\sqrt{2}}}{\sqrt{2}\,\Delta(x_0)}\, e^{x_0\gamma\,\mathrm{tg}\,\beta}\sqrt{\cos\beta}\times \\
&\qquad\times\left[-\tilde{f}_3(x_0)+\frac{2(1-\mu)}{x_0}\,a_2(x_0)+b_2(x_0)\right]\sqrt{\frac{\nu_0}{\nu_1}}\,\tilde{H}_e^2.
\end{aligned}
\right\}
\tag{51.15}
$$

Substituting (51.15) into (50.25), after a series of transformations founded upon utilization of (46.14), (46.15), we obtain

$$
\begin{aligned}
\int_{\nu_0}^{\nu_1}\theta_1^{*}\,d\nu &= \frac{1}{Eh}\,\frac{\sin^2\beta}{\cos\beta\sqrt{\cos\beta}}\sqrt{3(1-\mu^2)}\left\{\frac{P}{2\pi}\left[r_1(x_1)\sqrt{\frac{\nu_1}{h}}+\right.\right. \\
&\left.\left.+\,r_0(x_0)\sqrt{\frac{\nu_0}{h}}\right]-\frac{p(\nu_1^2-\nu_0^2)}{2}\,r_1(x_1)\sqrt{\frac{\nu_1}{h}}\right\}.
\end{aligned}
\tag{51.16}
$$

where

$$
r_1(x_1)=\frac{1+\dfrac{9}{8x_1}-\dfrac{2(2-\mu)}{x_1}}{\Delta(x_1)}\approx 1-\frac{1}{2x_1},
$$

$$
r_0(x_0)=\frac{1-\dfrac{9}{8x_0}+\dfrac{2(2-\mu)}{x_0}}{\Delta(x_0)}\approx 1+\frac{1}{2x_0}.
$$

It is easy to see that

$$
r_1(x_1)\sqrt{\frac{\nu_1}{h}}+r_0(x_0)\sqrt{\frac{\nu_0}{h}}=\sqrt{\frac{\nu_1}{h}}+\sqrt{\frac{\nu_0}{h}}.
$$

Thus, displacement of half the compensator is determined by the formula

$$\delta = \frac{1}{2Eh\sin\beta\cos^2\beta}\left(\frac{P}{\pi} + p\nu_0^2\right)\ln\frac{\nu_1}{\nu_0} + \frac{p(\nu_1^2 - \nu_0^2)}{2Eh\sin\beta}\left(\mu\cos^2\beta + \right.$$
$$\left. + \frac{3}{2}\operatorname{tg}^2\beta - \frac{1}{2}\right) + \frac{P}{2\pi Eh}\frac{\sin^2\beta}{\cos\beta\sqrt{\cos\beta}}\sqrt{3(1-\mu^2)}\left(\sqrt{\frac{\nu_1}{h}} + \sqrt{\frac{\nu_0}{h}}\right) -$$
$$- \frac{p(\nu_1^2 - \nu_0^2)}{2Eh}\frac{\sin^2\beta}{\cos\beta\sqrt{\cos\beta}}\sqrt[4]{3(1-\mu^2)}\left(1 - \frac{2}{x_1}\right)\sqrt{\frac{\nu_1}{h}}. \qquad (51.17)$$

Setting in (51.17) $p=0$, we find sag only under the action of compressive force:

$$\delta = \frac{P}{2\pi Eh\sin\beta\cos^2\beta}\times$$
$$\times\left[\ln\frac{\nu_1}{\nu_0} + \sin^2\beta\sqrt{\cos\beta}\sqrt[4]{3(1-\mu^2)}\left(\sqrt{\frac{\nu_1}{h}} + \sqrt{\frac{\nu_0}{h}}\right)\right]. \qquad (51.18)$$

Comparing (50.27) and (50.18), we note that because of the lack of angles of rotation of the edges $(\vartheta_1^0 = \vartheta_1^1 = 0)$ the sag of the truncated cone was diminished by almost twice.

§ 52. Truncated Conical Shell Under the Action of a Bending Load

In § 52 and subsequent sections of this chapter we will be limited to consideration of the strain of a truncated conical shell under a bending load. Apropos of the calculation of a conical shell containing an angle we must note the following: the question about the applicability of equations (16.13), (16.14) in this instance requires special analysis. The fact is that in their derivation in the left part of formulas (16.4)-(16.6) the term $\frac{h^2}{12}\frac{\sin^2\theta}{\nu^2} = \frac{h^2}{12}\frac{\cos^2\beta}{\nu^2}$ was dropped, which at $\nu=0$ is infinitely large. In order to evaluate the error made in this case, it is necessary to derive again the Meissner-type equations keeping these terms. It is only when the obtained equations will differ from (16.13), (16.14) only because of terms of a higher order of smallness, will it be possible to say that equations (16.13), (16.14) are adequate for the calculation of a shell with an angle.

Let us examine the truncated conical shell shown in Fig. 27.

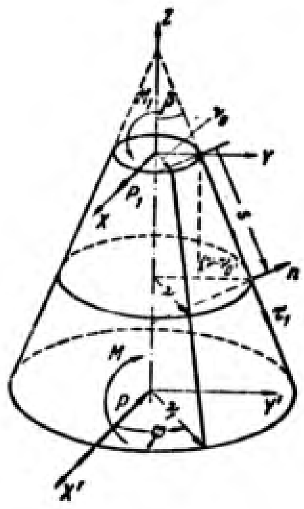


Fig. 27. Truncated conical shell under the action of a bending load.

We exchange the designationa $T_{1(1)}, T_{2(1)}, S_{(1)}, M_{1(1)}, M_{2(1)}, H_{(1)}, N_{1(1)}, N_{2(1)}$, accepted in Chapter II, for $t_1, t_2, s_{(1)}, m_1, m_2, h_{(1)}, n_1, n_2$ respectively. Setting into (15.21), (15.22)

$$\theta=\frac{\pi}{2}-\beta,\quad R_1\,d\theta=ds,\quad R_2=\frac{\nu}{\cos\beta},\quad ds=\frac{d\nu}{\sin\beta},\quad (\nu-\nu_0)=s\sin\beta. \tag{52.1}$$

We write two conditions of equilibrium for the section of shell contained between the edge section $\nu=\nu_0$ and the instantaneous section $\nu=\text{const}$:

$$\nu t_1\sin\beta+\nu n_1\cos\beta-\nu s_{(1)}-h_{(1)}\cos\beta+f(\nu)=0, \tag{52.2}$$

$$\nu m_1+\nu^2 t_1\cos\beta-\nu^2 n_1\sin\beta-h_{(1)}\nu\sin\beta+F(\nu)=0, \tag{52.3}$$

where

$$f(\nu)=\frac{P_1}{\pi}+\int^{\nu}(q_{1(1)}\sin\beta-q_{2(1)}+q_{n(1)}\cos\beta)\frac{\nu}{\sin\beta}\,d\nu. \tag{52.4}$$

$$F(\nu)=\frac{M_1}{\pi}+\frac{P_1}{\pi}(\nu-\nu_0)\operatorname{ctg}\beta-\int_{\nu}^{\nu}(q_{n(1)}\sin\beta-q_{1(1)}\cos\beta)\frac{\nu^2}{\sin\beta}\,d\nu+$$
$$+\int_{\nu_0}^{\nu}\cos\beta\left[\int_{\nu_0}^{\nu}(q_{1(1)}\sin\beta-q_{2(1)}+q_{n(1)}\cos\beta)\frac{\nu}{\sin\beta}\,d\nu\right]\frac{d\nu}{\sin\beta}. \tag{52.5}$$

Making the same transformations in expressions (15.34), (15.35) and comparing the result with (52.4), (52.5), it is simple to conclude that functions $f(v)$, $F(v)$ are connected with the functions of external loads f_0, f_1, F_0, F_1, introduced in § 15, in the following manner:

$$\left.\begin{aligned} F_0+F_1&=-\frac{f\cos\beta}{v}+\frac{F\sin\beta}{v^2}-\frac{\cos\beta}{v}\int_{v_0}^{v}\frac{q_{2(1)}v}{\sin\beta}dv,\\ f_0+f_1&=-\frac{f\sin\beta}{v}-\frac{F\cos\beta}{v^2}-\frac{1}{v}\int_{v_0}^{v}q_{2(1)}v\,dv.\end{aligned}\right\} \tag{52.6}$$

The left part of equation (52.2) is the sum of projections of all forces on section of shell (v_0, v), onto axis OX, the left part of equation (52.3) is the sum of the moments of all forces relative to the axis passing into the plane of section $v=\text{const}$ parallel to OY (Fig. 27). In accordance with this load functions $f(v)$, $F(v)$ can be presented in the form of the following sums:

$$\left.\begin{aligned} f(v)&=\frac{1}{\pi}(P_1+P_q),\\ F(v)&=\frac{1}{\pi}(M_1-P_1Z+M_q),\end{aligned}\right\} \tag{52.7}$$

where

$$Z=(v_0-v)\operatorname{ctg}\beta, \tag{52.8}$$

P_1, M_1 — force and moment on section v_0.

Holding the loads to be known functions of variable v, $P_q(v)$, $M_q(v)$ are easily found from equations

$$\left.\begin{aligned} P_q(v)&=\pi\int_{v_0}^{v}\left(q_{1(1)}-\frac{q_{2(1)}}{\sin\beta}+q_{n(1)}\operatorname{ctg}\beta\right)v\,dv,\\ M_q(v)&=-\pi\int_{v_0}^{v}(q_{n(1)}-q_{1(1)}\operatorname{ctg}\beta)v^2\,dv+\\ &+\operatorname{ctg}\beta\int_{v_0}^{v}\int_{v_0}^{v}(q_{1(1)}\sin\beta-q_{2(1)}+q_{n(1)}\cos\beta)v\,dv\,dv\end{aligned}\right\} \tag{52.9}$$

External loads applied to the shell as a whole should satisfy finite equilibrium relationships

$$\left.\begin{aligned} &P_1+P_q(v_1)-P=0, \\ &M_1+P_1L+M_q(v_1)-M=0 \qquad (L=(v_1-v_0)\operatorname{ctg}\beta), \end{aligned}\right\} \tag{52.10}$$

where L— height of truncated cone, P, M— resultant and total moment of loads applied to edge $v=v_1$.

From (52.7), (52.9), (52.10) it follows that

$$\left.\begin{aligned} &f(v_0)=\frac{P_1}{\pi}, \quad F(v_0)=\frac{M_1}{\pi}, \\ &f(v_1)=\frac{P}{\pi}, \quad F(v_1)=\frac{M}{\pi}. \end{aligned}\right\} \tag{52.11}$$

If the the intensities of loads $q_{1(1)}$, $q_{2(1)}$, $q_{n(1)}$ are constants, then P_q, M_q has the form

$$\left.\begin{aligned} &P_q=\pi\left(q_{1(1)}-\frac{q_{2(1)}}{\sin\beta}+q_{n(1)}\operatorname{ctg}\beta\right)\frac{(v^2-v_0^2)}{2}, \\ &M_q=-\pi(q_{n(1)}-q_{1(1)}\operatorname{ctg}\beta)\frac{(v^3-v_0^3)}{6}+ \\ &+\pi\operatorname{ctg}\beta\left(q_{1(1)}-\frac{q_{2(1)}}{\sin\beta}+q_{n(1)}\operatorname{ctg}\beta\right)\frac{(v-v_0)^2}{6}(v+2v_0). \end{aligned}\right\} \tag{52.12}$$

For a shell of constant thickness an illustration of such a load can be a weight load (axis OX is directed along the vertical to the ground)

$$\begin{aligned} &q_X=q, \qquad q=\rho h, \\ &q_{1(1)}=q\sin\beta, \quad q_{2(1)}=-q, \quad q_{n(1)}=q\cos\beta. \end{aligned}$$

Everything above refers to the equilibrium of a finite element of the shell.

The differential equations describing the equilibrium of a conical shell of varying thickness are derived from (16.13), (16.14), setting in them

$$\left.\begin{aligned} &R_1\,d\theta = ds, \quad b = \nu_1, \quad 4\gamma_0^4 = 12(1-\mu^2)\frac{\nu_1^2}{h_0^2},\\ &\theta = \frac{\pi}{2} - \beta, \quad R_1 = \infty. \end{aligned}\right\} \qquad (52.13)$$

We have

$$\left.\begin{aligned} &\frac{d^2\Psi_1}{ds^2} + \frac{d\Psi_1}{ds}\left(\frac{3}{\alpha}\frac{d\alpha}{ds} + \frac{\sin\beta}{\nu}\right) + \\ &+\Psi_1\left[\frac{3}{\alpha}\frac{d\alpha}{ds}\frac{(1+\mu)\sin\beta}{\nu} - \frac{2(1+\mu)\sin^2\beta}{\nu^2} - \frac{2(1-\mu)}{\nu^2}\right] + \\ &+V_1 4\gamma_0^4\left\{-\frac{1}{\alpha}\frac{\cos\beta}{\nu\nu_1} + \frac{(1-\mu^2)}{4\gamma_0^4}\left[\frac{2\cos\beta\sin\beta\nu_1}{\nu^2}\frac{d\alpha}{ds} + \right.\right.\\ &\qquad\left.\left. + \alpha\cos\beta\sin^2\beta\frac{\nu_1}{\nu^3}\right]\right\} = 4\gamma_0^4\frac{\Phi_3}{\alpha^3\nu_1^2},\\ &\frac{d^2V_1}{ds^2} + \frac{dV_1}{ds}\left(\frac{3}{\alpha}\frac{d\alpha}{ds} + \frac{\sin\beta}{\nu}\right) + \\ &+V_1\left[\frac{2}{\alpha}\frac{d^2\alpha}{ds^2} + \frac{1}{\alpha}\frac{d\alpha}{ds}(1+\mu)\frac{\sin\beta}{\nu} - \frac{2(1-\mu)\sin^2\beta}{\nu^2} - \right.\\ &\left.-\frac{2(1+\mu)}{\nu^2}\right] + \Psi_1\left[\frac{\cos\beta}{\alpha\nu\nu_1} + \frac{(1-\mu^2)}{4\gamma_0^4}\left(-2\frac{d\alpha}{ds}\cos\beta\sin\beta\frac{\nu_1}{\nu^2} - \right.\right.\\ &\qquad\left.\left. - \alpha\cos\beta\sin^2\beta\frac{\nu_1}{\nu^3}\right)\right] = \frac{1}{\alpha\nu_1}\Phi_4. \end{aligned}\right\} \qquad (52.14)$$

where

$$\alpha = \frac{h(\nu)}{h_0},$$

h_0 — thickness of wall of shell in section $\nu = \nu_0$; Φ_3, Φ_4 are determined by formulas (16.10), (16.11), where, inasmuch as the angle of the truncated cone is excluded from consideration, we can set

$$\Phi_3 = F_0 + F_1. \qquad (52.15)$$

§ 53. Approximate Solution of Meissner Type Equations

Introducing a complex combination of functions Ψ_1, V_1

$$\sigma_1 = \Psi_1 + 2i\gamma_0^2 V_1 \qquad (53.1)$$

taking into account that $ds = \frac{dv}{\sin\beta}$, we reduce system (52.14) to one equation of the second order, which after rejecting second-order terms (the order of $\frac{1}{\gamma_0^2}$, $\frac{1}{\gamma_0^4}$ in comparison with unity) is written thus:

$$\frac{d^2\sigma_1}{dv^2} + \frac{d\sigma_1}{dv}\left(\frac{1}{v} + \frac{3}{\alpha}\frac{d\alpha}{dv}\right) + 2i\gamma_0^2 \frac{\cos\beta}{\alpha v v_1 \sin^2\beta}\sigma_1 =$$
$$= 4\gamma_0^4 \frac{1}{\alpha^4 v_1^2 \sin^2\beta}(F_0 + F_1). \qquad (53.2)$$

The expressions for forces and bending moments through the introduced σ_1 function we obtain on the basis of (16.20), taking into account (52.1) and dropping quantities of order $\frac{1}{2\gamma_0^2}$ in comparison with unity:

$$\left.\begin{aligned}
t_1 &= \frac{v_1}{2\gamma_0^2}\frac{\alpha^2}{v}\operatorname{Im}\sigma_1\sin\beta + f_0 + f_1,\\
t_2 &= \frac{v_1}{2\gamma_0^2}\left[\operatorname{Im}\frac{d(\alpha^2\sigma_1)}{ds} + \frac{\alpha^2}{v}\operatorname{Im}\sigma_1\sin\beta\right] - \frac{\sin\beta}{v}f_2,\\
s_{(1)} &= \frac{v_1}{2\gamma_0^2}\frac{\alpha^2}{v}\operatorname{Im}\sigma_1 - \frac{1}{v}f_2,\\
m_1 &= \frac{\alpha^3 v_1^2}{4\gamma_0^4}\left[\operatorname{Re}\frac{d\sigma_1}{ds} + (1+\mu)\frac{\sin\beta}{v}\operatorname{Re}\sigma_1 + \right.\\
&\qquad\left. + (1-\mu^2)\frac{\cos\beta}{\alpha v}(f_0 + f_1)\right],\\
m_2 &= \frac{\alpha^3 v_1^2}{4\gamma_0^4}\left[\mu\operatorname{Re}\frac{d\sigma_1}{ds} + (1+\mu)\frac{\sin\beta}{v}\operatorname{Re}\sigma_1 - \right.\\
&\qquad\left. - (1-\mu^2)\frac{\sin\beta\cos\beta}{\alpha v^2}f_2\right],\\
h_{(1)} &= \frac{\alpha^3 v_1^2}{4\gamma_0^4}\left[-\frac{(1-\mu)}{v}\operatorname{Re}\sigma_1 - \frac{2(1-\mu^2)\cos\beta}{\alpha v^2}f_2\right],
\end{aligned}\right\} \qquad (53.3)$$

where

$$f_2 = \int_0^s q_{2(1)} v\, ds = \frac{1}{\sin\beta}\int_{v_0}^{v} q_{2(1)} v\, dv.$$

Let us note that the load terms in the expressions for moments can also be eliminated, since the bending stresses from these terms will have an order $\frac{h}{v}$ lower than tension stresses from the same terms in the expressions for forces. The general solution of equation (53.2) we look for in the form

$$\sigma_1 = \sigma_1^{*} + \tilde{\sigma}_1. \tag{53.4}$$

where σ_1^{*} — solution of the uniform equation, $\tilde{\sigma}_1$ — particular solution of the equation with a nonzero right side.

As the particular solution let us take the zero-moment solution

$$\sigma_1 = -i2\gamma_0^2 \frac{v(F_0 + F_1)}{a^2 \cos\beta v_1}. \tag{53.5}$$

Separating the real and imaginary parts (53.5), we obtain

$$\tilde{\Psi}_1 = 0, \quad \tilde{V}_1 = -\frac{(F_0 + F_1)v}{a^2 \cos\beta v_1}. \tag{53.6}$$

The forces of the zero-moment state have the form

$$\left.\begin{aligned}
\tilde{t}_1 &= -(F_0 + F_1)\operatorname{tg}\beta + (f_0 + f_1) = -\frac{F}{v^2\cos\beta}, \\
\tilde{s}_{(1)} &= -(F_0 + F_1)\frac{1}{\cos\beta} - \frac{1}{v}\int_{v_0}^{v} \frac{q_{2(1)}\, v\, dv}{\sin\beta} = \frac{f}{v} - \frac{F\operatorname{tg}\beta}{v^2}, \\
\tilde{t}_2 &= \frac{q_{n(1)} v}{\cos\beta}.
\end{aligned}\right\} \tag{53.7}$$

After separating the zero-moment part of the solution formulas (53.3) can be rewritten in the following form:

$$\left.\begin{aligned}
t_1 &= \frac{v_1}{2\gamma_0^2}\frac{a^2\sin\beta}{v}\operatorname{Im}\sigma_1^{*} + \tilde{t}_1, \\
t_2 &= \frac{v_1}{2\gamma_0^2}\sin\beta\left[\frac{d}{dv}\operatorname{Im}(a^2\sigma_1^{*}) + \frac{a^2}{v}\operatorname{Im}\sigma_1^{*}\right] + \tilde{t}_2, \\
s_{(1)} &= \frac{v_1}{2\gamma_0^2}\frac{a^2}{v}\operatorname{Im}\sigma_1^{*} + \tilde{s}_{(1)}, \\
m_1 &= \frac{a^3 v_1^2 \sin\beta}{4\gamma_0^4}\left[\operatorname{Re}\frac{d\sigma_1^{*}}{dv} + \frac{(1+\mu)}{v}\operatorname{Re}\sigma_1^{*}\right], \\
m_2 &= \frac{a^3 v_1^2 \sin\beta}{4\gamma_0^4}\left[\mu\operatorname{Re}\frac{d\sigma_1^{*}}{dv} + \frac{(1+\mu)}{v}\operatorname{Re}\sigma_1^{*}\right], \\
h_{(1)} &= -\frac{(1-\mu)a^3 v_1^2}{4\gamma_0^4 v}\operatorname{Re}\sigma_1^{*}.
\end{aligned}\right\} \tag{53.8}$$

Turn now to the solution of the uniform equation

$$\frac{d^2\sigma_1}{d\nu^2}+\frac{d\sigma_1}{d\nu}\left(\frac{1}{\nu}+\frac{3}{\alpha}\frac{d\alpha}{d\nu}\right)+2i\gamma_0^2\frac{\cos\beta}{\alpha\nu\nu_1\sin^2\beta}\sigma_1=0. \tag{53.9}$$

With the aid of replacement of variables, which easily is done, just as in § 19, we cancel in this equation the term with the first derivative. Instead of (53.9) we have the equation

$$\frac{d^2\tau}{dx^2}+\tau\left\{2i+\frac{1}{\gamma_0^2}\left[\frac{1}{\xi^2(\nu)\cdot u}\frac{d^2u}{d\nu^2}+\frac{1}{\xi^2\cdot u}\left(\frac{1}{\nu}+\frac{3}{\alpha}\frac{d\alpha}{d\nu}\right)\frac{du}{d\nu}\right]\right\}=0. \tag{53.10}$$

where

$$\sigma_1=\tau\cdot u,\quad u(\alpha,\nu)=\frac{1}{\alpha\sqrt[4]{\alpha\nu\cos\beta}}. \tag{53.11}$$

$$dx=\gamma_0\xi(\alpha,\nu)\,d\nu,\quad \xi(\alpha,\nu)=\frac{\sqrt{\cos\beta}}{\sin\beta\sqrt{\nu_1}}\frac{1}{\sqrt{\nu\alpha}}. \tag{53.12}$$

The coefficient of the unknown function τ in equation (53.10) can have a singularity of the form $\frac{A}{x^n}$, where the exponent n will depend on the form of function $\alpha(\nu)$. For example, for the case $\alpha=1$ (shell has constant thickness), equation (53.10) assumes the form

$$\frac{d^2\tau}{dx^2}+\tau\left(2i+\frac{1}{4x^2}\right)=0. \tag{53.13}$$

Just as equation (46.6), describing axisymmetric strain of a conical shell of constant thickness, equation (53.13) can be reduced to a Bessel equation, and its solution is represented in the form

$$\tau=\sqrt{x}\left[c_1I_0(x\sqrt{2i})+c_2H_0^{(1)}(x\sqrt{2i})\right]. \tag{53.14}$$

For a shell with linearly changing thickness $(\alpha=c\nu)$ equation (53.10) turns into an equation with constant coefficients

$$\frac{d^2\tau}{dx^2}+\tau\left(2i-\frac{1}{\gamma_0^2}\frac{9}{4}\frac{\sin^2\beta}{\cos\beta}c\nu_1\right)=0. \tag{53.15}$$

After the elimination of small terms it coincides with the resolvant equation for a cylindrical shell. Taking into consideration that we are considering only truncated thin-walled cone ($x \neq 0$, $\gamma_0^2 \gg 1$), in general equation (53.10) as the first approximation we also can eliminate terms of order $\frac{1}{\gamma_0^2}$ in comparison with unity, i.e., to replace it by the equation

$$\frac{d^2\tau}{dx^2} + \tau \cdot 2i = 0. \tag{53.16}$$

This does not mean that the conical shell is replaced in this case by a cylindrical of any equivalent radius, since the argument x is a function of ν determined in accordance with (53.12):

$$\left.\begin{aligned} x - x_0 &= \gamma_0 \int_{\nu_0}^{\nu} \xi(\nu)\, d\nu, \\ x_1 - x &= \gamma_0 \int_{\nu}^{\nu_1} \xi(\nu)\, d\nu. \end{aligned}\right| \tag{53.17}$$

For a long shell, when $(x_1 - x_0) > \pi$, the general solution of equation (53.16) is conveniently taken in the form

$$\begin{aligned} \tau = (A_1 - iB_1)[\theta(x - x_0) + i\zeta(x - x_0)] + \\ + (A_2 - iB_2)[\theta(x_1 - x) + i\zeta(x_1 - x)]. \end{aligned}$$

and correspondingly

$$\begin{aligned} \sigma_1^* = u(\alpha, \nu)\{(A_1 - iB_1)[\theta(x - x_0) + i\zeta(x - x_0)] + \\ + (A_2 - iB_2)[\theta(x_1 - x) + i\zeta(x_1 - x)]\}. \end{aligned} \tag{53.18}$$

where θ, ζ – functions introduced in § 27 (formulas (27.5), (27.6), Table 1 of the Appendix).

Earlier it was shown that elimination of terms of the order $\frac{1}{\gamma_0^2}$ in comparison with unity in the coefficient of the unknown function for an equation of type (53.10) leads to an error in the solution in terms of the degree $\frac{1}{\gamma_0}$ in comparison with the basic terms. Therefore upon replacing equation (53.10) by equation (53.16) in the solution of the latter we deprive ourselves of the possibility of keeping terms of such an order. This means that in making up the derivative of $\sigma_1^\bullet$. and also during the integration of $\sigma_1^\bullet$ function $u(\alpha, \nu)$ should be considered as a constant coefficient.

In this way we obtain

$$\left.\begin{aligned}
\mathrm{Re}\frac{d\sigma_1^\bullet}{d\nu} &= \gamma_0\xi(\nu)u(\nu)[-A_1\varphi(x-x_0)+B_1\psi(x-x_0)+\\
&\quad + A_2\varphi(x_1-x)-B_2\psi(x_1-x)],\\
\mathrm{Im}\frac{d\sigma_1^\bullet}{d\nu} &= \gamma_0\xi(\nu)u(\nu)[A_1\psi(x-x_0)+B_1\varphi(x-x_0)-\\
&\quad - A_2\psi(x_1-x)-B_2\varphi(x_1-x)].
\end{aligned}\right|$$

By (53.8), (53.18) and (53.19), ignoring quantities of order $\frac{1}{\gamma_0}$ in comparison with unity, we obtain the following expressions for forces and bending moments:

$$\left.\begin{aligned}
t_1-\tilde{t}_1 &= \frac{\nu_1}{2\gamma_0^2}\frac{\alpha^2\sin\beta}{\nu}u(\alpha,\nu)[A_1\zeta(x-x_0)-B_1\theta(x-x_0)+\\
&\quad + A_2\zeta(x_1-x)-B_2\theta(x_1-x)],\\
t_2-\tilde{t}_2 &= \frac{\nu_1\alpha^2}{2\gamma_0}\sin\beta u(\alpha,\nu)\xi(\nu)[A_1\psi(x-x_0)+\\
&\quad + B_1\varphi(x-x_0)-A_2\psi(x_1-x)-B_2\varphi(x_1-x_0)],\\
s_{(1)}-\tilde{s}_{(1)} &= \frac{\nu_1}{2\gamma_0^2}\frac{\alpha^2}{\nu}u(\alpha,\nu)[A_1\zeta(x-x_0)-B_1\theta(x-x_0)+\\
&\quad + A_2\zeta(x_1-x)-B_2\theta(x_1-x)],\\
m_1 &= \frac{\alpha^3\nu_1^2}{4\gamma_0^3}\sin\beta u(\alpha,\nu)\xi(\nu)[-A_1\varphi(x-x_0)+\\
&\quad + B_1\psi(x-x_0)+A_2\varphi(x_1-x)-B_2\psi(x_1-x)],\\
m_2 &= \mu m_1.
\end{aligned}\right| \qquad (53.20)$$

§ 54. Determination of Constants of Integration

Arbitrary constants A_1, B_1, A_2, B_2, figuring in (52.18), should be defined from the edge conditions. Let us assume that on the edges of the shell $\nu=\nu_0\ (x=x_0)$ and $\nu=\nu_1\ (x=x_1)$ are assigned forces $\left(s_{(1)}+\frac{2h_{(1)}}{R_2}\right)$, equal to s^0, s^1 respectively, and moments m_1, equal to m_1^0 and m_1^1. Let us note that on the basis of formulas (53.8) we can set

$$s_{(1)} + \frac{2h_{(1)}}{R_2} \approx s_{(1)}. \qquad (54.1)$$

With the same accuracy the quantity n_1 can be considered as the amplitude of the shearing force. Then, turning to (52.2), it is easy to see that the assignment of the quantity $s_{(1)}$ on the edge of the shell is equivalent to the assignment of the amplitude of radial force

$$h_e = t_1 \sin\beta + n_1 \cos\beta = \tilde{h}_e + \frac{\nu_1}{2\gamma_0^2}\frac{\alpha^2}{\nu}\operatorname{Im}\sigma_1^*, \qquad (54.2)$$

where $\tilde{h}_e = \tilde{t}_1 \sin\beta$. Therefore subsequently we hold that on the edges of the shell are assigned quantities $h_e^0, h_e^1, m_1^0, m_1^1$. Then, having in mind that $s_{(1)} - s_{(1)} = h_e - \tilde{h}_e$, and ignoring the mutual influence of the edges, for determination of A_1, B_1 and A_2, B_2 we derive two individual systems of equations:

$$\left.\begin{aligned} h_e^0 - \tilde{h}_e^0 &= -\frac{\nu_1}{2\gamma_0^2}\frac{u(1, \nu_0)}{\nu} B_1, \\ m_1^0 &= \frac{\nu_1^2 \sin\beta}{4\gamma_0^3} u(1, \nu_0)\xi(1, \nu_0)(-A_1 + B_1). \end{aligned}\right\} \qquad (54.3)$$

$$\left.\begin{aligned} h_e^1 - \tilde{h}_e^1 &= -\frac{\alpha_1^2}{2\gamma_0^2} u(\alpha_1, \nu_1) B_2, \\ m_1^1 &= \frac{\alpha_1^3\nu_1^2 \sin\beta}{4\gamma_0^3} u(\alpha_1, \nu_1)\xi(\alpha_1, \nu_1)(A_2 - B_2). \end{aligned}\right\} \qquad (54.4)$$

In this case we take into consideration that on the edge $x = x_0$ $\alpha = 1$, and on the edge $x = x_1$ $\alpha = h_1/h_0 = \alpha_1$.

Solving them, we have

$$\left.\begin{aligned} B_1 &= -(h_e^0 - \tilde{h}_e^0)\frac{2\gamma_0^2\nu}{\nu_1 u(1, \nu_0)}, \\ A_1 &= B_1 - \frac{m_1^0 \cdot 4\gamma_0^3}{\nu_1^2 \sin\beta\, u(1, \nu_0)\,\xi(1, \nu_0)}, \end{aligned}\right\} \qquad (54.5)$$

$$\left.\begin{aligned} B_2 &= -(h_e^1 - \tilde{h}_e^1)\frac{2\gamma_0^2}{\alpha_1^2 u(\alpha_1, \nu_1)}, \\ A_2 &= B_2 + \frac{m_1^1 4\gamma_0^3}{\alpha_1^3\nu_1^2 \sin\beta\, u(\alpha_1, \nu_1)\,\xi(\alpha_1, \nu_1)}. \end{aligned}\right\} \qquad (54.6)$$

Substituting the obtained values of constants into formulas (53.20), we obtain the final expressions for forces and moments. In order to give the equations symmetric form, we introduce new designations:

$$2\beta_0^2 = \sqrt{12(1-\mu^2)}\,\frac{\nu_0}{h_0}, \quad 2\beta_1^2 = \sqrt{12(1-\mu^2)}\,\frac{\nu_1}{h_1}. \tag{54.7}$$

where h_0, h_1 — values of wall thickness cn edges x_0 and x_1. Making the necessary calculations, we have

$$\left.\begin{aligned}
&t_2-\tilde{t}_2 = -\alpha^{1/4}\Big\{\left(\frac{\nu_0}{\nu}\right)^{1/4}\Big[(h_e^0-\tilde{h}_e^0)\,2\beta_0\sqrt{\cos\beta}\,\theta(x-x_0)+\\
&\qquad + m_1^0\frac{2\beta_0^2}{\nu_0}\psi(x-x_0)\Big] + \alpha_1^{-1/4}\left(\frac{\nu_1}{\nu}\right)^{3/4}\times\\
&\quad\times\Big[-(h_e^1-\tilde{h}_e^1)\,2\beta_1\sqrt{\cos\beta}\,\theta(x_1-x) + m_1^1\frac{2\beta_1^2}{\nu_1}\psi(x_1-x)\Big]\Big\}.\\
&h_e-\tilde{h}_e=\\
&=\alpha^{1/4}\Big\{\left(\frac{\nu_0}{\nu}\right)^{3/4}\Big[(h_e^0-\tilde{h}_e^0)\,\psi(x-x_0) - m_1^0\frac{2\beta_0}{\nu_0}\frac{1}{\sqrt{\cos\beta}}\zeta(x-x_0)\Big]+\\
&+\alpha_1^{-3/4}\left(\frac{\nu_1}{\nu}\right)^{3/4}\Big[(h_e^1-\tilde{h}_e^1)\,\psi(x_1-x) + \frac{2\beta_1}{\sqrt{\cos\beta}}\frac{m_1^1}{\nu_1}\zeta(x_1-x)\Big]\Big\}.\\
&m_1=\alpha^{1/4}\Big\{\left(\frac{\nu_0}{\nu}\right)^{1/4}\Big[(h_e^0-\tilde{h}_e^0)\frac{\nu_0}{\beta_0}\sqrt{\cos\beta}\,\zeta(x-x_0) + m_1^0\varphi(x-x_0)\Big]+\\
&+\alpha_1^{-1/4}\left(\frac{\nu_1}{\nu}\right)^{1/4}\Big[-(h_e^1-\tilde{h}_e^1)\frac{\nu_1}{\beta_1}\sqrt{\cos\beta}\,\zeta(x_1-x) + m_1^1\varphi(x_1-x)\Big]\Big\}.
\end{aligned}\right\} \tag{54.8}$$

Using (54.5), (54.6), (53.19), (53.20), we will write out also the expressions for quantities $\varepsilon_{2(1)}$ and $\Psi = \Psi_1/Eh_0$. Let us note that with accepted correctness of calculations it can be considered that

$$\varepsilon_{2(1)} - \tilde{\varepsilon}_{2(1)} = \frac{1}{Eh_0 a}(t_2 - \tilde{t}_2).$$

In this way we have

$$\left.\begin{aligned}
&Eh_0(\varepsilon_{2(1)}-\tilde{\varepsilon}_{2(1)})=\\
&\quad=-\alpha^{-1/4}\Big\{\left(\frac{\nu_0}{\nu}\right)^{1/4}\Big[(h_e^0-\tilde{h}_e^0)\,2\beta_0\sqrt{\cos\beta}\,\theta(x-x_0)+\\
&\qquad + m_1^0\frac{2\beta_0^2}{\nu_0}\psi(x-x_0)\Big] + \alpha_1^{-1/4}\left(\frac{\nu_1}{\nu}\right)^{3/4}\times\\
&\quad\times\Big[-(h_e^1-\tilde{h}_e^1)\,2\beta_1\sqrt{\cos\beta}\,\theta(x_1-x) + m_1^1\frac{2\beta_1^2}{\nu_1}\psi(x_1-x)\Big]\Big\}.
\end{aligned}\right\} \tag{54.9}$$

$$Eh_0\Psi = -\alpha^{-1/4}\left\{\left(\frac{\nu}{\nu_0}\right)^{1/4}\left[(h_e^0-\tilde{h}_e^0)\,2\beta_0^2\varphi(x-x_0)+\right.\right.$$
$$\left.+m_1^0\frac{4\beta_0^3}{\nu_0\sqrt{\cos\beta}}\,\theta(x-x_0)\right]+\alpha_1^{1/4}\left(\frac{\nu_1}{\nu}\right)^{1/4}\times$$
$$\left.\times\left[(h_e^1-\tilde{h}_e^1)\,2\beta_1^2\varphi(x_1-x)-m_1^1\frac{4\beta_1^3}{\nu_1\sqrt{\cos\beta}}\,\theta(x_1-x)\right]\right\}. \quad (54.9)\ \text{(Cont'd)}$$

On the edges exist the relationships:

$$\left.\begin{aligned} Eh_0(\varepsilon^0_{2(1)}-\tilde{\varepsilon}^0_{2(1)}) &= -(h_e^0-\tilde{h}_e^0)\,2\beta_0\sqrt{\cos\beta}-m_1^0\frac{2\beta_0^2}{\nu_0},\\ Eh_0\Psi^0 &= -(h_e^0-\tilde{h}_e^0)\,2\beta_0^2+m_1^0\frac{4\beta_0^3}{\nu_0\sqrt{\cos\beta}}.\end{aligned}\right\} \quad (54.10)$$

$$\left.\begin{aligned} Eh_1(\varepsilon^1_{2(1)}-\tilde{\varepsilon}^1_{2(1)}) &= (h_e^1-\tilde{h}_e^1)\,2\beta_1\sqrt{\cos\beta}-m_1^1\frac{2\beta_1^2}{\nu_1},\\ Eh_1\Psi'^1 &= -(h_e^1-\tilde{h}_e^1)\,2\beta_1^2+m_1^1\frac{4\beta_1^3}{\nu_1\sqrt{\cos\beta}}.\end{aligned}\right\} \quad (54.11)$$

If the edges of the shell are connected with rigid diaphragms, these equalities should hold:

$$\varepsilon^0_{2(1)}=\Psi^0=0,\quad \varepsilon^1_{2(1)}=\Psi^1=0.$$

Relationships (54.10), (54.11) allow easily finding forces and moments appearing in this case in the edge sections of the shell

$$h_e^0=\tilde{h}_e^0+\frac{Eh_0\tilde{\varepsilon}^0_{2(1)}}{\beta_0\sqrt{\cos\beta}},\quad m_1^0=-\frac{Eh_0\tilde{\varepsilon}^0_{2(1)}\nu_0}{2\beta_0^2}, \quad (54.12)$$

$$h_e^1=\tilde{h}_e^1-\frac{Eh_1\tilde{\varepsilon}^1_{2(1)}}{\beta_1\sqrt{\cos\beta}},\quad m_1^1=-\frac{Eh_1\tilde{\varepsilon}^1_{2(1)}\nu_1}{2\beta_1^2}. \quad (54.13)$$

§ 55. Determination of Displacements

From the found strains (54.9) it is easy to determine displacements in any section of the shell using two quadratures. Rewrite formulas (18.2), (18.5) allowing for (52.1). We obtain

$$\left.\begin{aligned} \Delta_{z(1)} &= \nu\left(\Psi-\int_{\nu_0}^{\nu}\varkappa_{1(1)}\frac{d\nu}{\sin\beta}\right)+D_1\nu,\\ \Delta_{e(1)} &= \nu\varepsilon_{2(1)}-\int_{\nu_0}^{\nu}\left(\gamma_{(1)}+\varepsilon_{2(1)}\sin\beta-\frac{\Delta_{z(1)}\cos\beta}{\nu}\right)\frac{d\nu}{\sin\beta}+D_2.\end{aligned}\right\} \quad (55.1)$$

Since between $\varkappa_{1(1)}$, Ψ and $\varepsilon_{2(1)}$ exists a connection determined by the first formula of (16.2), which in this case looks like:

$$\varkappa_{1(1)}=\sin\beta\frac{d\Psi}{dv}+\frac{1}{v}(\Psi\sin\beta+\varepsilon_{1(1)}\cos\beta), \qquad (55.2)$$

then the first formula of (55.1) can be rewritten in the form

$$\Delta_{z(1)}=-v\int_{v_0}^{v}(\Psi\sin\beta+\varepsilon_{1(1)}\cos\beta)\frac{dv}{v\sin\beta}+D_1v. \qquad (55.3)$$

The displacements which correspond to the zero-moment stressed state are equal to

$$\left.\begin{aligned}\tilde{\Delta}_{z(1)}&=-v\operatorname{ctg}\beta\int_{v_0}^{v}\frac{\tilde{\varepsilon}_{1(1)}dv}{v},\\ \tilde{\Delta}_{e(1)}&=v\tilde{\varepsilon}_{2(1)}-\frac{1}{\sin\beta}\int_{v_0}^{v}\left(\tilde{\gamma}_{(1)}+\tilde{\varepsilon}_{2(1)}\sin\beta-\frac{\tilde{\Delta}_{z(1)}\cos\beta}{v}\right)dv.\end{aligned}\right\} \qquad (55.4)$$

The edge effect gives the following displacements:

$$\left.\begin{aligned}\Delta_{z(1)}-\tilde{\Delta}_{z(1)}-D_1v&\approx-v\int_{v_0}^{v}\frac{\Psi}{v}dv,\\ \Delta_{e(1)}-\tilde{\Delta}_{e(1)}-D_1\operatorname{ctg}\beta(v-v_0)-D_2&=v(\varepsilon_{2(1)}-\tilde{\varepsilon}_{2(1)}).\end{aligned}\right\} \qquad (55.5)$$

In this case in the right part of (55.5) all terms of the order $\frac{1}{\gamma_0}$ in comparison with the remaining have been dropped.

In this way, the formulas for determination of total displacements assume the form

$$\left.\begin{aligned}\Delta_{z(1)}&=-v\int_{v_0}^{v}\frac{1}{v}(\Psi+\tilde{\varepsilon}_{1(1)}\operatorname{ctg}\beta)dv+D_1v,\\ \Delta_{e(1)}&=v\varepsilon_{2(1)}-\frac{1}{\sin\beta}\int_{v_0}^{v}\Bigg(\tilde{\gamma}_{(1)}+\tilde{\varepsilon}_{2(1)}\sin\beta+\\ &\quad+\frac{\cos^2\beta}{\sin\beta}\int_{v_0}^{v}\frac{\tilde{\varepsilon}_{1(1)}dv}{v}\Bigg)dv+D_1\operatorname{ctg}\beta(v-v_0)+D_2.\end{aligned}\right\} \qquad (55.6)$$

§ 56. Shell with Wall Thickness Linearly Changing Along the Meridian

As an example let us examine a shell with relative wall thickness changing by linear law

$$\alpha = a + c\nu, \tag{56.1}$$

$$a = 1 - \frac{\nu_0(\alpha_1 - 1)}{\nu_1 - \nu_0}, \quad c = \frac{\alpha_1 - 1}{\nu_1 - \nu_0}. \tag{56.2}$$

The edges of the shell are connected with rigid diaphragms. There is no distributed load. On the diaphragm of upper edge $\nu = \nu_0$ act force and moment P_1, M_1, and on the diaphragm of the lower edge act force and moment P, M, where because of the equilibrium of the shell as a whole their amounts satisfy the conditions:

$$P_1 = P, \quad M_1 + P_1(\nu_1 - \nu_0)\operatorname{ctg}\beta = M. \tag{56.3}$$

Our task is to determine the displacement and turn of edge $\nu = \nu_1$ relative to edge $\nu = \nu_0$ under the action of applied force and bending moment, i.e., to find the quantities a_{11}, a_{12}, a_{21}, a_{22} in the relationships:

$$\left.\begin{aligned} \Delta^1_{e(1)} &= Pa_{11} + Ma_{12}, \\ \omega_y = \frac{\Delta^1_{z(1)}}{\nu_1} &= Pa_{21} + Ma_{22}. \end{aligned}\right\} \tag{56.4}$$

We compute the values of displacements on edge $\nu = \nu_1$ using formulas (55.6). Substituting into them the strains of the zero-moment state

$$\left.\begin{aligned} \tilde{\varepsilon}_{1(1)} &= -\frac{1}{\pi E h_0 \alpha \cos\beta}\left[\frac{M_1}{\nu^2} + P_1\frac{(\nu-\nu_0)}{\nu^2}\operatorname{ctg}\beta\right], \\ \tilde{\varepsilon}_{2(1)} &= \frac{\mu}{\pi E h_0 \alpha \cos\beta}\left[\frac{M_1}{\nu^2} + P_1\frac{(\nu-\nu_0)}{\nu^2}\operatorname{ctg}\beta\right], \\ \tilde{\gamma}_{(1)} &= \frac{2(1+\mu)}{\pi E h_0 \alpha}\left(-\frac{M_1}{\nu^2}\operatorname{tg}\beta + P_1\frac{\nu_0}{\nu^2}\right) \end{aligned}\right\} \tag{56.5}$$

and the quantities Ψ, $\varepsilon_{z(1)}$, having the form of (54.9) and written allowing for (54.12), (54.13), after integration we obtain

$$\Delta^1_{r(1)} = \frac{(2+\mu)}{\pi E h_0 \cos\beta}(M_1 - P_1 v_0 \operatorname{ctg}\beta) I_2(v_1) -$$

$$- \frac{\mu}{\pi E h_0 \sin\beta} P_1 I_1(v_1) + \frac{\cos\beta}{\pi E h_0 \sin^2\beta}(M_1 - P_1 v_0 \operatorname{ctg}\beta) I_4(v_1) +$$

$$+ \frac{P_1}{\pi E h_0}\frac{\cos^2\beta}{\sin^3\beta} I_3(v_1) + D_1 \operatorname{ctg}\beta (v_1 - v_0) + D_2. \qquad (56.6)$$

$$\Delta^1_{z(1)} = \frac{v_1}{\pi E h_0 \sin\beta}[(M_1 - P_1 v_0 \operatorname{ctg}\beta) I_5(v_1) + P_1 \operatorname{ctg}\beta I_2(v_1)] +$$

$$+ \frac{\mu v_1 \sin\beta}{\pi E h_0 \cos^2\beta}\left[\frac{M_1}{v_0^2} - \frac{M_1 + P_1(v_1 - v_0)\operatorname{ctg}\beta}{v_1^2 \alpha_1}\right] + D_1 v. \qquad (56.7)$$

where the following designations have been introduced:

$$\left.\begin{aligned}
I_1(v) &= \int_{v_0}^{v} \frac{dv}{v\alpha} = \frac{1}{a}\ln\left(\frac{v}{v_0 \alpha}\right),\\
I_2(v) &= \int_{v_0}^{v} \frac{dv}{v^2 \alpha} = -\frac{1}{a}\left(\frac{1}{v} - \frac{1}{v_0}\right) - \frac{c}{a^2}\ln\left(\frac{v}{v_0 \alpha}\right),\\
I_3(v) &= \int_{v_0}^{v} I_2(v)\,dv = \frac{(v - v_0)}{a v_0} - \frac{c}{a^2}\ln\left(\frac{v}{v_0 \alpha}\right),\\
I_4(v) &= \int_{v_0}^{v} I_5(v)\,dv = \frac{1}{2a}\left(\frac{1}{v} - \frac{1}{v_0}\right) +\\
&\quad + \frac{(v - v_0)}{2a^2 v_0^2}(a - 2c v_0) + \frac{c\alpha}{a^3}\ln\left(\frac{v}{v_0 \alpha}\right),\\
I_5(v) &= \int_{v_0}^{v} \frac{dv}{v^3 \alpha} = -\frac{1}{2a}\left(\frac{1}{v^2} - \frac{1}{v_0^2}\right) +\\
&\quad + \frac{c}{a^2}\left(\frac{1}{v} - \frac{1}{v_0}\right) + \frac{c^2}{a^3}\ln\left(\frac{v}{v_0 \alpha}\right).
\end{aligned}\right\} \qquad (56.8)$$

During the integration of quantities connected with the edge effect it was assumed that the shell is long, and in accordance with this in the final result terms containing $\psi(x_1 - x_0)$, $\theta(x_1 - x_0)$, were dropped as small in comparison with unity. Considering that edge $v = v_0$ is not displaced and is not turned, we set $D_1 = D_2 = 0$ and on the basis of (56.6), (56.7), (56.3) find

$$\left.\begin{aligned}
\alpha_{11} &= \frac{1}{\pi E h_0}\left\{-\frac{(2+\mu) v_1}{\sin\beta} I_2(v_1) - \frac{\mu}{\sin\beta} I_1(v_1) +\right.\\
&\quad \left. + \frac{\cos^2\beta}{\sin^3\beta}[I_3(v_1) - v_1 I_4(v_1)]\right\},\\
\alpha_{12} &= \frac{1}{\pi E h_0}\left[\frac{(2+\mu)}{\cos\beta} I_2(v_1) + \frac{\cos\beta}{\sin^2\beta} I_4(v_1)\right],\\
\alpha_{21} &= \frac{\operatorname{ctg}\beta}{\pi E h_0}\left\{\frac{1}{\sin\beta}[I_2(v_1) - v_1 I_5(v_1)] - \frac{\mu \sin\beta}{v_0^2 \cos^2\beta}(v_1 - v_0)\right\},\\
\alpha_{22} &= \frac{1}{\pi E h_0}\left[\frac{I_5(v_1)}{\sin\beta} + \frac{\mu \sin\beta}{\cos^2\beta}\left(\frac{1}{v_0^2} - \frac{1}{v_1^2 \alpha_1}\right)\right].
\end{aligned}\right\} \qquad (56.9)$$

where $I_1(\nu_1), I_2(\nu_1), I_3(\nu_1), I_4(\nu_1), I_5(\nu_1)$ are found using formulas (56.8), in which it is necessary to set $\nu=\nu_1$.

For a conical shell of constant thickness we have

$$\left.\begin{aligned}
\alpha_{11}&=\frac{1}{\pi Eh}\left\{-\frac{2(1+\mu)}{\nu_0}s_1-\frac{\mu s_1}{\nu_1-\nu_0}\left[\ln\frac{\nu_1}{\nu_0}-\frac{\nu_1}{\nu_0}+1\right]+\right.\\
&\left.+\frac{\cos^2\beta s_1^3}{(\nu_1-\nu_0)^3}\left[2\frac{\nu_1}{\nu_0}-\frac{1}{2}\frac{\nu_1^2}{\nu_0^2}-\ln\frac{\nu_1}{\nu_0}-\frac{3}{2}\right]\right\},\\
\alpha_{12}&=\frac{1}{\pi Eh}\left[\frac{\cos\beta s_1^2}{2\nu_1\nu_0^2}+\frac{(2+\mu)\operatorname{tg}\beta\cdot s_1}{\nu_1\nu_0}\right],\\
\alpha_{21}&=\frac{1}{\pi Eh}\left[-\frac{\cos\beta s_1^2}{2\nu_1\nu_0^2}-\frac{\mu\operatorname{tg}\beta\cdot s_1}{\nu_0^2}\right],\\
\alpha_{22}&=\frac{1}{\pi Eh}\left[\frac{(\nu_1+\nu_0)}{2\nu_1^2\nu_0^2}s_1+\frac{\mu\sin\beta}{\cos^2\beta}\left(\frac{1}{\nu_0^2}-\frac{1}{\nu_1^2}\right)\right].
\end{aligned}\right\}\qquad(56.10)$$

where $s_1=\frac{\nu_1-\nu_0}{\sin\beta}$ — length of the generatrix of the middle cone. Letting s_1 remain constant and setting $\sin\beta=0$, $\cos\beta=1$, as the maximum change at $\nu_1\to\nu_0$ from (56.10) we obtain

$$\left.\begin{aligned}
\pi Eh\alpha_{11}&=-\frac{2(1+\mu)s_1}{\nu_0}-\frac{s_1^3}{3\nu_0^3},\\
\pi Eh\alpha_{12}&=\frac{s_1^2}{2\nu_0^3},\\
\pi Eh\alpha_{21}&=-\frac{s_1^2}{2\nu_0^3},\\
\pi Eh\alpha_{22}&=\frac{s_1}{\nu_0^3}.
\end{aligned}\right|\qquad(56.11)$$

Setting $s_1=L$, $\nu_0=R$, in this instance from (56.11), (56.4) we drive the formulas for a cylindrical shell:

$$\left.\begin{aligned}
\Delta_e^1&=\frac{1}{\pi EhR^3}\left[-\frac{PL^3}{3}-2(1+\mu)PLR^2+M\frac{L^2}{2}\right],\\
\omega_y&=\frac{1}{\pi EhR^3}\left(-P\frac{L^2}{2}+ML\right),
\end{aligned}\right\}\qquad(56.12)$$

which are distinguished from formulas (36.21), obtained earlier, only by the lack of terms of order $\frac{1}{\gamma}$ in comparison with the main terms.

CHAPTER V

THE SPHERICAL SHELL

§ 57. Axisymmetric Deformation of a Spherical Shell

Let us examine a shell of constant thickness whose middle surface is spherical. The principal radii of curvature of this surface are constant and mutually equal:

$$R_1 = R_2 = R. \qquad (57.1)$$

The radius of the parallel circle θ = const and coordinate Z of points of the middle surface which belong to the given circle, on selecting the system of coordinates shown in Fig. 28 are equal to

$$v = R \sin\theta, \quad Z = -R(1 - \cos\theta). \qquad (57.2)$$

On the shell acts an axisymmetric load. In Chapter II it was shown that the calculation of a shell of revolution for an axisymmetric load is reduced to solving system of equation (12.6). The condition of equilibrium of the finite side of the shell contained between two parallel sections θ, θ_0 (condition (11.3)) is also there. For a spherical shell it assumes the form

$$T_1 \sin\theta - N_1 \cos\theta = \frac{P_0}{2\pi R \sin\theta} + \frac{R}{\sin\theta} \int_{\theta_0}^{\theta} q_z \sin\theta \, d\theta. \qquad (57.3)$$

The condition of equilibrium of the shell as a whole is expressed by the equality

$$P+\int_0^{\theta_0} q_z 2\pi r R\, d\theta - P_0 = 0. \qquad (57.4)$$

where P – concentrated force applied in the top of the shell.

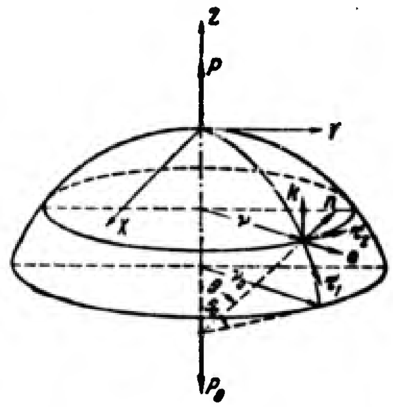


Fig. 28. Spherical shell.

Taking into account (57.4), equation (57.3) can be written in another form, namely:

$$T_1 \sin\theta - N_1 \cos\theta = \frac{P}{2\pi R \sin\theta} + \frac{R}{\sin\theta}\int_0^{\theta} q_z \sin\theta\, d\theta. \qquad (57.5)$$

Equation (57.5) is the condition of equilibrium of that part of the shell containing the top and bounded by section θ = const. In (57.5) making θ approach zero, it is easy to see that in the presence of a concentrated force in the top of the shell shearing force at $\theta = 0$ turns into infinity as $-\frac{P}{2\pi R \sin\theta}$. Selecting as constant parameter b the radius of the sphere and taking into account (57.1) and (57.2), the basic system of resolvant equations (12.6) we write in the form

$$\left.\begin{aligned} &\frac{d^2V_0}{d\theta^2} + \operatorname{ctg}\theta \frac{dV_0}{d\theta} + V_0(\mu - \operatorname{ctg}^2\theta) - \Psi_0 = \\ &\qquad = \frac{1}{R\sin\theta}\left(\mu \frac{d\Phi_1}{d\theta} + \Phi_1 \operatorname{ctg}\theta\right), \\ &\frac{d^2\Psi_0}{d\theta^2} + \operatorname{ctg}\theta \frac{d\Psi_0}{d\theta} - \Psi_0(\mu + \operatorname{ctg}^2\theta) + 4\gamma^4 V_0 = -\frac{4\gamma^4}{R\sin\theta}\Phi_2. \end{aligned}\right\} \qquad (57.6)$$

where

$$\left.\begin{aligned} &\frac{\Phi_1(\theta)}{R^2} = -\cos\theta \int_{\theta_0}^{\theta} q_r \sin\theta\, d\theta + \sin\theta\left(\frac{P_0}{2\pi R^2} + \int_{\theta_0}^{\theta} q_z \sin\theta\, d\theta\right), \\ &\qquad 4\gamma^4 = 12(1-\mu^2)\frac{R^2}{h^2}, \\ &\frac{\Phi_2(\theta)}{R^2} = -\sin\theta \int_{\theta_0}^{\theta} q_r \sin\theta\, d\theta - \cos\theta\left(\frac{P_0}{2\pi R^2} + \int_{\theta_0}^{\theta} q_z \sin\theta\, d\theta\right). \end{aligned}\right\} \qquad (57.7)$$

q_r, q_z — as earlier designate the components of distributed external load in the radial and axial directions. Forces and moments are expressed through power function V_0 and angle of rotation

$$\vartheta_1 = \frac{1}{Eh}\Psi_0 \tag{57.8}$$

according to the formulas

$$T_1 = V_0 \operatorname{ctg}\theta + \frac{\Phi_1}{R\sin\theta}, \quad T_2 = \frac{dV_0}{d\theta}, \quad N_1 = V_0 + \frac{\Phi_2}{R\sin\theta}. \tag{57.9}$$

$$\left.\begin{aligned} M_1 &= -\frac{D}{R}\left(\frac{d\vartheta_1}{d\theta} + \mu\vartheta_1 \operatorname{ctg}\theta\right), \\ M_2 &= -\frac{D}{R}\left(\vartheta_1 \operatorname{ctg}\theta + \mu\frac{d\vartheta_1}{d\theta}\right), \quad D = \frac{Eh^3}{12(1-\mu^2)}. \end{aligned}\right\} \tag{57.10}$$

We transform equation (57.6) with the aid of the replacement

$$V_0 = V_* - \frac{\Phi_2}{R\sin\theta}, \quad \Psi_0 = \Psi_*, \tag{57.11}$$

then we obtain

$$\left.\begin{aligned} &\frac{d^2V_*}{d\theta^2} + \operatorname{ctg}\theta\frac{dV_*}{d\theta} + V_*(\mu - \operatorname{ctg}^2\theta) - \Psi_* = H(\theta), \\ &\frac{d^2\Psi_*}{d\theta^2} + \operatorname{ctg}\theta\frac{d\Psi_*}{d\theta} - \Psi_*(\mu + \operatorname{ctg}^2\theta) + 4\gamma^4V_* = 0, \end{aligned}\right\} \tag{57.12}$$

where
$$H(\theta) = \frac{1}{R\sin\theta}\left[\mu\frac{d\Phi_1}{d\theta} + \Phi_1\operatorname{ctg}\theta + \mu\Phi_2 + \Phi_2 + \frac{d^2\Phi_2}{d\theta^2} - \operatorname{ctg}\theta\frac{d\Phi_2}{d\theta}\right].$$

This expression allowing for (57.7) is transformed to the form

$$H(\theta) = -(1+\mu)Rq_1 - R\frac{dq_n}{d\theta}, \tag{57.13}$$

where q_1, q_n — load components in the directions τ_1, n.

The right sides of equations (57.12) are favorably distinguished from the right sides of basic equations (57.6) by the fact that they contain functions bounded at θ = 0. This is achieved because of replacement (57.11), which allowed separating the zero-moment part of the solution:

$$V_0 = -\frac{\Phi_2}{R\sin\theta}, \quad \vartheta_1 = 0. \tag{57.14}$$

To solution (57.14) correspond forces and displacements of the zero-moment state:

$$
\left.
\begin{aligned}
T_1 &= \frac{1}{\sin^2\theta}\left(\frac{P_0}{2\pi R} + R\int_{\theta_0}^{\theta} q_z \sin\theta\, d\theta\right) = \\
&\qquad = \frac{1}{\sin^2\theta}\left(\frac{P}{2\pi R} + R\int_{0}^{\theta} q_z \sin\theta\, d\theta\right), \\
T_2 &= -\frac{1}{\sin^2\theta}\left(\frac{P_0}{2\pi R} + R\int_{\theta_0}^{\theta} q_z \sin\theta\, d\theta\right) + Rq_n, \\
N_1 &= M_1 = M_2 = 0.
\end{aligned}
\right\} \qquad (57.15)
$$

$$
\left.
\begin{aligned}
\Delta_e &= \frac{R^2 q_n}{Eh}\sin\theta - \frac{(1+\mu)}{2\pi Eh}\,\frac{P}{\sin\theta} - \frac{(1+\mu)R^2}{Eh\sin\theta}\int_{0}^{\theta} q_z \sin\theta\, d\theta, \\
\Delta_z &= C + \frac{\mu R^2}{Eh}\int_{\theta_0}^{\theta} q_n \sin\theta\, d\theta - \frac{(1+\mu)}{2\pi Eh} P \ln \operatorname{tg}\frac{\theta}{2} - \\
&\qquad - \frac{(1+\mu)R^2}{Eh}\int_{\theta_0}^{\theta}\frac{1}{\sin\theta}\int_{0}^{\theta} q_z \sin\theta\, d\theta\, d\theta.
\end{aligned}
\right\} \qquad (57.16)
$$

where C — axial displacement of the shell as a solid.

To construct the total stressed state to forces of the zero-moment solution (57.15) it is necessary to add the forces and moments calculated on the basis of the general solution of system (57.12).

§ 58. Particular Solution of Meissner Equations for Different Loads

System of equations (57.12) is heterogeneous. Let us construct the particular solution of this system for loads of different form.

Most frequently we find such loads, as a weight load, a load by hydrostatic pressure, linearly changing along axis OZ, and a load by centrifugal forces. To them correspond components of distributed surface load along directions k, n, e:

$$q_z = -\rho g h, \quad q_n = p - aZ, \quad q_e = \rho\omega^2 h v. \qquad (58.1)$$

where ρ — mass density of material of shell, ω — angular velocity of revolution.

The total load has components along axes τ_1, n

$$\left.\begin{aligned} q_1 &= \rho\omega^2 hR\sin\theta\cos\theta + \rho gh\sin\theta, \\ q_n &= p - aZ + \rho\omega^2 hR\sin^2\theta - \rho gh\cos\theta. \end{aligned}\right\} \tag{58.2}$$

For loads (58.2) function $H(\theta)$ has the form

$$H(\theta) = -[aR^2 + (2+\mu)\rho ghR]\sin\theta - (3+\mu)\rho\omega^2 hR^2\sin\theta\cos\theta. \tag{58.3}$$

The particular solution of equations (57.12) for this case are needed in the form [12]

$$\left.\begin{aligned} \Psi_* &= A_1\sin\theta + A_2\sin\theta\cos\theta, \\ V_* &= B_1\sin\theta + B_2\sin\theta\cos\theta. \end{aligned}\right\} \tag{58.4}$$

Substituting (58.4) into (57.12) and comparing the coefficients of similar terms, we obtain

$$\left.\begin{aligned} A_1 &= aR^2 + (2+\mu)\rho ghR, & B_1 &= \frac{1+\mu}{4\gamma^4}A_1, \\ A_2 &= (3+\mu)\rho\omega^2 hR^2\sin\theta\cos\theta, & B_2 &= \frac{(5+\mu)}{4\gamma^4}A_2. \end{aligned}\right\} \tag{58.5}$$

Note that during the calculation of A_1, A_2 terms of order h^2/R^2 in comparison with unity were dropped. From (58.4), (58.5) and (57.11) it follows that in comparison with the zero-moment solution (57.14) particular solution V_* calculated according to (58.4), can also be rejected as a quantity of order h^2/R^2 in comparison with unity. Particular solution Ψ_* and the corresponding value of angle of rotation

$$\vartheta_1 = Eh[aR^2 + (2+\mu)\rho ghR]\sin\theta + Eh(3+\mu)\rho\omega^2 hR^2\sin\theta\cos\theta \tag{58.6}$$

after calculating on its basis the bending moments lead to the values of flexural stress, of zero-moment order, multiplied by small quantity h/R. Therefore in zones far from the edges, flexural stresses from solution (58.6) also can be neglected. However, during the calculation of edge effect deletion of particular solution (58.6) can lead already to errors of order $\sqrt{\frac{h}{R}}$, in comparison with unity. A similar situation occurred during calculation of the particular solution of equation (25.27) for a cylindrical shell. Note that just as for a cylindrical shell for a sphere the most common load – uniform internal pressure – as the particular solution of basic equations (57.6) without damaging correctness can be taken zero-moment solution. Really, the right part of (58.6) does not contain pressure p. However, in the case of loads of general form in making up boundary conditions solution (58.6) should be taken into account.

§ 59. Linearly Independent Solutions of Uniform Meissner Equations for a Spherical Shell

Turn now to the solution of uniform system of equation (57.12). By the introduction of the complex combination

$$\sigma_{*}=\Psi_{*}-2i\gamma^{2}V_{*} \qquad (59.1)$$

the system is reduced to one equation in σ_{*}

$$\frac{d^{2}\sigma_{*}}{d\theta^{2}}+\operatorname{ctg}\theta\frac{d\sigma_{*}}{d\theta}-\mu\bar{\sigma}_{*}+\sigma_{*}(2i\gamma^{2}-\operatorname{ctg}^{2}\theta)=0.$$

Dropping in this equation $\mu\bar{\sigma}_{*}$ in comparison with $2i\gamma^{2}\sigma_{*}$, finally we obtain

$$\frac{d^{2}\sigma_{*}}{d\theta^{2}}+\operatorname{ctg}\theta\frac{d\sigma_{*}}{d\theta}+\sigma_{*}(2i\gamma^{2}-\operatorname{ctg}^{2}\theta)=0. \qquad (59.2)$$

By substitution

$$\sigma_{*}=\frac{d\sigma}{d\theta} \qquad (59.3)$$

equation (59.3) turns into

$$\frac{d}{d\theta}\left[\frac{d^{2}\sigma}{d\theta^{2}}+\operatorname{ctg}\theta\frac{d\sigma}{d\theta}+(2\gamma^{2}i+1)\sigma\right]=0. \qquad (59.4)$$

Equation (59.4) will be satisfied by any solution of the Legendre equation

$$\frac{d^{2}\sigma}{d\theta^{2}}+\operatorname{ctg}\theta\frac{d\sigma}{d\theta}+(n+1)n\sigma=0 \qquad (59.5)$$

with complex parameter

$$(n+1)n=2\gamma^{2}i+1.$$

The last equality we rewrite thus:

$$\left(n+\frac{1}{2}\right)^{2}=2i\gamma^{2}+\frac{5}{4}.$$

or, dropping $\frac{5}{4}$ in comparison with $2i\gamma^{2}$.

$$\left(n+\frac{1}{2}\right)^2 = 2i\gamma^2. \quad (59.6)$$

The general solution of equation (59.5) can be represented in the form

$$\sigma = AP_n(\cos\theta) + BH_n(\cos\theta). \quad (59.7)$$

where $P_n(\cos\theta)$ — solution of Legendre equation, regular in the pole of the sphere, i.e., at θ = 0 [142]. $H_n(\cos\theta)$ is a linear combination from the solution $P_n(\cos\theta)$ and a Lengendre function of the second kind $Q_n(\cos\theta)$, having singularity in the top

$$H_n(\cos\theta) = P_n(\cos\theta) - \frac{2i}{\pi} Q_n(\cos\theta). \quad (59.8)$$

Substituting (59.7) into (59.3), we obtain

$$\sigma_\theta = AP_n^1(\cos\theta) + B\left[P_n^1(\cos\theta) - \frac{2i}{\pi} Q_n^1(\cos\theta)\right] = \\ = AP_n^1(\cos\theta) + BH_n^1(\cos\theta), \quad (59.9)$$

where $P_n^1(\cos\theta)$, $Q_n^1(\cos\theta)$ — first associated Legendre functions:

$$\left.\begin{aligned} P_n^1(\cos\theta) &= -\sin\theta P_n'(\cos\theta), \\ Q_n^1(\cos\theta) &= -\sin\theta Q_n'(\cos\theta), \\ H_n^1(\cos\theta) &= -\sin\theta H_n'(\cos\theta), \end{aligned}\right\} \quad (59.10)$$

the prime indicates differentiation with respect to the argument in parentheses. Solution (59.9) is inconvenient for practical calculations, since there are no tables of Legendre functions with a complex parameter. Therefore usually to facilitate calculations we use an approximate representation of solution (59.9), employing the fact that for thin-walled shells parameter γ, and consequently, the modulus of the parameter n, is large in comparison with unity. The behavior of Legendre functions at $n \to \infty$ has been studied and can be expressed by the relationships:

$$\left.\begin{aligned} \lim_{n\to\infty}\left[n^{-1}P_n^1\left(\cos\frac{a}{n}\right)\right] &= -I_1(a), \\ \lim_{n\to\infty}\left[n^{-1}Q_n^1\left(\cos\frac{a}{n}\right)\right] &= \frac{\pi}{2} N_1(a), \end{aligned}\right\} \quad (59.11)$$

where $I_1(a)$, $N_1(a)$ — of Bessel and Neuman functions of the first order, taking into account that

$$I_1(a) + iN_1(a) = H_1^{(1)}(a).$$

where $H_1^{(1)}(a)$ — is a Hankel function of the first kind of the first order, it is easy to see also that

$$\lim_{n\to\infty}\left\{n^{-1}\left[P_n^1\left(\cos\frac{a}{n}\right)-\frac{2i}{\pi}Q_n^1\left(\cos\frac{a}{n}\right)\right]\right\}=-H_1^{(1)}(a). \qquad (59.12)$$

Using the indicated property of Legendre functions and setting $n=\gamma\sqrt{2}$.. we replace approximately

$$\left.\begin{array}{l} P_n^1(\cos\theta) \text{ by } -(\gamma\sqrt{2i})I_1(\gamma\theta\sqrt{2i}). \\ H_n^1(\cos\theta) \text{ by } -(\gamma\sqrt{2i})H_1^{(1)}(\gamma\theta\sqrt{2i}). \end{array}\right\} \qquad (59.13)$$

Such a replacement will be accurate enough for large γ, but small θ, such that $\gamma\theta$ is a magnitude of the order of unity. One can be certain of this on the basis of the following transformation of basic equation (59.2). Let us make the substitution

$$\sigma_0=\frac{\tau}{\sqrt{\sin\theta}}$$

we obtain in τ the equation

$$\frac{d^2\tau}{d\theta^2}+\left(2i\gamma^2+\frac{1}{2}-\frac{3}{4}\operatorname{ctg}^2\theta\right)\tau=0. \qquad (59.14)$$

We will consider that θ changes within limits from 0 to θ_0, while θ_0 can only insignificantly exceed $\frac{\pi}{2}$. At large θ the term $\frac{1}{2}-\frac{3}{4}\operatorname{ctg}^2\theta$ has the order of unity and does not play an essential role in comparison with $2i\gamma^2$, at small θ we have the approximate equality

$$\frac{1}{2}-\frac{3}{4}\operatorname{ctg}^2\theta\approx-\frac{3}{4}\frac{1}{\theta^2}.$$

In this way, instead of equation (59.14) we can consider the equation

$$\frac{d^2\tau}{d\theta^2}+\left(2i\gamma^2-\frac{3}{4\theta^2}\right)\tau=0,$$

the solution of which is expressed through known functions of Bessel and Hankel in the following manner:

$$\tau=A\sqrt{\theta}\,I_1(\gamma\theta\sqrt{2i})+B\sqrt{\theta}\,H_1^{(1)}(\gamma\theta\sqrt{2i}).$$

Thus, in this instance σ_0 has the form

$$\sigma_0=A\sqrt{\frac{\theta}{\sin\theta}}\,I_1(\gamma\theta\sqrt{2i})+B\sqrt{\frac{\theta}{\sin\theta}}\,H_1^{(1)}(\gamma\theta\sqrt{2i}) \qquad (59.15)$$

or near $\theta=0$

$$\sigma_0=AI_1(\gamma\theta\sqrt{2i})+BH_1^{(1)}(\gamma\theta\sqrt{2i}). \qquad (59.16)$$

The functions in this solution differ from (59.13) by only a constant factor. Note that representation (59.15) is an approximately accurate solution of equation (59.2) not only at small θ. The correctness of this representation can be evaluated by comparing the asymptotic expansion for Legendre functions for large n and for the Bessel functions at a large value of the argument $\gamma\theta\sqrt{2i}$ [79], [81]. In this case we will hold that $\gamma\sin\theta$ is sufficiently large in comparison with unity and in the asymptotic expansion we can be limited to holding only two higher terms $\left(\text{of the order of unity and } \frac{1}{\gamma}\right)$. For example,

$$P_n^1 = \sqrt{\frac{\gamma}{2\pi\sqrt{2}\sin\theta}}\left\{\left[\left(1-\frac{3\operatorname{ctg}\theta}{8\gamma}\right)\cos\left(\gamma\theta-\frac{\pi}{8}\right)-\sin\left(\gamma\theta-\frac{\pi}{8}\right)\right]-\right.$$
$$\left.-i\left[\cos\left(\gamma\theta-\frac{\pi}{8}\right)+\left(1-\frac{3\operatorname{ctg}\theta}{8\gamma}\right)\sin\left(\gamma\theta-\frac{\pi}{8}\right)\right]\right\}e^{\gamma\theta}. \qquad (59.17)$$

Furthermore, we have

$$-\gamma\sqrt{2i}\,I_1(\gamma\theta\sqrt{2i})=\gamma\sqrt{2i}\frac{d}{d(\theta\gamma\sqrt{2i})}[I_0(\gamma\theta\sqrt{2i})]=$$
$$=\gamma\sqrt{2}\left[\psi_1'(\gamma\theta\sqrt{2})+i\psi_2'(\gamma\theta\sqrt{2})\right]. \qquad (59.18)$$

Using the asymptotic representations for functions ψ_1' and ψ_1' given in Chapter IV, we obtain

$$-\gamma\sqrt{2i}\sqrt{\frac{\theta}{\sin\theta}}\,I_1(\gamma\theta\sqrt{2i})=\sqrt{\frac{\gamma}{2\pi\sqrt{2}\sin\theta}}\left\{\left[\left(1-\frac{3}{8\gamma\theta}\right)\cos\left(\gamma\theta-\frac{\pi}{8}\right)-\sin\left(\gamma\theta-\frac{\pi}{8}\right)\right]-\right.$$
$$\left.-i\left[\cos\left(\gamma\theta-\frac{\pi}{8}\right)+\left(1-\frac{3}{8\gamma\theta}\right)\sin\left(\gamma\theta-\frac{\pi}{8}\right)\right]\right\}e^{\gamma\theta} \qquad (59.19)$$

Comparing the right side of (59.17) and (59.19), we find that in the main terms they agree. In terms of the order of $\frac{1}{\gamma}$ in comparison with unity, the coincidence is good only at small θ.

In this way solution (59.15) can be used even for large θ, however, during calculations on the basis of this solution in this instance it is of no consequence to keep terms of order $\frac{1}{\gamma}$ in comparison with unity. Specifically, the factor $\sqrt{\frac{\theta}{\sin\theta}}$ during differentiation must be kept constant. If it is wished to obtain a result with the retention of small terms, it is necessary to use asymptotic representations for the Legendre functions. Let us designate the particular solutions from which (59.9) has been constructed through

$$P_n^1(\cos\theta)=\sigma_{\theta 1}, \quad H_n^1(\cos\theta)=\sigma_{\theta 2}$$

and use the asymptotic representations for $\sigma_{\circ 1}$, $\sigma_{\circ 2}$ and their derivatives. The asymptotic representation for $\sigma_{\circ 1}$ already has been given by equation (59.17). The representation for $\sigma_{\circ 2}$ has the form

$$\sigma_{\circ 2}=\sqrt{\frac{\sqrt{2}\,\gamma}{\pi\sin\theta}}\,e^{-\gamma\theta}\Big\{\Big[\cos\Big(\gamma\theta+\frac{\pi}{8}\Big)-\Big(1+\frac{3\,\mathrm{ctg}\,\theta}{8\gamma}\Big)\sin\Big(\gamma\theta+\frac{\pi}{8}\Big)\Big]+ + i\Big[\Big(1+\frac{3\,\mathrm{ctg}\,\theta}{8\gamma}\Big)\cos\Big(\gamma\theta+\frac{\pi}{8}\Big)+\sin\Big(\gamma\theta+\frac{\pi}{8}\Big)\Big]\Big\}. \quad (59.20)$$

To compose $\frac{d\sigma_{\circ 1}}{d\theta}$, $\frac{d\sigma_{\circ 2}}{d\theta}$ we use (59.3), (59.5) and (59.7); then we obtain

$$\frac{d\sigma_{\circ 1}}{d\theta}=\frac{d^2\sigma_1}{d\theta^2}=-\mathrm{ctg}\,\theta P_n^1(\cos\theta)-2i\gamma^2P_n(\cos\theta),$$

$$\frac{d\sigma_{\circ 2}}{d\theta}=\frac{d^2\sigma_2}{d\theta^2}=-\mathrm{ctg}\,\theta\Big[P_n^1(\cos\theta)-\frac{2i}{\pi}Q_n^1(\cos\theta)\Big]- -2i\gamma^2\Big[P_n(\cos\theta)-\frac{2i}{\pi}Q_n(\cos\theta)\Big]$$

whence, taking into account the expansion

$$P_n(\cos\theta)\approx \approx\frac{1}{\sqrt{2\pi\gamma\sqrt{2}\sin\theta}}e^{\gamma\theta}\Big\{\Big[\Big(1+\frac{\mathrm{ctg}\,\theta}{16\gamma}\Big)\cos\Big(\gamma\theta-\frac{\pi}{8}\Big)+\frac{\mathrm{ctg}\,\theta}{16\gamma}\sin\Big(\gamma\theta-\frac{\pi}{8}\Big)\Big]+ + i\Big[\frac{\mathrm{ctg}\,\theta}{16\gamma}\cos\Big(\gamma\theta-\frac{\pi}{8}\Big)-\Big(1+\frac{\mathrm{ctg}\,\theta}{16\gamma}\Big)\sin\Big(\gamma\theta-\frac{\pi}{8}\Big)\Big]\Big\}.$$

$$H_n(\cos\theta)=P_n(\cos\theta)-\frac{2i}{\pi}Q_n(\cos\theta)= =\sqrt{\frac{\sqrt{2}}{\pi\gamma\sin\theta}}\,e^{-\gamma\theta}\Big\{\Big[\sin\Big(\gamma\theta+\frac{\pi}{8}\Big)\Big(1-\frac{\mathrm{ctg}\,\theta}{16\gamma}\Big)-\frac{\mathrm{ctg}\,\theta}{16\gamma}\cos\Big(\gamma\theta+\frac{\pi}{8}\Big)\Big]- - i\Big[\Big(1-\frac{\mathrm{ctg}\,\theta}{16\gamma}\Big)\cos\Big(\gamma\theta+\frac{\pi}{8}\Big)+\frac{\mathrm{ctg}\,\theta}{16\gamma}\sin\Big(\gamma\theta+\frac{\pi}{8}\Big)\Big]\Big\}.$$

we have

$$\frac{d\sigma_{\circ 1}}{d\theta}\approx-2\gamma\sqrt{\frac{\gamma}{2\pi\sqrt{2}\sin\theta}}\times \times e^{\gamma\theta}\Big\{\Big[\frac{7\,\mathrm{ctg}\,\theta}{16\gamma}\cos\Big(\gamma\theta-\frac{\pi}{8}\Big)+\Big(1-\frac{7\,\mathrm{ctg}\,\theta}{16\gamma}\Big)\sin\Big(\gamma\theta-\frac{\pi}{8}\Big)\Big]+ + i\Big[\Big(1-\frac{7\,\mathrm{ctg}\,\theta}{16\gamma}\Big)\cos\Big(\gamma\theta-\frac{\pi}{8}\Big)-\frac{7\,\mathrm{ctg}\,\theta}{16\gamma}\sin\Big(\gamma\theta-\frac{\pi}{8}\Big)\Big]\Big\}. \quad (59.21)$$

$$\frac{d\sigma_{\circ 2}}{d\theta}=-2\gamma\sqrt{\frac{\gamma\sqrt{2}}{\pi\sin\theta}}\times \times e^{-\gamma\theta}\Big\{\Big[\Big(1+\frac{7\,\mathrm{ctg}\,\theta}{16\gamma}\Big)\cos\Big(\gamma\theta+\frac{\pi}{8}\Big)-\frac{7\,\mathrm{ctg}\,\theta}{16\gamma}\sin\Big(\gamma\theta+\frac{\pi}{8}\Big)\Big]+ + i\Big[\frac{7\,\mathrm{ctg}\,\theta}{16\gamma}\cos\Big(\gamma\theta+\frac{\pi}{8}\Big)+\Big(1+\frac{7\,\mathrm{ctg}\,\theta}{16\gamma}\Big)\sin\Big(\gamma\theta+\frac{\pi}{8}\Big)\Big]\Big\}. \quad (59.22)$$

The obtained equations are conveniently represented in the form

$$\left.\begin{aligned}\sigma_{\circ 1}&=\chi_1(\theta)+i\chi_2(\theta)\approx\sqrt{\frac{\gamma}{2\pi\sqrt{2}\sin\theta}}\,e^{\gamma\theta}[a_1(\theta)+ia_2(\theta)],\\ \frac{d\sigma_{\circ 1}}{d\theta}&=\chi_1'(\theta)+i\chi_2'(\theta)\approx\\ &\approx-2\gamma\sqrt{\frac{\gamma}{2\pi\sqrt{2}\sin\theta}}\,e^{\gamma\theta}[\tilde{a}_1(\theta)+i\tilde{a}_2(\theta)].\end{aligned}\right\} \quad (59.23)$$

$$\sigma_{e2} = \chi_3(\theta) + i\chi_4(\theta) = \sqrt{\frac{\gamma\sqrt{2}}{\pi \sin\theta}}\, e^{-\gamma\theta}\,[a_3(\theta) + i a_4(\theta)],$$
$$\frac{d\sigma_{e2}}{d\theta} = \chi_3'(\theta) + i\chi_4'(\theta) = -2\gamma\sqrt{\frac{\gamma\sqrt{2}}{\pi \sin\theta}}\, e^{-\gamma\theta}\,[\tilde{a}_3(\theta) + i \tilde{a}_4(\theta)]. \quad (59.24)$$

$$\sigma_1 = \omega_1 + i\omega_2 = \frac{1}{\sqrt{2\pi\gamma\sqrt{2}\sin\theta}}\, e^{\gamma\theta}\,[b_1(\theta) + i b_2(\theta)],$$
$$\sigma_2 = \omega_3 + i\omega_4 = \sqrt{\frac{\sqrt{2}}{\pi\gamma \sin\theta}}\, e^{-\gamma\theta}\,[b_3(\theta) + i b_4(\theta)]. \quad (59.25)$$

where

$$a_1(\theta) = \left(1 - \frac{3\,\mathrm{ctg}\,\theta}{8\gamma}\right)\cos\left(\gamma\theta - \frac{\pi}{8}\right) - \sin\left(\gamma\theta - \frac{\pi}{8}\right),$$
$$a_2(\theta) = -\left[\cos\left(\gamma\theta - \frac{\pi}{8}\right) + \left(1 - \frac{3\,\mathrm{ctg}\,\theta}{8\gamma}\right)\sin\left(\gamma\theta - \frac{\pi}{8}\right)\right],$$
$$\tilde{a}_1(\theta) = \frac{7\,\mathrm{ctg}\,\theta}{16\gamma}\cos\left(\gamma\theta - \frac{\pi}{8}\right) + \left(1 - \frac{7\,\mathrm{ctg}\,\theta}{16\gamma}\right)\sin\left(\gamma\theta - \frac{\pi}{8}\right),$$
$$\tilde{a}_2(\theta) = \left(1 - \frac{7\,\mathrm{ctg}\,\theta}{16\gamma}\right)\cos\left(\gamma\theta - \frac{\pi}{8}\right) - \frac{7\,\mathrm{ctg}\,\theta}{16\gamma}\sin\left(\gamma\theta - \frac{\pi}{8}\right),$$
$$a_3(\theta) = \cos\left(\gamma\theta + \frac{\pi}{8}\right) - \left(1 + \frac{3\,\mathrm{ctg}\,\theta}{8\gamma}\right)\sin\left(\gamma\theta + \frac{\pi}{8}\right),$$
$$a_4(\theta) = \left(1 + \frac{3\,\mathrm{ctg}\,\theta}{8\gamma}\right)\cos\left(\gamma\theta + \frac{\pi}{8}\right) + \sin\left(\gamma\theta + \frac{\pi}{8}\right),$$
$$\tilde{a}_3(\theta) = \left(1 + \frac{7\,\mathrm{ctg}\,\theta}{16\gamma}\right)\cos\left(\gamma\theta + \frac{\pi}{8}\right) - \frac{7\,\mathrm{ctg}\,\theta}{16\gamma}\sin\left(\gamma\theta + \frac{\pi}{8}\right),$$
$$\tilde{a}_4(\theta) = \frac{7\,\mathrm{ctg}\,\theta}{16\gamma}\cos\left(\gamma\theta + \frac{\pi}{8}\right) + \left(1 + \frac{7\,\mathrm{ctg}\,\theta}{16\gamma}\right)\sin\left(\gamma\theta + \frac{\pi}{8}\right),$$
$$b_1(\theta) = \left(1 + \frac{\mathrm{ctg}\,\theta}{16\gamma}\right)\cos\left(\gamma\theta - \frac{\pi}{8}\right) + \frac{\mathrm{ctg}\,\theta}{16\gamma}\sin\left(\gamma\theta - \frac{\pi}{8}\right),$$
$$b_2(\theta) = \frac{\mathrm{ctg}\,\theta}{16\gamma}\cos\left(\gamma\theta - \frac{\pi}{8}\right) - \left(1 + \frac{\mathrm{ctg}\,\theta}{16\gamma}\right)\sin\left(\gamma\theta - \frac{\pi}{8}\right),$$
$$b_3(\theta) = \left(1 - \frac{\mathrm{ctg}\,\theta}{16\gamma}\right)\sin\left(\gamma\theta + \frac{\pi}{8}\right) - \frac{\mathrm{ctg}\,\theta}{16\gamma}\cos\left(\gamma\theta + \frac{\pi}{8}\right),$$
$$b_4(\theta) = -\left[\left(1 - \frac{\mathrm{ctg}\,\theta}{16\gamma}\right)\cos\left(\gamma\theta + \frac{\pi}{8}\right) + \frac{\mathrm{ctg}\,\theta}{16\gamma}\sin\left(\gamma\theta + \frac{\pi}{8}\right)\right].$$

Summing up the above, we note that the general solution of uniform equation (59.2) allowing for the introduced designations, can be written thus:

$$\sigma_e = (A_1 - iB_1)(\chi_1 + i\chi_2) + (A_2 - iB_2)(\chi_3 + i\chi_4). \quad (59.26)$$

where at sufficiently large γ, but small θ

$$\chi_1 + i\chi_2 = -\gamma\sqrt{2i}\, I_1(\gamma\theta\sqrt{2i}),$$
$$\chi_3 + i\chi_4 = -\gamma\sqrt{2i}\, H_1^{(1)}(\gamma\theta\sqrt{2i}). \quad (59.27)$$

or after separating the real and imaginary parts

$$\chi_1 = \gamma\sqrt{2}\,\psi_1'(\gamma\theta\sqrt{2}), \quad \chi_2 = \gamma\sqrt{2}\,\psi_2'(\gamma\theta\sqrt{2}),$$
$$\chi_3 = \gamma\sqrt{2}\,\psi_3'(\gamma\theta\sqrt{2}), \quad \chi_4 = \gamma\sqrt{2}\,\psi_4'(\gamma\theta\sqrt{2}). \quad (59.28)$$

where $\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x)$ are the real and imaginary parts of functions $I_0(x\sqrt{i}), H_0^{(1)}(x\sqrt{i})$. Their representations in the form of a series are given in Chapter IV, § 46.

For large values of $\gamma\theta$

$$\left.\begin{aligned}\chi_1&=\gamma\sqrt{2}\sqrt{\frac{\theta}{\sin\theta}}\,\psi_1'(\gamma\theta\sqrt{2}),\\ \chi_2&=\gamma\sqrt{2}\sqrt{\frac{\theta}{\sin\theta}}\,\psi_2'(\gamma\theta\sqrt{2}),\\ \chi_3&=\gamma\sqrt{2}\sqrt{\frac{\theta}{\sin\theta}}\,\psi_3'(\gamma\theta\sqrt{2}),\\ \chi_4&=\gamma\sqrt{2}\sqrt{\frac{\theta}{\sin\theta}}\,\psi_4'(\gamma\theta\sqrt{2}),\end{aligned}\right\} \tag{59.29}$$

where in asymptotic expansion for functions $\psi_1', \psi_2', \psi_3', \psi_4'$ now one ought to keep only the main terms, of the order of unity. If it is desired to retain in calculations terms of order $1/\gamma$ in comparison with unity at sufficiently large $\gamma \sin\theta$ the solution must be taken in the form of (59.23), (59.24).

§ 60. Deformation of a Spherical Dome

Let us propose now that the zero-moment solution and general solution of system of equations (57.12) in some form have been constructed. Let us derive the formulas for calculation of forces and displacements in a spherical shell, not loaded in the top by a concentrated force, i.e., the case of a dome with summit at $P=0$.

In the neighborhood of the summit $\chi_3+i\chi_4$ behaves as $H_1^{(1)}(\gamma\theta\sqrt{2i})$, i.e., turns into infinity at $\theta = 0$ as $1/\theta$. In the absence of concentrated force in the summit the solution should be finite at $\theta = 0$. Therefore we must set $A_2=B_2=0$ and

$$\sigma_*=(A_1-iB_1)(\chi_1+i\chi_2). \tag{60.1}$$

In this case on the basis of formulas (57.8), (57.9), (57.10), (57.11) we obtain the following expressions for forces and moments:

$$\left.\begin{aligned}
&T_1=\tilde{T}_1-\frac{1}{2\gamma^2}\operatorname{ctg}\theta(A_1\chi_2-B_1\chi_1),\quad N_1=-\frac{1}{2\gamma^2}(A_1\chi_2-B_1\chi_1),\\
&H_\theta=T_1\cos\theta+N_1\sin\theta=\tilde{T}_1\cos\theta-\frac{1}{2\gamma^2\sin\theta}(A_1\chi_2-B_1\chi_1),\\
&T_2=\tilde{T}_2-\frac{1}{2\gamma^2}(A_1\chi_2'-B_1\chi_1'),\\
&M_1=-\frac{R}{4\gamma^4}\left[A_1(\chi_1'+\mu\operatorname{ctg}\theta\chi_1)+B_1(\chi_2'+\mu\operatorname{ctg}\theta\chi_2)\right]+\tilde{M}_1,\\
&M_2=-\frac{R}{4\gamma^4}\left[A_1(\operatorname{ctg}\theta\chi_1+\mu\chi_1')+B_1(\operatorname{ctg}\theta\chi_2+\mu\chi_2')\right]+\tilde{M}_2,
\end{aligned}\right\}\qquad (60.2)$$

where through $\tilde{T}_1$, $\tilde{T}_2$ we designate forces of the zero-moment state, determined according to (57.15) at $P=0$; $\tilde{M}_1$, $\tilde{M}_2$ — bending moments from particular solution (58.4), (58.5), computable using the formulas

$$\left.\begin{aligned}
&\tilde{M}_1=-\frac{R}{4\gamma^4}\left(\frac{d\Psi_*}{d\theta}+\mu\operatorname{ctg}\theta\Psi_*\right),\\
&\tilde{M}_2=-\frac{R}{4\gamma^4}\left(\operatorname{ctg}\theta\Psi_*+\mu\frac{d\Psi_*}{d\theta}\right),
\end{aligned}\right\}\qquad (60.3)$$

$$\chi_1'=\frac{d\chi_1}{d\theta},\quad \chi_2'=\frac{d\chi_2}{d\theta},\quad 4\gamma^4=12(1-\mu^2)\frac{R^2}{h^2}.\qquad (60.4)$$

According to the above we ignored during calculation of forces the particular solution V_* (the second equation of (58.4)). After determination of forces we easily find the displacements

$$\vartheta_1=\tilde{\vartheta}_1+\frac{1}{Eh}(A_1\chi_1+B_1\chi_2).\qquad (60.5)$$

$$\Delta_\theta=\tilde{\Delta}_\theta-\frac{R\sin\theta}{2\gamma^2Eh}\left[A_1(\chi_2'-\mu\operatorname{ctg}\theta\chi_2)-B_1(\chi_1'-\mu\operatorname{ctg}\theta\chi_1)\right].\qquad (60.6)$$

$$\begin{aligned}
\Delta_z=\tilde{\Delta}_z-\frac{R}{2\gamma^2Eh}\int_0^{\theta}\left[A_1(\chi_2\operatorname{ctg}\theta-\mu\chi_2')-B_1(\chi_1\operatorname{ctg}\theta-\mu\chi_1')\right]\sin\theta\,d\theta-\\
-\frac{R}{Eh}\int_0^{\theta}(A_1\chi_1+B_1\chi_2)\cos\theta\,d\theta+C,
\end{aligned}\qquad (60.7)$$

where $\tilde{\Delta}_\theta$ designates radial displacement in the zero-moment stressed state, computable in accordance with (57.16) at $P=0$. $\tilde{\Delta}_z$ is the sum of displacement in the zero-moment state and displacement corresponding to particular solution (58.4)

$$\begin{aligned}
\tilde{\Delta}_z=\frac{(1+\mu)R^2}{Eh}\int_0^{\theta}\frac{1}{\sin\theta}\int_0^{\theta}q_z\sin\theta\,d\theta\,d\theta-\\
-\frac{\mu R^2}{Eh}\int_0^{\theta}q_n\sin\theta\,d\theta-\frac{R}{Eh}\int_0^{\theta}\Psi_*\cos\theta\,d\theta.
\end{aligned}\qquad (60.8)$$

C – constant having the meaning of rigid displacement.

In the right side of (60.7) we can neglect the second term in comparison with the third. In this way, finally axial displacement assumes the form

$$\Delta_z = \tilde{\Delta}_z - \frac{R}{Eh}\int_0^{\theta}(A_1\chi_1 + B_1\chi_2)\cos\theta\, d\theta + C. \qquad (60.9)$$

Apropos of the calculation of the integrals in (60.9), we can say the following. Since

$$\chi_1 = \frac{d\omega_1}{d\theta}, \quad \chi_2 = \frac{d\omega_2}{d\theta},$$

then

$$\int \chi_1 \cos\theta\, d\theta = \omega_1 \cos\theta + \int \omega_1 \sin\theta\, d\theta,$$

$$\int \chi_2 \cos\theta\, d\theta = \omega_2 \cos\theta + \int \omega_2 \sin\theta\, d\theta.$$

But σ_1 satisfies equation (59.5) and, consequently,

$$(n+1)n\sigma_1 \sin\theta = -\frac{d}{d\theta}\left(\sin\theta \frac{d\sigma_1}{d\theta}\right).$$

$$\int \sigma_1 \cos\theta\, d\theta = \sigma_1 \cos\theta - \frac{\sin\theta}{n(n+1)}\frac{d\sigma_1}{d\theta}.$$

In the last expression the second term in comparison with the first is a quantity of order $\frac{1}{\gamma}$, since during differentiation $\frac{d\sigma_1}{d\theta}$ can increase as γ^1. If we disregard this term, then we obtain

$$\int \sigma_1 \cos\theta\, d\theta \approx \sigma_1 \cos\theta$$

consequently,

$$\int \chi_1 \cos\theta\, d\theta \approx \omega_1 \cos\theta, \quad \int \chi_2 \cos\theta\, d\theta \approx \omega_2 \cos\theta. \qquad (60.10)$$

In this way during calculations in the first approximation, i.e., rejecting terms of order $\frac{1}{\gamma}$ in comparison with unity, the slowly changing coefficients of σ_1, σ_2 during integration, just as during differentiation, can be considered as constants.

As an example of the application of the obtained equations we will examine the dome shown in Fig. 29. Let us propose from the beginning that there are no distributed loads, and the dome is loaded only along the edge by distance forces H_c^0 and bending moments M_1^0. Then forces of the zero-moment state and the particular

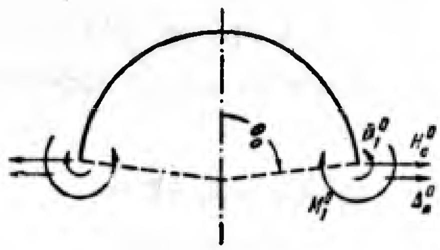


Fig. 29. Spherical dome.

solution of system (57.12) turn into zero. In this way, in formulas (60.2), (60.5), (60.6), (60.9) all quantities marked by a tilde vanish. To determine the constants A_1, B_1 according to radial force H_e^0 and bending moment M_1^0 assigned on edge θ_0 we derive the system of equations:

$$\left.\begin{aligned} &A_1 a_2(\theta_0) - B_1 a_1(\theta_0) = -2\gamma^2 \sin\theta_0 e^{-\gamma\theta_0} \sqrt{\frac{2\pi\sqrt{2}\sin\theta_0}{\gamma}}\, H_e^0, \\ &A_1\left[\tilde{a}_1(\theta_0) - \frac{\mu \operatorname{ctg}\theta_0}{2\gamma} a_1(\theta_0)\right] + \\ &+ B_1\left[\tilde{a}_2(\theta_0) - \frac{\mu \operatorname{ctg}\theta_0}{2\gamma} a_2(\theta_0)\right] = \frac{2\gamma^3}{R} e^{-\gamma\theta_0} \sqrt{\frac{2\pi\sqrt{2}\sin\theta_0}{\gamma}}\, M_1^0. \end{aligned}\right\} \quad (60.11)$$

Solving it, we find

$$\left.\begin{aligned} &A_1 = \frac{2\gamma^2 e^{-\gamma\theta_0}}{\Delta(\theta_0)} \sqrt{\frac{2\pi\sqrt{2}\sin\theta_0}{\gamma}} \left[\sin\theta_0 H_e^0\left(\tilde{a}_2 - \frac{\mu \operatorname{ctg}\theta_0}{2\gamma} a_2\right) - \frac{\gamma}{R} M_1^0 a_1\right], \\ &B_1 = -\frac{2\gamma^2 e^{-\gamma\theta_0}}{\Delta(\theta_0)} \sqrt{\frac{2\pi\sqrt{2}\sin\theta_0}{\gamma}} \left[\sin\theta_0 H_e^0\left(\tilde{a}_1 - \frac{\mu \operatorname{ctg}\theta_0}{2\gamma} a_1\right) + \frac{\gamma}{R} M_1^0 a_2\right], \end{aligned}\right\} \quad (60.12)$$

where $\Delta(\theta_0) = 1 - \frac{7}{8}\frac{\operatorname{ctg}\theta_0}{\gamma} + \frac{\mu \operatorname{ctg}\theta_0}{\gamma}$, and the values of functions $a_1, a_2, \tilde{a}_1, \tilde{a}_2$ are taken at $\theta = \theta_0$. Substituting (60.12) into formulas for the calculation of ϑ_1 and Δ_e, we find after a series of transformations the angle of rotation and radial displacement at the edge of the dome

$$\left.\begin{aligned} &Eh\vartheta_1^0 = 2\gamma^2 \sin\theta_0 g_1(\theta_0) H_e^0 - \frac{4\gamma^3}{R} g_1(\theta_0) M_1^0, \\ &Eh\Delta_e^0 = 2\gamma R \sin^2\theta_0 g_2(\theta_0) H_e^0 - 2\gamma^2 \sin\theta_0 g_1(\theta_0) M_1^0, \end{aligned}\right\} \quad (60.13)$$

where

$$g_1(\theta_0) = 1 + \frac{(1-2\mu)}{2\gamma} \operatorname{ctg}\theta_0, \quad g_2(\theta_0) = 1 - \frac{\mu \operatorname{ctg}\theta_0}{\gamma}.$$

At $\theta_0 = \frac{\pi}{2}$ these formulas coincide with formula (27.21) for a long cylindrical shell of radius R. If on the shell act distributed loads, then during the calculation of constants A_1, B_1 in formulas (60.12) and (60.13) it is necessary to replace H_e^0 by $H_e^0 - \tilde{T}_1^0 \cos\theta_0$ and M_1^0 by $M_1^0 - \tilde{M}_1^0$.

Substituting obtained expressions into (60.2), we will have the equations for determination of radial force and bending moments in any section of the shell:

$$\begin{aligned} H_e = \tilde{T}_1 \cos\theta + \frac{e^{-\gamma(\theta_0-\theta)}}{\Delta(\theta_0)} \sqrt{\frac{\sin\theta_0}{\sin\theta}} \Big\{ \frac{\sin\theta_0}{\sin\theta} \Big[\Big(1 - \frac{7\operatorname{ctg}\theta_0}{8\gamma}\Big) \cos\gamma(\theta_0-\theta) - \\ - \Big(1 - \frac{3\operatorname{ctg}\theta}{8\gamma}\Big) \sin\gamma(\theta_0-\theta) + \frac{\mu\operatorname{ctg}\theta_0}{\gamma} \cos\gamma(\theta_0-\theta)\Big] (H_e^0 - \tilde{T}_1^0 \cos\theta_0) + \\ + \frac{\gamma}{R\sin\theta} \Big[2\sin\gamma(\theta_0-\theta) - \frac{3}{8\gamma} (\operatorname{ctg}\theta_0 + \operatorname{ctg}\theta) \sin\gamma(\theta_0-\theta) + \\ + \frac{3}{8\gamma} (\operatorname{ctg}\theta_0 - \operatorname{ctg}\theta) \cos\gamma(\theta_0-\theta)\Big] (M_1^0 - \tilde{M}_1^0) \Big\}. \end{aligned} \tag{60.14}$$

$$\begin{aligned} M_1 = \tilde{M}_1 + \frac{e^{-\gamma(\theta_0-\theta)}}{\Delta(\theta_0)} \sqrt{\frac{\sin\theta_0}{\sin\theta}} \times \\ \times \Big\{ \frac{R\sin\theta_0}{\gamma} \Big[-\sin\gamma(\theta_0-\theta) - \frac{1}{\gamma}\Big(\frac{\mu}{2} - \frac{7}{16}\Big)(\operatorname{ctg}\theta + \operatorname{ctg}\theta_0) \sin\gamma(\theta_0-\theta) + \\ + \frac{1}{\gamma}\Big(\frac{\mu}{2} - \frac{7}{16}\Big)(\operatorname{ctg}\theta_0 - \operatorname{ctg}\theta) \cos\gamma(\theta_0-\theta)\Big] (H_e^0 - \tilde{T}_1^0 \cos\theta_0) + \\ + \Big[\Big(1 - \frac{7\operatorname{ctg}\theta}{8\gamma} + \frac{\mu\operatorname{ctg}\theta}{\gamma}\Big) \cos\gamma(\theta_0-\theta) + \\ + \Big(1 - \frac{3\operatorname{ctg}\theta_0}{8\gamma}\Big) \sin\gamma(\theta_0-\theta)\Big] (M_1^0 - \tilde{M}_1^0) \Big\}. \end{aligned} \tag{60.15}$$

Using the obtained formulas, we determine reaction force H_e^0 and reaction moment M_1^0 in a preassigned section of shell, loaded by internal pressure p. In this instance

$$\tilde{T}_1^0 = \frac{pR}{2}, \quad \tilde{M}_1^0 = 0$$

and in formula (60.13) it is necessary to replace

$$H_e^0 \text{ by } H_e^0 - \frac{pR}{2}\cos\theta_0,$$
$$\Delta_e^0 \text{ by } \Delta_e^0 - \frac{pR^2}{2Eh}(1-\mu)\sin\theta_0.$$

Then, setting $\Delta_e^0 = \vartheta_1^0 = 0$, we obtain for determination of desired amounts H_e^0, M_1^0 the following system of equations:

$$\left.\begin{aligned} R\sin\theta_0 g_1(\theta_0) H_e^0 - 2\gamma g_1(\theta_0) M_1^0 = \frac{pR^2}{2} g_1(\theta_0)\sin\theta_0\cos\theta_0, \\ 2R\sin\theta_0 g_2(\theta_0) H_e^0 - 2\gamma g_1(\theta_0) M_1^0 = \\ = -\frac{pR^2}{2\gamma}(1-\mu) + pR^2 g_2(\theta_0)\sin\theta_0\cos\theta_0; \end{aligned}\right\}$$

solving it, we have

$$H_e^0 = \frac{pR}{2}\cos\theta_0 - \frac{pR(1-\mu)}{2\gamma\sin\theta_0}\Big[1 + \frac{(1+2\mu)\operatorname{ctg}\theta_0}{2\gamma}\Big],$$
$$M_1^0 = -\frac{pR^2}{4\gamma^2}(1-\mu)\Big[1 + \frac{(1+2\mu)}{2\gamma}\operatorname{ctg}\theta_0\Big].$$

It is interesting to compare reaction forces in the closing of a hemisphere $\left(\theta_0 = \frac{\pi}{2}\right)$ and a long cyliner of radius R. In the hemisphere

$$H_e^0 = -\frac{pR(1-\mu)}{2\gamma} \quad M_1^0 = -\frac{pR^2}{4\gamma^2}(1-\mu).$$

In the cylinder

$$N_1^L = -\frac{pR(2-\mu)}{2\gamma}, \quad M_1^L = -\frac{pR^2}{4\gamma^2}(2-\mu).$$

§ 61. Coupling of a Hemisphere with a Long Cylindrical Shell

Formula (60.13) is handy for making up the conditions of coupling a spherical shell with shells of other forms. As an illustration we will examine the coupling of a hemisphere of radius R with a cylindrical shell of the same radius. The load is internal pressure. Our task will be to determine radial force and bending moment where the sphere turns into a cylinder. Figure 30 depicts the forces and moments applied to one of the involved parts of the reservoir (sphere, cylinder) and the replacing action of the rejected part. Also shown are the positive directions of radial displacement and the angle of rotation for both shells. The formulas (60.13) after replacing in them H_e^0 by $H_e^0 - \frac{pR}{2}\cos\theta_0$, Δ_e^0 by $\Delta_e^0 - \frac{pR^2}{2Eh}(1-\mu)\sin\theta_0$ at $\theta_0 = \frac{\pi}{2}$ give

$$\left.\begin{aligned} Eh\theta_1^0 &= 2\gamma^2 H_e^0 - \frac{4\gamma^3}{R} M_1^0, \\ Eh\Delta_e^0 &= 2\gamma R H_e^0 - 2\gamma^2 M_1^0 + pR^2\frac{(1-\mu)}{2}. \end{aligned}\right\} \qquad (61.1)$$

For the cylindrical part the formulas between θ_1^0, w^0 and N_1^0, M_1^0 are composed on the basis of (27.20), replacing in them w^0 and $w^0 - \frac{pR^2}{Eh}\left(1-\frac{\mu}{2}\right)$. In this case note that as a result of the equality of thickness and radii of the cylinder and sphere parameter $2\gamma^2$ in both cases has the same value. Thus, for the cylinder we have

$$\left.\begin{aligned} Eh\theta_1^0 &= 2\gamma^2 N_1^0 + \frac{4\gamma^3}{R} M_1^0, \\ Ehw^0 &= -2\gamma R N_1^0 - 2\gamma^2 M_1^0 + pR^2\left(1-\frac{\mu}{2}\right). \end{aligned}\right\} \qquad (61.2)$$

Let us write the conditions of continuity of forces and displacements at the joint:

$$\left.\begin{aligned} H_e^0 &= N_1^0, \quad M_{1\,\text{сф}}^0 = M_{1\,\text{цил}}^0, \\ \Delta_e^0 &= w^0, \quad \theta_{1\,\text{сф}}^0 = \theta_{1\,\text{цил}}^0. \end{aligned}\right\} \qquad (61.3)$$

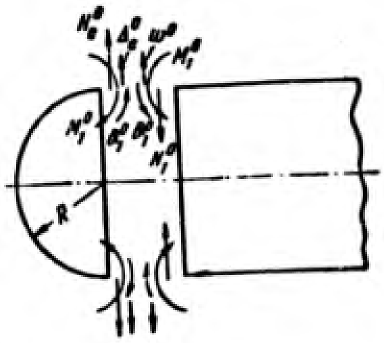


Fig. 30. Forces, moments and displacements at the junction of a hemisphere with a cylindrical shell.

Taking into account (61.1), (61.2), conditions (61.3) can be considered as a system of four algebraic equations for determining the four quantities H_e^0, N_1^0, $M_{1\,\text{сф}}^0$, $M_{1\,\text{шц}}^0$, solving which, we obtain

$$\left.\begin{aligned} M_{1\,\text{сф}}^0 &= M_{1\,\text{шц}}^0 = 0, \\ N_1^0 &= H_e^0 = \frac{pR}{8\gamma}. \end{aligned}\right\} \tag{61.4}$$

Radial displacement and angle of rotation at the connection are equal to

$$w^0 = \frac{pR^2}{4Eh}(3 - 2\mu), \quad \vartheta_1^0 = \frac{pR\gamma}{4Eh}. \tag{61.5}$$

We determin the axial displacement of point A of the reservoir relative to the connecting section. Setting in (60.9), (60.8) $\theta = \theta_0 = \frac{\pi}{2}$, we find constant of integration C from the condition that the connecting section has zero displacement, $\Delta_z\left(\frac{\pi}{2}\right) = 0$.

$$C = -\tilde{\Delta}_z\left(\frac{\pi}{2}\right) = 0.$$

Displacement in the angle of the sphere is equal to

$$\Delta_z(0) = \tilde{\Delta}_z(0) + \frac{R}{Eh}[A_1\omega_1(0) + B_1\omega_2(0)].$$

Since $\omega_1(0) = 1$, $\omega_2(0) = 0$ and, furthermore, A_1 contains small factor $e^{-\gamma\frac{\pi}{2}}$, we finally find

$$\Delta_z(0) \approx \tilde{\Delta}_z(0) = \frac{pR^2}{2Eh}(1 - \mu). \tag{61.6}$$

§ 62. Spherical Dome with Concentrated Force in the Top

Let us turn to analysis of the case when a shell is loaded in the top by concentrated force P. In the general solution of basic resolvant equation (59.2) it is now necessary to keep also the irregular part of the solution, i.e., to write it in the form of (59.26). Functions $\chi_1(\theta)$, $\chi_2(\theta)$, $\chi_3(\theta)$, $\chi_4(\theta)$ in the neighborhood of $\theta = 0$ lead as the first terms of the expansion

$$\left.\begin{aligned} \chi_1(\theta) &= -\frac{\gamma^4\theta^3}{4}+\ldots. & \chi_3(\theta) &= \frac{2\gamma^2}{\pi}\theta\ln\theta+\ldots.\\ \chi_2(\theta) &= -\gamma^2\theta+\ldots. & \chi_4(\theta) &= \frac{2}{\pi\theta}+\ldots. \end{aligned}\right\} \tag{62.1}$$

where the dots indicate term of a much higher order of smallness. According to (57.11), (59.1) and (59.26)

$$\left.\begin{aligned} V_0 &= -\frac{1}{2\gamma^2}[-B_1\chi_1(\theta)+A_1\chi_2(\theta)-B_2\chi_3(\theta)+A_2\chi_4(\theta)]-\frac{\Phi_2(\theta)}{R\sin\theta}.\\ \Psi_0 &= A_1\chi_1(\theta)+B_1\chi_2(\theta)+A_2\chi_3(\theta)+B_2\chi_4(\theta). \end{aligned}\right\} \tag{62.2}$$

In order to explain the behavior of function $\frac{\Phi_2(\theta)}{R\sin\theta}$ in the vicinity of $\theta = 0$, we represent it in the form

$$\frac{\Phi_2(\theta)}{R\sin\theta} = -R\int_{\theta_0}^{\theta} q_\theta\sin\theta\,d\theta - \operatorname{ctg}\theta\left(\frac{P}{2\pi R}+R\int_{0}^{\theta} q_z\sin\theta\,d\theta\right).$$

whence it is clear that at $\theta\approx 0$

$$\frac{\Phi_2(\theta)}{R\sin\theta} \approx -\frac{P}{2\pi R\theta}. \tag{62.3}$$

We select constants A_2, B_2, on the strength of the requirement of finiteness for angle of rotation ϑ_1 in the top and the condition that shearing force at $\theta = 0$ turn into infinity as $-\frac{P}{2\pi R\theta}$. From formulas (62.2), (62.3) and (57.8), (57.9) it is clear that these conditions are equivalent to the requirements of the boundedness of functions V_0 and Ψ_0 at $\theta = 0$. It is easy to see that these requirements are carried out if we take

$$A_2 = \frac{\gamma^2 P}{2R},\quad B_2 = 0.$$

Expressions (62.2) now assume the form

$$\left.\begin{aligned} V_0 &= -\frac{1}{2\gamma^2}\left[-B_1\chi_1(\theta)+A_1\chi_2(\theta)+\frac{P\gamma^2}{2R}\chi_4(\theta)\right]-\frac{\Phi_2(\theta)}{R\sin\theta}, \\ Eh\vartheta_1 &= A_1\chi_1(\theta)+B_1\chi_2(\theta)+\frac{P\gamma^2}{2R}\chi_3(\theta). \end{aligned}\right\} \quad (62.4)$$

Constants A_1, B_1 as earlier, should be defined according to boundary conditions on the edge $\theta = \theta_0$. At sufficiently large $\gamma\theta$ because of the asymptotic representations for σ_{*1} and σ_{*2} solution σ_{*3} during the determination of A_1, B_1, can be neglected. In this way, for A_1, B_1 in this instance they remain in force of formula (60.12). The stressed state in the neighborhood of the edge also is determined on the basis of (60.2). In other words, the term in the right part of (62.4), containing functions $\chi_1(\theta)$, $\chi_2(\theta)$, describe a simple edge effect, which determines the local stressed state at the edge $\theta=\theta_0$. Terms containing functions $\chi_3(\theta)$, $\chi_4(\theta)$, describe the singular edge effect induced by the presence of concentrated force and characterizing the local stressed state in the vicinity of the pole. In this case forces and moments are calculated by the formulas

$$\left.\begin{aligned} T_1 &= -\frac{P}{4R}\operatorname{ctg}\theta\chi_4(\theta)+\tilde{T}_1, \quad N_1=-\frac{P}{4R}\chi_4(\theta), \\ H_e &= -\frac{P}{4R}\frac{1}{\sin\theta}\chi_4(0)+\tilde{T}_1\cos\theta, \quad T_2=-\frac{P}{4R}\chi_4'(\theta)+\tilde{T}_2, \\ M_1 &= -\frac{P}{8\gamma^2}\left[\chi_3'(\theta)+\mu\operatorname{ctg}\theta\chi_3(\theta)\right]+\tilde{M}_1, \\ M_2 &= -\frac{P}{8\gamma^2}\left[\mu\chi_3'(\theta)+\operatorname{ctg}\theta\chi_3(\theta)\right]+\tilde{M}_2. \end{aligned}\right\} \quad (62.5)$$

In the absence of distributed loads the quantities marked with a tilde are equal to

$$\tilde{T}_1=-\tilde{T}_2=\frac{P}{2\pi R\sin^2\theta}, \quad \tilde{M}_1=\tilde{M}_2=0.$$

The obtained formulas allow explaining the behavior of forces and moments at $\theta = 0$. Taking into account the representation of functions χ_1, χ_2, χ_3, χ_4 in the vicinity of zero and formula (62.5), it is easy to see that forces T_1 and T_2 remain finite at $\theta = 0$ has the form

$$\left.\begin{aligned} N_1 &= -\frac{P}{2\pi R\theta}+\ldots, \\ M_1 &= -\frac{P}{4\pi}(1+\mu)\ln\theta+\ldots, \quad M_2=-\frac{P}{4\pi}(1+\mu)\ln\theta+\ldots \end{aligned}\right\} \quad (62.6)$$

In contrast to static quantities displacements in the point of application of force remain bounded. The boundedness of radial displacement directly follows from the boundedness of forces T_1 and T_2 at $\theta=0$, since

$$\Delta_r = \frac{R\sin\theta}{Eh}(T_2 - \mu T_1).$$

For clarification of the behavior of axial displacement we represent it in the form

$$\Delta_z = \int(-\varepsilon_1 R\sin\theta + \vartheta_1 R\cos\theta)\,d\theta + C = \frac{R}{Eh}\frac{P\gamma^2}{2R}\int\chi_3(\theta)\,d\theta + \ldots, \qquad (62.7)$$

where the dots indicate terms a higher order of smallness at $\theta=0$, and constants.

Taking into account (62.1) and integrating in (62.7), we obtain

$$\Delta_z = \frac{P\gamma^4}{2\pi Eh}\theta^2\ln\theta + \ldots \qquad (62.8)$$

Remembering that flexural rigidity D is $\frac{Eh^3}{12(1-\mu^2)}$, and introducing the designation $R\theta=s$, formula (62.8) can be represented in the form

$$\Delta_z = \frac{P}{8\pi D}s^2\ln s + \ldots \qquad (62.9)$$

Note that the change in quantities N_1, M_1, M_2, Δ_z in the sphere in the neighborhood of the application of the concentrated force completely coincides with the behavior of the same quantities in a flat plate.

§ 63. Stressed State of a Spherical Strip

In conclusion we will examine a spherical shell with a hole (Fig. 31). On the edges of the shell $\theta=\theta_1$ and $\theta=\theta_0$ are applied systems of forces M_1^1, H_r^1 and M_1^0, H_r^0. Furthermore, on the shell can act distributed loads and axial tension equivalent to forces P, P_0 In order to satisfy the edge conditions at both edges it is necessary to take the solution of the vasic resolvant equation in the form of (59.26). In this case in a sufficiently thin-walled and long shell $[\gamma(\theta_0-\theta_1)\gg 1]$ the solution $(A_1 - iB_1)\times(\chi_1+i\chi_2)$ will describe the stressed state in the neighborhood of edge θ_0, and the solution $(A_2 - iB_2)(\chi_3+i\chi_4)$ the state in the neighborhood of edge θ_1. Ignoring the mutual influence of the edges, we find that A_1, B_1 are determined according

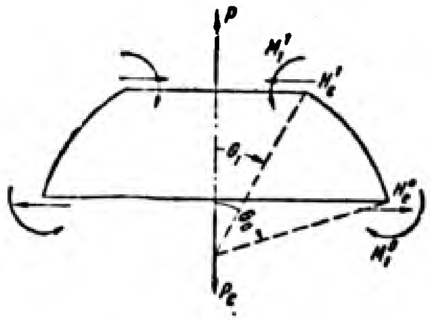


Fig. 31. Evenly distributed forces and moments applied to the edges of a spherical strip.

according to the formulas of (60.12). Correspondingly formulas (60.13), (60.14), (60.15) are valid. To determine A_2, B_2 we obtain the equations

$$\left.\begin{aligned} &A_2a_4(\theta_1)-B_2a_3(\theta_1)=-2\gamma^2\sin\theta_1 e^{\gamma\theta_1}\sqrt{\frac{\pi\sin\theta_1}{\gamma\sqrt{2}}}H_e^1,\\ &A_2\left[\tilde{a}_3(\theta_1)-\frac{\mu\operatorname{ctg}\theta_1}{2\gamma}a_3(\theta_1)\right]+B_2\left[\tilde{a}_4(\theta_1)-\frac{\mu\operatorname{ctg}\theta_1}{2\gamma}a_4(\theta_1)\right]=\\ &\qquad=\frac{2\gamma^3}{R}e^{\gamma\theta_1}\sqrt{\frac{\pi\sin\theta_1}{\gamma\sqrt{2}}}M_1^1.\end{aligned}\right\} \tag{63.1}$$

Solving them, we have

$$\left.\begin{aligned} &A_2=\frac{2\gamma^2e^{\gamma\theta_1}}{\Delta(\theta_1)}\sqrt{\frac{\pi\sin\theta_1}{\gamma\sqrt{2}}}\left[-\sin\theta_1H_e^1\left(\tilde{a}_4-\frac{\mu\operatorname{ctg}\theta_1}{2\gamma}a_4\right)+\right.\\ &\qquad\left.+\frac{\gamma}{R}M_1^1a_3\right].\\ &B_2=\frac{2\gamma^2e^{\gamma\theta_1}}{\Delta(\theta_1)}\sqrt{\frac{\pi\sin\theta_1}{\gamma\sqrt{2}}}\left[\sin\theta_1H_e^1\left(\tilde{a}_3-\frac{\mu\operatorname{ctg}\theta_1}{2\gamma}a_3\right)+\right.\\ &\qquad\left.+\frac{\gamma}{R}M_1^1a_4\right].\end{aligned}\right| \tag{63.2}$$

Here

$$\Delta(\theta_1)=1+\frac{7\operatorname{ctg}\theta_1}{8\gamma}-\frac{\mu\operatorname{ctg}\theta_1}{\gamma}$$

and $a_3, \tilde{a}_3, a_4, \tilde{a}_4$ express the values of these functions at $\theta=\theta_1$. In the presence of distributed loads and axial forces in (63.2) it is necessary to replace H_e^1 by $H_e^1-\tilde{T}_1^1\cos\theta_1$, M_1^1 by $M_1^1-\tilde{M}_1^1$. Forces, moments and angle of rotation ϑ_1 in edge θ_1 after determination of constants A_2, B_2 should be calculated according to the formulas which easily are obtained from (60.2) by the replacement $A_1\to A_2$, $B_1\to B_2$, $\chi_1\to\chi_3$, $\chi_2\to\chi_4$. For example.

$$T_1=\tilde{T}_1-\frac{1}{2\gamma^2}\operatorname{ctg}\theta(A_2\chi_4-B_2\chi_3),$$
$$\vartheta_1=\tilde{\vartheta}_1+\frac{1}{Eh}(A_2\chi_3+B_2\chi_4)$$

etc.

On edge θ_1 we have

$$
\left.
\begin{aligned}
&\vartheta_1^1=\tilde{\vartheta}_1^1+\frac{1}{Eh}\left[A_2\chi_3(\theta_1)+B_2\chi_4(\theta_1)\right],\\
&\Delta_e^1=\tilde{\Delta}_e^1-\frac{R\sin\theta_1}{2\gamma^2 Eh}\left\{A_2\left[\chi_4'(\theta_1)-\mu\operatorname{ctg}\theta_1\chi_4(\theta_1)\right]-\right.\\
&\qquad\qquad\left.-B_2\left[\chi_3'(\theta_1)-\mu\operatorname{ctg}\theta_1\chi_3(\theta_1)\right]\right\}.
\end{aligned}
\right\}
\qquad (63.3)
$$

Substituting here A_2 and B_2 in accordance with (63.2) we obtain for edge θ_1 formulas analogous to the formulas of (60.13)

$$
\left.
\begin{aligned}
&\vartheta_1^1-\tilde{\vartheta}_1^1=\frac{2\gamma^2}{Eh}\Big[(H_e^1-\tilde{T}_1\cos\theta_1)\sin\theta_1 f_1(\theta_1)+\\
&\qquad\qquad+\frac{2\gamma}{R}(M_1-\tilde{M}_1^1)f_1(\theta_1)\Big],\\
&\Delta_e^1-\tilde{\Delta}_e^1=\frac{2\gamma R\sin\theta_1}{Eh}\Big[-\sin\theta_1(H_e^1-\tilde{T}_1^1\cos\theta_1)f_2(\theta_1)-\\
&\qquad\qquad-\frac{\gamma}{R}(M_1^1-\tilde{M}_1^1)f_1(\theta_1)\Big].
\end{aligned}
\right\}
\qquad (63.4)
$$

where

$$
f_1(\theta_1)=1-\frac{1-2\mu}{2\gamma}\operatorname{ctg}\theta_1,\quad f_2(\theta_1)=1+\frac{\mu\operatorname{ctg}\theta_1}{\gamma}.
$$

Using formulas (60.13), (63.4), we will examine the stressed state of a spherical dome loaded in a certain parallel circle $\theta=\theta_1$ by distributed normal and tangential forces and bending moments of force p, t and m (Fig. 32).

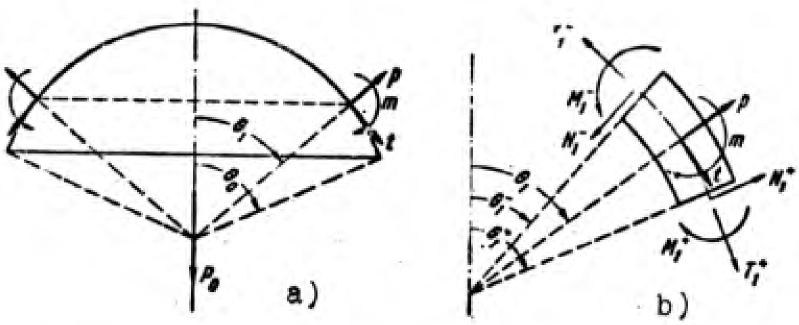


Fig. 32. a) Normal force and bending moment acting on a spherical dome evenly distributed along the parallel; b) external and internal forces and moments applied to a shell element bounded by sections θ_1^- and θ_1^+.

We propose that the shell is thin-walled $[\gamma(\theta_0-\theta_1)\gg 1]$ and sections $(0,\theta_1)$, (θ_1,θ_0) are rather "long"; then the mutual influence of edges θ_0 and θ_1 can be neglected. We are interested in only the stressed state near the line of load $\theta=\theta_1$. Crossing through this section internal forces in the shell T_1, N_1 and moment M_1 should jump;

$$\left.\begin{aligned} N_1^+ - N_1^- &= -p, \\ M_1^+ - M_1^- &= -m, \\ T_1^+ - T_1^- &= -t. \end{aligned}\right\} \qquad (63.5)$$

If we designate through H_e and V_z the radial and axial internal forces, where $V_z = N_1 \cos\theta - T_1 \sin\theta$, then we can write

$$\left.\begin{aligned} H_e^+ - H_e^- &= -p \sin\theta_1 - t\cos\theta_1 = -p_e, \\ V_z^+ - V_z^- &= -p\cos\theta_1 + t\sin\theta_1 = -p_z. \end{aligned}\right\} \qquad (63.6)$$

From the condition of equilibrium of the shell as a whole it follows that

$$P_0 = p_z \cdot 2\pi R \sin\theta_1.$$

The zero-moment solution also suffers a discontinuity going through the line of the load:

$$\tilde{T}_1 = -\tilde{T}_2 = \begin{cases} 0, & \theta < \theta_1, \\ \dfrac{P_0}{2\pi R \sin^2\theta}, & \theta > \theta_1. \end{cases} \qquad (63.7)$$

$$\tilde{\Delta}_e = \begin{cases} 0, & \theta < \theta_1, \\ -\dfrac{P_0(1+\mu)}{2\pi E h \sin\theta}, & \theta > \theta_1. \end{cases} \qquad (63.8)$$

Angle of rotation and total radial displacement should be continuous going through the loaded section

$$\theta_1^- = \theta_1^+, \quad \Delta_e^- = \Delta_e^+. \qquad (63.9)$$

For section θ_1^- we write relationships (60.13), and for section using formulas (63.4). We obtain

$$\left.\begin{aligned} Eh\theta_1^- &= 2\gamma^2 \sin\theta_1 g_1(\theta_1) H_e^- - \frac{4\gamma^3}{R} g_1(\theta_1) M_1^-, \\ Eh\Delta_e^- &= 2\gamma R \sin^2\theta_1 g_2(\theta_1) H_e^- - 2\gamma^2 \sin\theta_1 g_1(\theta_1) M_1^-. \end{aligned}\right\} \qquad (63.10)$$

$$\left.\begin{aligned} Eh\theta_1^+ &= 2\gamma^2 \sin\theta_1 f_1(\theta_1)\left(H_e^+ - p_z \operatorname{ctg}\theta_1\right) + \frac{4\gamma^3}{R} M_1^+ f_1(\theta_1), \\ Eh\Delta_e^+ &= -(1+\mu) p_z R - 2\gamma R \sin^2\theta_1 f_2(\theta_1) \times \\ &\quad \times \left(H_e^+ - p_z \operatorname{ctg}\theta_1\right) - 2\gamma^2 M_1^+ f_1(\theta_1). \end{aligned}\right\} \qquad (63.11)$$

Taking into account (63.5), (63.6), (63.10), (63.11), from relationships (63.9) we obtain two equations for determination of the two unknown quantities H_e^-, M_1^-:

$$\left.\begin{aligned}
&2\gamma^2 \sin\theta_1 g_1(\theta_1) H_e^- - \frac{4\gamma^3}{R} g_1(\theta_1) M_1^- = \\
&= +2\gamma^2 \sin\theta_1 f_1(\theta_1)\left(H_e^- - p_e - p_z \operatorname{ctg}\theta_1\right) + \frac{4\gamma^3}{R}\left(M_1^- - m\right), \\
&2\gamma R \sin^2\theta_1 g_2(\theta_1) H_e^- - 2\gamma^2 \sin\theta_1 g_1(\theta_1) M_1^- = \\
&= -(1+\mu) p_z R - 2\gamma R \sin^2\theta_1 f_2(\theta_1)\left(H_e^- - p_e - p_z \operatorname{ctg}\theta_1\right) - \\
&\qquad - 2\gamma^2\left(M_1^- - m\right) f_1(\theta_1).
\end{aligned}\right\} \quad (63.12)$$

We find the solution of system (63.12), ignoring quantities of order $\frac{1}{\gamma}$ in comparison with unity, i.e., setting $f_1(\theta_1) = f_2(\theta_1) = g_1(\theta_1) = g_2(\theta_1) = 1$. We obtain

$$\left.\begin{aligned}
H_e^- &= \frac{p_e}{2} + \frac{p_z}{2}\operatorname{ctg}\theta_1 - \frac{(1+\mu)p_z}{4\gamma\sin^2\theta_1} + \frac{\gamma m}{2R\sin\theta_1}, \\
M_1^- &= \frac{pR}{4\gamma} + \frac{m}{2};
\end{aligned}\right\} \quad (63.13)$$

on the basis of (63.5), (63.6) we will have

$$\left.\begin{aligned}
H_e^+ &= -\frac{p_e}{2} + \frac{p_z}{2}\operatorname{ctg}\theta_1 - \frac{(1+\mu)p_z}{4\gamma\sin^2\theta_1} + \frac{\gamma m}{2R\sin\theta_1}, \\
M_1^+ &= \frac{pR}{4\gamma} - \frac{m}{2}.
\end{aligned}\right\} \quad (63.14)$$

Note that so simple a problem of coupling two different sections of a shell is solved only when the line of the load θ_1 is far from the pole and from edge θ_0 and the shell itself is thin-walled ($\gamma\theta_1 \gg 1$, $\gamma(\theta_0 - \theta_1) \gg 1$), so that separate determination of the constants of integration on each of the edges (θ_1^-, θ_1^+, θ_0) and the use of asymptotic representations (59.23), (59.24) is justified. Otherwise, when separate determination of the constants is impossible, the problem is reduced to the common solution of six algebraic equations in six constants of integration - four coupling conditions on line θ_1

$$H_e^+ - H_e^- = -p_e, \quad M_1^+ - M_1^- = -m,$$
$$\vartheta_1^- = \vartheta_1^+, \quad \Delta_e^- = \Delta_e^+$$

two edge conditions

$$H_e(\theta_0) = H_e^0, \quad M_1(\theta_0) = M_1^0.$$

§ 64. Axisymmetric Deformation of a Spherical Shell of Small Curvature

In §§ 64, 65 we will consider an axisymmetrically loaded spherical shell of small curvature. A thin-walled shell of small curvature or a slightly curved plate subsequently will refer to a shell whose parameter $2\gamma^2=\sqrt{12(1-\mu^2)}\frac{R}{h}$ is great in comparison with unity and angle θ_0, characterizing the edge section, is so small that the quantity $q_0=\gamma\theta_0\sqrt{2}$ can only insignificantly exceed unity. In § 59 it was shown that at large γ and small θ the general solution of the uniform resolvant equation for a spherical shell can be expressed in Bessel functions (formulas (59.26)-(59.28)). Using the fact that parameter q has the order of unity in power expansion for functions $\psi_1, \psi_2, \psi_3, \psi_4$ and their derivatives (formulas § 46), we can be limited to only several first terms, for example, take

$$\left.\begin{aligned} &\psi_1(q)=1-\frac{q^4}{64}+\ldots,\quad \psi_1'(q)=-\frac{q^3}{16}\left(1-\frac{q^4}{1152}+\ldots\right),\\ &\psi_2(q)=-\frac{q^2}{4}\left(1-\frac{q^4}{576}+\ldots\right),\\ &\psi_2'(q)=-\frac{q}{2}\left(1-\frac{q^4}{192}+\ldots\right).\end{aligned}\right\}\quad(64.1)$$

$$\left.\begin{aligned} &\psi_3(q)=\frac{1}{2}-\frac{q^2}{2\pi}-\frac{q^4}{128}+\ldots+\frac{q^2}{2\pi}\ln\frac{\gamma_1 q}{2}+\ldots,\\ &\psi_3'(q)=-\frac{q}{2\pi}-\frac{q^3}{32}+\frac{q^5}{36\pi}+\ldots\\ &\qquad\ldots+\frac{q}{\pi}\left(1-\frac{q^4}{192}\right)\ln\frac{\gamma_1 q}{2}+\ldots,\\ &\psi_4(q)=\frac{2}{\pi}\left(1-\frac{q^4}{64}\right)\ln\frac{\gamma_1 q}{2}-\frac{q^2}{8}\left(1-\frac{3}{\pi}\frac{q^2}{8}\right)+\ldots,\\ &\psi_4'(q)=\frac{2}{\pi}\cdot\frac{1}{q}\left(1-\frac{q^2\pi}{8}\right)+\frac{q^3}{32\pi}\left(5-4\ln\frac{\gamma_1 q}{2}\right)+\ldots\end{aligned}\right\}\quad(64.2)$$

Furthermore, because of the smallness of θ it is possible to set $\cos\theta\approx 1$, $\sin\theta\approx\theta$, $\sqrt{\frac{\theta}{\sin\theta}}\approx 1$. The radius of the parallel circle of the edge section of the shell is designated through a. In accordance with the above assumption about the smallness of θ_0 we have $a\approx R\theta_0$. In Table 6 are given values of θ_0 calculated at $\mu^2=0.1$ for different values of q_0 and $\frac{a}{h}$ on the basis of the relationship [12]

$$\theta_0=\frac{q_0^2}{2\gamma^2\theta_0}=0.306\,\frac{q_0^2 h}{a}.$$

Let us examine a slightly curved plate loaded by a concentrated force in the center and by edge forces and moments H_e^0, M_1^0. During the

Table 6.

θ_0 \ a/h	25	20	15	10
0,75	0,0069	0,0086	0,0115	0,0172
1,00	0,0122	0,0153	0,0205	0,0306
1,50	0,0276	0,0343	0,0405	0,0690
2,0	0,0490	0,0612	0,0880	0,1224

determination of constants A_1, B_1, in solution (62.4), it is now not possible to drop terms containing $\chi_3(\theta)$, $\chi_4(\theta)$, since at $\theta=\theta_0$ they have values comparable with the remaining terms of the solution. Therefore during the composition of expressions for H_e, M_1 we combine equations (60.2), (62.5). Taking into account the smallness of θ_0 for determination of A_1, B_1, we obtain the equations:

$$\begin{aligned} -\frac{1}{2\gamma^2}\frac{1}{\theta_0}\left[A_1\chi_2(\theta_0)-B_1\chi_1(\theta_0)\right] &= H_e^0-\tilde{T}_1^0+\frac{P}{4R}\frac{1}{\theta_0}\chi_4(\theta_0), \\ -\frac{R}{4\gamma^4}\left\{A_1\left[\chi_1'(\theta_0)+\frac{\mu}{\theta_0}\chi_1(\theta_0)\right]+B_1\left[\chi_2'(\theta_0)+\frac{\mu}{\theta_0}\chi_2(\theta_0)\right]\right\} &= \\ = M_1^0+\frac{P}{8\gamma^2}\left[\chi_3'(\theta_0)+\frac{\mu}{\theta_0}\chi_3(\theta_0)\right]. \end{aligned} \qquad (64.3)$$

where

$$\tilde{T}_1^0=\frac{P}{2\pi R\theta_0^2}.$$

Having in mind that in this case χ_l $(l=1, 2, 3, 4)$ are determined through ψ_l $(l=1, 2, 3, 4)$ using formulas (59.28) and

$$\chi_1'(\theta)+\frac{\mu}{\theta}\chi_1(\theta)=2\gamma^2\left[\psi_2(q)-\frac{(1-\mu)}{q}\psi_1'(q)\right],$$

$$\chi_2'(\theta)+\frac{\mu}{\theta}\chi_2(\theta)=-2\gamma^2\left[\psi_1(q)+\frac{(1-\mu)}{q}\psi_2'(q)\right],$$

$$\chi_3'(\theta)+\frac{\mu}{\theta}\chi_3(\theta)=2\gamma^2\left[\psi_4(q)-\frac{(1-\mu)}{q}\psi_3'(q)\right],$$

we solve system (64.3) in A_1, B_1. We obtain

$$\begin{aligned} A_1 &= \frac{1}{\Delta}\left\{\left(H_e^0-\tilde{T}_1^0+\frac{P\gamma^2}{2Rq_0}\psi_4'\right)\left(\psi_1+\frac{1-\mu}{q_0}\psi_2'\right)-\right. \\ &\left.\quad -\frac{2\gamma^2}{R}\left[M_1+\frac{P}{4}\left(\psi_4-\frac{1-\mu}{q_0}\psi_3'\right)\right]\frac{\psi_1'}{q_0}\right\}, \\ B_1 &= \frac{1}{\Delta}\left\{\left(H_e^0-\tilde{T}_1^0+\frac{P\gamma^2}{2Rq_0}\psi_4'\right)\left(\psi_2-\frac{1-\mu}{q_0}\psi_1'\right)-\right. \\ &\left.\quad -\frac{2\gamma^2}{R}\left[M_1+\frac{P}{4}\left(\psi_4-\frac{1-\mu}{q_0}\psi_3'\right)\right]\frac{\psi_2'}{q_0}\right\}. \end{aligned} \qquad (64.4)$$

where

$$\Delta = \frac{1}{q_0^2}\left(\psi_1'^2 + \psi_2'^2\right)\left[q_0 \frac{\psi_1'\psi_2 - \psi_2'\psi_1}{\psi_1'^2 + \psi_2'^2} - (1-\mu)\right]. \tag{64.5}$$

In formulas (64.4) functions ψ_i $(i = 1, 2, 3, 4)$ and their derivatives are calculated for values of the argument $q_0 = \gamma\sqrt{2}\theta_0$. For subsequent calculations it is convenient to introduce the following combinations of functions ψ_i, ψ_i':

$$\left.\begin{aligned} \Psi_1(q) &= q\,\frac{\psi_1'\psi_2 - \psi_2'\psi_1}{\psi_1'^2 + \psi_2'^2}, \quad & \Psi_2(q) &= q\,\frac{\psi_1\psi_1' + \psi_2\psi_2'}{\psi_1'^2 + \psi_2'^2}, \\ \Psi_3(q) &= q^2\,\frac{\psi_1^2 + \psi_2^2}{\psi_1'^2 + \psi_2'^2}, \quad & \Psi_4(q) &= \frac{1}{q}\left(\psi_1'^2 + \psi_2'^2\right). \end{aligned}\right\} \tag{64.6}$$

Using expansion (64.1), we can write

$$\left.\begin{aligned} \Psi_1(q) &= 2 + \frac{q^4}{96} + \ldots, \quad & \Psi_2(q) &= \frac{q^2}{4}\left(1 - \frac{q^4}{128} + \ldots\right), \\ \Psi_3(q) &= 4 + \frac{5}{48}q^4 + \ldots, \quad & \Psi_4(q) &= \frac{q}{4}\left(1 + \frac{q^4}{192} + \ldots\right). \end{aligned}\right\} \tag{64.7}$$

In order to determine radial displacement and angle of rotation of the edge Δ_e^0, ϑ_1^0, we will use equations

$$\vartheta_1 = \frac{1}{Eh}\left[A_1\chi_1(\theta) + B_1\chi_2(\theta)\right] + \frac{P\gamma^2}{2EhR}\chi_3(\theta),$$

$$\Delta_e = \bar{\Delta}_e - \frac{R\theta}{2\gamma^2 Eh}\left\{A_1\left[\chi_2'(\theta) - \frac{\mu}{\theta}\chi_2(\theta)\right] - B_1\left[\chi_1'(\theta) - \frac{\mu}{\theta}\chi_1(\theta)\right] + \right.$$
$$\left. + \frac{P\gamma^2}{2R}\left[\chi_4'(\theta) - \frac{\mu}{\theta}\chi_4(\theta)\right]\right\}.$$

$$\bar{\Delta}_e = -(1+\mu)\frac{P}{2\pi\theta_0 Eh}$$

and the found values of constants A_1 and B_1. Making the necessary calculation, we obtain

$$\left.\begin{aligned} Eh\vartheta_1^0 = {} & \frac{P\gamma^3\sqrt{2}}{2R}\psi_3' + \frac{2\gamma^2\theta_0}{\Psi_1 - (1-\mu)}\left\{\left(H_e^0 - \bar{T}_1^0 + \right.\right. \\ & \left.\left. + \frac{P\gamma^2}{2Rq_0}\psi_4'\right)\Psi_2 - \frac{2\gamma^2}{R}\left[M_1^0 + \frac{P}{4}\left(\psi_4 - \frac{1-\mu}{q}\psi_3'\right)\right]\right\}, \\ Eh\Delta_e^0 = {} & -\frac{(1+\mu)P}{2\pi\theta_0} + \frac{P\theta\gamma^2}{2}\left(\psi_3 + \frac{1+\mu}{q}\psi_4'\right) + \\ & + \frac{R\theta_0}{\Psi_1 - (1-\mu)}\left\{\left(H_e^0 - \bar{T}_1^0 + \frac{P\gamma^2}{2Rq_0}\psi_4'\right)\left[\Psi_3 - 2\Psi_1 + (1-\mu^2)\right] - \right. \\ & \left. - \frac{2\gamma^2}{R}\left[M_1^0 + \frac{P}{4}\left(\psi_4 - \frac{1-\mu}{q}\psi_3'\right)\right]\Psi_2\right\}. \end{aligned}\right\} \tag{64.8}$$

In the equations it is understood that all ψ_i, Ψ_i $(i=1, 2, 3, 4)$ and their derivatives are calculated at $q=q_0$. Using representations of (64.2), we note that

$$-\bar{T}_1^0+\frac{P\gamma^2}{2Rq_0}\psi_4'(q_0)=\left[-\frac{q_0}{4}+\frac{q_0^3}{32\pi}\left(5-4\ln\frac{\gamma_1 q_0}{2}\right)+\ldots\right]\frac{P\gamma^2}{2Rq_0}$$

$$\psi_4(q_0)-\frac{1-\mu}{q_0}\psi_3'(q_0)=\frac{1-\mu}{2\pi}-\frac{(3+\mu)q_0^2}{32}+\frac{(11+16\mu)q_0^4}{\pi\cdot 4\cdot 9\cdot 16}+\ldots$$

$$\ldots+\frac{1+\mu}{\pi}\ln\frac{\gamma_1 q_0}{2}-\frac{(5+\mu)}{192\pi}q_0^4\ln\frac{\gamma_1 q_0}{2}+\ldots$$

$$\Psi_1(q_0)-(1-\mu)=1+\mu+\frac{q_0^4}{96}+\ldots$$

etc. Then we obtain

$$\left.\begin{aligned}
\vartheta_1^0&=\frac{H_e^0a^2\theta_0}{4D(1+\mu)}\,\frac{1-\frac{q_0^4}{128}}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}}-\frac{M_1^0a}{D(1+\mu)}\,\frac{1}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}}-\\
&\quad-\frac{P}{4\pi D}\,\frac{a}{1+\mu}\left(1+\frac{q_0^4}{48}\ln q_0+\ldots\right),\\
\Delta_e^0&=\frac{(1-\mu)H_e^0a}{Eh}\,\frac{1+\frac{1}{1-\mu^2}\frac{q_0^4}{12}}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}}-\\
&\quad-\frac{M_1^0a^2\theta_0}{4D(1+\mu)}\,\frac{1-\frac{q_0^4}{128}}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}}-\frac{Pa^2\theta_0}{64\pi D}\,\frac{5+\mu}{1+\mu}.
\end{aligned}\right\}\qquad(64.9)$$

Here

$$D=\frac{Eh^3}{12(1-\mu^2)}.$$

Note that in formulas (64.9) in terms containing P, the members contain q_0^4, have been dropped, since during calculation starting from (64.2) it is not possible to keep sequentially all terms of such an order.

In the limit at $\theta_0\to 0$, $q_0\to 0$ (64.9) coincide with the formulas for a flat plate:

$$\vartheta_1^0=-\frac{M_1^0a}{D(1+\mu)}-\frac{Pa}{4\pi D}\,\frac{1}{1+\mu},\qquad \Delta_e^0=\frac{(1-\mu)H_e^0a}{Eh}.$$

Let us determine the axial displacement of a slightly curved plate under the action of applied force P and edge load M_1^0, H_e^0. Let us show first that also in the presence of concentrated force axial displacement can be determined by approximate formula:

$$\Delta_z \approx \int_{\theta_0}^{\theta} (R\vartheta_1 \cos\theta + \ldots)\, d\theta + C =$$

$$= \frac{R}{Eh} \int_{\theta_0}^{\theta} \left[A_1\chi_1(q) + B_1\chi_2(q) + \frac{P\gamma^3\sqrt{2}}{2R} \psi_3'(q) \right] d\theta + C. \qquad (64.10)$$

For this let us write out the term eliminated in subintegral expression

$$-R\varepsilon_1 \sin\theta \approx -R\varepsilon_1\theta = -\frac{R\theta}{Eh}\left[\frac{1}{2\gamma^2} A_1\left(\mu\chi_2' - \frac{1}{\theta}\chi_2\right) - \right.$$

$$\left. - \frac{1}{2\gamma^2} B_1\left(\mu\chi_1' - \frac{1}{\theta}\chi_1\right) + \tilde{T}_1 - \mu\tilde{T}_2 - \frac{P}{4R}\left(\mu\chi_4' - \frac{1}{\theta}\chi_4\right)\right] =$$

$$= -\frac{R\theta}{Eh}\left[\frac{1}{2\gamma^2} A_1\left(\mu\chi_2' - \frac{1}{\theta}\chi_2\right) - \frac{1}{2\gamma^2} B_1\left(\mu\chi_1' - \frac{1}{\theta}\chi_1\right)\right] +$$

$$+ \frac{P\gamma\sqrt{2}}{4}\left\{(1+\mu)\left[\psi_4'(q) - \frac{2}{\pi}\frac{1}{q}\right] + \mu q \psi_3(q)\right\}. \qquad (64.11)$$

Comparing (64.11) with the subintegral expression in (64.10), we are convinced that $R\varepsilon_1\theta$ is a quantity of order $\frac{1}{2\gamma^2}$ in comparison with $R\vartheta_1$, if we take the latter as unity. With the accepted correctness of calculations this confirms approximation equation (64.10). Reading axial displacement from the edge of the plate $\theta = \theta_0$, we set $C = 0$ and, integrating, we obtain

$$\Delta_z = \frac{R}{Eh}\left\{A_1[\psi_1(q) - \psi_1(q_0)] + B_1[\psi_2(q) - \psi_2(q_0)] + \right.$$

$$\left. + \frac{P\gamma^2}{2R}[\psi_3(q) - \psi_3(q_0)]\right\}. \qquad (64.12)$$

Displacement in the center of the plate, i.e., at $\theta = 0$, is equal to

$$\Delta_z(0) = \frac{R}{Eh}\{A_1[1 - \psi_1(q_0)] - B_1\psi_2(q_0)\} + \frac{P\gamma^2}{2Eh}\left[\frac{1}{2} - \psi_3(q_0)\right]. \qquad (64.13)$$

Here is is taken into account that

$$\psi_1(0) = 0, \quad \psi_2(0) = 0, \quad \psi_3(0) = \frac{1}{2}.$$

Substituting into (64.13) expressions for the constants of integration in accordance with (64.4), we have

$$\Delta_z(0) = \frac{R}{Eh}\left(H_e^0 - \tilde{T}_1 + \frac{P\gamma^2}{2Rq_0}\psi_4'\right)\frac{\left[-\Psi_3 + (1-\mu)\left(\Psi_1 + \frac{\psi_2'}{\Psi_4}\right) + \frac{q_0\psi_1}{\Psi_4}\right]}{\Psi_1 - (1-\mu)} -$$

$$- \frac{2\gamma^2}{Eh}\left[M_1^0 + \frac{P}{4}\left(\psi_4 - \frac{1-\mu}{q_0}\psi_3'\right)\right]\frac{\left(\frac{\psi_1'}{\Psi_4} - \Psi_2\right)}{\Psi_1 - (1-\mu)} + \frac{\gamma^2 P}{2Eh}\left(\frac{1}{2} - \psi_3\right). \qquad (64.14)$$

Here also the values of all functions ψ_i, Ψ_i and their derivatives should be calculated at $q=q_0$; for small q_0, using expansion (64.2) and making the necessary calculations, we obtain

$$\Delta_z(0) = -\frac{H_e^0 a^3 \theta_0}{32D(1+\mu)} \frac{5+\mu}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}} + \frac{M_1^0 a^2}{2D(1+\mu)} \frac{1-\frac{q_0^4}{144}}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}} + \\ + \frac{Pa^2}{16\pi D}\frac{3+\mu}{1+\mu}\left[1-\frac{11(1+\mu)}{576(3+\mu)} q_0^4 \ln q_0\right]. \qquad (64.15)$$

Axial displacement in the center of a circular flat plate is found by setting in (64.15) $q_0=0$, $\theta_0=0$:

$$\Delta_z(0) = \frac{M_1^0 a^2}{2D(1+\mu)} + \frac{Pa^2}{16\pi D}\frac{3+\mu}{1+\mu}.$$

In conclusion let us note that at very small q_0 and θ_0 in equations for angle of rotation and axial displacement it is possible, keeping terms containing $H_e^0\theta_0$, to eliminate everywhere higher orders of q in comparison with unity. Formulas (64.9), (64.15) then will take the form

$$\left.\begin{aligned} \theta_1^0 &= -\frac{M_1^0 a}{D(1+\mu)} + \frac{H_e^0 a^2 \theta_0}{4D(1+\mu)} - \frac{P}{4\pi D}\frac{a}{1+\mu}, \\ \Delta_e' &= \frac{(1-\mu) H_e^0 a}{Eh}, \\ \Delta_z(0) &= \frac{M_1^0 a^2}{2D(1+\mu)} + \frac{Pa^2}{16\pi D}\frac{3+\mu}{1+\mu} - \frac{H_e^0 a^3 \theta_0}{32D}\frac{5+\mu}{1+\mu}. \end{aligned}\right\} \qquad (64.16)$$

In order to understand the meaning of the simplification, turn to the basic resolvant equation (59.2). Let us assume that angle θ (even its maximum value θ_0) is so small that despite the considerable magnitude of parameter $2\gamma^2$, the following holds:

$$2\gamma^2 \ll \operatorname{ctg}^2 \theta. \qquad (64.17)$$

In this instance, if we eliminate the term $2i\gamma^2$ in the coefficient of σ_*, we find that equation (59.2) breaks into two independent equations, each of which will contain only one unknown function V_*, Ψ_*. This will correspond to replacing the shell by completely flat plate and to consideration separately of the plane problem and the assignment of bend. However, not always does replacing the shell by a flat plate, despite satisfying condition (64.17), give acceptable results. It is necessary to take into account also the character of the loading on the shell. For example, in a rotating shell the centrifugal forces create a bend in the shell, no matter how small its curvature. This forces us to take into account the influence of the plane stressed state on the bend of the plate, which is caused by the initial curvature [85]. In this case (59.2) should be simplified in the following manner. Substituting into it the expression for σ_* through real and imaginary parts of this function (59.1), we discard in the left side of the resulting equation only the quantity $\Psi_* 2i\gamma^2$ in comparison with $-\sigma_* \mathrm{ctg}^2\theta$, having kept, however, the term $4\gamma^4 V_*$. Then we obtain the following two equations in Ψ_* and V_*:

$$\frac{d^2V_*}{d\theta^2}+\mathrm{ctg}\,\theta\frac{dV_*}{d\theta}-V_*\,\mathrm{ctg}^2\theta=0,$$
$$\frac{d^2\Psi_*}{d\theta^2}+\mathrm{ctg}\,\theta\frac{d\Psi_*}{d\theta}-\Psi_*\,\mathrm{ctg}^2\theta+4\gamma^4V_*=0.$$

Going to variable $v=r$ and setting approximately $\cos\theta=1$, $\sin\theta=\theta$, $R\,d\theta=dr$, we obtain

$$\left.\begin{aligned}\frac{d^2V_*}{dr^2}+\frac{1}{r}\frac{dV_*}{dr}-\frac{V_*}{r^2}=0,\\ \frac{d^2\Psi_*}{dr^2}+\frac{1}{r}\frac{d\Psi_*}{dr}-\frac{\Psi_*}{r^2}+\frac{Eh}{D}V_*=0.\end{aligned}\right\} \qquad (64.18)$$

The first equation of (64.18) describes the plane stressed state, the second corresponds to the problem of bend allowing for the plane stressed state, since function V_* plays here the role of the load term. If we return to the basic system of equations which describe axisymmetric deformation of a shell of revolution (11.1), (11.8), then in accordance with the indicated approach in the second equation of (11.1) and in equation (11.8) we must set $R_1=R$, $R\,d\theta=dr$, $\cos\theta\approx1$, $\sin\theta\approx0$, and in the remaining equations of equilibrium take $\sin\theta\approx\theta$. Then we obtain two groups of equations, to which it is necessary to add another relationship of elasticity and approximate expressions for deformations. They will have the form

$$\left.\begin{aligned}&\frac{d}{dr}(rT_1)-T_2=0,\quad \frac{d}{dr}(r\varepsilon_2)-\varepsilon_1=0,\\ &T_1=\frac{Eh}{1-\mu^2}(\varepsilon_1+\mu\varepsilon_2),\quad T_2=\frac{Eh}{1-\mu^2}(\varepsilon_2+\mu\varepsilon_1),\\ &\varepsilon_1=\frac{du}{dr},\quad \varepsilon_2=\frac{u}{r}.\end{aligned}\right\} \qquad (64.19)$$

$$\left.\begin{aligned}&\frac{d}{dr}(rM_1)-M_2=rN_1, \quad T_1\theta-N_1=0,\\&M_1=D(\varkappa_1+\mu\varkappa_2), \quad M_2=D(\varkappa_2+\mu\varkappa_1),\\&\varkappa_1=-\frac{d\vartheta_1}{dr}, \quad \varkappa_2=-\frac{\vartheta_1}{r}, \quad \vartheta_1=\frac{dw}{dr}.\end{aligned}\right\} \tag{64.20}$$

Group of equations (64.19) represents equations of the plane problem in the case of axisymmetric loading, and (64.20) are distinguished from the equations for the bend of a plane plate only by the presence of the term $T_1\theta$. Introducing stress function V by the formula

$$T_1=\frac{V}{r}, \quad T_2=\frac{dV}{dr},$$

systems (64.19), (64.20) are reduced to two resolvant equations:

$$\left.\begin{aligned}&\frac{d^2V}{dr^2}+\frac{1}{r}\frac{dV}{dr}-\frac{V}{r^2}=0,\\&\frac{d^2\vartheta_1}{dr^2}+\frac{1}{r}\frac{d\vartheta_1}{dr}-\frac{\vartheta_1}{r^2}=-\frac{V\theta}{rD}.\end{aligned}\right\} \tag{64.21}$$

Solving equations (64.21) under the edge conditions on the edge of the plate

$$T_1=H_e^0, \quad M_1=M_1^0$$

taking into account the requirement of boundedness of forces and angle of rotation in the center, we arrive for the calculation of a weakly arched plate to the already obtained formulas (64.16).

§ 65. Forces and Moments in the Center and at the Edge of an Axisymmetrically Arched Shell of Small Curvature

We determine forces and moments in the center and at the edge of weakly arched plat loaded by edge loads H_e^0, M_1^0. There are no concentrated force in the summit and distributed loads. With the accepted (§ 64) simplifications on the edge of the shell exist of relationships

$$T_1^0\theta_0-N_1^0=0, \quad T_1^0+N_1^0\theta_0=H_e^0;$$

hence, ignoring the quantity θ_0^2 in comparison with unity, we obtain

$$H_e^0\approx T_1^0, \quad N_1^0\approx H_e^0\theta_0. \tag{65.1}$$

From the relationship

$$\Delta_e^0=\frac{a}{Eh}(T_2^0-\mu T_1^0)$$

we find that

$$T_2^0=\mu T_1^0+\frac{Eh\Delta_e^0}{a}. \qquad (65.2)$$

In exactly the same manner, replacing in relationships (57.10) $\cos\theta$ by 1 and $\sin\theta$ by θ we obtain the expression for M_2^0 through known quantities M_1^0 and ϑ_1^0

$$M_2^0=\mu M_1^0-\frac{Eh^3}{12a}\vartheta_1^0. \qquad (65.3)$$

Substituting into (65.2), (65.3) the quantities Δ_e^0, ϑ_1^0, determined by formulas (64.9), and setting $P=0$, we will have

$$T_2^0=H_e^0\left[1+\frac{q_0^4}{12(1+\mu)}\,\frac{1-\frac{1-\mu}{8}}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}}\right]-$$
$$-\frac{3}{h^2}M_1^0a\theta_0(1-\mu)\frac{1-\frac{q_0^4}{128}}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}} \qquad (65.4)$$

$$M_2^0=M_1^0\left[1-\frac{(1-\mu)}{(1+\mu)}\frac{q_0^4}{96}\,\frac{1}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}}\right]-$$
$$-\frac{H_e^0a\theta_0}{4}(1-\mu)\frac{1-\frac{q_0^4}{128}}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}}. \qquad (65.5)$$

Using formulas (60.2), (62.5) and (62.1), we find that forces and bending moments in the center of the plate are equal to

$$\left.\begin{aligned}T_1(0)=T_2(0)=\frac{A_1}{2},\\ M_1(0)=M_2(0)=\frac{R}{4v^4}(1+\mu)B_1.\end{aligned}\right\} \qquad (65.6)$$

Remembering the expressions for A_1, B_1, according to (64.4), we can write

$$
\left.
\begin{aligned}
T_2(0) &= \frac{H_e^0}{2\Delta(q_0)}\left[\psi_1(q_0)+\frac{1-\mu}{q_0}\psi_2'(q_0)\right]- \\
&\qquad -\sqrt{3(1-\mu^2)}\,\frac{M_1^0}{h}\,\frac{\psi_1'(q_0)}{q_0\Delta(q_0)}, \\
M_1(0) &= \frac{H_e^0 h}{4\Delta(q_0)}\sqrt{\frac{1+\mu}{3(1-\mu)}}\left[\psi_2(q_0)-\frac{1-\mu}{q_0}\psi_1'(q_0)\right]- \\
&\qquad -\frac{1+\mu}{2}M_1^0\,\frac{\psi_2'(q_0)}{q_0\Delta(q_0)}.
\end{aligned}
\right\}
\tag{65.7}
$$

Substituting into (65.7) expansion (64.1), we obtain

$$
T_2(0) = H_e^0\frac{1-\frac{q_0^4}{96}\frac{(3+\mu)}{(1+\mu)}}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}}+\sqrt{\frac{3(1-\mu)}{1+\mu}}\,\frac{\gamma^2 M_1^0\theta_0^2}{2h}\,\frac{1-\frac{7q_0^4}{1152}}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}}, \tag{65. }
$$

$$
M_1(0) = -\frac{H_e^0 a\theta_0(3+\mu)}{8}\,\frac{1-\frac{25+7\mu}{3+\mu}\frac{q_0^4}{1152}}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}}+M_1^0\frac{1-\frac{q_0^4}{96}}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}}. \tag{65.9}
$$

We determine with the aid of the obtained equations the bend and force in the center and on the edges of a slightly arched heavy plate on a smooth base. On the plate acts distributed load

$$
q_n \approx q_z = -\rho h, \tag{65.10}
$$

where ρ – specific weight of material.

By formulas (57.15) we calculate forces of the zero-moment state and radial displacement

$$
\left.
\begin{aligned}
\tilde{T}_1 &= \frac{\rho h R(\cos\theta-1)}{\sin^2\theta}\approx-\frac{\rho h R}{2}, \\
\tilde{T}_2 &= -\frac{\rho h R}{2}, \quad \tilde{\Delta}_e = -\frac{a\rho R}{2E}(1-\mu).
\end{aligned}
\right\}
\tag{65.11}
$$

Since on the edge of the plate $T_1^0=H_e^0=0$, then from (57.3), (57.4) it follows that shearing force on the edge is

$$
N_1^0=\frac{\rho h R\theta_0}{2}.
$$

During the calculations Δ_e^0 and M_2^0 by formulas (64.9) and (65.5) it is not necessary to replace Δ_e^0 by $\Delta_e^0-\tilde{\Delta}_e^0$, H_e^0 by $H_e^0-\tilde{T}_1^0$, and after this set in them $H_e^0=M_1^0=0$. As a result we obtain

$$\left.\begin{aligned}
\Delta_e^0 &= \frac{\rho a^4\theta_0(1-\mu^2)}{16Eh^2}\,\frac{7+\mu}{1+\mu}\,\frac{1}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}},\\
T_2^0 &= \frac{Eh}{a}\,\Delta_e^0 = \frac{\rho a^3\theta_0(1-\mu^2)}{16h}\,\frac{7+\mu}{1+\mu}\,\frac{1}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}},\\
M_2^0 &= -\frac{\rho h a^2}{8}(1-\mu)\frac{1-\frac{q_0^4}{128}}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}}.
\end{aligned}\right\} \tag{65.12}$$

Axial displacement in the center of the plate is determined by the formula (64.15)

$$\Delta_z(0) = -\frac{\rho h a^4}{64D}\,\frac{5+\mu}{1+\mu}\,\frac{1}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}}. \tag{65.13}$$

The minus sign in the last formula is explained by the fact that a positive direction of Δ_z has been accepted along axis OZ, and the load - intrinsic weight - is directed to the opposite side. In a flat plate we would have correspondingly

$$\Delta_e^0 = 0,\quad T_2^0 = 0,\quad M_2^0 = -\frac{\rho h a^2}{8}(1-\mu),\quad \Delta_z(0) = -\frac{\rho h a^4}{64D}\,\frac{5+\mu}{1+\mu}.$$

The presence of initial curvature gives rise to circumferential forces T_2^0 and decreases the circumferential bending moment by a factor of $\dfrac{1+\frac{1}{1+\mu}\frac{q_0^4}{96}}{1-\frac{q_0^4}{128}}$ and axial displacement by a factor of $\left(1+\frac{1}{1+\mu}\frac{q_0^4}{96}\right)$. There is also a decrease in bending moment in the center of the plate. Really, replacing in (65.8), (65.9), $T_2(0)$ by $T_2(0)-\tilde{T}_2(0)$, H_e^0 by $H_e^0-\tilde{H}_e^0$ and setting $H_e^0 = M_1^0 = 0$, we find

$$\left.\begin{aligned}
T_2(0) &= -\frac{\rho a^3\theta_0}{16h}\,\frac{(4+\mu)(1-\mu)}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}},\\
M_1(0) = M_2(0) &= -\frac{\rho h a^2(3+\mu)}{16}\,\frac{1-\frac{25+7\mu}{3+\mu}\frac{q_0^4}{1152}}{1+\frac{1}{1+\mu}\frac{q_0^4}{96}},
\end{aligned}\right\} \tag{65.14}$$

while in a flat plate

$$T_2 = 0,\quad M_2(0) = -\frac{\rho h a^2(3+\mu)}{16}.$$

§ 66. Deformation of a Spherical Shell Under the Action of a Bending Load

The deformation of an arbitrary shell of revolution under the action of a bending load of form

$$\left.\begin{aligned} q_1 &= q_{1(1)} \cos\varphi, \\ q_2 &= q_{2(1)} \sin\varphi, \\ q_n &= q_{n(1)} \cos\varphi \end{aligned}\right\} \qquad (66.1)$$

was considered in §§ 14-18 of Chapter III. Subsequently let us pass from the designations $T_{1(1)}, T_{2(1)}, S_{(1)}, H_{(1)}, M_{1(1)}, M_{2(1)}, N_{1(1)}, N_{2(1)}$, used for the static quantities in Chapter II, to the designations $t_1, t_2, s_1, h_1, m_1, m_2, n_1, n_2$.

Conditions of equilibrium of the finite part of the spherical shell contained between edge section θ_0 and instantaneous section θ, are obtained from (15.21), (15.22), if in them we set $R_1 = R$, $v = R\sin\theta$, and have the form

$$\left.\begin{aligned} & t_1 R\sin\theta\cos\theta + n_1 R\sin^2\theta - s_1 R\sin\theta - h_1\sin\theta + f(\theta) = 0, \\ & n_1 R^2\sin^2\theta\cos\theta + h_1 R\sin\theta\cos\theta - m_1 R\sin\theta - \\ & \qquad - t_1 R^2\sin^3\theta - F(\theta) = 0, \end{aligned}\right\} \qquad (66.2)$$

where we have introduced the designations

$$\left.\begin{aligned} f(\theta) &= \frac{P_0}{\pi} - \frac{P_q(\theta)}{\pi}, \\ F(\theta) &= \frac{M_0}{\pi} - \frac{P_0(Z - Z_0)}{\pi} + \frac{M_q(\theta)}{\pi}, \end{aligned}\right\} \qquad (66.3)$$

$$\left.\begin{aligned} & P_q(\theta) = -\pi \int_{\theta_0}^{\theta} (q_{1(1)}\cos\theta - q_{2(1)} + q_{n(1)}\sin\theta) R^2\sin\theta\, d\theta, \\ & M_q(\theta) = -\int_{\theta_0}^{\theta} P_q(\theta) R\sin\theta\, d\theta - \\ & \qquad -\pi \int_{\theta_0}^{\theta} (q_{n(1)}\cos\theta - q_{1(1)}\sin\theta) R^3\sin^2\theta\, d\theta, \\ & Z = -R(1 - \cos\theta), \quad Z_0 = -R(1 - \cos\theta_0). \end{aligned}\right\} \qquad (66.4)$$

$P_q(\theta)$, $M_q(\theta)$ — resultant and composite moment of distributed external load (66.1), applied on shell section (θ_0, θ), calculated relative to the axes parallel to OX and OY in section $\theta = \mathrm{const}$. In order that the equilibrium of the shell as a whole would be held, the load should satisfy the conditions:

$$\left.\begin{aligned} P_* - P_0 + P_q(0) = 0, \\ -M_* + M_0 + P_0 Z_0 + M_q(0) = 0. \end{aligned}\right\} \qquad (66.5)$$

P_*, M_* — concentrated actions in the pole (Fig. 33). Taking into account these conditions, it is simple to calculate the values of functions (66.3) at $\theta = 0$:

$$f(0) = \frac{P_*}{\pi}, \quad F(0) = \frac{M_*}{\pi}. \qquad (66.6)$$

The basic resolvant equations of problem (16.15), (16.16) (Meissner equations type) for a spherical shell of constant thickness have the form

$$\left.\begin{aligned} &\frac{d^2\Psi_1}{d\theta^2} + \mathrm{ctg}\,\theta\frac{d\Psi_1}{d\theta} + \Psi_1\left(1 + \mu - \frac{4}{\sin^2\theta}\right) - 4\gamma^4 V_1 + \\ &\qquad + V_1(1 - \mu^2)\left(\frac{2}{\sin^2\theta} - 3\right) = 4\gamma^4\Phi_3(\theta), \\ &\frac{d^2V_1}{d\theta^2} + \mathrm{ctg}\,\theta\frac{dV_1}{d\theta} + V_1\left(1 - \mu - \frac{4}{\sin^2\theta}\right) + \Psi_1 - \\ &\qquad - \Psi_1\frac{(1-\mu^2)}{4\gamma^4}\left(\frac{2}{\sin^2\theta} - 3\right) = R\Phi_4(\theta), \\ &4\gamma^4 = \frac{12(1-\mu^2)R^2}{h^2}. \end{aligned}\right\} \qquad (66.7)$$

Load functions $F_0(\theta) + F_1(\theta)$, $f_0(\theta) + f_1(\theta)$, used in §§ 15, 16 (formulas (15.34) (15.35)), are associated with the newly introduced functions $f(\theta)$, $F(\theta)$ by simple relationships

$$\left.\begin{aligned} F_0(\theta) + F_1(\theta) &= \frac{F(\theta)\cos\theta - f(\theta)R\sin^2\theta}{R^2\sin^2\theta} - \frac{f_2(\theta)}{R}, \\ f_0(\theta) + f_1(\theta) &= -\frac{F(\theta) + f(\theta)R\cos\theta}{R^2\sin\theta} - \frac{\cos\theta}{R\sin\theta}f_2(\theta), \end{aligned}\right\} \qquad (66.8)$$

where

$$f_2(\theta) = R^2\int_{\theta_0}^{\theta} q_{2(1)}\sin\theta\, d\theta. \qquad (66.9)$$

The right side of equations (66.7) are equal to

$$\left.\begin{aligned}\Phi_3(\theta) &= \frac{1}{R^2\sin^2\theta}(F\cos\theta - fR\sin^2\theta) - \frac{f_2}{R},\\ R\Phi_4(\theta) &= -\frac{\mu}{R^2}\frac{1}{\sin\theta}\frac{d}{d\theta}(F + fR\cos\theta) -\\ &- \frac{(1-\mu)\cos\theta}{R^2\sin^2\theta}(F + fR\cos\theta) +\\ &+ \frac{(1-\mu)}{R}\frac{\cos\theta}{\sin\theta}\frac{df_2}{d\theta} + \left(1-\mu-\frac{4}{\sin^2\theta}\right)\frac{f_2}{R}.\end{aligned}\right\} \qquad (66.10)$$

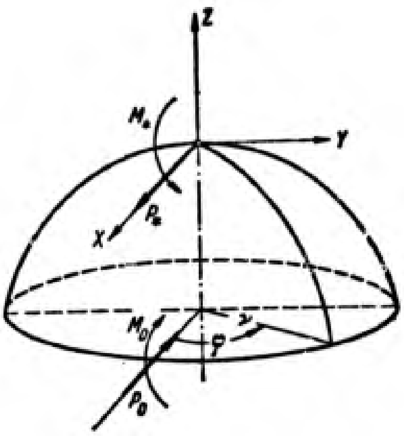


Fig. 33. Concentrated effects in the pole, main vector and main moment of loads applied to the edge of a spherical dome.

Forces and bending moments are determined through functions V_1 and Ψ_1 by formulas (16.6), (16.7), in which it is necessary to set

$$R_1 = R, \quad \nu = R\sin\theta, \quad V = V_1R, \quad \Psi Eh = \Psi_1. \qquad (66.11)$$

Introducing complex unknown

$$\sigma_1 = \Psi_1 + 2i\gamma^2 V_1, \qquad (66.12)$$

we replace equations (66.7) by one equation in σ_1, which after rejecting in the left side quantites of the order $\frac{1}{2\gamma^2}$ in comparison with unity, assumes the form

$$L_1(\sigma_1) + \sigma_1\left(2i\gamma^2 - \frac{4}{\sin^2\theta}\right) = 4\gamma^4\Phi_3(\theta) + 2i\gamma^2 R\Phi_4(\theta), \qquad (66.13)$$

where

$$L_1(\sigma_1) = \frac{d^2\sigma_1}{d\theta^2} + \operatorname{ctg}\theta\frac{d\sigma_1}{d\theta}. \qquad (66.14)$$

Note that the right side of equation (66.13) is not regular in point $\theta = 0$ even in the absence of concentrated effects. In order to get rid of the irregularity of this kind, we make the replacement

$$\Psi_1 = \Psi_2, \quad V_1 = V_2 + \frac{1}{R}f_2(\theta). \qquad (66.15)$$

then relative to

$$\sigma_2 = \Psi_2 + 2i\gamma^2 V_2 \tag{66.16}$$

we will have the equation

$$L_1(\sigma_2) + \sigma_2\left(2i\gamma^2 - \frac{4}{\sin^2\theta}\right) = 4\gamma^4 \frac{F\cos\theta - fR\sin^2\theta}{R^2\sin^2\theta} + \\ + 2i\gamma^2\left[-\frac{\mu}{R^2\sin\theta}\frac{d}{d\theta}(F + fR\cos\theta) - \right. \\ \left. - \frac{(1-\mu)\cos\theta}{R^2\sin^2\theta}(F + fR\cos\theta) - Rq_{2(1)}\cos\theta(1+\mu)\right]. \tag{66.17}$$

To the simplifications made during the transition from system of equations (66.7) to equation (66.13), allowing for substitution (66.15) corresponds the following variant of expressions for forces and moments through introduced Ψ_2, V_2: functions

$$\left.\begin{aligned} t_1 &= V_2\operatorname{ctg}\theta - \frac{1}{R^2\sin\theta}(F + fR\cos\theta) - \frac{1}{4\gamma^4}\frac{d\Psi_2}{d\theta}, \\ t_2 &= \frac{dV_2}{d\theta} + V_2\operatorname{ctg}\theta + Rq_{2(1)}\sin\theta - \frac{\mu}{4\gamma^4}\frac{d\Psi_2}{d\theta}, \\ s_1 &= \frac{V_2}{\sin\theta}, \quad h_e \approx -\frac{f(\theta)}{R\sin\theta} + s_1, \end{aligned}\right\} \tag{66.18}$$

$$\left.\begin{aligned} m_1 &= \frac{R}{4\gamma^4}\left[\frac{d\Psi_2}{d\theta} + (1+\mu)\Psi_2\operatorname{ctg}\theta\right], \\ m_2 &= \frac{R}{4\gamma^4}\left[\mu\frac{d\Psi_2}{d\theta} + (1+\mu)\Psi_2\operatorname{ctg}\theta\right], \\ h_1 &= -\frac{R(1-\mu)}{4\gamma^4}\frac{1}{\sin\theta}\Psi_2, \\ n_1 &= \frac{1}{4\gamma^4}\left(\frac{d^2\Psi_2}{d\theta^2} + 2\operatorname{ctg}\theta\frac{d\Psi_2}{d\theta} - \frac{2\Psi^2}{\sin_2\theta}\right). \end{aligned}\right\} \tag{66.19}$$

§ 67. Particular Solution of Equations of Meissner Type

Let us rewrite the right term of equation (66.17), changing in it only the order of the terms. We will have

$$4\gamma^4\frac{F\cos\theta - fR\sin^2\theta}{R^2\sin^2\theta} - 2i\gamma^2(1-\mu)\frac{f\cos^2\theta}{R\sin^2\theta} - 2i\gamma^2(1-\mu)\frac{F\cos\theta}{R^2\sin^2\theta} + \\ + 2i\gamma^2\left[-\frac{\mu}{R^2\sin\theta}\frac{d}{d\theta}(F + fR\cos\theta) - Rq_{2(1)}(1+\mu)\cos\theta\right]. \tag{67.1}$$

At large θ the first term in (67.1) is great in comparison with the remaining, as $2\gamma^2$ in comparison with unity, and is the only essential term. When θ is close to zero, it is necessary to estimate (67.1) more thoroughly. The last term in the brackets at θ = 0 remains bounded, and the first three turn into infinity as $\frac{1}{\sin^2\theta}$. However, the third term is in comparison with the first quantity of order $\frac{1}{2\gamma^2}$

at all θ including θ = 0, and therefore always can be dropped. In this way, at large θ equation (66.17) can be simplified and written in the form

$$L_1(\sigma_2) + \sigma_2\left(2i\gamma^2 - \frac{4}{\sin^2\theta}\right) = 4\gamma^4 \frac{F\cos\theta - fR\sin^2\theta}{R^2\sin^2\theta}. \qquad (67.2)$$

At small θ we can use the equation

$$L_1(\sigma_2) + \sigma_2\left(2i\gamma^2 - \frac{4}{\sin^2\theta}\right) = 4\gamma^4 \frac{F\cos\theta - fR\sin^2\theta}{R^2\sin^2\theta} - 2i\gamma^2(1-\mu)\frac{f\cos^2\theta}{R\sin^2\theta}. \qquad (67.3)$$

Within the limits of this section we will be limited to large θ and will construct the particular solution of equation (67.2). To simplify the form of the right side of (67.2) we will make one substitution of variables:

$$\left.\begin{aligned} &\Psi_2 = \Psi_3 - \frac{\gamma^4}{R^2}(F\cos\theta - fR\sin^2\theta), \quad V_2 = V_3, \\ &\sigma_2 = \sigma_3 - \frac{\gamma^4}{R^2}(F\cos\theta - fR\sin^2\theta), \end{aligned}\right\} \qquad (67.4)$$

where $\sigma_3 = \Psi_3 + 2i\gamma^2 V_3$. Then we obtain

$$L_1(\sigma_3) + \sigma_3\left(2i\gamma^2 - \frac{4}{\sin^2\theta}\right) = 2i\gamma^2 G(\theta) + \ldots, \qquad (67.5)$$

where

$$G(\theta) = \frac{\gamma^4}{R^2}[F(\theta)\cos\theta - f(\theta)R\sin^2\theta], \qquad (67.6)$$

and the dots indicate the discarded small term $\frac{\gamma^4}{R^2}L_1(F\cos\theta - fR\sin^2\theta)$. Let us examine the particular case when there is no distributed load. There is only a system of loads applied on edge θ_0, statically equivalent to force P_0 and to moment M_0, which are balanced by concentrated effects P_*, M_* in the pole. In this instance function $G(\theta)$ has the form

$$G(\theta) = A + B\cos\theta, \qquad (67.7)$$

$$A = -\frac{\gamma^4 P_*}{\pi R}, \quad B = \frac{\gamma^4}{\pi}\left(\frac{M_*}{R^2} + \frac{P_*}{R}\right). \qquad (67.8)$$

We are looking for the particular solution of equation (67.5) when the right side of (67.7) is far from the pole in the form

$$\left.\begin{aligned} \Psi'_3 &= a_1 + \frac{a_2}{\sin^2\theta} + b_1\cos\theta + \frac{b_2\cos\theta}{\sin^2\theta}, \\ V'_3 &= c_1 + \frac{c_2}{\sin^2\theta} + d_1\cos\theta + d_2\frac{\cos\theta}{\sin^2\theta}. \end{aligned}\right\} \qquad (67.9)$$

Substituting the above expressions into equation (67.5) and comparing coefficients of like terms, we derive a system of algebraic equations for determination of the unknown coefficients:

$$\left.\begin{aligned} a_1 &= A, & -2c_2 - 4c_1 + a_2 &= 0, \\ b_1 - 2d_1 &= B, & -4d_1 + b_2 &= 0, \\ -4\gamma^4 c_1 &= 0, & -2a_2 - 4a_1 - 4\gamma^4 c_2 &= 0, \\ -2b_1 - 4\gamma^4 d_1 &= 0, & -4b_1 - 4\gamma^4 d_2 &= 0. \end{aligned}\right\} \qquad (67.10)$$

Solving it, we have

$$\left.\begin{aligned} a_1 &= A, & a_2 &= -\frac{2A}{\gamma^4\left(1+\frac{1}{\gamma^4}\right)}, \\ b_1 &= \frac{B}{1+\frac{1}{\gamma^4}}, & d_1 &= -\frac{2B}{4\gamma^4}\,\frac{1}{1+\frac{1}{\gamma^4}}, \\ c_2 &= -\frac{4A}{4\gamma^4\left(1+\frac{1}{\gamma^4}\right)}, & b_2 &= -2B\,\frac{1}{1+\frac{1}{\gamma^4}}, \\ c_1 &= 0, & d_2 &= -\frac{B}{\gamma^4\left(1+\frac{1}{\gamma^4}\right)}. \end{aligned}\right\} \qquad (67.11)$$

Dropping $\frac{1}{\gamma^4}$ in comparison with unity, we obtain

$$\Psi_3 = A - \frac{2A}{\gamma^4}\,\frac{1}{\sin^2\theta} + B\cos\theta - \frac{2B}{\gamma^4}\,\frac{\cos\theta}{\sin^2\theta},$$

$$V_3 = -\frac{A}{\gamma^4}\,\frac{1}{\sin^2\theta} - \frac{B}{2\gamma^4}\cos\theta - \frac{B}{\gamma^4}\,\frac{\cos\theta}{\sin^2\theta},$$

or

$$\sigma_3 \approx (A + B\cos\theta)\left(1 + \frac{4}{2i\gamma^2\sin^2\theta}\right). \qquad (67.12)$$

It is easy to see that we can come to the same result if during differentiation we consider $A + B\cos\theta$ as constant. We can assert that for any slowly changing function $G(\theta)$ the particular solution can be taken in the form

$$\sigma_3 = G(\theta)\left(1 + \frac{4}{2i\gamma^2\sin^2\theta}\right). \qquad (67.13)$$

$$\Psi_3 = \frac{\gamma^4}{R^2}(F\cos\theta - fR\sin^2\theta), \quad V_3 = -\frac{1}{R^2}\,\frac{F\cos\theta - fR\sin^2\theta}{\sin^2\theta}.$$

The quantities σ_2 corresponding to this solution is

$$\tilde{\sigma}_2 = -2i\gamma^2\,\frac{F\cos\theta - fR\sin^2\theta}{R^2\sin^2\theta}. \qquad (67.14)$$

Solution (67.14) is marked by a tilde ∿ to indicate that it gives the zero-moment stressed state. Really, substituting (67.14) into equations (66.18), (66.19), we obtain

$$\left.\begin{aligned}\tilde{t}_1 &= -\frac{F(\theta)}{R^2\sin^3\theta}, \quad \tilde{s}_1 = -\frac{F\cos\theta - fR\sin^2\theta}{R^2\sin^3\theta},\\ \tilde{t}_2 &= \frac{F(\theta)}{R^2\sin^3\theta} + q_{n\,(1)}R. \quad \tilde{m}_1 = \tilde{m}_2 = \tilde{h}_1 = \tilde{n}_1 = 0.\end{aligned}\right\} \tag{67.15}$$

In this way we arrived at the conclusion that far from places of pronounced change in loads, i.e., smoothly changing right side, as the particular solution of the basic resolvant equation (67.2) we can take zero-moment solution (67.14). Note that when there are no concentrated effects in the pole ($P_* = M_* = 0$). and there are only a distributed load and load on edge θ_0, zero-moment solution (67.14) and forces (67.15) because of (66.6) remain bounded even at $\theta = 0$. In this instance zero-moment solution (67.14) can be used as the particular solution of the basic equations also for small θ.

§ 68. Solution of Uniform Resolvant Equation

Let us pass now to the solution of uniform equation

$$\frac{d^2\sigma_2}{d\theta^2} + \operatorname{ctg}\theta\frac{d\sigma_2}{d\theta} + \sigma_2\left(2i\gamma^2 - \frac{4}{\sin^2\theta}\right) = 0. \tag{68.1}$$

By the substitution

$$\sigma_2 = \frac{1}{\sin^2\theta}\sigma_*, \quad 2i\gamma^2 = n(n+1),$$

$$\xi = \cos\theta, \quad \frac{d^2\sigma_*}{d\xi^2} = y(\xi)$$

it can be brought to the Legendre equation

$$(\xi^2 - 1)y'' + 2\xi y' - n(n+1)y = 0.$$

Use of Legendre functions with a complex parameter can be avoided if, in the same way as in § 59 in examining axisymmetric deformation, we approximately express the solution of the involved equation through Bessel functions. Setting

$$\sigma_2 = \tau/\sqrt{\sin\theta}, \tag{68.2}$$

instead of (68.1), we obtain

$$\frac{d^2\tau}{d\theta^2} + \tau\left(2i\gamma^2 - \frac{15}{4}\frac{\cos^2\theta}{\sin^2\theta}\right) = 0. \tag{68.3}$$

Then with the substitution

$$\tau_1\sqrt{\theta}=\tau, \quad \frac{\cos^2\theta}{\sin^2\theta}\approx\frac{1}{\theta^2}, \quad x=\sqrt{2i}\,\gamma\theta \tag{68.4}$$

we bring (68.3) to the Bessel equation

$$\frac{d^2\tau_1}{dx^2}+\frac{1}{x}\frac{d\tau_1}{dx}+\tau_1\left(1-\frac{4}{x^2}\right)=0. \tag{68.5}$$

the general solution of which can be taken in the form

$$\tau_1=(A_1-iB_1)T_2(\sqrt{2i}\,\gamma\theta)+(A_2-iB_2)H_2^{(1)}(\sqrt{2i}\,\gamma\theta). \tag{68.6}$$

In this way, the general solution (68.1) is approximately representable thus:

$$\sigma_2=\sqrt{\frac{\theta}{\sin\theta}}\left[(A_1-iB_1)I_2(\sqrt{2i}\,\gamma\theta)+(A_2-iB_2)H_2^{(1)}(\sqrt{2i}\,\gamma\theta)\right]. \tag{68.7}$$

The correctness of the representation of solutions of the Legendre equation (or equation (68.1) leading to it) through Bessel functions was discussed in § 59. At large values γ, but small θ solution (68.7) could be used for calculation, holding in this case terms of order $\frac{1}{\gamma}$ in comparison with unity. At large γ the retention of these terms becomes inadmissable.

Let us examine a spherical shell without an opening. In the summit of the shell there are no concentrated effects $(P_*=M_*=0)$. The shell is thin-walled and the value of angle θ_0, characterizing the edge section, is found within the limits of $0\ll\theta_0\ll\pi$. Inasmuch as there are no concentrated effects and σ_2 should be bounded in the pole, we set $A_2=B_2=0$. In this way,

$$\sigma_2=(A_1-iB_1)[\zeta_1(\theta)+i\zeta_2(\theta)]. \tag{68.8}$$

where we have introduced the designations

$$\zeta_1(\theta)=\sqrt{\frac{\theta}{\sin\theta}}\,\mathrm{Re}\,I_2(\sqrt{2i}\,\gamma\theta), \quad \zeta_2(\theta)=\sqrt{\frac{\theta}{\sin\theta}}\,\mathrm{Im}\,I_2(\sqrt{2i}\,\gamma\theta). \tag{68.9}$$

We use the asymptotic representations for functions $I_2(\sqrt{2i}\,\gamma\theta)$ and its derivative given in Chapter IV (formulas (46.14), (46.15), (46.9)), keeping in them only principal terms; then we obtain

$$\left.\begin{aligned}
\zeta_1(\theta) &= -\frac{e^{\gamma\theta}}{\sqrt{2\pi\gamma\sin\theta}\sqrt{2}}\cos\left(\gamma\theta-\frac{\pi}{8}\right),\\
\zeta_2(\theta) &= \frac{e^{\gamma\theta}}{\sqrt{2\pi\gamma\sin\theta}\sqrt{2}}\sin\left(\gamma\theta-\frac{\pi}{8}\right),\\
\zeta_1'(\theta) &= -\frac{\gamma e^{\gamma\theta}}{\sqrt{2\pi\gamma\sin\theta}\sqrt{2}}\left[\cos\left(\gamma\theta-\frac{\pi}{8}\right)-\sin\left(\gamma\theta-\frac{\pi}{8}\right)\right],\\
\zeta_2'(\theta) &= \frac{\gamma e^{\gamma\theta}}{\sqrt{2\pi\gamma\sin\theta}\sqrt{2}}\left[\cos\left(\gamma\theta-\frac{\pi}{8}\right)+\sin\left(\gamma\theta-\frac{\pi}{8}\right)\right].
\end{aligned}\right\} \quad (68.10)$$

Having the solution of the uniform equation and the particular solution corresponding to the right side, obtained in the previous section, we compute the forces and moments by the formulas:

$$\left.\begin{aligned}
t_1 &= \tilde{t}_1 + \frac{\operatorname{ctg}\theta}{2\gamma^2}[A_1\zeta_2(\theta)-B_1\zeta_1(\theta)],\\
t_2 &= \tilde{t}_2 + \frac{1}{2\gamma^2}[A_1\zeta_2'(\theta)-B_1\zeta_1'(\theta)],\\
s_1 &= \tilde{s}_1 + \frac{1}{2\gamma^2}\frac{1}{\sin\theta}[A_1\zeta_2(\theta)-B_1\zeta_1(\theta)],\\
m_1 &= \frac{R}{4\gamma^4}[A_1\zeta_1'(\theta)+B_1\zeta_2'(\theta)], \quad m_2=\mu m_1,\\
h_1 &= -\frac{R(1-\mu)}{4\gamma^4}[A_1\zeta_1(\theta)+B_1\zeta_2(\theta)].
\end{aligned}\right\} \quad (68.11)$$

Circumferential deformation $\varepsilon_{2(1)}$ and function Ψ_2 have the form

$$\left.\begin{aligned}
\varepsilon_{2(1)} &= \tilde{\varepsilon}_{2(1)} + \frac{1}{Eh}\frac{1}{2\gamma^2}[A_1\zeta_2'(\theta)-B_1\zeta_1'(\theta)],\\
\Psi_2 &= A_1\zeta_1(\theta)+B_1\zeta_2(\theta).
\end{aligned}\right\} \quad (68.12)$$

where

$$\tilde{\varepsilon}_{2(1)} = \frac{1}{Eh}(\tilde{t}_2-\mu\tilde{t}_1).$$

In accordance wtih the accepted accuracy of calculations during the derivation of formulas (68.11) we dropped secondary terms, for example we took

$$m_1=\frac{R}{4\gamma^4}\frac{d\Psi_2}{d\theta}, \quad m_2=\frac{R}{4\gamma^4}\mu\frac{d\Psi_2}{d\theta} \text{ etc.}$$

Radial force h_e we determine with the aid of the first equation of statics (66.2)

$$h_e = t_1\cos\theta+\left(n_1+\frac{h_1}{\nu}\right)\sin\theta = -\frac{f(\theta)}{R\sin\theta}+\left(s_1+\frac{2h_1}{R}\right),$$

where, taking into account formulas (68.11), (66.19), we can set

$$n_1+\frac{h_1}{\nu}\approx n_1, \quad s_1+\frac{2h_1}{R}\approx s_1 \quad (68.13)$$

and

$$h_e=\tilde{h}_e+\frac{1}{2\gamma^2}\frac{1}{\sin\theta}[A_1\zeta_2(\theta)-B_1\zeta_1(\theta)].$$

where

$$\tilde{h}_e = \tilde{t}_1 \cos\theta. \qquad (68.14)$$

§ 69. Determination of Constants of Integration

With the aid of formulas (68.11), (68.12), (68.14) we determine the stressed state in the neighborhood of edge θ_0. On the edge act radial forces and bending moments with preassigned amplitude h_e^0 and m_1^0. The shell is loaded by a distributed load smoothly changing along the meridian. Inasmuch as we are looking for the stressed state near the edge, where it is held that $\gamma\theta_0 \gg 1$, then we have no interest in whether or not there are concentrated effects in the angle. The case when $\gamma\theta_0$ has the order of unity and it is necessary to take into account the mutual influence of the edge and pole in which there are concentrated effects, was examined in § 72. Requiring that at shell edge θ_0 the equality

$$h_e = h_e^0, \quad m_1 = m_1^0, \qquad (69.1)$$

would hold, according to (68.14) and to the fourth equation of (68.11), we obtain for determination of constants A_1, B_1 the system of equations:

$$\left.\begin{aligned} &A_1\zeta_2(\theta_0) - B_1\zeta_1(\theta_0) = 2\gamma^2 \sin\theta_0 (h_e^0 - \tilde{h}_e^0), \\ &A_1\zeta_1'(\theta_0) + B_1\zeta_2'(\theta_0) = \frac{4\gamma^4}{R} m_1^0. \end{aligned}\right\} \qquad (69.2)$$

Solving it, we find

$$\left.\begin{aligned} A_1 = e^{-\gamma\theta_0}\sqrt{2\pi\gamma\sin\theta_0}\sqrt{2}\,\Big\{&2\gamma^2\sin\theta_0(h_e^0 - \tilde{h}_e^0)\Big[\cos\left(\gamma\theta_0 - \frac{\pi}{8}\right) + \\ &+ \sin\left(\gamma\theta_0 - \frac{\pi}{8}\right)\Big] - \frac{4\gamma^3}{R} m_1^0 \cos\left(\gamma\theta_0 - \frac{\pi}{8}\right)\Big\}, \\ B_1 = e^{-\gamma\theta_0}\sqrt{2\pi\gamma\sin\theta_0}\sqrt{2}\,\Big\{&2\gamma^2\sin\theta_0(h_e^0 - \tilde{h}_e^0)\Big[\cos\left(\gamma\theta_0 - \frac{\pi}{8}\right) - \\ &- \sin\left(\gamma\theta_0 - \frac{\pi}{8}\right)\Big] + \frac{4\gamma^3}{R} m_1^0 \sin\left(\gamma\theta_0 - \frac{\pi}{8}\right)\Big\}. \end{aligned}\right\} \qquad (69.3)$$

Substituting these values of the constants into equations (68.11), we find the forces and moments in any cut of shell

$$\left.\begin{aligned} h_e = \tilde{h}_e + \frac{1}{\sin\theta}\sqrt{\frac{\sin\theta_0}{\sin\theta}}\,\Big\{&(h_e^0 - \tilde{h}_e^0)\sin\theta_0\psi[\gamma(\theta_0 - \theta)] + \\ &+ \frac{2\gamma}{R} m_1^0 \zeta[\gamma(\theta_0 - \theta)]\Big\}, \\ m_1 = \sqrt{\frac{\sin\theta_0}{\sin\theta}}\,\Big\{&-\frac{R}{\gamma}(h_e^0 - \tilde{h}_e^0)\sin\theta_0\zeta[\gamma(\theta_0 - \theta)] + \\ &+ m_1\varphi[\gamma(\theta_0 - \theta)]\Big\}. \end{aligned}\right\} \qquad (69.4)$$

$$t_1 = \tilde{t}_1 + \operatorname{ctg}\theta \sqrt{\frac{\sin\theta_0}{\sin\theta}} \left\{ (h_e^0 - \tilde{h}_e^0) \sin\theta_0 \psi\, [\gamma(\theta_0 - \theta)] + \right.$$
$$\left. + \frac{2\gamma}{R} m_1^0 \zeta\, [\gamma(\theta_0 - \theta)] \right\},$$
$$t_2 = \tilde{t}_2 + 2\gamma \sqrt{\frac{\sin\theta_0}{\sin\theta}} \left\{ (h_e^0 - \tilde{h}_e^0) \sin\theta_0 \theta\, [\gamma(\theta_0 - \theta)] - \right.$$
$$\left. - \frac{\gamma}{R} m_1^0 \psi\, [\gamma(\theta_0 - \theta)] \right\}. \qquad \text{(69.4 cont.)}$$

and also deformation $\varepsilon_{2(1)}$ and the quantity Ψ_2:

$$\varepsilon_{2(1)} = \tilde{\varepsilon}_{2(1)} + \frac{2\gamma}{Eh} \sqrt{\frac{\sin\theta_0}{\sin\theta}} \left\{ (h_e^0 - \tilde{h}_e^0) \sin\theta_0 \theta\, [\gamma(\theta_0 - \theta)] - \right.$$
$$\left. - \frac{\gamma}{R} m_1^0 \psi\, [\gamma(\theta_0 - \theta)] \right\},$$
$$\Psi'_2 = 2\gamma^2 \sqrt{\frac{\sin\theta_0}{\sin\theta}} \left\{ -(h_e^0 - \tilde{h}_e^0) \sin\theta_0 \varphi\, [\gamma(\theta_0 - \theta)] + \right.$$
$$\left. + \frac{2\gamma}{R} m_1^0 \theta\, [\gamma(\theta_0 - \theta)] \right\}. \qquad (69.5)$$

Here have been introduced functions $\theta(x)$, $\zeta(x)$, $\psi(x)$, already found earlier during the calculation of cylindrical shells (formulas (27.5), (27.6)), which should be calculated at $x = \gamma(\theta_0 - \theta)$. Comparing equations (69.4) with the principal terms of equations (60.14), (60.15), it is simple to be convinced that in the principal terms the analogy between stressed states during axisymmetric and bending loads is kept. Formulas (69.5) at $\theta = \theta_0$ give

$$\varepsilon_{2(1)}^0 - \tilde{\varepsilon}_{2(1)}^0 = \frac{2\gamma}{Eh} \left[(h_e^0 - \tilde{h}_e^0) \sin\theta_0 - \frac{\gamma}{R} m_1^0 \right],$$
$$\Psi_2^0 = 2\gamma^2 \left[-(h_e^0 - \tilde{h}_e^0) \sin\theta_0 + \frac{2\gamma}{R} m_1^0 \right]. \qquad (69.6)$$

Formulas (69.6) coincide with (60.13) if in the latter we discard quantities of order $\frac{1}{\gamma}$, and in (69.6) replace

$(h_e^0 - \tilde{h}_e^0)$ by H_e^0, $\quad \varepsilon_2^0 - \tilde{\varepsilon}_2^0$ by $\frac{\Delta_e^0}{R \sin\theta_0}$,

m_1^0 by M_1^0, $\quad \Psi_2^0$ by $-Eh\vartheta_1^0$.

§ 70. The Stressed State of a Heavy Hemisphere with Horizontal Axis

We will use formulas (69.6) to determine the bending moment and radial force in a fixed section of hemisphere $\left(\theta_0 = \frac{\pi}{2}\right)$, subjected to a distributed weight load $q_x = q$. The total reaction force and moment in fixed section P_0 and M_0 are determined from conditions of equilibrium of the shell as a whole. At $q_x = q$

$$\left.\begin{aligned}
&q_{1(1)} = q\cos\theta, \quad q_{2(1)} = -q, \quad q_{z(1)} = q\sin\theta, \\
&P_q(\theta) = 2\pi qR^2(\cos\theta - \cos\theta_0), \\
&M_q(\theta) = 2\pi qR^3\left[\frac{1}{2}(\sin^2\theta - \sin^2\theta_0) + \cos\theta(\cos\theta - \cos\theta_0)\right], \\
&P_0 = 2\pi qR^2(1 - \cos\theta_0), \\
&M_0 = -P_0Z_0 - M_q(0) = \\
&\qquad = -2\pi qR^3\left[(1 - \cos\theta_0)\cos\theta_0 - \frac{1}{2}\sin^2\theta_0\right].
\end{aligned}\right\} \tag{70.1}$$

at $\theta_0 = \frac{\pi}{2}$

$$P_0 = 2\pi qR^2, \quad M_0 = \pi qR^3.$$

Forces of the zero-moment state are equal to

$$\left.\begin{aligned}
&\tilde{T}_1^0 = -\frac{M_0}{\pi R^2}, \quad \tilde{T}_2^0 = \frac{M_0}{\pi R^2} + qR, \\
&\tilde{\varepsilon}_2^0 = \frac{(1+\mu)}{Eh}\frac{M_0}{\pi R^2} + \frac{qR}{Eh}, \quad \tilde{h}_e^0 = 0.
\end{aligned}\right\} \tag{70.2}$$

In the fixed cut should be held the equality

$$\varepsilon_2^0 = \Psi_2^0 = 0,$$

or

$$\left.\begin{aligned}
&\frac{2\gamma}{Eh}\left(h_e^0 - \frac{\gamma}{R}m_1^0\right) = -\frac{(1+\mu)}{Eh}\frac{M_0}{\pi R^2} - \frac{qR}{Eh}, \\
&-h_e^0 + \frac{2\gamma}{R}m_1^0 = 0,
\end{aligned}\right\} \tag{70.3}$$

whence we have

$$h_e^0 = -\frac{1}{\gamma}\left[(1+\mu)\frac{M_0}{\pi R^2} + qR\right], \quad m_1^0 = -\frac{1}{2\gamma^2}\left[(1+\mu)\frac{M_0}{\pi R} + qR^2\right]. \tag{70.4}$$

A local rise in stress near the edge of the shell induced by the introduction of bending moment m_1^0 is simple to estimate by comparing the amplitude of bending stress

$$\sigma_{1(1)} = \pm\frac{6m_1^0}{h^2} = \mp\sqrt{\frac{3}{1-\mu^2}}\left[(1+\mu)\frac{M_0}{\pi R^2 h} + \frac{qR}{h}\right] \tag{70.5}$$

with the amplitude of stress of the zero-moment state

$$\tilde{\sigma}_{1(1)} = \frac{\tilde{T}_1}{h} = -\frac{M_0}{\pi R^2 h}, \quad \tilde{\sigma}_{2(1)} = \frac{\tilde{T}_2}{h} = \frac{M_0}{\pi R^2 h} + \frac{qR}{h}. \tag{70.6}$$

§ 71. A Spherical Dome Under the Action of Concentrated Force and Bending Moment in the Pole

Let us designate the stressed state in the neighborhood of the pole of a shell in which are applied concentrated tangential force P_* and bending moment M_* [91], [92] (Fig. 33). For simplicity we assume that distributed loads are absent $P_q(0)=M_q(0)\equiv 0$. On edge θ_0 acts a system of edge loads statically equivalent to force P_0 and to moment M_0, which are connected with P_*, M_* by relationships (66.5). Inasmuch as now we must construct a solution adequate in the neighborhood of the pole, we use the basic resolvant equation in the form (67.3). Let us pass from equation (67.3) to an equation with a regular right part using the replacement

$$\left.\begin{aligned}\Psi_2&=\Psi_3-\frac{4\gamma^4}{R^2}\frac{1}{4}(F\cos\theta-fR\sin^2\theta),\\ V_2&=V_3+\frac{(1-\mu)}{4R^2}fR\cos^2\theta.\end{aligned}\right| \qquad (71.1)$$

As a result we obtain

$$\begin{aligned}&L_1(\sigma_3)+\sigma_3\left(2i\gamma^2-\frac{4}{\sin^2\theta}\right)=\\ &=\frac{\gamma^4}{R^2}2i\gamma^2(F\cos\theta-fR\sin^2\theta)+4\gamma^4\frac{(1-\mu)}{R^2}\frac{fR\cos^2\theta}{4}+\\ &\qquad+\frac{\gamma^4}{R^2}L_1(F\cos\theta-fR\sin^2\theta).\end{aligned} \qquad (71.2)$$

The last term of the right side of (71.2) can be discarded in comparison with the first as a small quantity. The second term cannot be neglected. It has a significant value in the neighborhood of $\theta=0$. Really, since the term containing $f(\theta)$, in the first item at $\theta=0$ turns into zero, the main role in this instance is played by the second item. We will consider the equation with the right side equal to $2\gamma^2 iG_1(\theta)$, where

$$G_1(\theta)=\frac{\gamma^4}{R^2}(F\cos\theta-fR\sin^2\theta)+\frac{\gamma^4}{R^2}\frac{1}{2i\gamma^2}(1-\mu)fR\cos^2\theta. \qquad (71.3)$$

$$(F\cos\theta - fR\sin^2\theta)_{\theta=\gamma} = \frac{M_0}{\pi}, \quad (f\cos^2\theta)_{\theta=\gamma} = \frac{P_0}{\pi}. \qquad (71.4)$$

The general solution of the equation we write out, keeping in the solution of the uniform equation the term having singularity in the vertex namely:

$$\sigma_3 = (A_1 - iB_1)(\zeta_1 + i\zeta_2) + (A_2 - iB_2)(\zeta_3 + i\zeta_4) + \\ + \left(1 + \frac{4}{2i\gamma^2\sin^2\theta}\right)\frac{\gamma^4}{R^2}\left[F\cos\theta - fR\sin^2\theta + \frac{1}{2i\gamma^2}(1-\mu)fR\cos^2\theta\right]. \qquad (71.5)$$

The following disignations have been introduced here:

$$\left.\begin{aligned} \zeta_3(\theta) &= \sqrt{\frac{\theta}{\sin\theta}}\,\mathrm{Re}\,H_2^{(1)}(q\sqrt{i}) = -\sqrt{\frac{\theta}{\sin\theta}}\left[\psi_3(q) + \frac{2}{q}\psi_4'(q)\right], \\ \zeta_4(\theta) &= \sqrt{\frac{\theta}{\sin\theta}}\,\mathrm{Im}\,H_2^{(1)}(q\sqrt{i}) = -\sqrt{\frac{\theta}{\sin\theta}}\left[\psi_4(q) - \frac{2}{q}\psi_3'(q)\right], \\ q &= \gamma\theta\sqrt{2}. \end{aligned}\right\} \qquad (71.6)$$

Constants A_2 and B_2 we determine from the condition of finiteness of the solution at $\theta=0$. Since

$$H_2^{(1)}(x) = I_2(x) + \frac{i}{\pi}\left[2I_2(x)\ln\frac{x}{2} - \left(1 + \frac{4}{x^2}\right) + \dots\right],$$

where the points indicate terms of the order x^2 and above, then to execute the imposed condition it is necessary that

$$(A_2 - iB_2)\frac{i}{\pi} = G_1(\theta),$$

and, consequently,

$$A_2 = -2\gamma^2\frac{(1-\mu)}{R^2}\frac{P_0R}{4}, \quad B_2 = \frac{\gamma^4}{R^2}M_0. \qquad (71.7)$$

As a result of the choice σ_3 behaves in the vertex as $\frac{\gamma^4}{R^2}\frac{M_0}{\pi}\frac{2i\gamma^2\theta^2}{4}\ln\left(\frac{\sqrt{2i}\gamma\theta}{2}\right)$. In this way, $\sigma_3(0)=0$.

Returning to variables Ψ_2, V_2, it can be said that

$$\Psi_2(0) = -\frac{\gamma^4}{R^2}\frac{M_0}{\pi}, \quad V_2(0) = \frac{(1-\mu)}{4R^2}\frac{P_0 R}{\pi}. \tag{71.8}$$

Taking into account this, on the basis of (66.18), (66.19) we find that forces and bending moments at $\theta=0$ have singularities of the form

$$\left.\begin{aligned} t_1 &= -\frac{(3+\mu)}{4}\frac{P_0}{\pi R\theta}+\dots, \quad t_2 = \frac{(1-\mu)}{4}\frac{P_0}{\pi R\theta}+\dots, \\ s_1 &= \frac{1-\mu}{4}\frac{P_0}{\pi R\theta}+\dots, \end{aligned}\right\} \tag{71.9}$$

$$\left.\begin{aligned} m_1 &= -\frac{(1+\mu)M_0}{4\pi R\theta}+\dots, \quad m_2 = -\frac{(1+\mu)M_0}{4\pi R\theta}+\dots, \\ h_1 &= \frac{(1-\mu)M_0}{4\pi R\theta}+\dots, \quad n_1 = \frac{1}{2}\frac{M_0}{\pi R^2\theta^2}+\dots \end{aligned}\right\} \tag{71.10}$$

In expressions (71.9) terms of the form $\frac{M_0}{\pi R\theta}$, have not been written out, since for their consecutive calculation it is necessary to keep in the right side of (66.17) the term $-\frac{(1-\mu)\cos\theta}{R^2\sin^2\theta}F(0)$ and to take a non-simplified variant of the expressions for forces through functions of V_2 and Ψ_2. For example,

$$t_1 = -\frac{1}{4\gamma^4}\left[\frac{d\Psi_2}{d\theta}+(1+\mu)\frac{\cos\theta}{\sin\theta}\Psi_2\right]+V_2\operatorname{ctg}\theta-\frac{F+fR\cos\theta}{R^2\sin\theta}$$

$$s_1 = \frac{V_2}{\sin\theta}+\frac{2(1-\mu)}{4\gamma^4}\Psi_2 \quad \text{etc.}$$

Then in expressions for t_1, t_2, s_1 will appear the terms

$$-\frac{M_0}{2\pi R^2\theta}, \quad \frac{(1-\mu)M_0}{4\pi R^2\theta}, \quad -\frac{(1-\mu)M_0}{4\pi R^2\theta}$$

correspondingly. By direct substitution of the obtained expressions into the first equation of (66.2) one can be certain that allowing for these terms in the neighborhood of $\theta=0$ it is satisfied accurately.

However, during the calculation of stress the consideration of these items gives a correction in the amount of $\frac{h}{R}$ in comparison with unity, for example

$$\sigma_{1(1)}=\frac{t_1}{h}\pm\frac{6m_1}{h^2}=\frac{M_0}{2\pi R^2 h\theta}\left[-1\mp(1+\mu)\frac{R}{h}\right],$$

therefore keeping them during the calculation by the theory of thin shells is not necessary.

We will explain the character of the change in displacements in the neighborhood of the point of application of concentrated forces. From (16.1) and (66.11), (66.15) it follows that

$$\frac{R}{Eh}\Psi_2=-\frac{dw_{(1)}}{d\theta}+w_{(1)}\operatorname{ctg}\theta.$$

or

$$\frac{R}{Eh}\Psi_2=-\sin\theta\frac{d}{d\theta}\left(\frac{w_{(1)}}{\sin\theta}\right). \qquad (71.11)$$

On the basis of (71.11) we easily compute the amplitude of the displacement normal to the middle surface,

$$w_{(1)}=-\frac{R}{Eh}\sin\theta\int\frac{\Psi_2}{\sin\theta}d\theta+KR\sin\theta. \qquad (71.12)$$

Here KR designates the constant of integration.

Taking into account (71.1), we have

$$w_{(1)}=-\frac{R\sin\theta}{Eh}\int\left[\frac{\Psi_3}{\sin\theta}-\frac{\gamma^4}{R^2}\frac{F\cos\theta-fR\sin^2\theta}{\sin\theta}\right]d\theta+KR\sin\theta. \qquad (71.13)$$

Since $\Psi_{3(0)}$ at small θ changes in proportion to $\theta^2\ln\theta$, then it is easy to see that for a clarification of the behavior of $w_{(1)}$ in the neighborhood of $\theta=0$ it is necessary to focus the basic attention on the second term in the subintegral expression of (71.13).

Integrating it by parts, we obtain

$$\int F(\theta)\,\mathrm{ctg}\,\theta\,d\theta = F(\theta)\ln\theta - \int \frac{dF}{d\theta}\ln(\sin\theta)\,d\theta$$

and finally

$$\frac{Eh}{R}w_{(1)} = \frac{\gamma^4}{R^2}F(\theta)\sin\theta\ln\theta - \sin\theta\int\frac{\Psi_3(\theta)}{\sin\theta}d\theta - \\ - \frac{\gamma^4}{R^2}\sin\theta\int\left[\frac{dF}{d\theta}\ln(\sin\theta) + fR\sin\theta\right]d\theta + KR\sin\theta. \qquad (71.14)$$

Since $\frac{dF}{d\theta}$ is proportional to $\sin\theta$, then from (71.14) we find that at small θ $w_{(1)}$ has the form

$$w_{(1)} = \frac{M_0}{4\pi D}R\theta\ln\theta + \ldots + KR\theta, \qquad (71.15)$$

where

$$D = \frac{Eh^3}{12(1-\mu^2)}.$$

The term containing constant of integration K. is kept, since because of the replacement of this constant (71.15) can be brought to the form

$$w_{(1)} = \frac{M_0}{4\pi D}R\theta\ln(R\theta) + \ldots + K_1R\theta. \qquad (71.16)$$

The amplitude of displacement in the meridian direction is found using the relationship

$$\frac{1}{R}\frac{du_{(1)}}{d\theta} = \varepsilon_{1(1)} - \frac{w_{(1)}}{R}. \qquad (71.17)$$

Integrating it, taking into account Hooke's law and formula (71.9), (71.14), we find

$$u_{(1)} = -\frac{P_0}{\pi Eh}\frac{(3-\mu)(1+\mu)}{4}\ln(R\theta) + \ldots + KR\cos\theta + K_2R. \qquad (71.18)$$

where K_2 – new constant of integration.

The amplitude of radial displacement is calculated easily using the first formula of (17.3)

$$\Delta_{r(1)} = u_{(1)} \cos\theta + w_{(1)} \sin\theta = \\ = -\frac{P_0}{4\pi Eh}(3-\mu)(1+\mu)\ln(R\theta) + \ldots + KR + K_2 R\cos\theta. \qquad (71.19)$$

The amplitude of circumferential displacement has the form

$$v_{(1)} = \varepsilon_{2(1)} R\sin\theta - \Delta_{r(1)} = \frac{P_0}{4\pi Eh}(3-\mu)(1+\mu)\ln(R\theta) + \ldots \qquad (71.20)$$

For clarification of the sense of the constants of integration K and K_2 we compute even axial displacement

$$\Delta_{z(1)} = w_{(1)}\cos\theta - u_{(1)}\sin\theta = \frac{M_0}{4\pi D} R\theta\ln(R\theta) + \\ + \frac{P_0}{4\pi EhR}(3-\mu)(1+\mu) R\theta\ln(R\theta) + \ldots - K_2 R\sin\theta.$$

From (71.19), (71.20) it is clear that K_2 is the rigid rotation of the shell around axis OY, and constant KR is rigid displacement in the direction of axis OX (see § 18).

Introducing the designation $R\theta = r$, we note that in expansions (71.9), (71.10), (71.18), (71.20) the main terms coincide with analogous expressions for forces, moments and displacements in the plane problem and the problem of bending for a flat plate during the action in one case a concentrated force in the plane of the plate, and in the other of the concentrated moment causing its bend. For both a sphere and plate is obtained

$$t_1 = -(3+\mu)\frac{P_0}{4\pi r} + \ldots, \quad t_2 = (1-\mu)\frac{P_0}{4\pi r} + \ldots,$$
$$s_1 = (1-\mu)\frac{P_0}{4\pi r} + \ldots, \quad u_{(1)} = -\frac{P_0}{4\pi Eh}(3-\mu)(1+\mu) r\ln r + \ldots,$$
$$v_{(1)} = \frac{P_0}{4\pi Eh}(3-\mu)(1+\mu) r\ln r + \ldots,$$
$$m_1 = -(1+\mu)\frac{M_0}{4\pi r} + \ldots, \quad m_2 = -(1+\mu)\frac{M_0}{4\pi r} + \ldots,$$
$$h_{(1)} = \frac{(1-\mu) M_0}{4\pi r} + \ldots, \quad n_1 = \frac{1}{2}\frac{M_0}{\pi r^2} + \ldots, \quad w_{(1)} = \frac{M_0}{4\pi D} r\ln r.$$

We examined the behavior of the solution (71.5) in the neighborhood $\theta=0$. It is not necessary to speak of the behaves at a great distance from the top. At large $\gamma\theta$ functions $\zeta_3(\theta)$, $\zeta_4(\theta)$ decrease with an increase in θ as $e^{-\gamma\theta}$, and the particular solution differs from the zero-moment solution by only a small member. Really, the particular solution is equal to

$$\sigma_3=\frac{\gamma^4}{R^2}(F\cos\theta-fR\sin^2\theta)-(1-\mu)\frac{f\cos^2\theta}{R\sin^2\theta}-$$
$$-2i\gamma^2\left[\frac{F\cos\theta-fR\sin^2\theta}{R^2\sin^2\theta}-(1-\mu)\frac{1}{4R}f\cos^2\theta\right]. \qquad (71.21)$$

Going to function σ_2, on the basis of (71.1) obtain

$$\Psi_2=-(1-\mu)\frac{f\cos^2\theta}{R\sin^2\theta},\quad V_2=-\frac{F\cos\theta-fR\sin^2\theta}{R^2\sin^2\theta},$$
$$\sigma_2=-(1-\mu)\frac{f\cos^2\theta}{R\sin^2\theta}-2i\gamma^2\frac{F\cos\theta-fR\sin^2\theta}{R^2\sin^2\theta}.$$

When $\sin\theta\gg0$ (71.21) differs from σ_2 by only the small term.

The constants A_1, B_1, in solution (71.5) are determined from conditions on the edge $\theta=\theta_0$. Part of the solution $(A_1-iB_1)(\zeta_1+i\zeta_2)$ describes the edge effect connected with edge θ_0. The member

$$\left[-2\gamma^2(1-\mu)\frac{P_0}{4R}-i\frac{\gamma^4}{R^2}M_0\right](\zeta_3+i\zeta_4)$$

represents a singular edge effect in the neighborhood of $\theta=0$, connected with the presence of concentrated action in the top. If the shell is very thin and $\gamma\theta_0$ is rather large, then the edge effects localize themselves each in their neighborhood. In this instance constants A_1, B_1 can be determined by setting $\zeta_3(\theta_0)=\zeta_4(\theta_0)=0$, and the stressed state in the neighborhood of edge $\theta=\theta_0$ can be computed with the formula of (69.4).

§ 72. Consideration of the Mutual Influence of the Edge of the Dome and a Concentrated Effect Applied in the Pole

For a shell which is insufficiently thin (γ small) or flat (θ is small) the determination of constants A_1, B_1 from the edge conditions at $\theta=\theta_0$ must use the general solution in the form

$$\begin{aligned}\sigma_3=\Psi_3+2i\gamma^2V_3=(A_1-iB_1)(\zeta_1+i\zeta_2)+\\ +\left[-2\gamma^2(1-\mu)\frac{P_*}{4R}-i\frac{\gamma^4}{R^2}M_*\right](\zeta_3+i\zeta_4)+\\ +\frac{\gamma^4}{R^2}\left(1+\frac{4}{2i\gamma^2\sin^2\theta}\right)\left[F\cos\theta-fR\sin^2\theta+\frac{(1-\mu)fR\cos^2\theta}{2i\gamma^2}\right].\end{aligned} \tag{72.1}$$

With the aid of (71.1) going to functions Ψ_2, V_2, we have

$$\begin{aligned}\Psi_2=A_1\zeta_1+B_1\zeta_2-2\gamma^2(1-\mu)\frac{P_*}{4R}\zeta_3+\\ +\frac{\gamma^4}{R^2}M_*\zeta_4-(1-\mu)\frac{f\cos^2\theta}{R\sin^2\theta}.\end{aligned} \tag{72.2}$$

$$\begin{aligned}2\gamma^2V_2=A_1\zeta_2-B_1\zeta_1-2\gamma^2(1-\mu)\frac{P_*}{4R}\zeta_4-\\ -\frac{\gamma^4}{R^2}M_*\zeta_3-\frac{4}{2\gamma^2\sin^2\theta}\frac{\gamma^4}{R^2}[F(0)\cos\theta-f(0)R\sin^2\theta].\end{aligned} \tag{72.3}$$

Apropos of the last member in the right side of (72.2) we can say the following. It is essential only in the neighborhood of $\theta=0$ at $P_*\neq 0$. At large θ and no tangential force this member in comparison with the zero-moment solution is small, as $\frac{1}{2\gamma^2}$ in comparison with unity. This lets us, without hurting the correctness, replace in this case

$$\frac{(1-\mu)f(\theta)\cos^2\theta}{R\sin^2\theta} \text{ by } (1-\mu)\frac{P_*}{\pi R\theta^2}.$$

In examining the behavior of $V_2(\theta)$ in the neighborhood of $\theta=0$ one should bear in mind that at $M_*=0$, $P_*\neq 0$ the quantity $\pi(F\cos\theta-fR\sin^2\theta)=P_*Z$

has the mechanical sense of moment of force P_* calculated relative to the center of instantaneous section $\theta=\text{const}$. Therefore at $\theta=0$ this quantity is identical to zero. To avoid misunderstandings it is convenient to combine the last two members in the right side of (72.3). Taking into account our remarks, we have

$$\left.\begin{aligned}\Psi_2 &= A_1\zeta_1(\theta)+B_1\zeta_2(\theta)+\frac{\gamma^4}{R^2}M_*\zeta_1(\theta)-\\&\qquad -2\gamma^2\frac{(1-\mu)}{4R}P_*\zeta_3(\theta)-(1-\mu)\frac{P_*}{\pi R\theta^2},\\ 2\gamma^2V_2 &= A_1\zeta_2(\theta)-B_1\zeta_1(\theta)-2\gamma^2(1-\mu)\frac{P_*}{4R}\zeta_4(\theta)-\\&-\frac{1}{\sin^2\theta}\left[\frac{\gamma^4}{R^2}M_*\sin^2\theta\zeta_3(\theta)+\frac{2\gamma^2}{R^2}(F\cos\theta-fR\sin^2\theta)\right].\end{aligned}\right\}\quad(72.4)$$

In this case it is useful to remember that in the neighborhood of $\theta=0$ exist the representations

$$\zeta_3(\theta)=-\frac{4}{2\pi\gamma^2\theta^2}+\ldots,\quad \zeta_4(\theta)=-\frac{1}{\pi}+\ldots$$

On the basis of (72.4) and quantities (66.18), (66.19) let us compute forces and bending moments in the shell. In this case in quantities (66.18) we discard the terms $\frac{1}{4\gamma^4}\frac{d\Psi_2}{d\theta}$ and $\frac{\mu}{4\gamma^4}\frac{d\Psi_2}{d\theta}$, since when θ is small, they are of the order of $\frac{1}{2\gamma^2}$ in comparison with the basic terms, taken as unit. In either case, taking into account the correctness of the solution, they cannot be held.

We have

$$\left.\begin{aligned}t_1 &= \frac{\operatorname{ctg}\theta}{2\gamma^2}\Big\{A_1\zeta_2(\theta)-B_1\zeta_1(\theta)-2\gamma^2(1-\mu)\frac{P_*}{4R}\zeta_4(\theta)-\\&-\frac{1}{\sin^2\theta}\left[\frac{\gamma^4}{R^2}M_*\zeta_3(\theta)\sin^2\theta+\frac{2\gamma^2}{R^2}(F\cos\theta-fR\sin^2\theta)\right]\Big\}-\\&\qquad-\frac{F+fR\cos\theta}{R^2\sin\theta},\end{aligned}\right|\quad(72.5)$$

$$
\left.\begin{aligned}
t_2 = {} & A_1 \frac{1}{2\gamma^2}\left[\zeta_2'(\theta) + \zeta_2(\theta)\operatorname{ctg}\theta\right] - B_1 \frac{1}{2\gamma^2}\left[\zeta_1'(\theta) + \zeta_1(\theta)\operatorname{ctg}\theta\right] - \\
& - \frac{(1-\mu)}{4R} P_*\left[\zeta_4'(\theta) + \operatorname{ctg}\theta\,\zeta_4(\theta)\right] - \frac{\gamma^2}{2R^2} M_*\left[\zeta_3'(\theta) + \operatorname{ctg}\theta\,\zeta_3(\theta)\right] + \\
& + \frac{F(\theta)}{R^2\sin^3\theta} + Rq_{n(1)}, \\
s_1 = {} & \frac{1}{\sin\theta}\frac{1}{2\gamma^2}\Big\{ A_1\zeta_2(\theta) - B_1\zeta_1(\theta) - 2\gamma^2(1-\mu)\frac{P_*}{4R}\zeta_4(\theta) - \\
& - \frac{1}{\sin^2\theta}\left[\frac{\gamma^4}{R^2} M_*\zeta_3(\theta)\sin^2\theta + \frac{2\gamma^2}{R^2}(F\cos\theta - fR\sin^2\theta)\right]\Big\}, \\
h_e = {} & -\frac{f(\theta)}{R\sin\theta} + s_1.
\end{aligned}\right\} \qquad (72.5)
$$

$$
\left.\begin{aligned}
m_1 = {} & \frac{R}{4\gamma^4}\Big\{ A_1\left[\zeta_1'(\theta) + (1+\mu)\operatorname{ctg}\theta\,\zeta_1(\theta)\right] + \\
& + B_1\left[\zeta_2'(\theta) + (1+\mu)\operatorname{ctg}\theta\,\zeta_2(\theta)\right] + \\
& + \frac{\gamma^4}{R^2} M_*\left[\zeta_4'(\theta) + (1+\mu)\operatorname{ctg}\theta\,\zeta_4(\theta)\right] - \\
& - 2\gamma^2(1-\mu)\frac{P_*}{4R}\left[\zeta_3'(\theta) + (1+\mu)\operatorname{ctg}\theta\,\zeta_3(\theta)\right] + \\
& + (1-\mu)\frac{P_*}{\pi R}\left[\frac{2}{\theta^3} - (1+\mu)\frac{\operatorname{ctg}\theta}{\theta^2}\right]\Big\}, \\
m_2 = {} & \frac{R}{4\gamma^4}\Big\{ A_1\left[\mu\zeta_1'(\theta) + (1+\mu)\operatorname{ctg}\theta\,\zeta_1(\theta)\right] + \\
& + B_1\left[\mu\zeta_2'(\theta) + (1+\mu)\operatorname{ctg}\theta\,\zeta_2(\theta)\right] + \\
& + \frac{\gamma^4}{R^2} M_*\left[\mu\zeta_4'(\theta) + (1+\mu)\operatorname{ctg}\theta\,\zeta_4(\theta)\right] - \\
& - 2\gamma^2(1-\mu)\frac{P_*}{4R}\left[\mu\zeta_3'(\theta) + (1+\mu)\operatorname{ctg}\theta\,\zeta_3(\theta)\right] + \\
& + (1-\mu)\frac{P_*}{\pi R}\left[\frac{2\mu}{\theta^3} - (1+\mu)\frac{\operatorname{ctg}\theta}{\theta^2}\right]\Big\}, \\
h_1 = {} & -\frac{R(1-\mu)}{4\gamma^4\sin\theta}\Big[A_1\zeta_1(\theta) + B_1\zeta_2(\theta) + \frac{\gamma^4}{R^2} M_*\zeta_4(\theta) - \\
& - 2\gamma^2(1-\mu)\frac{P_*}{4R}\zeta_3(\theta) - (1-\mu)\frac{P_*}{\pi R}\frac{1}{\theta^2}\Big].
\end{aligned}\right\} \qquad (72.6)
$$

Let us stop in more detail on the calculation of shearing force n_1. On the basis of (66.19), taking into account differential equation (66.17), we can write

$$
n_1 = V_2 + \frac{1}{\sin^2\theta}(F\cos\theta - fR\sin^2\theta) + \frac{1}{4\gamma^4}\left(\operatorname{ctg}\theta\frac{d\Psi_2}{d\theta} + \frac{2\Psi_2}{\sin^2\theta}\right).
$$

Substituting into this formula the expressions for V_2, Ψ_2 in accordance with (72.4), and dropping unessential members (for example, $\frac{A_1}{4\gamma^4}\left[\operatorname{ctg}\theta\,\zeta_1'(\theta) + \frac{2\zeta_1(\theta)}{\sin^2\theta}\right]$ in comparison with $\frac{1}{2\gamma^2}A_1\zeta_2(\theta)$ etc.,), we obtain

$$n_1 = \frac{1}{2\gamma^2}[A_1\zeta_2(\theta) - B_1\zeta_1(\theta)] - (1-\mu)\frac{P_*}{4R}\zeta_4(\theta) -$$

$$-\frac{\gamma^2}{2R^2}M_*\zeta_3(\theta) + \frac{M_*}{4R^2\sin^2\theta}[\sin\theta\cos\theta\zeta_4'(\theta) + 2\zeta_4(\theta)]. \quad (72.7)$$

Functions $\zeta_i(\theta)$ and $\zeta_i'(\theta)$ are expressed through $\psi_i(q)$, $\psi_i'(q)$ $(q=\gamma\theta\sqrt{2})$ $(i=1, 2, 3, 4)$ in the following manner:

$$\left.\begin{aligned}
\zeta_1 &= -\sqrt{\frac{\theta}{\sin\theta}}\left[\psi_1(q) + \frac{2}{q}\psi_2'(q)\right],\\
\zeta_2 &= -\sqrt{\frac{\theta}{\sin\theta}}\left[\psi_2(q) - \frac{2}{q}\psi_1'(q)\right],\\
\zeta_3 &= -\sqrt{\frac{\theta}{\sin\theta}}\left[\psi_3(q) + \frac{2}{q}\psi_4'(q)\right],\\
\zeta_4 &= -\sqrt{\frac{\theta}{\sin\theta}}\left[\psi_4(q) - \frac{2}{q}\psi_3'(q)\right],
\end{aligned}\right\} \quad (72.8)$$

$$\left.\begin{aligned}
\zeta_1'(\theta) &= -\gamma\sqrt{2}\sqrt{\frac{\theta}{\sin\theta}}\left[\psi_1'(q) - \frac{2}{q}\psi_1(q) - \frac{4}{q^2}\psi_2'(q)\right],\\
\zeta_2'(\theta) &= -\gamma\sqrt{2}\sqrt{\frac{\theta}{\sin\theta}}\left[\psi_2'(q) - \frac{2}{q}\psi_2(q) + \frac{4}{q^2}\psi_1'(q)\right],\\
\zeta_3'(\theta) &= -\gamma\sqrt{2}\sqrt{\frac{\theta}{\sin\theta}}\left[\psi_3'(q) - \frac{2}{q}\psi_3(q) - \frac{4}{q^2}\psi_4'(q)\right],\\
\zeta_4'(\theta) &= -\gamma\sqrt{2}\sqrt{\frac{\theta}{\sin\theta}}\left[\psi_4'(q) - \frac{2}{q}\psi_4(q) + \frac{4}{q^2}\psi_3'(q)\right].
\end{aligned}\right\} \quad (72.9)$$

§ 73. Weakly Distorted Circular Plate Under the Action of a Bending Load

In §§ 64 and 65 we considered the deformation of a thin spherical shell of small curvature during the action of an axisymmetric load. With the same assumptions in geometric dimensions we will find the stressed state and deformation of shells under bending load. Let us assume that the shell is loaded by distributed load $q_{1(1)}\cos\varphi$, $q_{2(1)}\sin\varphi$, $q_{n(1)}\cos\varphi$ and by edge effects, equivalent to force P_0 and to moment M_γ. A concentrated influence in the center is absent $(P_* = M_* = 0)$. On edge $\theta=\theta_0$ are assigned the amplitudes of radial force and bending moment

$$h_r = h_r^0, \quad m_1 = m_1^0. \quad (73.1)$$

The basic functions V_2, Ψ_2, through which are expressed all forces and moments (formulas (66.18), (66.19), have the form

$$\left.\begin{aligned}\Psi_2 &= A_1\tilde{\varphi}_1(\theta) + B_1\tilde{\zeta}_2(\theta),\\ 2\gamma^2 V_2 &= A_1\tilde{\varphi}_2(\theta) - B_1\tilde{\varphi}_1(\theta) + 2\gamma^2\tilde{V}_2.\end{aligned}\right\} \qquad (73.2)$$

where $\tilde{V}_2$ is the zero-moment solution computable by (67.14), (66.16), functions $\tilde{\varphi}_1, \tilde{\varphi}_2$ are determined by the expressions (72.8), in which now we can set $\sqrt{\theta/\sin\theta}\approx 1$, and constants A_1, B_1 are determined according to boundary conditions on the edge $\theta=\theta_0$. Setting $\cos\theta\approx 1$, $\sin\theta\approx\theta$, We have

$$\left.\begin{aligned}m_1 &= \frac{R\gamma\sqrt{2}}{4\gamma^4}[-A_1 g_{11}(q) - B_1 g_{12}(q)],\\ h_e &= \tilde{h}_e + \frac{1}{2\gamma^2\theta}[-A_1 g_{21}(q) + B_1 g_{22}(q)],\end{aligned}\right\} \qquad (73.3)$$

$$\left.\begin{aligned}&\tilde{h}_e = \tilde{t}_1,\\ &g_{11}(q) = \psi_1'(q) - \frac{(1-\mu)}{q}\psi_1(q) - \frac{2(1-\mu)}{q^2}\psi_2'(q),\\ &g_{12}(q) = \psi_2'(q) - \frac{(1-\mu)}{q}\psi_2(q) + \frac{2(1-\mu)}{q^2}\psi_1'(q),\\ &g_{21}(q) = \psi_2(q) - \frac{2}{q}\psi_1'(q), \quad g_{22}(q) = \psi_1(q) + \frac{2}{q}\psi_2'(q),\end{aligned}\right\} \qquad (73.4)$$

where $\tilde{t}_1$ is calculated using the first formula of (67.15). Note that load functions $F(\theta)$, $f(\theta)$, determined by the relationships (66.3), (66.4), in the involved case use differences of close amounts. Therefore during the calculation of load terms more accurate representations of $\cos\theta = 1 - \frac{\theta^2}{2} + \ldots$, $\sin\theta = \theta - \frac{\theta^3}{2\cdot 3} + \ldots$

Edge conventions (73.1) give for determination of A_1, B_1 the system of equations

$$\left.\begin{aligned}A_1 g_{11}(q_0) + B_1 g_{12}(q_0) &= -\frac{4\gamma^3}{\sqrt{2}R} m_1^0,\\ -A_1 g_{21}(q_0) + B_1 g_{22}(q_0) &= (h_e^0 - \tilde{h}_e^0)\, 2\gamma^2\theta_0.\end{aligned}\right\} \qquad (73.5)$$

where

$$q_0 = \gamma\theta_0 \sqrt{2}, \quad \tilde{h}_e^0 \approx -\frac{M_0}{\pi R^2 \theta_0^4} = -\frac{M_0 R}{\pi a^4}. \tag{73.6}$$

Solving it, we have

$$\left.\begin{aligned} A_1 &= \frac{1}{\Delta(q_0)}\left[-\frac{4\gamma^3}{\sqrt{2}R} m_1^0 g_{22}(q_0) - (h_e^0 - \tilde{h}_e^0)\, 2\gamma^2\theta_0 g_{12}(q_0)\right], \\ B_1 &= \frac{1}{\Delta(q_0)}\left[(h_e^0 - \tilde{h}_e^0)\, 2\gamma^2\theta_0 g_{11}(q_0) - \frac{4\gamma^3}{\sqrt{2}R} m_1^0 g_{21}(q_0)\right], \\ \Delta(q_0) &= g_{11}(q_0) g_{22}(q_0) + g_{12}(q_0) g_{21}(q_0). \end{aligned}\right\} \tag{73.7}$$

When q_0 insignificantly exceeds unity, using expansion (64.1) it is simple to calculate that

$$\left.\begin{aligned} g_{11}(q) &= -\frac{q^3\cdot 2(5+\mu)}{2^2\cdot 4^2\cdot 3} + \frac{q^7\cdot 4(9+\mu)}{2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdot 5} + \ldots, \\ g_{12}(q) &= -\frac{q(3+\mu)}{2^2\cdot 2} + \frac{q^5\cdot 3(7+\mu)}{2^2\cdot 4^2\cdot 6^2\cdot 4} - \ldots, \\ g_{21}(q) &= -\frac{q^2}{2^2\cdot 2} + \frac{q^6\cdot 3}{2^2\cdot 4^2\cdot 6^2\cdot 4} - \ldots, \\ g_{22}(q) &= -\frac{q^4\cdot 2}{2^2\cdot 4^2\cdot 3} + \frac{q^8\cdot 4}{2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdot 5} - \ldots, \\ \Delta(q) &= \frac{q^5(3+\mu)}{64}\left(1 + \frac{q^4}{4^2\cdot 6^2}\,\frac{5+\mu}{3+\mu}\right). \end{aligned}\right\} \tag{73.8}$$

In terms of the found values of constants A_1, B_1 we compute the values of function $\psi = \frac{1}{Eh}\Psi_2$ and circumferential deformation $\varepsilon_{2(1)}$ on the edge $\theta = \theta_0$ using formula (73.2) and the formula

$$\begin{aligned} Eh\varepsilon_{2(1)} = Eh\tilde{\varepsilon}_{2(1)} + \frac{1}{2\gamma^2}\Big\{A_1\left[\zeta_2'(\theta) + \frac{(1-\mu)}{\theta}\zeta_2(\theta)\right] - \\ - B_1\left[\zeta_1'(0) + \frac{(1-\mu)}{\theta}\zeta_1(0)\right]\Big\}. \end{aligned} \tag{73.9}$$

where

$$\left.\begin{aligned} &\zeta_2'(\theta) + \frac{(1-\mu)}{\theta}\zeta_2(\theta) = \\ &= -\gamma\sqrt{2}\left[\psi_2'(q) - \frac{(1+\mu)}{q}\psi_2(q) + \frac{2(1+\mu)}{q^2}\psi_1'(q)\right] \approx \\ &\approx -\gamma\sqrt{2}\left[-\frac{q(3-\mu)}{2^2\cdot 2} + \frac{q^5\cdot 3(7-\mu)}{2^2\cdot 4^2\cdot 6^2\cdot 4} + \ldots\right], \\ &\zeta_1'(0) + \frac{(1-\mu)}{\theta}\zeta_1(0) = \end{aligned}\right\} \tag{73.10}$$

$$= -\gamma\sqrt{2}\left[\psi_1'(q) - \frac{(1+\mu)}{q}\psi_1(q) - \frac{2(1+\mu)}{q^2}\psi_2'(q)\right] \approx$$

$$\approx -\gamma\sqrt{2}\left[-\frac{q^3(5-\mu)\cdot 2}{2^2\cdot 4^2\cdot 3} + \frac{q^7\cdot(9-\mu)4}{2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdot 5}\right],$$

$$\tilde{\varepsilon}_{2(1)} = \frac{1}{Eh}(\tilde{t}_2 - \mu\tilde{t}_1). \qquad (73.10 \text{ cont'd})$$

At $q=q_0$ we have

$$\Psi^0 = \frac{m_1^0 a}{D(3+\mu)} \frac{1+\frac{q_0^4}{576}}{1+\frac{q_0^4}{576}\frac{5+\mu}{3+\mu}} - \frac{(h_e^0-\tilde{h}_e^0)a^2\theta_0}{6D(3+\mu)} \frac{1+\frac{q_0^4}{960}}{1+\frac{q_0^4}{576}\frac{5+\mu}{3+\mu}},$$

$$\varepsilon_{2(1)}^0 - \tilde{\varepsilon}_{2(1)}^0 = -\frac{m_1^0 a^2}{6D(3+\mu)R} \frac{1+\frac{q_0^4}{960}}{1+\frac{q_0^4}{576}\frac{5+\mu}{3+\mu}} +$$

$$+ \frac{(h_e^0-\tilde{h}_e^0)(3-\mu)}{Eh} \frac{1+\frac{q_0^4}{576}\frac{37-\mu^2}{9-\mu^2}}{1+\frac{q_0^4}{576}\frac{5+\mu}{3+\mu}}. \qquad (73.11)$$

Relationships (73.11) can be solved in m_1^0, $h_e^0 - \tilde{h}_e^0$:

$$h_e^0 - \tilde{h}_e^0 = \frac{Eh(\varepsilon_{2(1)}^0 - \tilde{\varepsilon}_{2(1)}^0)}{(3-\mu)} \frac{1+\frac{q_0^4(4+\mu)}{288(3+\mu)}}{1+\frac{q_0^4}{288}\frac{15-\mu^2}{9-\mu^2}} +$$

$$+ \frac{Eh\Psi^0\theta_0}{6(3-\mu)} \frac{1+\frac{q_0^4(9+2\mu)}{720(3+\mu)}}{1+\frac{q_0^4}{288}\frac{15-\mu^2}{9-\mu^2}},$$

$$m_1^0 = \frac{D(3+\mu)}{a}\Psi^0 \frac{1+\frac{q_0^4}{288}\frac{26-\mu-\mu^2}{9-\mu^2}}{1+\frac{q_0^4}{288}\frac{15-\mu^2}{9-\mu^2}} +$$

$$+ Eh(\varepsilon_{2(1)}^0 - \tilde{\varepsilon}_{2(1)}^0)\frac{\theta_0 a}{6(3-\mu)} \frac{1+\frac{q_0^4(9+2\mu)}{720(3+\mu)}}{1+\frac{q_0^4}{288}\frac{15-\mu^2}{9-\mu^2}}. \qquad (73.12)$$

Formulas (73.12) are handy for determination of bending moment and radial force in a shell with assigned deformation of the edge. For example, for a rigidly fixed edge, when $\varepsilon_{2(1)}^0 = \Psi^0 = 0$, we have

$$h_e^0 = \tilde{h}_e^0 - \frac{Eh\tilde{\varepsilon}^0_{2\,(1)}}{3-\mu} \frac{1+\frac{q_0^4(4+\mu)}{288(3+\mu)}}{1+\frac{q_0^4}{288}\frac{15-\mu^2}{9-\mu^2}},$$
$$m_1^0 = -\frac{Eh\tilde{\varepsilon}^0_{2\,(1)}\theta_0 a}{6(3-\mu)} \frac{1+\frac{q_0^4(9+2\mu)}{720(3+\mu)}}{1+\frac{q_0^4}{288}\frac{15-\mu^2}{9-\mu^2}}. \qquad (73.13)$$

Let us return to a shell with forces h_e^0 and moments m_1^0 assigned on the edge. We compute the bending moment and radial force in the pole of the sphere, i.e., at $\vartheta=0$ $(q=0)$. On the basis of (73.4), (73.8), (67.15) it is easy to see that

$$m_1(0)=0, \quad h_e(0)=\tilde{h}_e(0)=0,$$

since in the absence of concentrated forces in the pole the equalities $f(0)=0$, $F(0)=0$ exist. For calculation of $t_2(0)$ we use the expression

$$t_2=\tilde{t}_2+\frac{1}{2\gamma^2}\left[A_1\left(\zeta_2'+\frac{1}{\theta}\zeta_2\right)-B_1\left(\zeta_1'+\frac{1}{\theta}\zeta_1\right)\right]. \qquad (73.14)$$

where

$$\zeta_2'+\frac{1}{\theta}\zeta_2=-\gamma\sqrt{2}\left(-\frac{q\cdot 3}{2^2\cdot 2}+\frac{q^5\cdot 3\cdot 7}{2^2\cdot 4^2\cdot 6^2\cdot 4}\right),$$
$$\zeta_1'+\frac{1}{\theta}\zeta_1=-\gamma\sqrt{2}\left(-\frac{q^3\cdot 2\cdot 5}{2^2\cdot 4^2\cdot 3}+\frac{q^7\cdot 4\cdot 9}{2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdot 5}\right). \qquad (73.15)$$

We derive

$$t_2(0)=\tilde{t}_2(0)=q_{n\,(1)}R.$$

Using (73.14) and (73.7), we compute the value of the amplitude of circumferential force at $\theta=\theta_0$

$$t_2^0 = \bar{t}_2^0 - \frac{m_1^0\, 2(1-\mu^2)\, a\theta_0}{h^2(3+\mu)} \frac{1+\frac{q_0^4}{960}}{1+\frac{q_0^4}{576}\frac{5+\mu}{3+\mu}} +$$

$$+ 3(h_e^0 - \bar{h}_e^0) \frac{1+q_0^4 \frac{(37+5\mu)}{1728(3+\mu)}}{1+\frac{q_0^4}{576}\frac{5+\mu}{3+\mu}}. \quad (73.16)$$

From formulas (66.19) it follows that

$$m_2^0 = \mu m_1^0 + \frac{R}{4\gamma^4}(1-\mu^2)\Psi_2^0 \operatorname{ctg}\theta_0 \approx \mu m_1^0 + \frac{Eh^3}{12a}\Psi^0,$$

consequently,

$$m_2^0 = m_1^0 \left[\mu + \frac{1-\mu^2}{3+\mu} \frac{1+\frac{q_0^4}{576}}{1+\frac{q_0^4}{576}\frac{5+\mu}{3+\mu}} \right] -$$

$$- \frac{(h_e^0 - \bar{h}_e^0)(1-\mu^2)\theta_0 a}{6(3+\mu)} \frac{1+\frac{q_0^4}{960}}{1+\frac{q_0^4}{576}\frac{5+\mu}{3+\mu}}. \quad (73.17)$$

Determine the amplitude of normal displacement $w_{(1)}$. Setting in (16.1) $R_1 = R_2 = R$ and taking into account that $v = R\sin\theta$, we obtain

$$\left.\begin{aligned} \Psi &= -\frac{\sin\theta}{R}\frac{d}{d\theta}\left(\frac{w_{(1)}}{\sin\theta}\right), \\ \frac{w_{(1)}}{\sin\theta} &= -\int_0^\theta \frac{R\Psi}{\sin\theta}\, d\theta + K, \end{aligned}\right\} \quad (73.18)$$

where K - constant of integration.

Because of the smallnesses of θ we have

$$\int_0^\theta \frac{\Psi}{\sin\theta}\, d\theta \approx \int_0^\theta \frac{\Psi}{\theta}\, d\theta = \frac{1}{Eh}\operatorname{Re}\int_0^{\gamma\theta\sqrt{2}} \frac{1}{q}(A_1 - iB_1)\, I_2(q\sqrt{i})\, dq. \quad (73.19)$$

Taking into consideration that

$$\int_0^q \frac{1}{q} I_2(q\sqrt{i})\,dq = -\frac{I_1(q\sqrt{i})}{q\sqrt{i}} + \frac{1}{2} = \frac{1}{iq}\frac{d}{dq}[I_0(q\sqrt{i})] + \frac{1}{2} = \frac{1}{2} + \frac{\psi_2'(q)}{q} - i\frac{\psi_1'(q)}{q}.$$

we bring (73.19) to the form

$$\int_0^{\theta} \frac{\psi}{\sin\theta}\,d\theta \approx \frac{1}{Eh}\left\{A_1\left[\frac{1}{2} + \frac{\psi_2'(q)}{q}\right] - B_1\frac{\psi_1'(q)}{q}\right\}. \tag{73.20}$$

From (73.18), (73.20) it follows that

$$\frac{w_{(1)}}{\theta} = -\frac{R}{Eh}\left\{A_1\left[\frac{1}{2} + \frac{\psi_2'(q)}{q}\right] - B_1\frac{\psi_1'(q)}{q}\right\} + K.$$

Selecting the constant K so that $w_{(1)}(\theta_0)$ would be zero, finally we obtain

$$w_{(1)} = \frac{R}{Eh}\theta\left\{A_1\left[\frac{\psi_2'(q_0)}{q_0} - \frac{\psi_2'(q)}{q}\right] - B_1\left[\frac{\psi_1'(q_0)}{q_0} - \frac{\psi_1'(q)}{q}\right]\right\}. \tag{73.21}$$

The values of constants A_1, B_1, in this formula are calculated in accordance with (73.7), and the values of functions $\psi_1'(q)$, $\psi_2'(q)$ can be taken from tables. If we are limited during calculations to the roughest approximation, i.e., take

$$\frac{\psi_2'(q_0)}{q_0} - \frac{\psi_2'(q)}{q} = \frac{q_0^4 - q^4}{2^2\cdot 4^2\cdot 6}, \quad \frac{\psi_1'(q_0)}{q_0} - \frac{\psi_1'(q)}{q} = -\frac{(q_0^2 - q^2)}{2^2\cdot 4},$$

and in expansions (73.8) keep only the first term, then for the calculation of $w_{(1)}$ we obtain an approximate formula, valid for small q (or in any case insignificantly exceeding unity)

$$w_{(1)} = m_1^0\frac{R^2\theta\theta_0}{2D(3+\mu)}\left(1 - \frac{q^2}{q_0^2}\right) - \frac{R^3\theta\cdot\theta_0^3(h_e^0 - \bar{h}_e^0)}{6\cdot 8\cdot(3+\mu)D}(7+\mu)\left(1 - \frac{3+\mu}{7+\mu}\frac{q^2}{q_0^2}\right)\left(1 - \frac{q^2}{q_0^2}\right) \tag{73.22}$$

or, taking into consideration that $R\theta = r,\ R\theta_0 = a,\ \frac{q}{q_0} = \frac{r}{a}$,

$$w_{(1)} = m_1^0 \frac{ra}{2(3+\mu)D}\left(1 - \frac{r^2}{a^2}\right) - \\ -(h_e^0 - \tilde{h}_e^0)\frac{ra^3(7+\mu)}{48(3+\mu)DR}\left(1 - \frac{r^2}{a^2}\right)\left(1 - \frac{3+\mu}{7+\mu}\frac{r^2}{a^2}\right). \qquad (73.23)$$

§ 74. The Bending of a Vertical Weakly Distorted Plate by Its Own Weight

As an example let us examine a weighable vertically positioned spherical shell (axis OX is directed along the vertical to the ground). The components of the distributed load are equal to

$$q_{1(1)} = p\cos\theta,\quad q_{2(1)} = -p,\quad q_{n(1)} = p\sin\theta, \qquad (74.1)$$

where $p = \rho h,\ \rho$ – specific weight of the material of the shell. The load functions have the form

$$f(\theta) = \frac{P_0}{\pi} + 2pR^2(\cos\theta_0 - \cos\theta),$$
$$F(\theta) = \frac{M_0}{\pi} + \frac{P_0 R}{\pi}(\cos\theta_0 - \cos\theta) + pR^3(\cos\theta_0 - \cos\theta)^2.$$

From the conditions $f(0) = 0,\ F(0) = 0$ we determine force P_0 and moment M_0, which balance the weight load:

$$P_0 = 2\pi pR^2(1 - \cos\theta_0) \approx p\pi a^2,$$
$$M_0 = \pi pR^3(1 - \cos\theta_0)^2 \approx \frac{\pi p a^4}{4R}.$$

Let us assume that on the edge we have the values

$$\left.\begin{aligned} h_e^0 &= t_1^0\cos\theta_0 + \left(n_1^0 + \frac{h_{(1)}^0}{a}\right)\sin\theta_0 = -\frac{P_0}{\pi a},\\ m_1^0 &= -\frac{M_0}{\pi a}. \end{aligned}\right\} \qquad (74.2$$

These conditions mean that a reaction force on the edge $-P_0$ is created only because of the radial forces $h_e^0 \cos\varphi$.

$$\int_0^{2\pi} h_e^0 \cos\varphi \cdot \cos\varphi a \, d\varphi = -P_0,$$

and reaction moment M_0 is created only because of bending moments $m_1^0 \cos\varphi$. From equality (66.2) and conditions (74.2) it follows that in this case on the edge the combination of tangential force and twisting the edge is equal to zero-moment.

$$s_1^0 + \frac{2h_1^0}{R} = 0,$$

and the amplitude of the axial force

$$\left(n_1^0 + \frac{h_1^0}{a}\right)\cos\theta_0 - t_1^0 \sin\theta_0 = 0.$$

Thus, it is necessary to calculate the shell under a load (74.1) and edge conditions (74.2). In the pole of the shell forces and moments are equal to zero

$$t_1(0) = t_2(0) = m_1(0) = 0.$$

Circumferential force and circumferential bending moment in section $\theta = \theta_0$ is computed according to (73.16), (73.17). In this case one ought to have in mind that in the involved case

$$h_e^0 = -pa, \quad \tilde{h}_e^0 \approx \tilde{t}_1^0 = -\frac{M_0 R}{\pi a^3} = -\frac{pa}{4}, \quad h_e^0 - \tilde{h}_e^0 = -\frac{3}{4} pa,$$
$$m_1^0 = -\frac{pa^2}{4R}, \quad \tilde{t}_2^0 = \frac{M_0 R}{\pi a^3} + pa = \frac{5}{4} pa.$$

Taking into consideration that

$$\frac{12(1-\mu^2) a^2 \theta_0}{Rh^2} = q_0^4,$$

and assuming

$$\frac{1+\frac{q_0^4(37+5\mu)}{1728(3+\mu)}}{1+\frac{q_0^4}{576}\frac{5+\mu}{3+\mu}} \approx 1+\frac{q_0^4}{1728}\frac{22+2\mu}{3+\mu},$$

we have

$$t_2^0 = -pa\left[1-\frac{q_0^4}{384}\frac{5-\mu}{3+\mu}\right]. \qquad (74.3)$$

$$m_2^0 = -\frac{pa^3}{4R}\left(\mu+\frac{1-\mu^2}{3+\mu}\right)+\frac{pa^3}{8R}\frac{1-\mu^2}{3+\mu}\left[1+\frac{q_0^4(2-\mu)}{1440(3+\mu)}\right]. \qquad (74.4)$$

As a second example let us examine the same shell loaded by load (74.1), but fixed at the edge. Reaction forces h_e^0 and moments m_1^0 are determined from formulas (73.13). Taking into consideration that

$$\tilde{\varepsilon}_{2(1)}^0 = \frac{1}{Eh}\frac{pa}{4}(5+\mu),$$

we obtain

$$h_e^0 = -\frac{2pa}{3-\mu}\left(1-\frac{5+\mu}{3-\mu}\frac{q_0^4}{2304}\right). \qquad (74.5)$$

$$m_1^0 = -\frac{pa^3}{24R}\frac{5+\mu}{3-\mu}\frac{1+\frac{q_0^4(9+2\mu)}{720(3-\mu)}}{1+\frac{q_0^4}{288}\frac{15-\mu^2}{9-\mu^2}}. \qquad (74.6)$$

The obtained results mean that apart from radial force and bending moment in the fixed section there also appear tangential and axial forces equal to

$$s_1^0+\frac{2h_1^0}{R} = pa+h_e^0 = \frac{pa(1-\mu)}{3-\mu}\left[1+\frac{5+\mu}{(3-\mu)(1-\mu)}\frac{q_0^4}{1152}\right]. \qquad (74.7)$$

$$\left(n_1^0+\frac{h_1^0}{a}\right)\cos\theta_0-t_1^0\sin\theta_0=\frac{pa^2}{4R}+\frac{m_1^0}{a}=$$

$$=\frac{pa^2(13-7\mu)}{24R(3-\mu)}\left[1+\frac{(5+\mu)(21+6\mu-\mu^2)}{(13-7\mu)(9-\mu^2)}\frac{q_0^4}{1440}\right]. \qquad (74.8)$$

§ 75. Weakly Distorted Circular Plate with Concentrated Tangential Force and Bending Moment in the Center

In §§ 73 and 74 we considered a shell not loaded by concentrated loads in the center. Now let $P_*\neq 0$, $M_*\neq 0$, i.e., in the center are applied concentrated a tangential force and concentrated bending moment. We will solve the problem approximately by dividing it in half: 1) the solution to the plane problem, 2) the solution to the bending problem allowing for the presence of the plane stressed state [94]. The correctness of such a solution was already discussed during analysis of axisymmetric deformation (§ 64).

The equations of equilibrium of the plane problem have the form of

$$\left.\begin{aligned}&\frac{d}{dr}(rt_1)-t_2+s_1+q_{1(1)}r=0,\\&\frac{d}{dr}(rs_1)+s_1-t_2+q_{2(1)}r=0.\end{aligned}\right\} \qquad (75.1)$$

Forces t_1, t_2, s_1 are connected with deformations by Hooke's law

$$\varepsilon_{2(1)}=\frac{1}{Eh}(t_2-\mu t_1),\quad \varepsilon_{1(1)}=\frac{1}{Eh}(t_1-\mu t_2),\quad \gamma_{(1)}=\frac{2(1+\mu)}{Eh}s_1. \qquad (75.2)$$

Deformations are expressed through displacements in the following manner:

$$\left.\begin{aligned}&\varepsilon_{1(1)}=\frac{du_{(1)}}{dr},\quad \varepsilon_{2(1)}=\frac{u_{(1)}+v_{(1)}}{r},\\&\gamma_{(1)}=\frac{dv_{(1)}}{dr}-\frac{u_{(1)}+v_{(1)}}{r}.\end{aligned}\right\} \qquad (75.3)$$

System of equilibrium equations (75.1) admits a decrease in order by one unit. Really, from (75.1) it follows that

$$r(t_1 - s_1) + f(r) = 0, \quad (75.4)$$

where

$$f(r) = \frac{P_0}{\pi} + \int_a^r (q_{1(1)} - q_{2(1)}) r\, dr. \quad (75.5)$$

Load function $f(r)$ and relationship (75.4) correspond to the first equation of (66.2) for a shell. But if in the first equation of (66.2) and in the expression for $f(0)$ we set $n_1 = 0$, $h_1 = 0$, $\cos\theta = 1$, $\sin\theta = 0$, $R\sin\theta = r$, then they will coincide with (75.4), (75.5). Instead of (75.1) now we can consider, for example, this system:

$$\left.\begin{aligned} \frac{d}{dr}(rt_1) - t_2 + s_1 + q_{1(1)} r = 0, \\ r(t_1 - s_1) + f(r) = 0. \end{aligned}\right\} \quad (75.6)$$

These equations can be satisfied identically by introducing function of forces V in the following manner:

$$\left.\begin{aligned} rt_1 = V - f(r), \quad t_2 = \frac{dV}{dr} + \frac{V}{r} + q_{2(1)} r, \\ rs_1 = V. \end{aligned}\right\} \quad (75.7)$$

The equation of continuity of deformations is derived easily from (75.3)

$$\frac{d}{dr}(r\varepsilon_{2(1)}) = \gamma_{(1)} + \varepsilon_{2(1)} + \varepsilon_{1(1)}, \quad (75.8)$$

moreover with the aid of (75.2) it can be written in forces

$$\frac{d}{dr}(rt_2 - \mu rt_1) - 2(1+\mu)s_1 - (1-\mu)(t_1 + t_2) = 0. \quad (75.9)$$

Substituting expressions for forces (75.7) into (75.9), we obtain the equation for determination of functions of forces V

$$r^2 \frac{d^2V}{dr^2} + r\frac{dV}{dr} - 4V =$$
$$= -(1-\mu) f(r) - \mu r \frac{df}{dr} - r\left[\frac{d}{dr}(q_{2(1)} r^2) - (1-\mu) q_{2(1)} r\right]. \qquad (75.10)$$

Equation (75.10) must be solved under the edge condition

$$t_1 = t_1^0. \qquad (75.11)$$

where t_1^0 is a given value.

If $t_1^0 = 0$, then this implies that total force is realized only because of tangential forces s_1^0, the amount of which is simple to find from (75.4), setting $r = a$.

Condition (75.11) easily can be expressed through function of forces

$$V(a) = V^0 = at_1^0 + \frac{P_0}{\pi}. \qquad (75.12)$$

As a second edge condition we have the requirement of boundedness of $V(0)$. We will write the solution of equation (75.10) for a particular form of distributed load

$$q_{1(1)} = p, \quad q_{2(1)} = -p.$$

The right side of (75.10) in this instance is equal to

$$(1-\mu)\left(-\frac{P_0}{\pi} + qa^2\right) = -(1-\mu)\frac{P_0}{\pi}. \qquad (75.13)$$

The general solution of (75.10) has the form

$$V = C_1 r^2 + C_2 \frac{1}{r^2} + \frac{(1-\mu)}{4}\frac{P_0}{\pi}.$$

Setting $C_2 = 0$ and satisfying edge condition (75.12), obtain

$$V = \frac{t_1^0 r^2}{a} + \frac{P_0 r^2}{\pi a^2} + \frac{(1-\mu)}{4}\frac{P_0}{\pi}\left(1 - \frac{r^2}{a^2}\right). \qquad (75.14)$$

Hence, taking into account the condition of equilibrium (75.13), we obtain expressions for forces:

$$
\left.
\begin{aligned}
t_1 &= t_1^0 \frac{r}{a} - \frac{(3+\mu)}{4\pi} P_*\left(\frac{1}{r} - \frac{r}{a^2}\right), \\
t_2 &= t_1^0 \frac{3r}{a} + \frac{3}{4}(3+\mu)\frac{P_0}{\pi a^2} r + (1-\mu)\frac{P_*}{4\pi r} + 2pr, \\
s_1 &= t_1^0 \frac{r}{a} + \frac{P_0 r}{\pi a^2} + (1-\mu)\frac{P_*}{4\pi}\left(\frac{1}{r} - \frac{r}{a^2}\right).
\end{aligned}
\right\}
\qquad (75.15)
$$

In the particular case when concentrated force is absent $P_* = 0$, $P_0 = p\pi a^2$, under the edge condition

$$t_1^0 = -\frac{P_0}{\pi a} = -pa$$

(which corresponds to condition (74.2) for a shell) using formulas (75.15) obtain

$$
\left.
\begin{aligned}
t_1^0 &= -pa, \qquad s_1^0 = 0, \\
t_2^0 &= -pa, \quad t_1(0) = t_2(0) = s_1(0) = 0.
\end{aligned}
\right\}
\qquad (75.16)
$$

Comparing (75.16) with (74.3), it is easy to see that the absolute value of the amplitude of circumferential stretching forces on the edge of the shell amounts to $\left(1 - \frac{q_0^4}{384}\frac{5-\mu}{3+\mu}\right)$ of the same amount in a flat plate. When the edge of the plate is fixed, the following condition should hold

$$t_2^0 - \mu t_1^0 = 0.$$

Forces on the edge are equal to

$$t_1^0 = -\frac{2pa}{3-\mu}, \quad t_2^0 = -\mu\frac{2pa}{3-\mu}, \quad s_1^0 = \frac{(1-\mu)}{3-\mu} pa. \qquad (75.17)$$

It is easy to see that the right parts of (75.17) coincide with the principal problems in (74.5), (74.7). In the last equations members containing q_0^4 are the correction to the plane stressed state which corresponds to consideration of curvature.

Let us pass now to the solution of bending problems allowing for the presence of the plane stressed state. Equations of equilibrium have the form

$$\left.\begin{aligned}&\frac{d}{dr}(rn_1)+n_2-(t_1+t_2)\theta+q_{n(1)}r=0,\\&rn_1=\frac{d}{dr}(rm_1)-m_2+h_1,\quad rn_2=r\frac{dh_1}{dr}+2h_1-m_2,\end{aligned}\right\} \tag{75.18}$$

where $\theta=\frac{r}{R}$.

From bending of the plane plate they differ only in the presence of the term $-(t_1+t_2)\theta$, where t_1, t_2 are already known functions of the radius, found earlier with the solution to the plane problem (75.1) and taking into account relationship (75.4), it is simple to exclude from (75.18) the quantities t_2, n_2, m_2. In this case it turns out to be possible to integrate the obtained differential relationship which connects the quantities n_1, m_1, h_1, t_1 and external loads. As a result we obtain

$$r^2n_1-rm_1-rh_1=\frac{r^2t_1}{R}+F(r), \tag{75.19}$$

where

$$F(r)=\frac{M_0}{\pi}+\frac{1}{R}\int_a^r rf(r)\,dr-\int_a^r\left[q_{n(1)}r^2-\frac{q_{1(1)}r^2}{R}\right]dr. \tag{75.20}$$

Equation (75.19) coincides with the second equation of (66.2) if in the latter we set $R\sin\theta=r$, $\cos\theta=1$. Introducing function of displacements

$$\Psi=-\frac{dw_{(1)}}{dr}+\frac{w_{(1)}}{r}, \tag{75.21}$$

we express the components of deformation $\varkappa_{1(1)}, \varkappa_{2(1)}, \tau_{(1)}$ through Ψ:

$$\varkappa_{1(1)}=\frac{d\Psi}{dr}+\frac{\Psi}{r}, \quad \varkappa_{2(1)}=-\tau_{(1)}=\frac{\Psi}{r}. \tag{75.22}$$

Bending moments m_1, m_2 and twisting moment h_1 are equal to

$$\left.\begin{aligned} m_1&=D\left[\frac{d\Psi}{dr}+(1+\mu)\frac{\Psi}{r}\right];\\ m_2&=D\left[\mu\frac{d\Psi}{dr}+(1+\mu)\frac{\Psi}{r}\right], \quad h_1=-D(1-\mu)\frac{\Psi}{r}. \end{aligned}\right\} \tag{75.23}$$

Excluding from (75.19) and the second equation of (75.18) the quantity n_1 and in the obtained equation we substitute for m_1, m_2, h_1 their expressions through function ψ in accordance with (75.23). We obtain an equation in the unknown function

$$r^2\frac{d^2\Psi}{dr^2}+r\frac{d\Psi}{dr}-4\Psi=\frac{1}{D}\left[\frac{r^3 t_1}{R}+F(r)\right]. \tag{75.24}$$

Equation (75.24) should be solved under the boundary condition

$$(m_1)_{r=a}=m_1^0. \tag{75.25}$$

As a second boundary condition we have the condition of boundedness of function ψ at $r=0$. Let us write out the right part of (75.24) for a load

$$q_{n(1)}=p\sin\theta\simeq p\frac{r}{R}, \quad q_{1(1)}=-q_{2(1)}=p$$

and the corresponding plane stressed state (75.15)

$$\left.\begin{aligned} F(r)&=\frac{M_0}{\pi}+\frac{P_0}{2\pi R}(r^2-a^2)+\frac{p}{4R}(r^2-a^2)^2,\\ \frac{1}{D}\left[\frac{r^3 t_1}{R}+F(r)\right]&=\frac{1}{D}(A_0+A_1r^2+A_2r^4),\\ A_0&=\frac{M_0}{\pi}, \quad A_1=-\frac{(1+\mu)}{4\pi R}P_0,\\ A_2&=\frac{t_1^0}{Ra}+\frac{P_0}{a^2}\frac{(3+\mu)}{4\pi R}+\frac{p}{4R}. \end{aligned}\right\} \tag{75.26}$$

In this case we take into account the condition of equilibrium as a whole

$$M_* = M_0 - \frac{P_* a^2}{2R} - \frac{\pi p a^4}{4R}. \qquad (75.27)$$

Going to dimensionless coordinate $\xi = \frac{r}{a}$ and taking into account (75.26), we represent the general solution of equation (75.24) in the following manner:

$$\Psi = C_3 \xi^2 + C_4 \xi^{-2} - \frac{A_0}{4D} + \frac{A_1 a^2}{4D} \xi^2 \ln \xi + \frac{A_2 a^4}{12D} \xi^4.$$

Setting $C_4 = 0$ and determining C_3 from condition (75.25), we obtain

$$\Psi = \frac{m_1^0 a}{D(3+\mu)} \xi^2 - \frac{A_0}{4D}\left[1 - \frac{(1+\mu)}{(3+\mu)} \xi^2\right] + \frac{A_1 a^2}{4D}\left[\xi^2 \ln \xi - \frac{\xi^2}{(3+\mu)}\right] + \frac{A_2 a^4}{12D}\left[\xi^4 - \frac{(5+\mu)}{(3+\mu)} \xi^2\right]. \qquad (75.28)$$

On the basis of (75.28), (75.26) and (75.23) obtain expressions for the bending moments:

$$m_1 = m_1^0 \xi + \frac{M_*(1+\mu)}{4\pi a}\left(-\frac{1}{\xi} + \xi\right) - \frac{(1+\mu)(3+\mu) P_* a}{16\pi R} \xi \ln \xi + \left[\frac{t_1^0 a}{R} + \frac{(3+\mu)}{4\pi R} \frac{P_*}{a^2} + \frac{p}{4R}\right] \frac{(5+\mu) a^2}{12} (\xi^3 - \xi). \qquad (75.29)$$

$$m_2 = m_1^0 \frac{1+3\mu}{3+\mu} \xi + \frac{M_*(1+\mu)}{4\pi a}\left[-\frac{1}{\xi} + \frac{(1+3\mu)}{3+\mu} \xi\right] - \frac{(1+\mu)(3+\mu) P_* a}{16\pi R}\left[(1+3\mu) \xi \ln \xi - \frac{(1-\mu^2)}{(3+\mu)} \xi\right] + \frac{a^3}{12}\left[\frac{t_1^0}{Ra} + \frac{(3+\mu) P_*}{4\pi R a^2} + \frac{p}{4R}\right]\left[\xi^3 (1+5\mu) - \frac{(5+\mu)(1+3\mu)}{(3+\mu)} \xi\right]. \qquad (75.30)$$

The amplitude of displacement $w_{(1)}$ normal to the middle surface is found easily using known function Ψ:

$$\frac{w_{(1)}}{r}=\int_r^a \frac{\Psi}{r}\,dr. \qquad (75.31)$$

in this case rigid rotation of the plate as a whole is selected in such a way that $w_{(1)}$ it would turn into zero at $r=a$. Substituting (75.28) into (75.31) and integrating, we obtain

$$\frac{w_{(1)}}{r}=\frac{m_1^0 a}{2D(3+\mu)}(1-\xi^2)+\frac{M_*}{4\pi D}\left[\ln\xi+\frac{(1+\mu)}{2(3+\mu)}(1-\xi^2)\right]+ \\ +\frac{(1+\mu)P_* a^2}{32\pi RD}\left[\xi^2\ln\xi+\frac{(5+\mu)}{2(3+\mu)}(1-\xi^2)\right]- \\ -\frac{(7+\mu)}{(3+\mu)}\frac{a^4}{48D}\left[\frac{f_1^0}{Ra}+\frac{(3+\mu)P_*}{4\pi Ra^2}+\frac{p}{4R}\right](1-\xi^2)\left(1-\frac{3+\mu}{7+\mu}\xi^2\right). \qquad (75.32)$$

At $P_*=M_*=0$, assuming in (75.28) and (75.29) $\xi=1$, we obtain expressions for the circumferential bending moment and function Ψ on the edge of the plate

$$\left.\begin{aligned} m_2^0&=m_1^0\frac{1+3\mu}{3+\mu}-\frac{a^2(1-\mu^2)}{6R(3+\mu)}\left(f_1^0+\frac{pa}{4}\right),\\ \Psi^0&=\frac{m_1^0 a}{D(3+\mu)}-\frac{a^3}{6DR(3+\mu)}\left(f_1^0+\frac{pa}{4}\right).\end{aligned}\right\} \qquad (75.33)$$

The right side of formulas (75.33) coincide with the principal terms of the right parts of (73.11), (73.17). Formula (75.32) at $P_*=M_*=0$ completely coincides with (73.23). Figure 34 depicts a normal bend $w=w_{(1)}\cos\varphi$, taking place along the vertical diameter of a weighable spherical shell, calculated by formula (75.32). On the lower half of the diameter $w=w_{(1)}$, on the upper half $-w=-w_{(1)}$. The shell had dimensions and load: $\sin\theta_0=\frac{a}{R}=0.344$, $\theta_0=0.351$, $q_1=q=\rho h$,

or

$$q_{1(1)}=q,\quad q_{2(1)}=-q,\quad q_{n(1)}=q\frac{r}{R},$$
$$M_*=-\frac{1}{4}\pi qa^2Ra_*,\quad a_*=0.0230,\quad P_*=-\frac{1}{4}Q,$$
$$M_0=\frac{3}{4}\pi qa^2Ra_0,\quad a_0=0.01286,\quad P_0=\frac{3}{4}Q,$$

$Q\approx\pi qa^2$ – general weight of shell.

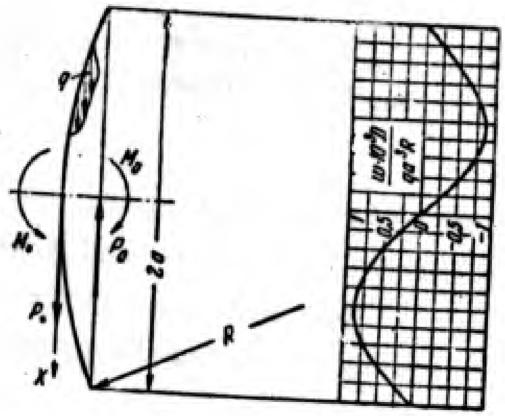


Fig. 34. Normal bend of a weakly twisted weighable spherical plate taking place on the vertical diameter.

§ 76. Deformation of a Spherical Dome Under a Self-Balancing Load on the Edge

In the majority of cases a distributed load acting on a shell is axisymmetric or varies according to the law of $\cos\varphi\,(\sin\varphi)$, i.e., belongs to the class of non-self-balancing loads. The asymmetry of shell deformation, i.e., the appearance in the stress and deformations of components proportional to $\cos k\varphi\,(\sin k\varphi)$, $k \neq 0, 1$, is caused usually by the asymmetry of the edge conditions. In this case the influence of the asymmetric component on proper working order of the shell can be very considerable. As an example we can give a ponderable shell located in the vertical plane. The weight of the shell is balanced by radial forces and bending moments applied on the lower semicircumference of the contour. As a result of such an asymmetry of weight distribution in the edge forces, apart from the axisymmetric component $(k=0)$ and the component proportional to $\cos\varphi\,(k=1)$, there are components of the form $\cos k\varphi\,(k=2, 4, 6, \ldots)$. In this case the amplitude of the bend at $k=2$ can in magnitude exceed all the remaining components.

We will examine the problem about deformation of a spherical shell under an asymmetric edge load which varies according to the law of $\cos k\varphi$ $(\sin k\varphi)$ $(k \geqslant 2)$ [82], [95]. At such a load all displacements, deformations, internal forces and moments in the shell vary in the circumferential direction by the same law:

$$
\begin{gathered}
(u,\ w,\ \varepsilon_1,\ \varepsilon_2,\ \varkappa_1,\ \varkappa_2,\ T_1,\ T_2,\ M_1,\ M_2,\ N_1) = \\
= (u_{(k)},\ w_{(k)},\ \varepsilon_{1(k)},\ \varepsilon_{2(k)},\ \varkappa_{1(k)},\ \varkappa_{2(k)},\ t_1,\ t_2,\ m_1,\ m_2,\ n_1)\cos k\varphi, \\
(v,\ \gamma,\ \tau,\ S,\ H,\ N_2) = (v_{(k)},\ \gamma_{(k)},\ \tau_{(k)},\ s,\ h_{(k)},\ n_2)\sin k\varphi.
\end{gathered} \tag{76.1}
$$

We will write out the basic system of equations, setting $R_1 = R_2 = R,\ \nu = R\sin\theta$ and using for the amplitudes of forces and moments lower case letters, as shown in (76.1). In the absence of distributed loads $(q_{1(k)} = q_{2(k)} = q_{n(k)} = 0)$ the system of equations of equilibrium has the form

$$
\left.
\begin{aligned}
&\frac{d}{d\theta}(\sin\theta t_1) - t_2\cos\theta + k\left(s + \frac{h_{(k)}}{R}\right) + n_1\sin\theta = 0, \\
&\frac{d}{d\theta}\left[\sin\theta\left(s + \frac{h_{(k)}}{R}\right)\right] - kt_2 + \left(s + \frac{h_{(k)}}{R}\right)\cos\theta + n_2\sin\theta = 0, \\
&\frac{d}{d\theta}(n_1\sin\theta) + kn_2 - (t_1 + t_2)\sin\theta = 0,
\end{aligned}
\right\} \tag{76.2}
$$

$$
\left.
\begin{aligned}
n_1 R\sin\theta &= \frac{dm_1}{d\theta}\sin\theta + (m_1 - m_2)\cos\theta + kh_{(k)}, \\
n_2 R\sin\theta &= \frac{dh_{(k)}}{d\theta}\sin\theta + 2h_{(k)}\cos\theta - km_2.
\end{aligned}
\right\} \tag{76.3}
$$

The relationships of elasticity and equations of the connection between components of deformation and displacements correspondingly are written thus:

$$
\left.
\begin{aligned}
t_1 &= B[\varepsilon_{1(k)} + \mu\varepsilon_{2(k)}], & m_1 &= D[\varkappa_{1(k)} + \mu\varkappa_{2(k)}], \\
t_2 &= B[\varepsilon_{2(k)} + \mu\varepsilon_{1(k)}], & m_2 &= D[\varkappa_{2(k)} + \mu\varkappa_{1(k)}], \\
s &= B\frac{1-\mu}{2}\gamma_{(k)}, & h_{(k)} &= D(1-\mu)\tau_{(k)},
\end{aligned}
\right\} \tag{76.4}
$$

$$
\left.
\begin{aligned}
\varepsilon_{1(k)} &= \frac{1}{R}\left(\frac{du_{(k)}}{d\theta} + w_{(k)}\right), \\
\varepsilon_{2(k)} &= \frac{1}{R}\left[w_{(k)} + \frac{1}{\sin\theta}(kv_{(k)} + u_{(k)}\cos\theta)\right], \\
\gamma_{(k)} &= \frac{1}{R}\left[\frac{dv_{(k)}}{d\theta} - \frac{1}{\sin\theta}(ku_{(k)} + v_{(k)}\cos\theta)\right], \\
\varkappa_{1(k)} &= -\frac{1}{R^2}\left[\frac{d^2w_{(k)}}{d\theta^2} - \frac{du_{(k)}}{d\theta}\right], \\
\varkappa_{2(k)} &= \frac{1}{R^2}\left[-\frac{\cos\theta}{\sin\theta}\frac{dw_{(k)}}{d\theta} + \frac{k^2w_{(k)}}{\sin^2\theta} + \frac{u_{(k)}\cos\theta + kv_{(k)}}{\sin\theta}\right], \\
\tau_{(k)} &= \frac{1}{R^2}\left[\frac{k}{\sin\theta}\frac{dw_{(k)}}{d\theta} - \frac{k\cos\theta}{\sin^2\theta}w_{(k)} + \right. \\
&\qquad \left. + \frac{dv_{(k)}}{d\theta} - \frac{ku_{(k)} + v_{(k)}\cos\theta}{\sin\theta}\right].
\end{aligned}
\right\} \tag{76.5}
$$

Equations (76.2) – (76.5) form a system of the eighth order in the usual derivatives, which it is necessary to solve under edge conditions of the form of (8.25) or (8.26). Subsequently we will consider a spherical shell without an aperature (dome), on the edge of which is system of forces

$$\left.\begin{aligned} &T_1^0 = t_1^0 \cos k\varphi, \quad S^0 + \frac{2H^0}{R} = s^0 \sin k\varphi, \\ &N_1^0 + \frac{1}{R\sin\theta_0}\frac{\partial H^0}{\partial\varphi} = n^0 \cos k\varphi, \quad M_1^0 = m_1^0 \cos k\varphi. \end{aligned}\right\} \quad (76.6)$$

System of equations (76.2) – (76.5) can be reduced to three equations relative to the amplitudes of displacements $u_{(k)}, v_{(k)}, w_{(k)}$:

$$\begin{aligned} &L^{i1}(u_{(k)}) + L^{i2}(v_{(k)}) + L^{i3}(w_{(k)}) + \\ &\quad + \frac{1}{4\gamma^4}[N^{i1}(u_{(k)}) + N^{i2}(v_{(k)}) + N^{i3}(w_{(k)})] = 0, \\ &\quad i = 1, 2, 3. \end{aligned} \quad (76.7)$$

where L^{ij}, N^{ij} – differential operators of the following orders:

L^{ij} ($i\downarrow$, $j\rightarrow$)			N^{ij} ($i\downarrow$, $j\rightarrow$)		
2	1	1	2	1	3
1	2	0	1	1	2
1	0	0	3	2	4

System (76.7) is a system of equations with variable coefficients, where several of them have in the top of the shell a singularity of the form $\frac{1}{\sin^n\theta}$ $(n = 1, 2, 3, 4)$. Development of a solution to this system is facilitated by that fact that as a rule the stressed state in the shell is made up of a slowly changing part and a rapidly changing component, which corresponds to the edge effect. To evaluate the slowly changing component we will start from equations of the zero-moment theory. A solution of the edge effect type can be found approximately, assuming that it has the form

$$\left.\begin{aligned} &u_{(k)} = \frac{1}{\gamma}F_1 e^{\gamma f_k(\theta)}, \quad v_{(k)} = \frac{1}{\gamma^2}F_2 e^{\gamma f_k(\theta)}, \\ &w_{(k)} = F_3 e^{\gamma f_k(\theta)}. \end{aligned}\right\} \quad (76.8)$$

For now we exclude the part of the sphere containing the top. Then, substituting (76.8) into (76.7) and dropping quantities of order $\frac{1}{\gamma}$ in comparison with unity, we obtain a system characterizing the first asymptotic approximation :

$$\left.\begin{aligned} &L_0^{11}(u) + L_0^{13}(w) = 0,\\ &L_0^{21}(u) + L_0^{22}(v) + L_0^{23}(w) = 0,\\ &L_0^{31}(u) + L_0^{33}(w) + \frac{1}{4\gamma^4} N_0^{33}(w) = 0, \end{aligned}\right\} \qquad (76.9)$$

where L_0^{ij} is a term with a higher-order derivative in the operator L^{ij}. For simplicity subsequently we omit in the designation of displacements the common coefficient (k), having in mind that everywhere we are speaking about the amplitudes of displacements and deformations.

Simplifications allowed during the transition from (76.7) to (76.9) will not contradict the following variant of the system in displacements:

$$\left.\begin{aligned} &L_0^{11}(u) + L_0^{13}(w) = 0,\\ &L^{21}(u) + L^{22}(v) + L^{23}(w) = 0,\\ &L_0^{31}(u) + L_0^{33}(w) + \frac{1}{4\gamma^4} N^{33}(w) = 0. \end{aligned}\right\} \qquad (76.10)$$

On the other hand, the last system differs from (in 76.9) in that it keeps all terms containing the factors $\frac{1}{\sin^n\theta}$. Consideration of these terms is necessary in order that the solution remains valid even in the area of the top. From the first and third equations of (76.10) we easily obtain a resolvant equation in w:

$$\{N^{33} + 4\gamma^4[L_0^{33} - L_0^{31}L_0^{13}(L_0^{11})^{-1}]\}\, w = 0. \qquad (76.11)$$

By a direct check one can ascertain that to system (76.10) corresponds to the following simplification of the basic system of equations (76.2) – (76.5)

$$\begin{aligned}&\frac{d}{d\theta}(\sin\theta t_1)=0\to t_1=0\to \varepsilon_1=-\mu\varepsilon_2,\\&\frac{d}{d\theta}(\sin\theta s)-kt_2+s\cos\theta=0,\\&\frac{d}{d\theta}(\sin\theta n_1)+kn_2-t_2\sin\theta=0,\end{aligned}\qquad(76.12)$$

$$\begin{aligned}&t_2=Eh\varepsilon_2, \quad m_1=D(\varkappa_1+\mu\varkappa_2),\\&s=B\frac{(1-\mu)}{2}\gamma,\quad m_2=D(\varkappa_2+\mu\varkappa_1),\quad h_{(k)}=D(1-\mu)\tau,\end{aligned}\qquad(76.13)$$

$$\begin{aligned}&\varepsilon_1=\frac{1}{R}\left(\frac{du}{d\theta}+w\right),\quad \varepsilon_2=\frac{w}{R},\\&\gamma=\frac{1}{R}\left(\frac{dv}{d\theta}-\frac{ku+v\cos\theta}{\sin\theta}\right),\\&\varkappa_1=-\frac{1}{R^2}\frac{d^2w}{d\theta^2},\quad \varkappa_2=\frac{1}{R^2}\left(-\frac{\cos\theta}{\sin\theta}\frac{dw}{d\theta}+\frac{k^2w}{\sin^2\theta}\right),\\&\tau=\frac{1}{R^2}\left(\frac{k}{\sin\theta}\frac{dw}{d\theta}-\frac{k\cos\theta}{\sin^2\theta}w\right).\end{aligned}\qquad(76.14)$$

From the system of equations it is evident that the resolvant equation in w should be analogous to the bending equation for a plate and differs from it only in the consideration of the "elastic base," which appears because of the curvature of the shell and the presence in it the circumferential forces t_2. Therefcre subsequently we will call this the bending solution and accompany, where required, by the upper coefficient $^{(u)}$ (for example, $w^{(u)}$, $u^{(u)}$ etc.) Note that as a result of the consideration of t_2 this solution far from the top possess the features of edge effect.

In explicit form equation (76.11), while keeping in the coefficients only the principal singularities, has the form of

$$\frac{d^4w}{d\theta^4}+2\frac{\cos\theta}{\sin\theta}\frac{d^3w}{d\theta^3}-\frac{2k^2+1}{\sin^2\theta}\frac{d^2w}{d\theta^2}+\frac{2k^2+1}{\sin^3\theta}\frac{dw}{d\theta}+ \\ +w\left(4\gamma^4+\frac{k^4-4k^2}{\sin^4\theta}\right)=0. \qquad (76.15)$$

Designating by L the operator

$$L=\frac{d^2}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{d}{d\theta}-\frac{k^2}{\sin^2\theta} \qquad (76.16)$$

and introducing new function

$$\zeta=\frac{1}{2\gamma^3}L(w). \qquad (76.17)$$

instead of (76.15) we obtain

$$\left.\begin{array}{l} L(\zeta)+2\gamma^2 w=0, \\ L(w)-2\gamma^2\zeta=0. \end{array}\right\} \tag{76.18}$$

This system is easily brought to one equation of the second order in complex function

$$\sigma = w + i\zeta, \tag{76.19}$$

$$\frac{d^2\sigma}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{d\sigma}{d\theta}+\sigma\left(2i\gamma^2-\frac{k^2}{\sin^2\theta}\right)=0. \tag{76.20}$$

Let us note that in the work [82] an accurate solution is constructed for the total system of differential equations of a spherical shell and possesses the features of edge effect. In this case a resolvant equation of the following form is obtained:

$$\Delta U+(1+il)U=0,$$

where Δ designates the operator $\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\right]$ and $l^2=\left(1+12\frac{R^2}{h^2}\right)(1-\mu^2)-1$. Setting in this equation $U=U_{(k)}\cos k\varphi$ and ignoring quantities of order $\frac{h^2}{R^2}$ in comparison with unity, we obtain the equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dU_{(k)}}{d\theta}\right)-\frac{k^2}{\sin^2\theta}U_{(k)}+2\gamma^2 iU_{(k)}=0,$$

coinciding with (76.20). Approximately solving (76.20) by the method already used in §§ 59, 68, we obtain

$$\sigma=C_1\sqrt{\frac{\theta}{\sin\theta}}\,I_k(\sqrt{2i}\,\gamma\theta)+C_2\sqrt{\frac{\theta}{\sin\theta}}\,H_k^{(1)}(\sqrt{2i}\,\gamma\theta). \tag{76.21}$$

where I_k, $H_k^{(1)}$ – Bessel and Hankel functions of the first kind, number k. Setting $C_2=0$, we obtain a solution finite in the angle:

$$\sigma = C_1 \sqrt{\frac{\theta}{\sin\theta}} I_k(\sqrt{2i}\,\gamma\theta). \qquad (76.22)$$

§ 77. "Zero-Moment" and "Purely Moment" Solution of the Basic Equations for a Sphere $k \geqslant 2$

We will write out now the equations of equilibrium of zero-moment theory. Assuming in (76.2), (76.3)

$$m_1 = m_2 = h_{(s)} = n_1 = n_2 = 0,$$

we have

$$\left.\begin{aligned} &\frac{d}{d\theta}(\sin\theta t_1) - t_2\cos\theta + ks = 0,\\ &\frac{d}{d\theta}(\sin\theta s) - kt_2 + s\cos\theta = 0,\\ &t_1 + t_2 = 0. \end{aligned}\right\} \qquad (77.1)$$

From the third equation follows $t_1 = -t_2 = t$, and the two others can be brought to the form

$$\left.\begin{aligned} &\frac{d}{d\theta}(\sin^2\theta t) + ks\sin\theta = 0,\\ &\frac{d}{d\theta}(\sin^2\theta s) + kt\sin\theta = 0. \end{aligned}\right\} \qquad (77.2)$$

Let us make a replacement of variables:

$$\sin^2\theta t = T, \quad \sin^2\theta s = S, \qquad (77.3)$$

$$\sin\theta = \frac{1}{\operatorname{ch}\alpha}, \qquad (77.4)$$

where from (77.4) it follows that

$$\frac{d\alpha}{d\theta} = -\frac{1}{\sin\theta}, \quad e^{\alpha} = \operatorname{ctg}\frac{\theta}{2}.$$

Then instead of (77.2) we obtain a system of equations with constant coefficients

$$\left.\begin{aligned} \frac{dT}{d\alpha} - kS = 0, \\ \frac{dS}{d\alpha} - kT = 0, \end{aligned}\right\}$$

solving which, we will have

$$T = A_1 \operatorname{ctg}^k \frac{\theta}{2} + A_2 \operatorname{ctg}^{-k} \frac{\theta}{2}, \quad S = A_1 \operatorname{ctg}^k \frac{\theta}{2} - A_2 \operatorname{ctg}^{-k} \frac{\theta}{2}.$$

Returning to the previous variable and setting $A_1 = 0$, $A_2 = \frac{Eh}{R} K$, we obtain a solution of system (77.1), finite at $\theta = 0$:

$$\left.\begin{aligned} \tilde{t}_1 = -\tilde{s} = -\tilde{t}_2 = \frac{Eh}{R} K \frac{1}{\sin^2\theta} \operatorname{tg}^k \frac{\theta}{2}, \\ \tilde{\varepsilon}_1 = \frac{1}{R} K (1+\mu) \frac{1}{\sin^2\theta} \operatorname{tg}^k \frac{\theta}{2}, \quad \tilde{\varepsilon}_2 = -\tilde{\varepsilon}_1 = \frac{\tilde{\gamma}}{2}. \end{aligned}\right\} \quad (77.5)$$

Using the found zero-moment forces and relationships (76.4), (76.5), we write out the system of equations for determination of displacements:

$$\left.\begin{aligned} \frac{du}{d\theta} + w = \frac{R}{Eh} (\tilde{t}_1 - \mu \tilde{t}_2), \\ \frac{kv}{\sin\theta} + \frac{u\cos\theta + w\sin\theta}{\sin\theta} = \frac{R}{Eh} (\tilde{t}_2 - \mu \tilde{t}_1), \\ \frac{dv}{d\theta} - \frac{ku}{\sin\theta} - \frac{v\cos\theta}{\sin\theta} = 2\frac{R}{Eh} (1+\mu) \tilde{s}. \end{aligned}\right\} \quad (77.6)$$

The particular solution of this system, corresponding to the right parts, we designate $\tilde{u}, \tilde{v}, \tilde{w}$, and the solution of the uniform system through $u^{(M)}$, $v^{(M)}$, $w^{(M)}$. The meaning of the (M) will be explained later. Both these solutions are found easily. We exclude from (77.6) displacement w, and the remaining two equations we give using the substitution

$$\frac{u}{\sin\theta}=U, \quad \frac{v}{\sin\theta}=V$$

and a change of (77.4) to the form

$$\left.\begin{aligned}\frac{dU}{d\alpha}+kV&=-2e^{-k\alpha}\operatorname{ch}^2\alpha(1+\mu)K,\\ \frac{dV}{d\alpha}+kU&=2e^{-k\alpha}\operatorname{ch}^2\alpha(1+\mu)K.\end{aligned}\right\}$$

This system of equations is equivalent to one equation

$$\frac{d^2U}{d\alpha^2}-k^2U=[-e^{-(k-2)\alpha}+e^{-(k+2)\alpha}](1+\mu)K,$$

solving which, we have

$$\begin{aligned}U&=Ne^{-k\alpha}+\left[\frac{1}{4(k-1)}e^{-(k-2)\alpha}+\frac{1}{4(k+1)}e^{-(k+2)\alpha}\right](1+\mu)K,\\ V&=Ne^{-k\alpha}+\left[\frac{(k-2)}{4k(k-1)}e^{-(k-2)\alpha}+\right.\\ &\qquad\left.+\frac{(k+2)}{4k(k+1)}e^{-(k+2)\alpha}-\frac{2}{k}e^{-k\alpha}\operatorname{ch}^2\alpha\right](1+\mu)K.\end{aligned}$$

Here, just as earlier, we dropped the solution of the uniform equation which is irregular in the angle. Returning to the previous variable, we obtain

$$\left.\begin{aligned}\tilde{u}&=(1+\mu)K\sin\theta\left[\frac{1}{4(k-1)}\operatorname{tg}^{k-2}\frac{\theta}{2}+\frac{1}{4(k+1)}\operatorname{tg}^{k+2}\frac{\theta}{2}\right],\\ \tilde{v}&=(1+\mu)K\sin\theta\left[\frac{k-2}{4k(k-1)}\operatorname{tg}^{k-2}\frac{\theta}{2}+\frac{k+2}{4k(k+1)}\operatorname{tg}^{k+2}\frac{\theta}{2}-\right.\\ &\qquad\left.-\frac{2}{k}\frac{1}{\sin^2\theta}\operatorname{tg}^{k}\frac{\theta}{2}\right],\\ \tilde{w}&=(1+\mu)K\left[-\frac{(\cos\theta+k-2)}{4(k-1)}\operatorname{tg}^{k-2}\frac{\theta}{2}-\frac{(\cos\theta+k+2)}{4(k+1)}\operatorname{tg}^{k+2}\frac{\theta}{2}+\right.\\ &\qquad\left.+\frac{1}{\sin^2\theta}\operatorname{tg}^{k}\frac{\theta}{2}\right].\end{aligned}\right\}\quad(77.7)$$

$$\left.\begin{aligned} u^{(M)} &= v^{(M)} = N \sin\theta \, \mathrm{tg}^k \frac{\theta}{2}, \\ w^{(M)} &= -N(k+\cos\theta)\, \mathrm{tg}^k \frac{\theta}{2}. \end{aligned}\right\} \qquad (77.8)$$

Displacements (77.8) cause a purely moment stressed state in the shell, since they satisfy the equations

$$\varepsilon_1 = \varepsilon_2 = \gamma = 0$$

and to them correspond the zero values of forces. Calculating the changes in curvatures corresponding to (77.8), we have

$$\varkappa_1^{(M)} = -\varkappa_2^{(M)} = -\tau^{(M)} = \frac{N}{R^2} \frac{k(k^2-1)}{\sin^2\theta} \mathrm{tg}^k \frac{\theta}{2}$$

on the basis of the relationships the elasticity obtain

$$m_1^{(M)} = -m_2^{(M)} = -h_{(k)}^{(M)} = D(1-\mu)\frac{N}{R^2} \frac{k(k^2-1)}{\sin^2\theta} \mathrm{tg}^k \frac{\theta}{2}. \qquad (77.9)$$

Substituting these expressions into (76.3), we are convinced that

$$n_1^{(M)} = n_2^{(M)} = 0.$$

In this way, with displacements (77.8) in the shell there exists a purely moment stressed state. In accordance with this displacements (77.8) are noted by (M). Note that, having taken for the basic displacements (77.8), we made the actual calculation of deformations and then forces and moments. In this case it turns out that the equations of statics (76.2) – (76.3) are satisfied. This means that the quantities with (M) give an accurate particular solution of total system of equations (76.2) – (76.5). At $k=0,1$ a purely moment state in the shell does not appear, since in this instance we have

$$m_1^{(M)} = m_2^{(M)} = h_{(k)}^{(M)} = 0.$$

Displacements (77.8) now are the displacements of a solid. Really, at $k=0$

$$u^{(M)} = v^{(M)} = N \sin\theta, \quad w^{(M)} = -N\cos\theta.$$

Calculating axial displacement, we have

$$\Delta_z^{(M)} = w^{(M)}\cos\theta - u^{(M)}\sin\theta = -N.$$

Circumferential displacement

$$v^{(M)} = \frac{N}{R}\cdot\nu$$

determines the rotation around axis OZ through an angle $\frac{N}{R}$. At $k=1$

$$u_{(1)}^{(M)} = v_{(1)}^{(M)} = N(1-\cos\theta), \quad w_{(1)}^{(M)} = -N\sin\theta.$$

The amplitudes of axial and radial displacements are equal to

$$\Delta_{z(1)}^{(M)} = -N\sin\theta,$$
$$\Delta_{r(1)}^{(M)} = u_{(1)}^{(M)}\cos\theta + w_{(1)}^{(M)}\sin\theta = -N(1-\cos\theta).$$

The displacements themselves along axes OZ and OX are equal to

$$\Delta_z^{(M)} = -\frac{N}{R}\nu\cos\varphi,$$
$$\Delta_x^{(M)} = \Delta_r^{(M)}\cos\varphi - v^{(M)}\sin\varphi = -\frac{N}{R}(1-\cos\theta)R.$$

It is easy to see that such displacements appear during the rotation of the shell as a whole through an angle $\frac{N}{R}$ around axis OY, passing through the center of the sphere.

Returning to the case of $k \geqslant 2$, we note that finally displacements of the shell should be calculated as the sum of three displacements

$$u = u^{(и)} + \tilde{u} + u^{(M)},$$
$$v = v^{(и)} + \tilde{v} + v^{(M)},$$
$$w = w^{(и)} + \tilde{w} + w^{(M)}.$$

As already was indicated, displacements $u^{(M)}, v^{(M)}, w^{(M)}$ are the accurate particular solution of a system of equations in displacements (76.7). To them correspond the deformation and forces

$$\left.\begin{aligned} &\varepsilon_1 = \varepsilon_2 = \gamma = 0,\\ &\varkappa_1 = -\varkappa_2 = -\tau = N\frac{k(k^2-1)}{\sin^2\theta}\operatorname{tg}^k\frac{\theta}{2},\\ &m_1 = -m_2 = -h_{(k)} = D\frac{(1-\mu)}{R^2}N\frac{k(k^2-1)}{\sin^2\theta}\operatorname{tg}^k\frac{\theta}{2},\\ &t_1 = t_2 = s = 0,\quad n_1 = n_2 = 0.\end{aligned}\right\} \tag{77.10}$$

Using a static-geometric analogy, we note that there should exist also an accurate solution of the basic system, possessing the form

$$\left.\begin{aligned} &m_1 = m_2 = h_{(k)} = 0,\\ &t_2 = -t_1 = s = A\frac{1}{\sin^2\theta}\operatorname{tg}^k\frac{\theta}{2},\\ &-\varepsilon_2 = \varepsilon_1 = -\frac{\gamma}{2} = -\frac{A}{Eh}(1+\mu)\frac{1}{\sin^2\theta}\operatorname{tg}^k\frac{\theta}{2},\\ &\varkappa_2 = \varkappa_1 = \tau = 0,\quad n_1 = n_2 = 0.\end{aligned}\right\} \tag{77.11}$$

Comparing (77.11) and zero-moment solution (77.5), we observe that the expressions for forces and deformations in them coincide. Consequently, they can differ from one another only in state (77.10). In this way, zero-moment solution for a spherical shell is the second accurate particular solution of the basic system of equations.

Displacements $u^{(и)}, v^{(и)}, w^{(и)}$ are calculated on the basis of solution (76.22) by the approximate relationships (76.14), (76.13), and forces and moments satisfy the simplified equations of statics (76.12). In the left part of the edge conditions (76.6) are the sum magnitudes from three stressed states – bending, zero-moment and purely moment – while the solution contains a sufficient number of arbitrary constants (one complex constant C and two real – K, N) to satisfy all four edge conditions. Calculation of the flexural component of the solution does not give rise to doubts when θ is sufficiently large. We can verify the contribution of this component in the general solution when θ is small. If we calculate on the basis of (76.22), (76.12)-(76.14) forces and moments of the "flexural state," using because of the smallness of θ the following approximate equality:

$$I_k(\sqrt{2i}\,\gamma\theta)\approx\theta^k-\frac{i2\gamma^2\theta^{k+2}}{4(k+1)},$$

and keep in the zero-moment and purely moment solutions, having decomposed them preliminarily in a series in powers of θ, also only the first terms, then it is simple to obtain the corresponding expressions for forces and moments, valid in the neighborhood of $\theta\approx 0$. On substituting these expressions into the first equation of equilibrium (76.2) we see that it is satisfied in the main terms because of the zero-moment solution. The same can be said even about the second equation. In the third equation of statics the principal terms will be members which correspond to the flexural solution, while the equation also is satisfied. This fact will agree with the fact that in the neighborhood of the angle a certain part of the sphere behaves as a flat plate in which there is a breakdown of the stressed state into the purely flexural and plane state. To determine the boundary of this part of the sphere one could use equation (76.20), from which it follows that upon execution of inequality

$$\frac{k^2}{\sin^2\theta}\gg 2\gamma^2$$

we can neglect curvature and consider the sphere as a flat plate.

§ 78. Deformation of a Spherical Shell at $k=2$

Let us examine in more detail the case of $k=2$ [92]. On the basis of (76.22), (76.19) we have

$$\left.\begin{aligned}
&\sigma=(-A_1+iB_1)\sqrt{\frac{\theta}{\sin\theta}}\,I_2(q\sqrt{i}), \qquad q=\gamma\theta\sqrt{2}.\\
&w^{(u)}=\operatorname{Re}\sigma=\sqrt{\frac{\theta}{\sin\theta}}\Big\{A_1\Big[\psi_1(q)+\frac{2}{q}\psi_2'(q)\Big]+\\
&\qquad\qquad+B_1\Big[\psi_2(q)-\frac{2}{q}\psi_1'(q)\Big]\Big\},\\
&\zeta=\operatorname{Im}\sigma=\sqrt{\frac{\theta}{\sin\theta}}\Big\{A_1\Big[\psi_2(q)-\frac{2}{q}\psi_1'(q)\Big]-\\
&\qquad\qquad-B_1\Big[\psi_1(q)+\frac{2}{q}\psi_2'(q)\Big]\Big\}.
\end{aligned}\right\} \qquad (78.1)$$

For calculation of forces and bending moments on the basis of (78.1), (76.12)-(76.14) we derive the formulas:

$$\left.\begin{aligned} m_1^{(u)} &= -\frac{D}{R^2}\left[2\gamma^2\zeta - (1-\mu)\left(\operatorname{ctg}\theta\frac{dw^{(u)}}{d\theta} - \frac{4w^{(u)}}{\sin^2\theta}\right)\right], \\ n_1^{(u)} &= -\frac{D}{R^3}\left[2\gamma^2\frac{d\zeta}{d\theta} + (1-\mu)\frac{dw^{(u)}}{d\theta}\right], \\ \frac{2h_{(2)}^{(u)}}{R\sin\theta} &= \frac{D}{R^3}\,\frac{4(1-\mu)}{\sin^2\theta}\left(\frac{dw^{(u)}}{d\theta} - \operatorname{ctg}\theta\, w^{(u)}\right), \\ t_1^{(u)} &= 0, \quad t_2^{(u)} = \frac{Eh}{R}w^{(u)}. \end{aligned}\right\} \qquad (78.2)$$

Tangential force $s^{(u)}$ is found from the second equation of (76.12) and the last relationship of (78.2)

$$s^{(u)} = \frac{1}{\sin^2\theta}\left[\frac{2Eh}{R}\int w^{(u)}\sin\theta\, d\theta + C\right],$$

while the constant C should be defined from the condition of boundedness of $s^{(u)}$ at $\theta = 0$. For small θ we can write

$$s^{(u)} = \frac{1}{\theta^2}\left[\frac{2Eh}{R}\int \theta w^{(u)}\, d\theta + C\right] =$$
$$= \frac{2Eh}{R}\operatorname{Re}\left\{(-A_1 + iB_1)\frac{1}{(\sqrt{i}q)^2}\int I_2(q\sqrt{i})(q\sqrt{i})\,d(q\sqrt{i}) + \frac{C_1}{\theta^2}\right\}.$$

Making the subsequent calculations, in the course of which we use the following relationship between Bessel functions:

$$\int xI_2(x)\,dx = -\int\left[2I_0'(x) + xI_0(x)\right]dx = -2I_0(x) + xI_0'(x),$$

we obtain

$$s^{(u)} = \frac{2Eh}{R}\left\{-A_1\frac{1}{q}\left[\psi_2'(q) - \frac{2}{q}\psi_2(q)\right] + \right.$$
$$\left. + B_1\frac{1}{q}\left[\psi_1'(q) - \frac{2}{q}\psi_1(q) + \frac{2}{q}\right]\right\}. \qquad (78.3)$$

For large θ and correspondingly large values of q the calculations can of $s^{(u)}$ can be done in exactly the same manner, assuming that

$$\int \sin\theta\, w^{(u)}\, d\theta \approx \frac{\sin\theta}{\theta}\int \theta w^{(u)}\, d\theta,$$

since $\sin\theta/\theta$ is a smoothly changing quantity and with the accepted correctness of calculations can be held constant. Dropping quantities of order $1/\gamma$ in comparison with unity, we can write

$$s^{(u)} = \frac{2Eh}{R}\left(\frac{\theta}{\sin\theta}\right)^{1/2}\left[-\frac{A_1}{q}\psi_2'(q) + \frac{B_1}{q}\psi_1'(q)\right]. \qquad (78.4)$$

Formulas (78.2) for shearing force and bending moments also can be written in two forms: in one – for small θ, where in secondary terms it is possible to assume $\cos\theta \approx 1, \sin\theta \approx 0$. in the other – for large values of q. In the second case for the calculation of $\psi_1(q)$, $\psi_2(q)$ and their derivatives one could use asymptotic representations, keeping in them only the first term, and during calculations sequentially drop all members of the order $1/\gamma$ with respect to the basic members. Making the combinations $s + \frac{2h_{(2)}}{R}$, t_1, $n_1 + \frac{2h_{(2)}}{R\sin\theta}$, m_1, summarizing in this case flexural, zero-moment and purely moment solutions and subordinating them to boundary conditions (76.6), we derive a system of algebraic equations for determination of constants of integration A_1, B_1, N and K. In Tables 7 and 8 this system is written out, while Table 7 corresponds to the case of small θ and q, Table 8 – to the case of large values of q.

The algebraic system of Table 8 is easily solved in the unknowns. From the first equation is found the constant which belongs to the zero-moment solution

$$K = \frac{\sin^2\theta_0}{\operatorname{tg}^2\frac{\theta_0}{2}} \frac{t_1^0 R}{Eh}. \tag{78.5}$$

From the second and fourth are excluded the constants of the edge effect A_1, B_1, and from the obtained relationship is found the constant of the purely moment state

$$N = \frac{\sin^3\theta_0}{\left(\frac{\sin^2\theta_0}{2} - 1\right)\operatorname{tg}^2\frac{\theta_0}{2}} \frac{4\gamma^4 R}{Eh\cdot 12(1-\mu)}\left[n_1^0 - \frac{\sin\theta_0}{2}(s^0 + t_1^0)\right]. \tag{78.6}$$

After this are found the constants of the edge effect

$$\left.\begin{aligned} A_1 &= \left(\frac{\sin\theta_0}{\theta_0}\right)^{1/2} \frac{2\gamma^2}{\Delta_0} \frac{R}{Eh}\left[-\psi_1'(q_0)\frac{m_1^{0*}}{R} + \frac{1}{\gamma\sqrt{2}}\psi_1(q_0) n_1^{0*}\right], \\ B_1 &= \left(\frac{\sin\theta_0}{\theta_0}\right)^{1/2} \frac{2\gamma^2}{\Delta_0} \frac{R}{Eh}\left[-\psi_2'(q_0)\frac{m_1^{0*}}{R} + \frac{1}{\gamma\sqrt{2}}\psi_2(q_0) n_1^{0*}\right], \end{aligned}\right\} \tag{78.7}$$

where

$$\left.\begin{aligned} \Delta_0 &= \frac{e^{2\gamma\theta_0}}{4\pi\gamma\theta_0}, \\ \frac{m_1^{0*}}{R} &= \frac{m_1^0}{R} - \frac{\sin\theta_0}{\sin^2\theta_0 - 2}\left[n_1^0 - \frac{\sin\theta_0}{2}(s^0 + t_1^0)\right], \\ n_1^{0*} &= n_1^0 + \frac{2}{\sin^2\theta_0 - 2}\left[n_1^0 - \frac{\sin\theta_0}{2}(s^0 + t_1^0)\right]. \end{aligned}\right\} \tag{78.8}$$

$$
\left.
\begin{aligned}
\psi_1(q_0) &= \frac{e^{\gamma\theta_0}}{\sqrt{2\pi\gamma\theta_0\sqrt{2}}}\cos\left(\gamma\theta_0-\frac{\pi}{8}\right),\\
\psi_2(q_0) &= \frac{-e^{\gamma\theta_0}}{\sqrt{2\pi\gamma\theta_0\sqrt{2}}}\sin\left(\gamma\theta_0-\frac{\pi}{8}\right),\\
\psi_1'(q_0) &= \frac{e^{\gamma\theta_0}}{2\sqrt{\pi\gamma\theta_0\sqrt{2}}}\left[\cos\left(\gamma\theta_0-\frac{\pi}{8}\right)-\sin\left(\gamma\theta_0-\frac{\pi}{8}\right)\right],\\
\psi_2'(q_0) &= \frac{-e^{\gamma\theta_0}}{2\sqrt{\pi\gamma\theta_0\sqrt{2}}}\left[\cos\left(\gamma\theta_0-\frac{\pi}{8}\right)+\sin\left(\gamma\theta_0-\frac{\pi}{8}\right)\right]
\end{aligned}
\right\}
\qquad (78.9)
$$

Table 7.

A_1	B_1	N	K	
0	0	0	$\frac{\mathrm{tg}^2\frac{\theta_0}{2}}{\sin^2\theta_0}$	$=\frac{t_1^0R}{Eh}$
$-\frac{2}{q_0}\left(\psi_2'-\frac{2}{q_0}\psi_2\right)_{q=q_0}$	$\frac{2}{q_0}\left(\psi_1'-\frac{2}{q}\psi_1+\frac{2}{q}\right)_{q=q_0}$	$-\frac{12(1-\mu)}{4\gamma^4}\frac{\mathrm{tg}^2\frac{\theta_0}{2}}{\sin^2\theta_0}$	$-\frac{\mathrm{tg}^2\frac{\theta_0}{2}}{\sin^2\theta_0}$	$=\frac{s^0R}{Eh}$
$-2\gamma^2\left[\left(\psi_2-\frac{2}{q}\psi_1'\right)-\right.$ $\left.-\frac{(1-\mu)}{q}\left(\psi_1'-\frac{6}{q}\psi_1-\frac{12}{q^2}\psi_2'\right)\right]_{q=q_0}$	$2\gamma^2\left[\left(\psi_1+\frac{2}{q}\psi_2'\right)+\right.$ $\left.+\frac{(1-\mu)}{q}\left(\psi_2'-\frac{6}{q}\psi_2+\frac{12}{q^2}\psi_1'\right)\right]_{q=q_0}$	$6(1-\mu)\frac{\mathrm{tg}^2\frac{\theta_0}{2}}{\sin^2\theta_0}$	0	$=\frac{m_1^0}{Eh}4\gamma^4$
$-2\gamma^3\sqrt{2}\left[\left(\psi_2'-\frac{2}{q}\psi_2+\frac{4}{q^2}\psi_1'\right)-\right.$ $\left.-\frac{4(1-\mu)}{q^2}\left(\psi_1'-\frac{3}{q}\psi_1-\frac{6}{q^2}\psi_2'\right)\right]_{q=q_0}$	$2\gamma^3\sqrt{2}\left[\left(\psi_1'-\frac{2}{q}\psi_1-\frac{4}{q^2}\psi_2'\right)+\right.$ $\left.+\frac{4(1-\mu)}{q^2}\left(\psi_2'-\frac{3}{q}\psi_2+\frac{6}{q^2}\psi_1'\right)\right]_{q=q_0}$	$-\frac{12(1-\mu)}{\sin\theta_0}\frac{\mathrm{tg}^2\frac{\theta_0}{2}}{\sin^2\theta_0}$	0	$=\frac{n_1^0R}{Eh}4\gamma^4$

Table 8.

A_1	B_1	N	K	
0	0	0	$\frac{\mathrm{tg}^2\frac{\theta_0}{2}}{\sin^2\theta_0}$	$=\frac{t_1^0R}{Eh}$
$-\frac{2}{q_0}\left(\frac{\theta_0}{\sin\theta_0}\right)^{\frac{3}{2}}\psi_2'(q_0)$	$\frac{2}{q_0}\left(\frac{\theta_0}{\sin\theta_0}\right)^{\frac{3}{2}}\psi_1'(q_0)$	$-\frac{12(1-\mu)}{4\gamma^4}\frac{\mathrm{tg}^2\frac{\theta_0}{2}}{\sin^2\theta_0}$	$-\frac{\mathrm{tg}^2\frac{\theta_0}{2}}{\sin^2\theta_0}$	$=\frac{s^0R}{Eh}$
$-2\gamma^2\left(\frac{\theta_0}{\sin\theta_0}\right)^{\frac{1}{2}}\psi_2(q_0)$	$2\gamma^2\left(\frac{\theta_0}{\sin\theta_0}\right)^{\frac{1}{2}}\psi_1(q_0)$	$6(1-\mu)\frac{\mathrm{tg}^2\frac{\theta_0}{2}}{\sin^2\theta_0}$	0	$=\frac{m_1^0}{Eh}4\gamma^4$
$-2\gamma^3\sqrt{2}\left(\frac{\theta_0}{\sin\theta_0}\right)^{\frac{1}{2}}\psi_2'(q_0)$	$2\gamma^3\sqrt{2}\left(\frac{\theta_0}{\sin\theta_0}\right)^{\frac{1}{2}}\psi_1'(q_0)$	$-\frac{12(1-\mu)}{\sin\theta_0}\frac{\mathrm{tg}^2\frac{\theta_0}{2}}{\sin^2\theta_0}$	0	$=\frac{n_1^0R}{Eh}4\gamma^4$

In this way, when the flexural solution has a clearly expressed character of the edge effect, the purely moment state is determined outside the dependence on edge effect directly according to the amount of the edge forces n_1^0, s^0, t_1^0. Constant N is equal to zero only when the external edge loads will satisfy the relationship

$$n_1^0 - \frac{\sin\theta_0}{2}(s^0 + t_1^0) = 0. \qquad (78.10)$$

The edge effect in this instance will be determined directly by the quantities m_1^0, n_1^0. Using the obtained values of constants A_1, B_1, K, N, we write the equations for determination of the angle of rotation and the normal displacement of the edge ϑ_1^0 and $w^{(0)}$ (amplitudes of the angle of rotation and displacement are expressed). We have

$$\left.\begin{aligned} \vartheta_1^{0\,(u)} &\approx \left(\frac{1}{R}\frac{dw^{(u)}}{d\theta}\right)_{\theta_0} = \left(\frac{\theta_0}{\sin\theta_0}\right)^{1/2}\frac{\gamma\sqrt{2}}{R}\left[A_1\psi_1'(q_0) + B_1\psi_2'(q_0)\right], \\ w^{0\,(u)} &= \left(\frac{\theta_0}{\sin\theta_0}\right)^{1/2}\left[A_1\psi_1(q_0) + B_1\psi_2(q_0)\right]. \end{aligned}\right\} \qquad (78.11)$$

Let us introduce for $\vartheta_1^{0\,(u)}$, $w^{0\,(u)}$ new designations ϑ_1^{0*}, w^{0*}, considering them as the difference:

$$\vartheta_1^{0*} = \vartheta_1^0 - (\tilde{\vartheta}_1^0 + \vartheta_1^{0\,(M)}), \quad w^{0*} = w^0 - (\tilde{w}^0 + w^{0\,(M)}), \qquad (78.12)$$

where

$$\tilde{\vartheta}_1^0 = \frac{(1+\mu)K}{3R}\,\mathrm{tg}^2\frac{\theta_0}{2}\left[\frac{2(2+\cos\theta_0)}{\sin\theta_0} - \sin\theta_0\right],$$

$$\vartheta_1^{0\,(M)} = \frac{1}{R}\left(\frac{dw^{(M)}}{d\theta} - u^{(M)}\right)_{\theta_0} = -\frac{N}{R}\,\frac{2(2+\cos\theta_0)}{\sin\theta_0}\,\mathrm{tg}^2\frac{\theta_0}{2},$$

$$\tilde{w}^{(0)} = (1+\mu)K\,\frac{(2+\cos\theta_0)}{3}\,\mathrm{tg}^2\frac{\theta_0}{2},$$

$$w^{0\,(M)} = -N(2+\cos\theta_0)\,\mathrm{tg}^2\frac{\theta_0}{2}.$$

On the basis of (78.7)-(78.9, (78.11) we obtain

$$w^{0*} = \frac{2\gamma^2}{Eh}\left(-m_1^{0*} + \frac{R}{\gamma}n_1^{0*}\right), \quad \vartheta_1^{0*} = \frac{2\gamma^2}{Eh}\left(-\frac{2\gamma}{R}m_1^{0*} + n_1^{0*}\right). \qquad (78.13)$$

In many practically important cases the proper working order of the shell is determined not by stress but by the amount of normal displacement. During the determination of w one ought to keep in mind that in general, when the edge load does not satisfy relationship (78.10), the greatest member in the total bend is the purely moment member. Actually,

$$w^0 = \frac{2\gamma^2}{Eh}\left(-m_1^{0*} + \frac{R}{\gamma}n_1^{0*}\right) + \frac{(1+\mu)}{3}(2+\cos\theta_0)\sin^2\theta_0\,\frac{t_1^0 R}{Eh} - \\ - \frac{(2+\cos\theta_0)\sin^3\theta_0}{6(1-\mu)(\sin^2\theta_0 - 2)}\,4\gamma^4\,\frac{R}{Eh}\left[n_1^0 - \frac{\sin\theta_0}{2}(s^0 + t_1^0)\right]$$

or, dropping quantities of lower orders,

$$w^0 = -\frac{2\gamma^2}{Eh} m_1^0 - \\ -\frac{(2+\cos\theta_0)\sin^3\theta_0}{6(1-\mu)(\sin^2\theta_0-2)} \frac{4\gamma^4 R}{Eh}\left[n_1^0 - \frac{\sin\theta_0}{2}(s^0+t_1^0)\right]. \qquad (78.14)$$

For a very flat shell this does not occur. At small angels θ_0 the constant of the purely moment state cannot be determined outside the dependence on edge effect and with a decrease in q_0 the portion of flexural component $w^{(u)}$ in the overall bend increases. Numerical calculations made for a shell with geometric dimensions

$$\theta_0 = 0{,}351; \quad 2\gamma^2 = 40{,}1; \quad \mu = 0{,}25; \quad q_0 = 2{,}22$$

under the edge conditions

$$\frac{t_1^0 R}{Eh} = R\mathcal{E}\cos\theta_0, \quad \frac{n_1^0 R}{Eh} = R\mathcal{E}\sin\theta_0,$$

$$\frac{m_1^0}{Eh} = 0{,}01285 R\mathcal{E}, \quad s^0 = 0,$$

where $\mathcal{E}$ is a certain scale coefficient of the edge load, showed that the bending of the shell is determined basically by the flexural component. Determination of the amplitude of the bend, made after solving the system of equations of Table 8, by the formula

$$w_{(2)} = A_1\left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}}\left[\psi_1(q) + \frac{2}{q}\psi_2'(q)\right] + B_1\left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}}\left[\psi_2(q) - \frac{2}{q}\psi_1'(q)\right] + \\ + (1+\mu) K \frac{(2+\cos\theta)}{3}\operatorname{tg}^2\frac{\theta}{2} - N(2+\cos\theta)\operatorname{tg}^2\frac{\theta}{2} \qquad (78.15)$$

gave the following amplitude on the edge:

$$w_{(2)}^0 = 2{,}215 R\mathcal{E}.$$

The solution of the truncated system of Table 7, corresponding to the consideration of the flexural component of the solution, under the edge conditions

$$\frac{n_1^0 R}{Eh} = R\mathcal{E}\sin\theta_0, \quad \frac{m_1^0}{Eh} = R\mathcal{E}\,0{,}01285$$

leads to the quantity

$$w_{(?)}^0 = 1{,}96 R\mathcal{E}.$$

Calculation of a simply flat plate with a given value of shearing force and bending moment on the edge gave the result

$$w_{(2)}^0 = 3{,}28 R\mathcal{E}.$$

One ought to remember that formulas (78.14), (78.15) serve for determination of the amplitude of the bend. In order to determine the bend in any point $\varphi \neq 0$, it is necessary to multiply the value of the amplitude by $\cos 2\varphi$, since in the examined illustration

$$w = w_{(2)} \cos 2\varphi.$$

§ 79. A Spherical Shell Under Action of Concentrated Normal Force in an Arbitrary Point $\theta = \theta_1$, $\varphi = 0$

In §§ 71, 62 were obtained solutions for cases when in the pole of a shell there are concentrated forces – bending moment, normal and tangential force. These solutions could be used to define the stressed state of a spherical shell loaded by concentrated forces applied in a certain point $\theta = \theta_1$, $\varphi = 0$ [93] (Fig. 35). Let us connect with this point, as with the pole, a system of geographical coordinates (ψ, β_1). The special edge effect induced by the presence in this point of concentrated forces is easily found from formulas §§ 71 and 62 (71.5), (62.5), in which it is necessary only to replace the designations (θ, φ) by (ψ, β_1). Let us examine the event when on the shell acts normal concentrated force P, applied in point $\theta = \theta_1$, $\varphi = 0$. Using formulas (62.5) and taking into consideration that

$$\chi_3(\psi) + i\chi_4(\psi) = H_n^1(\cos\psi) = -\sin\psi H_n'(\cos\psi),$$
$$\chi_3'(\psi) + i\chi_4'(\psi) = -\cos\psi H_n'(\cos\psi) + \sin^2\psi H_n''(\cos\psi),$$
$$n(n+1) = 2i\gamma^2 + 1,$$

we have

$$\left.\begin{aligned}
T_1^{(\psi,\beta_1)} &= \frac{P}{4R}\cos\psi \operatorname{Im} H_n'(\cos\psi) + \frac{P}{2\pi R}\frac{1}{\sin^2\psi},\\
T_2^{(\psi,\beta_1)} &= \frac{P}{4R}\left[\cos\psi \operatorname{Im} H_n'(\cos\psi) - \sin^2\psi \operatorname{Im} H_n''(\cos\psi)\right] - \\
&\qquad - \frac{P}{2\pi R}\frac{1}{\sin^2\psi},\\
S_{12}^{(\psi,\beta_1)} &= 0, \quad N_2^{(\psi,\beta_1)} = 0,\\
N_1^{(\psi,\beta_1)} &= \frac{P}{4R}\sin\psi \operatorname{Im} H_n'(\cos\psi),\\
M_1^{(\psi,\beta_1)} &= \frac{P}{8\gamma^2}\left[(1+\mu)\cos\psi \operatorname{Re} H_n'(\cos\psi) - \sin^2\psi \operatorname{Re} H_n''(\cos\psi)\right],\\
M_2^{(\psi,\beta_1)} &= \frac{P}{8\gamma^2}\left[(1+\mu)\cos\psi \operatorname{Re} H_n'(\cos\psi) - \mu\sin^2\psi \operatorname{Re} H_n''(\cos\psi)\right],\\
H_{12}^{(\psi,\beta_1)} &= 0.
\end{aligned}\right\} \qquad (79.1)$$

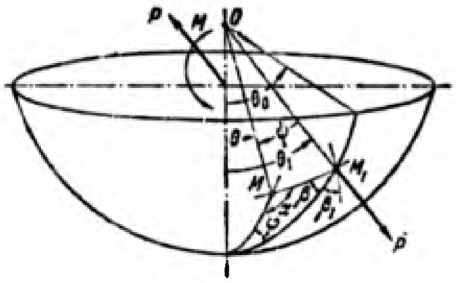


Fig. 35. Spherical shell loaded by a concentrated force applied in an arbitrary point.

If the point of application of the forces is considerably far from the edge of the shell and the shell is thin, then the special edge effect connected with the presence of concentrated force and decreasing in proportion to the distance from point M_1, practically has no effect on the stressed state of the edge of the shell $\theta=\theta_0$. The components of the zero-moment stressed state

$$\tilde{T}_1^{(\psi,\beta_1)}=-\tilde{T}_2^{(\psi,\beta_1)}=\frac{P}{2\pi R}\frac{1}{\sin^2\psi} \tag{79.2}$$

at $\theta=\theta_0$ balance force P and moment M, to which are equivalent the loads applied on the edge θ_0. If the distribution of the edge loads is such that the zero-moment stressed state (79.2) does not satisfy the edge conventions in every point (θ_0, φ), then this means that, apart from the forces of the zero-moment state, on the edge acts a certain self-balancing system of edge forces and moments, which causes the usual edge effect, decreasing in proportion to the distance from the edge θ_0. In a sufficiently thin shell the imposition of these two different edge effects does not occur, and the stressed state in the neighborhood of M_1 is defined only by formulas (79.1).

In a shell which is insufficiently thin or when force P is applied near the edge, the stressed state in the neighborhood of point M_1 is made up of state (79.1) and the usual edge effect connected with edge θ_0. In order to be able to construct this ordinary edge effect, i.e., satisfy the edge conditions on edge θ_0, it is necessary to pass from system of coordinates ψ, β_1 system of coordinates θ, φ, bound with pole O_1. In this case state (79.1), axisymmetric in system ψ, β_1, no longer will be axisymmetric in system θ, φ.

As it is easy to see from Fig. 36, formulas for the conversion of forces and bending moments during the transition from one system

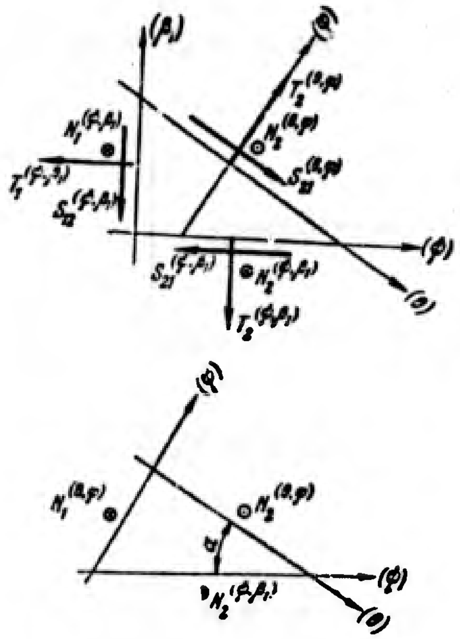


Fig. 36. Illustrations explaining the formulas for conversion of forces and moments during the transition from one system of curvilinear coordinates to the other.

of coordinates to the other analogous to formulas for the conversion of stresses in the plane problem of the theory of elasticity

$$\left.\begin{aligned}
T_1^{(\theta,\varphi)} &= T_1^{(\psi,\beta_1)}\cos^2\alpha + T_2^{(\psi,\beta_1)}\sin^2\alpha - 2S_{12}^{(\psi,\beta_1)}\sin\alpha\cos\alpha,\\
T_2^{(\theta,\varphi)} &= T_1^{(\psi,\beta_1)}\sin^2\alpha + T_2^{(\psi,\beta_1)}\cos^2\alpha + 2S_{12}^{(\psi,\beta_1)}\sin\alpha\cos\alpha,\\
S_{12}^{(\theta,\varphi)} &= \left(T_1^{(\psi,\beta_1)} - T_2^{(\psi,\beta_1)}\right)\sin\alpha\cos\alpha + S_{12}^{(\psi,\beta_1)}\left(\cos^2\alpha - \sin^2\alpha\right),\\
N_2^{(\theta,\varphi)} &= N_1^{(\psi,\beta_1)}\sin\alpha + N_2^{(\psi,\beta_1)}\cos\alpha,\\
N_1^{(\theta,\varphi)} &= N_1^{(\psi,\beta_1)}\cos\alpha - N_2^{(\psi,\beta_1)}\sin\alpha,
\end{aligned}\right\} \qquad (79.3)$$

$$\left.\begin{aligned}
M_1^{(\theta,\varphi)} &= M_1^{(\psi,\beta_1)}\cos^2\alpha + M_2^{(\psi,\beta_1)}\sin^2\alpha - 2H^{(\psi,\beta_1)}\sin\alpha\cos\alpha,\\
M_2^{(\theta,\varphi)} &= M_1^{(\psi,\beta_1)}\sin^2\alpha + M_2^{(\psi,\beta_1)}\cos^2\alpha + 2H^{(\psi,\beta_1)}\sin\alpha\cos\alpha,\\
H^{(\theta,\varphi)} &= \left(M_1^{(\psi,\beta_1)} - M_2^{(\psi,\beta_1)}\right)\sin\alpha\cos\alpha + H^{(\psi,\beta_1)}\left(\cos^2\alpha - \sin^2\alpha\right).
\end{aligned}\right\} \qquad (79.4)$$

In this case it is taken into account that in a spherical shell exist the equalities

$$S_{12} = S_{21}, \quad H_{12} = H_{21} = H$$

and the quantities N_1, N_2 are correctly the shearing forces.

We will present also the formulas of spherical trigonometry, which will be needed subsequently:

$$\left.\begin{aligned}
&\cos\psi = \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos\varphi,\\
&\sin\alpha\sin\psi = \sin\theta_1\sin\varphi, \quad \sin\beta\sin\psi = \sin\theta\sin\varphi,\\
&\cos\alpha\sin\psi = \cos\theta_1\sin\theta - \cos\theta\sin\theta_1\cos\varphi,\\
&\cos\beta\sin\psi = \cos\theta\sin\theta_1 - \sin\theta\cos\theta_1\cos\varphi.
\end{aligned}\right\} \qquad (79.5)$$

Bringing the stressed state (79.1) to coordinates (θ, φ), with the aid of formulas (79.3)-(79.5) we obtain

$$\left.\begin{aligned}
T_1^{(\theta,\varphi)} &= \tilde{T}_1^{(\theta,\varphi)} + \\
&+ \frac{P}{4R}\left[\cos\psi \,\mathrm{Im}\, H_n'(\cos\psi) - \sin^2\theta_1 \sin^2\varphi \,\mathrm{Im}\, H_n''(\cos\psi)\right], \\
T_2^{(\theta,\varphi)} &= \tilde{T}_2^{(\theta,\varphi)} + \\
&+ \frac{P}{4R}\left[\cos\psi \,\mathrm{Im}\, H_n'(\cos\psi) + (\sin^2\theta_1 \sin^2\varphi - \sin^2\psi) \,\mathrm{Im}\, H_n''(\cos\psi)\right], \\
S_{12}^{(\theta,\varphi)} &= \tilde{S}_{12}^{(\theta,\varphi)} + \\
&+ \frac{P}{4R}(\cos\theta_1 \sin\theta - \sin\theta_1 \cos\theta \cos\varphi) \sin\theta_1 \sin\varphi \,\mathrm{Im}\, H_n''(\cos\psi).
\end{aligned}\right\} \quad (79.6)$$

$$\left.\begin{aligned}
N_1^{(\theta,\varphi)} &= \frac{P}{4R}(\cos\theta_1 \sin\theta - \cos\theta \sin\theta_1 \cos\varphi) \,\mathrm{Im}\, H_n'(\cos\psi), \\
M_1^{(\theta,\varphi)} &= \frac{P}{8\gamma^2}\{(1+\mu)\cos\psi \,\mathrm{Re}\, H_n'(\cos\psi) + \\
&\quad + [(1-\mu)\sin^2\theta_1 \sin^2\varphi - \sin^2\psi] \,\mathrm{Re}\, H_n''(\cos\psi)\}, \\
M_2^{(\theta,\varphi)} &= \frac{P}{8\gamma^2}\{(1+\mu)\cos\psi \,\mathrm{Re}\, H_n'(\cos\psi) - \\
&\quad - [(1-\mu)\sin^2\theta_1 \sin^2\varphi + \mu \sin^2\psi] \,\mathrm{Re}\, H_n''(\cos\psi)\}, \\
H^{(\theta,\varphi)} &= -\frac{P}{8\gamma^2}(1-\mu)\sin\theta_1 \sin\varphi \times \\
&\quad \times (\cos\theta_1 \sin\theta - \cos\theta \sin\theta_1 \cos\varphi) \,\mathrm{Re}\, H_n''(\cos\psi).
\end{aligned}\right\} \quad (79.7)$$

where

$$\left.\begin{aligned}
\tilde{T}_1^{(\theta,\varphi)} &= -\tilde{T}_2^{(\theta,\varphi)} = \frac{P}{2\pi R}\left(\frac{1}{\sin^2\psi} - \frac{2\sin^2\theta_1 \sin^2\varphi}{\sin^4\psi}\right), \\
\tilde{S}_{12}^{(\theta,\varphi)} &= \frac{P}{\pi R}(\cos\theta_1 \sin\theta - \sin\theta_1 \cos\theta \cos\varphi)\frac{\sin\theta_1 \sin\varphi}{\sin^4\psi}
\end{aligned}\right\} \quad (79.8)$$

are the forces of the zero-moment state, and the remaining terms in formulas (79.6), (79.7) show the singular "edge effect" from concentrated force P. We will agree subsequently to consider only such a shell for which is held the condition that

$$\theta_1 + \theta_0 < \pi. \quad (79.9)$$

In this instance the inequality

$$\psi < \pi$$

always is held, and the zero-moment solution (79.8) is regular everywhere, with the exception of point $\psi = 0$.

The stressed state corresponding to the edge effect connected with the edge $\theta=\theta_0$. which it is necessary to impose on state (79.6), (79.7) in such a way thet the edge conditions on the edge are satisfied, usually is looked for in the form of trigonometric series in coordinate ϕ. In this case forces and moments (79.6), (79.7) must be represented also in the form of series.

Because of symmetry relative to plane $\phi = 0$ they are representable by series of form

$$T_1(T_2, N_1, M_1, M_2)=\sum_{k=0}^{\infty} T_{1(k)}(T_{2(k)}, N_{1(k)}, M_{1(k)}, M_{2(k)}) \cos k\varphi, \qquad (79.10)$$

$$S_{12}(H, N_2)=\sum_{k=0}^{\infty} S_{12(k)}(H_{(k)}, N_{2(k)}) \sin k\varphi,$$

where

$$T_{1(0)}=\frac{1}{2\pi}\int_0^{2\pi} T_1 d\varphi, \quad T_{1(k)}=\frac{1}{\pi}\int_0^{2\pi} T_1 \cos k\varphi\, d\varphi,$$

$$S_{12(k)}=\frac{1}{\pi}\int_0^{2\pi} S_{12} \sin k\varphi\, d\varphi$$

etc.

§ 80. Equilibrium of a Finite Part of a Dome During the Action of a Normal Concentrated Force Applied in an Arbitrary Point

Let us draw the section $\theta = \mathrm{const}$ and, having discarded part of the shell (θ, θ_0), we replace its action on the remaining part by a system of internal forces and moments, shown in Fig. 37.

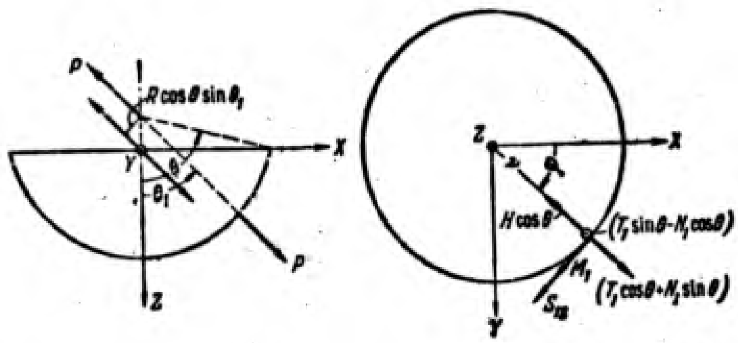


Fig. 37. Internal and external forces acting on parts of a spherical shell $(0, \theta)$.

On the strength of the fact that the involved bounded parts of the shell is in equilibrium, internal forces and moments in section $\theta = \mathrm{const}$ should satisfy the conditions:

$$
\left.\begin{aligned}
&\int_0^{2\pi} (T_1 \sin\theta - N_1 \cos\theta) v \, d\varphi = \begin{cases} P\cos\theta_1, & \theta > \theta_1, \\ 0, & \theta < \theta_1, \end{cases} \\
&\int_0^{2\pi} [S_{12} \sin\varphi - (T_1 \cos\theta + N_1 \sin\theta)\cos\varphi] v \, d\varphi = \\
&\qquad\qquad = \begin{cases} P\sin\theta_1, & \theta > \theta_1, \\ 0, & \theta < \theta_1, \end{cases} \\
&\int_0^{2\pi} [(T_1 \sin\theta - N_1 \cos\theta) v \cos\varphi + M_1 \cos\varphi - H\cos\theta\sin\varphi] v \, d\varphi = \\
&\qquad\qquad = \begin{cases} PR\cos\theta\sin\theta_1, & \theta > \theta_1, \\ 0, & \theta < \theta_1. \end{cases}
\end{aligned}\right\} \quad (80.1)
$$

The first two equations of (80.1) are the condition of equilibriu of the chosen part of the shell written in projections onto axes OZ, OX the third is the equation of moments relative to axis OY.

Taking into account expansion (79.10), from (80.1) we obtain

$$\left.\begin{array}{l}
T_{1(0)}\sin\theta - N_{1(0)}\cos\theta = \begin{cases} \dfrac{P\cos\theta_1}{2\pi R\sin\theta}, & \theta > \theta_1, \\ 0, & \theta < \theta_1, \end{cases} \\
T_{1(1)}\cos\theta + N_{1(1)}\sin\theta - S_{12(1)} = \begin{cases} -\dfrac{P\sin\theta_1}{\pi R\sin\theta}, & \theta > \theta_1, \\ 0, & \theta < \theta_1, \end{cases} \\
(N_{1(1)}\cos\theta - T_{1(1)}\sin\theta)\nu - M_{1(1)} + H_{(1)}\cos\theta = \\
\qquad = \begin{cases} -\dfrac{P\cos\theta\sin\theta_1}{\pi\sin\theta}, & \theta > \theta_1, \\ 0, & \theta < \theta_1. \end{cases}
\end{array}\right\} \quad (80.2)$$

It is easy to show that conditions (80.2) should satisfy the zero-moment part of the solution taken individually, i.e., at any θ the following should be valid:

$$\left.\begin{array}{l}
\tilde{T}_{1(0)}\sin\theta = \begin{cases} \dfrac{P\cos\theta_1}{2\pi R\sin\theta}, & \theta > \theta_1, \\ 0, & \theta < \theta_1, \end{cases} \\
\tilde{T}_{1(1)}\cos\theta - \tilde{S}_{12(1)} = \begin{cases} -\dfrac{P\sin\theta_1}{\pi R\sin\theta}, & \theta > \theta_1, \\ 0, & \theta < \theta_1, \end{cases} \\
-\tilde{T}_{1(1)}\sin\theta\cdot\nu = \begin{cases} -\dfrac{P\cos\theta\sin\theta_1}{\pi\sin\theta}, & \theta > \theta_1, \\ 0, & \theta < \theta_1. \end{cases}
\end{array}\right\} \quad (80.3)$$

This means that the remaining part of the solution, which is equations (79.6), (79.7) and which corresponds to a special edge effect, is a self-balancing stressed state. Physically this is completely clear, since the special edge effect fades in proportion to the distance from the point of application of force and cannot take part in providing equilibrium of finite shell element $(\theta, 0)$, if θ is considerably far from θ_1. This fact may also be formally verified. For example, comparing the expression $(T_{1(0)} - \tilde{T}_{1(0)})\sin\theta - N_{1(0)}\cos\theta$, using (79.5) and (79.6) and taking into consideration that

$$\frac{dH_n(\cos\psi)}{d\varphi} = \frac{dH_n(\cos\psi)}{d(\cos\psi)}\frac{d(\cos\psi)}{d\varphi} = -\sin\theta\sin\theta_1\sin\varphi H_n'(\cos\psi),$$

we obtain

$$(T_{1(0)} - \tilde{T}_{1(0)})\sin\theta - N_{1(0)}\cos\theta =$$
$$= \frac{1}{2\pi}\frac{P}{4R}\int_0^{2\pi}\left[\sin\theta_1\cos\varphi\,\mathrm{Im}\,H_n'(\cos\psi) -\right.$$
$$\left. - \sin^2\theta_1\sin\theta\sin^2\varphi\,\mathrm{Im}\,H_n''(\cos\psi)\right]d\varphi =$$
$$= -\frac{1}{2\pi}\frac{P}{4R\sin\theta}\int_0^{2\pi}\frac{d^2 f_n(\cos\psi)}{d\varphi^2}\,d\varphi = 0.$$

We verify also the execution of the first equality of (80.3). For calculation

$$\tilde{T}_{1(0)} = \frac{1}{2\pi}\frac{P}{2\pi R}\int_0^{2\pi}\left(\frac{1}{\sin^2\psi} - \frac{2\sin^2\theta_1\sin^2\varphi}{\sin^4\psi}\right)d\varphi$$

we use the equation introduced in [81].

$$\int_0^{\pi}\frac{\cos m\varphi}{a + b\cos\varphi}\,d\varphi = \frac{\pi}{\sqrt{a^2 - b^2}}\left(\frac{-a + \sqrt{a^2 - b^2}}{b}\right)^m. \qquad (80.4)$$

where $\sqrt{a^2 - b^2}$ has such a value that

$$\left|\frac{-a + \sqrt{a^2 - b^2}}{b}\right| < 1.$$

Setting in (80.4) one time $m = 0$,

$$a = 1 + \cos\theta\cos\theta_1, \quad b = \sin\theta\sin\theta_1,$$

and some other time $m = 0$, $a = 1 - \cos\theta\cos\theta_1$, $b = -\sin\theta\sin\theta_1$, we have

$$\int_0^{\pi}\frac{d\varphi}{1 + \cos\psi} = \frac{\pi}{\cos\theta + \cos\theta_1},$$
$$\int_0^{\pi}\frac{d\varphi}{1 - \cos\psi} = \begin{cases}\dfrac{\pi}{\cos\theta_1 - \cos\theta}, & \theta > \theta_1, \\[2mm] \dfrac{\pi}{\cos\theta - \cos\theta_1}, & \theta < \theta_1.\end{cases}$$

rther, we easily compute the integral

$$\int_0^{2\pi} \frac{d\varphi}{\sin^2\psi} = \int_0^{\pi} \frac{d\varphi}{1+\cos\psi} + \int_0^{\pi} \frac{d\varphi}{1-\cos\psi} = \begin{cases} \dfrac{2\pi\cos\theta_1}{\cos^2\theta_1 - \cos^2\theta}, & \theta > \theta_1. \\ \dfrac{2\pi\cos\theta}{\cos^2\theta - \cos^2\theta_1}, & \theta < \theta_1. \end{cases}$$

t is somewhat longer but just as simple to calculate the integral

$$\int_0^{2\pi} \frac{\sin^2\varphi}{\sin^4\psi}\, d\varphi = \begin{cases} \dfrac{\pi\cos\theta_1}{\sin^2\theta(\cos^2\theta_1 - \cos^2\theta)}, & \theta > \theta_1. \\ \dfrac{\pi\cos\theta}{\sin^2\theta_1(\cos^2\theta - \cos^2\theta_1)}, & \theta < \theta_1. \end{cases}$$

nd finally

$$\tilde{T}_{1(0)} = \begin{cases} \dfrac{P}{2\pi R}\dfrac{\cos\theta_1}{\sin^2\theta}, & \theta > \theta_1. \\ 0, & \theta < \theta_1. \end{cases}$$

Similarly we can verify also the remaining equality (80.3).

§ 81. Representation of the Solution in the Form of a Trigonometric Series in the Coordinate φ. Conditions on the Edge

Before going to the question about satisfaction of the edge conditions on edge θ_0. let us transform formulas (79.6), excluding $H''_n(\cos\psi)$ with the aid of equation (59.4). Copying it in the form

$$\sin^2\psi H''_n(\cos\psi) - 2\cos\psi H'_n(\cos\psi) + (2i\gamma^2 + 1) H_n(\cos\psi) = 0,$$

we find that

$$\begin{aligned} \operatorname{Re} H''_n(\cos\psi) &= \frac{2\cos\psi}{\sin^2\psi}\operatorname{Re} H'_n(\cos\psi) + \\ &+ \frac{1}{\sin^2\psi}[2\gamma^2 \operatorname{Im} H_n(\cos\psi) - \operatorname{Re} H_n(\cos\psi)] \approx \\ &\approx \frac{2\cos\psi}{\sin^2\psi}\operatorname{Re} H'_n(\cos\psi) + \frac{2\gamma^2}{\sin^2\psi}\operatorname{Im} H_n(\cos\psi), \\ \operatorname{Im} H''_n(\cos\psi) &\approx \frac{2\cos\psi}{\sin^2\psi}\operatorname{Im} H'_n(\cos\psi) - \frac{2\gamma^2}{\sin^2\psi}\operatorname{Re} H_n(\cos\psi). \end{aligned}$$

Formulas (79.6), (79.7) then assume the form

$$\left.\begin{aligned}T_1&=\tilde{T}_1+\frac{P}{4R}\Big[\Big(1-\frac{2\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\Big)\cos\psi\,\mathrm{Im}\,H'_n(\cos\psi)+\\&\qquad+2\gamma^2\frac{\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\,\mathrm{Re}\,H_n(\cos\psi)\Big],\\T_2&=\tilde{T}_2+\frac{P}{4R}\Big[-\Big(1-\frac{2\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\Big)\cos\psi\,\mathrm{Im}\,H'_n(\cos\psi)+\\&\qquad+2\gamma^2\Big(1-\frac{\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\Big)\mathrm{Re}\,H_n(\cos\psi)\Big],\\S_{12}&=\tilde{S}_{12}+\frac{P}{4R}(\cos\theta_1\sin\theta-\sin\theta_1\cos\theta\cos\varphi)\times\\&\qquad\times\frac{\sin\theta_1\sin\varphi}{\sin^2\psi}\big[2\cos\psi\,\mathrm{Im}\,H'_n(\cos\psi)-2\gamma^2\,\mathrm{Re}\,H_n(\cos\psi)\big].\end{aligned}\right\}\quad(81.1)$$

$$\left.\begin{aligned}N_1&=\frac{P}{4R}(\cos\theta_1\sin\theta-\cos\theta\sin\theta_1\cos\varphi)\,\mathrm{Im}\,H'_n(\cos\psi),\\M_1&=-\frac{P}{8\gamma^2}\Big\{(1-\mu)\Big(1-\frac{2\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\Big)\cos\psi\,\mathrm{Re}\,H'_n(\cos\psi)+\\&\qquad+\Big[1-(1-\mu)\frac{\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\Big]2\gamma^2\,\mathrm{Im}\,H_n(\cos\psi)\Big\},\\M_2&=\frac{P}{8\gamma^2}\Big\{(1-\mu)\Big(1-\frac{2\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\Big)\cos\psi\,\mathrm{Re}\,H'_n(\cos\psi)-\\&\qquad-\Big[\mu+(1-\mu)\frac{\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\Big]2\gamma^2\,\mathrm{Im}\,H_n(\cos\psi)\Big\},\\H&=-\frac{P}{8\gamma^2}\frac{(1-\mu)\sin\theta_1\sin\varphi}{\sin^2\psi}(\cos\theta_1\sin\theta-\cos\theta\sin\theta_1\cos\varphi)\times\\&\qquad\times\big[2\cos\psi\,\mathrm{Re}\,H'_n(\cos\psi)+2\gamma^2\,\mathrm{Im}\,H_n(\cos\psi)\big].\end{aligned}\right\}\quad(81.2)$$

where $\tilde{T}_1, \tilde{T}_2, \tilde{S}_{12}$ are determined as before with the aid of (79.8). Setting in (81.1), (81.2) $\theta=\theta_0$, we obtain the forces and moments $T_1(\theta_0, \varphi)$, $T_2(\theta_0, \varphi)$, ..., which we designate through $T_1^*, S_{12}^*, M_1^*, H^*$, etc. Taking into consideration that the edge conditions on the tangential and shearing forces can be satisfied only in combination with the twisting moment, we obtain on the basis of (81.1), (81.2) system of forces

$$\left.\begin{aligned}&T_1(\theta_0, \varphi)=T^*(\varphi), \quad S_{12}(\theta_0, \varphi)+\frac{1}{R}H(\theta_0, \varphi)=\bar{S}^*(\varphi),\\&M_1(\theta_0, \varphi)=M_1^*(\varphi), \quad N_1(\theta_0, \varphi)+\frac{1}{\nu}\frac{\partial}{\partial\varphi}H(\theta_0, \varphi)=N^*(\varphi).\end{aligned}\right\}\quad(81.3)$$

Let us assume that on the edge of the shell we have system of forces

$$T(\varphi), \quad \bar{S}(\varphi), \quad M(\varphi), \quad N(\varphi). \qquad (81.4)$$

which are different from (81.3). Systems (81.3), (81.4) are statically equivalent, i.e., have identical principal vector and principal moment

$$P = -P\sin\theta_1 i - P\cos\theta_1 k, \quad M = PR\cos\theta_0 \sin\theta_1 j.$$

If we add to (81.3) the self-balancing system of forces

$$\left.\begin{aligned} T^{**} = T - T^{*}, \quad \bar{S}^{**} = \bar{S} - \bar{S}^{*}, \\ M^{**} = M - M^{*}, \quad N^{**} = N - N^{*}, \end{aligned}\right\} \qquad (81.5)$$

then the total stressed state which satisfies the assigned conditions (81.4) will be made up of state (81.1), (81.2) and the edge effect induced by the self-balancing load (81.5). To construct this edge effect (81.5) must be represented in the form of trigonometric series in coordinate φ. Then we can satisfy the edge conditions separately for every harmonic by means of imposition on the state $\{T_{1(k)}, T_{2(k)}, N_{1(k)}, M_{1(k)}, M_{2(k)}\}\cos k\varphi, (S_{12(k)}, N_{2(k)}, H_{(k)})\sin k\varphi$ (formulas (81.1), (81.2), (79.10)) the stressed state caused by the edge load $\{T^{**}_{(k)}, N^{**}_{(k)}, M^{**}_{(k)}\}\cos k\varphi; \bar{S}^{**}_{(k)}\sin k\varphi$. Issue about creation of the last stressed state at $k \geqslant 2$ was examined in §§ 76, 77. Furthermore, inasmuch as the load (81.5) is self-balanced the equality

$$\left.\begin{aligned} \bar{S}^{**}_{(0)} = 0, \quad T^{**}_{(0)}\sin\theta_0 - N^{**}_{(0)}\cos\theta_0 = 0, \\ T^{**}_{(1)}\cos\theta_0 + N^{**}_{(1)}\sin\theta_0 - \bar{S}^{**}_{(1)} = 0, \\ (T^{**}_{(1)}\sin\theta_0 - N^{**}_{(1)}\cos\theta_0)\nu_0 + M^{**}_{(1)} = 0 \end{aligned}\right\} \qquad (81.6)$$

should be held with the continuation of the text, and correspondingly the number of independent edge conditions at $k = 0, 1$ drops from four to two. Essential in this instance are the conditions containing radial force and meridian bending moment (§§ 60, 69).

§ 82. A Spherical Shell Loaded Along the Parallel by a Distributed Normal Load of Constant Intensity

Representation of state (81.1), (81.2) in the form of trigonometric series is necessary only to satisfy the edge conditions. If the shell is thin-walled ($2\gamma^2 \gg 1$), the force is rather far from the edge and the imposition of special and simple edge effects does not take place, then the stressed state in the vicinity of the force can be directly calculated on the basis of (81.1), (81.2) or, which is even simpler, according to (79.1).

However, expansion in a series can prove to be useful also for other purposes. Taking into account formal expansion of the delta function in a trigonometric series

$$\left.\begin{aligned} &P\delta(\varphi-0)=p_0+\sum_{k=1}^{\infty}p_k\cos k\varphi,\\ &p_0=\frac{P}{2\pi R\sin\theta_1},\quad p_k=\frac{P}{\pi R\sin\theta_1}\end{aligned}\right\} \tag{82.1}$$

and using the expansion of stressed state (81.1), (81.2) in a series (79.10), it is simple to obtain a solution to the problem about a spherical shell loaded along the parallel $\theta=\theta_1$ by a normal linear load of intensity $p_k\cos k\varphi$. Thus, for instance, the zero harmonic in the expansion of forces and moments (81.1), (81.2) in a series in φ will describe the stressed state in a shell loaded along the parallel θ by an axisymmetrically distributed normal linear load of intensity

$$p_0=\frac{P}{2\pi R\sin\theta_1}. \tag{82.2}$$

This load is balanced by a system of edge loads with principal vector $P\cos\theta_1\,\boldsymbol{k}$.

Let us assume that the external loads assigned on edge θ_0, also have axisymmetric distribution and the edge conditions have the form of

$$\left.\begin{aligned} &H_{e(0)}=T_{1(0)}\cos\theta_0+N_{1(0)}\sin\theta_0=p_0\frac{\cos\theta_1\sin\theta_1\cos\theta_0}{\sin^2\theta_0},\\ &M^{*}_{1(0)}=0.\end{aligned}\right\} \tag{82.3}$$

On the basis of the first equality of (80.3) it is easy to see that the zero-moment axisymmetric state satisfies conditions (82.3). We will explain now the character of the fading effect of the axisymmetric component of the remaining part of the solution, which is a term in formulas (81.1), (81.2). Let us write out, for example, the expression for the meridian bending moment

$$M_{1(0)}=-\frac{p_0R\sin\theta_1}{8\gamma^2}\int_0^{2\pi}\left\{(1-\mu)\cos\psi\left(1-\frac{2\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\right)\times\right.$$

$$\left.\times\operatorname{Re}H_n'(\cos\psi)+2\gamma^2\left[1-\frac{(1-\mu)\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\right]\operatorname{Im}H_n(\cos\psi)\right\}d\varphi. \qquad (82.4)$$

Let us examine the integral $\int_0^{2\pi}H_n(\cos\psi)\,d\varphi$, the imaginary part of which is a term in the right part of (82.4).

By law of composition for Legendre functions we have [142]

$$H_n(\cos\psi)=h_0+2\sum_{m=1}^{\infty}h_m\cos m\varphi,$$

where

$$h_m=\begin{cases}\dfrac{\Pi(n-m)}{\Pi(n+m)}H_n^m(\cos\theta)P_n^m(\cos\theta_1), & \theta>\theta_1,\\[2mm] \dfrac{\Pi(n-m)}{\Pi(n+m)}H_n^m(\cos\theta_1)P_n^m(\cos\theta), & \theta<\theta_1.\end{cases}$$

Here $\Pi(x)=\Gamma(x+1)=x!$, while this series converges evenly in φ. During integration the convergence of the series is not impaired and during practical calculation of the involved definite integral we can be limited to a finite number of terms $m<|n|$. If we now use asymptotic presentations for associated Legendre functions at large n with respect to absolute value and at $m<|n|$, then, setting $n\approx\gamma(1+i)$, we obtain

$$h_m\gamma\pi\sqrt{2\sin\theta\sin\theta_1}=\begin{cases}e^{-\gamma(\theta-\theta_1)}e^{i\left[\gamma(\theta-\theta_1)-\frac{\pi}{4}\right]}, & \theta>\theta_1,\\[2mm] e^{-\gamma(\theta_1-\theta)}e^{i\left[\gamma(\theta_1-\theta)-\frac{\pi}{4}\right]}, & \theta<\theta_1.\end{cases}$$

In this way, in both cases the involved integral decreases as $e^{-\gamma|\theta-\theta_1|}$ during the increase of $|\theta_1-\theta|$. The same change must characterize the remaining terms in (82.4), since, in essence, they all amount to the "edge effect" induced by the presence of a line of distortion $\theta=\theta_1$.

Proposing that the dimensions of the shell are such that $e^{-\gamma(\theta_0-\theta_1)} \ll 1$, we note that edge conditions (82.3) are satisfied in this instance automatically (because of the zero-moment part of the solution). The edge effect connected with edge θ_0 is absent, and in the neighborhood of the line of loading the forces and moments can be calculated as the null terms of expansion (79.10) with the aid of formulas (81.1), (81.2). The latter is valid not only under special edge conditions (82.3), but in all cases when the mutual influence of the edge and the line of distortion θ_1 can be neglected.

Practical calculations of forces and bending moments using formulas (81.1), (81.2), (79.10) are difficult because of the lack of tables of Legendre functions for a complex value n. This can be circumvented if we use the approximate equality

$$\left.\begin{aligned}
&H_n(\cos\psi) \approx \left(\frac{\psi}{\sin\psi}\right)^{1/2} H_0^{(1)}(q\sqrt{i}) = \left(\frac{\psi}{\sin\psi}\right)^{1/2} [\psi_3(q) + i\psi_4(q)],\\
&H_n'(\cos\psi) = -\frac{1}{\sin\psi} H_n^{\cdot}(\cos\psi) \approx \frac{n}{\sin\psi}\left(\frac{\psi}{\sin\psi}\right)^{1/2} H_1^{(1)}(q\sqrt{i}) =\\
&\qquad = -\frac{\gamma\sqrt{2}}{\sin\psi}\left(\frac{\psi}{\sin\psi}\right)^{1/2} [\psi_3'(q) + i\psi_4'(q)],\\
&n = \gamma\sqrt{2i}, \quad q = \gamma\psi\sqrt{2},
\end{aligned}\right\} \tag{82.5}$$

the correctness of which already was discussed above (§ 59). Taking into account representation (82.5), let us write the final expressions for forces and bending moments:

$$\left.\begin{aligned}
T_{1(0)}(\theta) = \frac{p_0\sin\theta_1}{2}\int_0^{\pi}\Bigg\{&\left(1 - \frac{2\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\right)\times\\
&\times\left[\frac{2}{\pi}\frac{1}{\sin^2\psi} - \frac{\gamma\sqrt{2}\cos\psi}{\sin\psi}\left(\frac{\psi}{\sin\psi}\right)^{1/2}\psi_4'(\gamma\psi\sqrt{2})\right] +\\
&+ 2\gamma^2\frac{\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\left(\frac{\psi}{\sin\psi}\right)^{1/2}\psi_3(\gamma\psi\sqrt{2})\Bigg\}\,d\varphi,\\
T_{2(0)}(\theta) = \frac{p_0\sin\theta_1}{2}\int_0^{\pi}\Bigg\{&\left(1 - \frac{2\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\right)\times\\
&\times\left[-\frac{2}{\pi}\frac{1}{\sin^2\psi} + \frac{\gamma\sqrt{2}\cos\psi}{\sin\psi}\left(\frac{\psi}{\sin\psi}\right)^{1/2}\psi_4'(\gamma\psi\sqrt{2})\right] +\\
&+\left(1 - \frac{\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\right)2\gamma^2\left(\frac{\psi}{\sin\psi}\right)^{1/2}\psi_3(\gamma\psi\sqrt{2})\Bigg\}\,d\varphi.
\end{aligned}\right\} \tag{82.6}$$

$$M_{1(0)}(\theta)=\frac{p_0 R\sin\theta_1}{2}\int_0^{\pi}\left(\frac{\psi}{\sin\psi}\right)^{1/2}\times$$
$$\times\left\{\frac{(1-\mu)}{\gamma\sqrt{2}}\left(1-\frac{2\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\right)\frac{\cos\psi}{\sin\psi}\psi_3'(\gamma\psi\sqrt{2})-\right.$$
$$\left.-\left[1-\frac{(1-\mu)\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\right]\psi_4(\gamma\psi\sqrt{2})\right\}d\varphi.$$
$$M_{2(0)}(\theta)=-\frac{p_0 R\sin\theta_1}{2}\int_0^{\pi}\left(\frac{\psi}{\sin\psi}\right)^{1/2}\times$$
$$\times\left\{\frac{(1-\mu)}{\gamma\sqrt{2}}\left(1-\frac{2\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\right)\frac{\cos\psi}{\sin\psi}\psi_3'(\gamma\psi\sqrt{2})+\right.$$
$$\left.+\left[\mu+\frac{(1-\mu)\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\right]\psi_4(\gamma\psi\sqrt{2})\right\}d\varphi.$$
$$S_{12(0)}=H_{(0)}=0.$$
$$N_{1(0)}(\theta)=\frac{p_0\sin\theta_1}{2}\int_0^{\pi}(\cos\theta_1\sin\theta-\cos\theta\sin\theta_1\cos\varphi)\times$$
$$\times\left[\frac{2}{\pi}\frac{1}{\sin^2\psi}-\frac{\gamma\sqrt{2}}{\sin\psi}\left(\frac{\psi}{\sin\psi}\right)^{1/2}\psi_4'(\gamma\psi\sqrt{2})\right]d\varphi+f_0(\theta).$$
$$f_0(\theta)=-\frac{p_0\sin\theta_1}{\pi}\int_0^{\pi}(\cos\theta_1\sin\theta-\cos\theta\sin\theta_1\cos\varphi)\frac{d\varphi}{\sin^2\psi}. \qquad (82.6)\ \text{(Cont'd)}$$

It is easy to see that forces $T_{1(0)}$, $T_{2(0)}$ and bending moments $M_{1(0)}$, $M_{2(0)}$ are everywhere continuous; a discontinuity during the transition through parallel θ_1 happens only to shearing force, moreover jump $|N_{1(0)}|$ happens because of the discontinuity in $f_0(0)$ and in absolute value is equal to p_0.

Really, calculating $f_0(0)$ with the aid of (80.4), we obtain

$$\int_0^{2\pi}\frac{\cos\varphi}{\sin^2\psi}d\varphi=\int_0^{\pi}\frac{\cos\varphi}{1+\cos\psi}d\varphi+\int_0^{\pi}\frac{\cos\varphi}{1-\cos\psi}d\varphi=$$
$$=\begin{cases}\dfrac{2\pi\cos\theta(1-\cos^2\theta_1)}{(\cos^2\theta_1-\cos^2\theta)\sin\theta\sin\theta_1}, & \theta>\theta_1,\\[2ex] \dfrac{2\pi\cos\theta_1(1-\cos^2\theta)}{(\cos^2\theta-\cos^2\theta_1)\sin\theta\sin\theta_1}, & \theta<\theta_1.\end{cases}$$
$$f_0(\theta)=\begin{cases}-p_0\dfrac{\sin\theta_1}{\sin\theta}, & \theta>\theta_1,\\ 0, & \theta<\theta_1.\end{cases} \qquad (82.7)$$

§ 83. Example of Calculation by Formulas (82.6)

The definite integrals in (82.6) can be calculated by any formula for approximate calculation. In this case it is essential that the subintegral expressions in formulas for forces are always finite, even at $\psi=0$, and in the formulas for moments contain only logarithmic singularities of the form $-\frac{(1+\mu)}{\pi}\ln\psi$. It is simple to be satisfied in this if one considers that in the neighborhood of $q=\psi\gamma\sqrt{2}=0$ $\psi_3(q)$, $\psi_4(q)$ and their derivatives have representations (64.2), and

$$\lim_{\varphi\to 0}\left(\frac{\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\right)_{\theta=\theta_1}=1.$$

Because of the presence of a logarithmic singularity in the subintegral expressions for moments, during the calculation of $M_{1(0)}(\theta_1)$, $M_{2(0)}(\theta_1)$ it is necessary to break the interval of integration in half:

$$0\leqslant\varphi\leqslant\varphi_*,\quad \varphi_*\leqslant\varphi\leqslant\pi,$$

and take into account that

$$-\frac{(1+\mu)}{\pi}\int_0^{\varphi_*}\ln\left(\gamma_1\frac{\gamma\psi\sqrt{2}}{2}\right)d\varphi=-\frac{(1+\mu)}{\pi}\varphi_*\ln\left(\gamma_1\frac{\gamma\psi_*\sqrt{2}}{2}\right),$$

since when $\theta=\theta_1$ and φ is small there exists the equality

$$\psi\approx\sin\theta_1\cdot\varphi.$$

The choice of φ_* should conform to the correctness of calculations. As an example let us calculate the bending moment in section θ_1 for a shell with parameters $\gamma\sqrt{2}=20$, $\mu=0.25$, $\theta_0=60°$, loaded along the parallel circle $\theta_1=20°$ $(\sin\theta_1=0.342)$ by normal forces of intensity p_0.

At $\varphi_*=0.015$, $\gamma\psi_*\sqrt{2}=0.100$

$$-\frac{(1+\mu)}{\pi}\varphi_*\ln\left(\gamma_1\frac{\gamma\psi_*\sqrt{2}}{2}\right)=-\frac{1.25}{3.14}0.015(0.577-2.995)=0.0145.$$

The calculation of the quantity

$$\int_0^{\pi}\left(\frac{\psi}{\sin\psi}\right)^{1/2}\left\{\frac{(1-\mu)}{\gamma\sqrt{2}}\left(1-\frac{2\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\right)\frac{\cos\psi}{\sin\psi}\psi_3'(\gamma\psi\sqrt{2})-\right.$$
$$\left.-\left[1-\frac{(1-\mu)\sin^2\theta_1\sin^2\varphi}{\sin^2\psi}\right]\psi_1(\gamma\psi\sqrt{2})\right\}d\varphi=0.1103$$

is reduced into Table 9, where when φ and ψ are small the subintegral expression is approximately equal to $-\frac{(1-\mu)}{\psi\gamma\sqrt{2}}\psi_3'(\gamma\psi\sqrt{2})-\mu\psi_4(\gamma\psi\sqrt{2})$, and calculation of the definite follows the formula for trapezoids. Finally $M_{1(0)}(\theta_1)=\frac{p_0R\sin\theta_1}{2}0.1103=p_0R\cdot 0.0188$. By formulas (63.13), setting in them $m=0$, we have $M_1(\theta_1)=\frac{p_0R}{4\gamma}=p_0R0.0177$. The inaccuracy of formulas (63.13) has the order $\frac{1}{\gamma\sqrt{2}}$ in comparison with unity, which in this case amounts to 5%. The divergence between both the results rests within the limits of this inaccuracy. Let us remember that formulas (63.13) are valid only when $q(\theta_1)=\gamma\theta_1\sqrt{2}$ is great and asymptotic equations can be used for calculation of functions $\psi_i(q_1)$ and their derivatives, while expressions (82.6) are adequate for calculations with any arbitrarily small θ_1. Furthermore, they have still the advantage that they allow calculation of any internal force or moment at any θ (not too close to the edge) independently of the remaining quantities. It is especially simple to find forces and moments in point $\theta=0$. Since in this instance $\psi=\theta_1$ and integration in (82.6) is accurate, we have

$$T_{1(0)}(0)=T_{2(0)}(0)=\frac{p_0\pi\sin\theta_1\gamma^2}{2}\left(\frac{\theta_1}{\sin\theta_1}\right)^{1/2}\psi_3(\gamma\theta_1\sqrt{2}). \qquad (83.1)$$

$$M_{1(0)}(0)=M_{2(0)}(0)=-\frac{p_0\pi}{4}R(1+\mu)\sin\theta_1\left(\frac{\theta_1}{\sin\theta_1}\right)^{1/2}\psi_4(\gamma\theta_1\sqrt{2}). \qquad (83.2)$$

$$N_{1(0)}=f_0(0)=0. \qquad (83.3)$$

Remembering expansion (64.2), at small θ_1 (83.1), (83.2) can be rewritten thus:

$$T_{1(0)}(0)=T_{2(0)}(0)=\frac{p_0\pi}{2}\sin\theta_1\gamma^2\left[\frac{1}{2}+\frac{\gamma^2\theta_1^2}{2\pi}\ln\left(\gamma_1\frac{\gamma\theta_1\sqrt{2}}{2}\right)+\ldots\right],$$
$$M_{1(0)}(0)=M_{2(0)}(0)=$$
$$=-\frac{p_0\pi}{4}R(1+\mu)\sin\theta_1\left[\frac{2}{\pi}\ln\left(\gamma_1\frac{\gamma\theta_1\sqrt{2}}{2}\right)-\frac{\gamma^2\theta_1^2}{4}+\ldots\right].$$

Table 9.

φ	$\cos\varphi$	$\cos\psi=\cos\varphi+\cos^2\theta_1(1-\cos\varphi)$	ψ	$\gamma\psi\sqrt{2}$	$\psi_3'(\gamma\psi\sqrt{2})$	$\psi_4(\gamma\psi\sqrt{2})$	(1) Подынтегральное выражение $a(\varphi)\approx-\frac{(1-\mu)}{\gamma\sqrt{2}}\frac{\psi_3'}{\psi}-\mu\psi_4$	
$\varphi_0=0{,}015$	0,99989	0,99999	=0,005	0,100	−0,0929	−1,541	1,082	$\int_0^{\varphi_0} a(\varphi)\,d\varphi=0{,}0145$
0,055	0,99849	0,99982	0,019	0,380	−0,1932	−0,707	0,558	$\int_{\varphi_0}^{0{,}175} a(\varphi)\,d\varphi=\frac{1}{2}\,0{,}040\,[1{,}082+2\,(0{,}558+0{,}374+0{,}243)+0{,}162]=0{,}0720$
0,095	0,99549	0,99947	0,0326	0,652	−0,2248	−0,397	0,374	
0,135	0,99090	0,99895	0,0458	0,916	−0,2273	−0,223	0,243	
0,175	0,98473	0,99823	0,0595	1,190	−0,2136	−0,107	0,162	
0,349	0,9397	0,9929	0,119	2,38	−0,105	0,042	0,044	$\int_{0{,}175}^{\pi} a(\varphi)\,d\varphi=\frac{1}{2}\,\frac{3{,}14}{18}\,[0{,}162+2\,(0{,}044+0{,}015)+0{,}003]=0{,}0238$
0,523	0,866	0,9843	0,177	3,54	−0,031	0,033	0,015	
0,698	0,766	0,9726	0,235	4,70	−0,002	0,011	0,003	
					0	0	0	
					0	0	0	

KEY: (1) Subintegral expression.

Subtending the circumference of the load in point $\theta=0$, i.e., making θ_1 in (83.1), (83.2) approach zero, so that $p_0 2\pi R\sin\theta_1$ would give P, as the passage to the limit we obtain the same result which can be reached on the basis of formulas (62.5):

$$T_1(0)=T_2(0)=\frac{P\gamma^2}{8R},$$

$$M_1(0)=M_2(0)=\left[-\frac{P}{4\pi}(1+\mu)\ln\left(\frac{\gamma\theta_1\sqrt{2}}{2}\right)\right]_{\theta_1\to 0}.$$

In order to explain the speed of the tendency to infinity of the quantity $N_{1(0)}(0)$, it is necessary to use now the expression for $f(\theta)$ at $\theta>\theta_1$ Then we obtain

$$N_{1(0)}(0)=\left[-\frac{p_0\sin\theta_1}{\sin\theta}\right]_{\substack{\theta=0\\ \theta_1\to 0}}=-\left(\frac{P}{2\pi R\sin\theta}\right)_{\theta=0}.$$

§ 84. A Shell Loaded Along the Parallel by a Normal Load Varying According to the Law of $\cos k\varphi$

Let us write out the formulas which correspond to the k-th harmonic in the expansion of forces and moments (81.1), (81.2) in a trigonometric series in coordinate φ. They will describe the stressed state in a shell loaded by a normal load distributed along the parallel θ_1 of intensity:

$$p_k \cos k\varphi, \quad p_k = \frac{P}{\pi R \sin\theta_1}, \tag{84.1}$$

$$\begin{aligned}
T_{1(k)}(\theta) &= \frac{p_k \sin\theta_1}{2} \int_0^\pi \Big\{\Big(1 - \frac{2\sin^2\theta_1 \sin^2\varphi}{\sin^2\psi}\Big) \times \\
&\times \Big[\frac{2}{\pi}\frac{1}{\sin^2\psi} - \frac{\gamma\sqrt{2}\cos\psi}{\sin\psi}\Big(\frac{\psi}{\sin\psi}\Big)^{1/2} \psi_4'(\gamma\psi\sqrt{2})\Big] + \\
&+ \frac{\sin^2\theta_1 \sin^2\varphi}{\sin^2\psi} 2\gamma^2 \Big(\frac{\psi}{\sin\psi}\Big)^{1/2} \psi_3(\gamma\psi\sqrt{2})\Big\} \cos k\varphi \, d\varphi. \\
T_{2(k)}(\theta) &= \frac{p_k \sin\theta_1}{2} \int_0^\pi \Big\{\Big(1 - \frac{2\sin^2\theta_1 \sin^2\varphi}{\sin^2\psi}\Big) \times \\
&\times \Big[-\frac{2}{\pi}\frac{1}{\sin^2\psi} + \frac{\gamma\sqrt{2}\cos\psi}{\sin\psi}\Big(\frac{\psi}{\sin\psi}\Big)^{1/2} \psi_4'(\gamma\psi\sqrt{2})\Big] + \\
&+ \Big(1 - \frac{\sin^2\theta_1 \sin^2\varphi}{\sin^2\psi}\Big) 2\gamma^2 \Big(\frac{\psi}{\sin\psi}\Big)^{1/2} \psi_3(\gamma\psi\sqrt{2})\Big\} \cos k\varphi \, d\varphi, \\
S_{12(k)}(\theta) &= p_k \sin^2\theta_1 \int_0^\pi (\cos\theta_1 \sin\theta - \sin\theta_1 \cos\theta \cos\varphi) \times \\
&\times \Big[\frac{2}{\pi}\frac{1}{\sin^2\psi} - \frac{\gamma\sqrt{2}\cos\psi}{\sin\psi}\Big(\frac{\psi}{\sin\psi}\Big)^{1/2} \psi_4'(\gamma\psi\sqrt{2}) - \\
&- \gamma^2 \Big(\frac{\psi}{\sin\psi}\Big)^{1/2} \psi_3(\gamma\psi\sqrt{2})\Big] \frac{\sin\varphi \sin k\varphi}{\sin^2\psi} \, d\varphi,
\end{aligned} \tag{84.2}$$

$$\begin{aligned}
M_{1(k)}(\theta) &= \frac{p_k R \sin\theta_1}{2} \int_0^\pi \Big(\frac{\psi}{\sin\psi}\Big)^{1/2} \times \\
&\times \Big\{\frac{(1-\mu)}{\gamma\sqrt{2}}\Big(1 - \frac{2\sin^2\theta_1 \sin^2\varphi}{\sin^2\psi}\Big) \frac{\cos\psi}{\sin\psi} \psi_3'(\gamma\psi\sqrt{2}) - \\
&- \Big[1 - \frac{(1-\mu)\sin^2\theta_1 \sin^2\varphi}{\sin^2\psi}\Big] \psi_4(\gamma\psi\sqrt{2})\Big\} \cos k\varphi \, d\varphi, \\
M_{2(k)}(\theta) &= -\frac{p_k R \sin\theta_1}{2} \int_0^\pi \Big(\frac{\psi}{\sin\psi}\Big)^{1/2} \times \\
&\times \Big\{\frac{(1-\mu)}{\gamma\sqrt{2}}\Big(1 - \frac{2\sin^2\theta_1 \sin^2\varphi}{\sin^2\psi}\Big) \frac{\cos\psi}{\sin\psi} \psi_3'(\gamma\psi\sqrt{2}) + \\
&+ \Big[\mu + \frac{(1-\mu)\sin^2\theta_1 \sin^2\varphi}{\sin^2\psi}\Big] \psi_4(\gamma\psi\sqrt{2})\Big\} \cos k\varphi \, d\varphi.
\end{aligned} \tag{84.3}$$

$$H_{(k)}(\theta) = -\frac{p_k R \sin^2\theta_1 (1-\mu)}{2}\int_0^{\pi}(\cos\theta_1 \sin\theta - \sin\theta_1 \cos\theta\cos\varphi)\times$$
$$\times\left(\frac{\psi}{\sin\psi}\right)^{1/2}\left[\psi_1(\gamma\psi\sqrt{2}) - \frac{2\cos\psi}{\gamma\sqrt{2}\sin\psi}\psi_3'(\gamma\psi\sqrt{2})\right]\frac{\sin\varphi\sin k\varphi}{\sin^2\psi}\,d\varphi,$$
$$N_{1(k)}(\theta) = \frac{p_k \sin\theta_1}{2}\int_0^{\pi}(\cos\theta_1 \sin\theta - \cos\theta\sin\theta_1\cos\varphi)\times$$
$$\times\left[\frac{2}{\pi}\frac{1}{\sin^2\psi} - \frac{\gamma\sqrt{2}}{\sin\psi}\left(\frac{\psi}{\sin\psi}\right)^{1/2}\psi_4'(\gamma\psi\sqrt{2})\right]\cos k\varphi\,d\varphi + f_k(\theta).$$ (84.3) (Cont'd)

where

$$f_k(\theta) = -\frac{p_k \sin\theta_1}{\pi}\int_0^{\pi}(\cos\theta_1\sin\theta - \cos\theta\sin\theta_1\cos\varphi)\frac{\cos k\varphi}{\sin^2\psi}\,d\varphi. \quad (84.4)$$

Noticing that

$$\int_0^{\pi}\frac{\cos m\varphi\,d\varphi}{\sin^2\psi} = \frac{(-1)^m\pi}{2(\cos\theta+\cos\theta_1)}\frac{(1-\cos\theta)^m(1-\cos\theta_1)^m}{\sin^m\theta\sin^m\theta_1} +$$
$$+\begin{cases}\dfrac{\pi}{2(\cos\theta_1-\cos\theta)}\dfrac{(1+\cos\theta)^m(1-\cos\theta_1)^m}{\sin^m\theta\sin^m\theta_1}, & \theta>\theta_1,\\[2ex] \dfrac{\pi}{2(\cos\theta-\cos\theta_1)}\dfrac{(1+\cos\theta_1)^m(1-\cos\theta)^m}{\sin^m\theta\sin^m\theta_1}, & \theta<\theta_1,\end{cases}$$

we have

$$f_k(\theta) = -\frac{p_k\sin\theta_1}{2\sin\theta}\left\{(-1)^k\left[\frac{(1-\cos\theta)(1-\cos\theta_1)}{\sin\theta\sin\theta_1}\right]^k + \right.$$
$$\left.+\left[\frac{(1+\cos\theta)(1-\cos\theta_1)}{\sin\theta\sin\theta_1}\right]^k\right\}, \quad \theta>\theta_1,$$
$$f_k(\theta) = -\frac{p_k\sin\theta_1}{2\sin\theta}\left\{(-1)^k\left[\frac{(1-\cos\theta)(1-\cos\theta_1)}{\sin\theta\sin\theta_1}\right]^k - \right.$$
$$\left.-\left[\frac{(1+\cos\theta_1)(1-\cos\theta)}{\sin\theta\sin\theta_1}\right]^k\right\}, \quad \theta<\theta_1,$$
$$f_k(\theta_1^+) - f_k(\theta_1^-) = -p_k. \quad (84.5)$$

§ 85. A Shell Loaded Along Parallel by Bending Moments of Intensity m_0 and $m_1\cos\varphi$

Let us examine now a shell loaded in point A_1 by concentrated moment M (Fig. 38). Formulas (72.8), if we set $A_1 = B_1 = 0$, $P_* = 0$ and replace θ by ψ and φ by $\beta_1 = -\beta$, let us determine the bending and

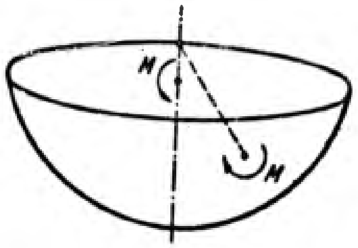


Fig. 38. Spherical shell loaded by a concentrated bending moment.

twisting moments in a certain point A in system of coordinates (ψ, β_1)

$$\left.\begin{aligned} M_1^{(\psi, \beta_1)} &= \frac{M}{4R}\left[\zeta_4'(\psi) + (1+\mu)\operatorname{ctg}\psi\,\zeta_4(\psi)\right]\cos\beta, \\ M_2^{(\psi, \beta_1)} &= \frac{M}{4R}\left[\mu\zeta_4'(\psi) + (1+\mu)\operatorname{ctg}\psi\,\zeta_4(\psi)\right]\cos\beta, \\ H^{(\psi, \beta_1)} &= \frac{M}{4R}\,\frac{(1-\mu)}{\sin\psi}\,\zeta_4(\psi)\sin\beta. \end{aligned}\right\} \quad (85.1)$$

Using transformation formulas (79.4), in system of coordinates θ, φ we will have

$$\left.\begin{aligned} M_1 = \frac{M}{4R}\Big[&(1+\mu)\operatorname{ctg}\psi\cos\beta\,\zeta_4(\psi) + \\ &+(\cos^2\alpha + \mu\sin^2\alpha)\cos\beta\,\zeta_4'(\psi) - 2(1-\mu)\frac{\sin\alpha\cos\alpha\sin\beta}{\sin\psi}\zeta_4(\psi)\Big], \\ M_2 = \frac{M}{4R}\Big[&(1+\mu)\operatorname{ctg}\psi\cos\beta\,\zeta_4(\psi) + \\ &+(\sin^2\alpha + \mu\cos^2\alpha)\cos\beta\,\zeta_4'(\psi) + 2(1-\mu)\frac{\sin\alpha\cos\alpha\sin\beta}{\sin\psi}\zeta_4(\psi)\Big]. \end{aligned}\right\} \quad (85.2)$$

In this case, just as earlier, we will not deal with the simple edge effect, connected with edge θ_0, considering that the stressed state in the neighborhood of the point of application of a concentrated moment is wholly described by formulas of form (85.2). Having in mind representation

$$\left.\begin{aligned} M\delta(\varphi - 0) &= m_0 + \sum_{k=1}^{\infty} m_k\cos k\varphi, \\ m_0 = \frac{M}{2\pi R\sin\theta_1}, &\quad m_k = \frac{M}{\pi R\sin\theta_1}, \end{aligned}\right\} \quad (85.3)$$

we find at first the axisymmetric distribution of bending moments in a shell loaded by a bending moment of intensity m_0 distributed along the parallel θ_1.

Writing out the zero term in the expansion of moments (85.2) in a trigonometric series in φ and expressing M through m_0 using the second formula of (85.3), we obtain

$$\left.\begin{aligned} M_{1(0)} &= \frac{m_0 \sin\theta_1}{2} \int_0^{\pi} \Big\{ (1+\mu)\frac{\cos\beta}{\sin\psi}\left[\cos\psi\,\zeta_4(\psi)+\frac{1}{\pi}\right] + \\ &\quad + [1-(1-\mu)\sin^2\alpha]\cos\beta\,\zeta_4'(\psi) - \\ &\quad - 2(1-\mu)\frac{\sin\alpha\cos\alpha\sin\beta}{\sin\psi}\left[\zeta_4(\psi)+\frac{1}{\pi}\right]\Big\}\,d\varphi + r_{1(0)}(\theta), \\ M_{2(0)} &= \frac{m_0 \sin\theta_1}{2} \int_0^{\pi} \Big\{ (1+\mu)\frac{\cos\beta}{\sin\psi}\left[\cos\psi\,\zeta_4(\psi)+\frac{1}{\pi}\right] + \\ &\quad + [\mu+(1-\mu)\sin^2\alpha]\cos\beta\,\zeta_4'(\psi) + \\ &\quad + 2(1-\mu)\left[\zeta_4(\psi)+\frac{1}{\pi}\right]\frac{\sin\alpha\cos\alpha}{\sin\psi}\sin\beta\Big\}\,d\varphi + r_{2(0)}(\theta). \end{aligned}\right\} \qquad (85.4)$$

$$\left.\begin{aligned} r_{1(0)}(\theta) &= \\ &= -\frac{m_0\sin\theta_1}{2\pi}\int_0^{\pi}\left[(1+\mu)\frac{\cos\beta}{\sin\psi} - 2(1-\mu)\sin\alpha\cos\alpha\frac{\sin\beta}{\sin\psi}\right]d\varphi, \\ r_{2(0)}(\theta) &= \\ &= -\frac{m_0\sin\theta_1}{2\pi}\int_0^{\pi}\left[(1+\mu)\frac{\cos\beta}{\sin\psi} + 2(1-\mu)\sin\alpha\cos\alpha\frac{\sin\beta}{\sin\psi}\right]d\varphi. \end{aligned}\right\} \qquad (85.5)$$

Calculating the integrals

$$\int_0^{\pi}\frac{\cos\beta}{\sin\psi}\,d\varphi = \int_0^{\pi}\frac{1}{\sin^2\psi}(\cos\theta\sin\theta_1 - \sin\theta\cos\theta_1\cos\varphi)\,d\varphi = \begin{cases} 0, & \theta>\theta_1, \\ \dfrac{\pi}{\sin\theta_1}, & \theta<\theta_1, \end{cases}$$

$$\int_0^{\pi}\sin\alpha\cos\alpha\frac{\sin\beta}{\sin\psi}\,d\varphi = \int_0^{\pi}\frac{1}{\sin^4\psi}(\sin^2\theta\sin\theta_1\cos\theta_1\sin^2\varphi - \\ - \sin^2\theta_1\sin\theta\cos\theta\sin^2\varphi\cos\varphi)\,d\varphi = \begin{cases} \dfrac{\pi}{2}\dfrac{\sin\theta_1}{\sin^2\theta}, & \theta>\theta_1, \\ 0, & \theta<\theta_1, \end{cases}$$

we obtain

$$\left.\begin{aligned} r_{1(0)}(\theta) &= \begin{cases} (1-\mu)\dfrac{m_0}{2}\dfrac{\sin^2\theta_1}{\sin^2\theta}, & \theta>\theta_1, \\ -(1+\mu)\dfrac{m_0}{2}, & \theta<\theta_1, \end{cases} \\ r_{2(0)}(\theta) &= \begin{cases} -(1-\mu)\dfrac{m_0}{2}\dfrac{\sin^2\theta_1}{\sin^2\theta}, & \theta>\theta_1, \\ -(1+\mu)\dfrac{m_0}{2}, & \theta<\theta_1. \end{cases} \end{aligned}\right\} \qquad (85.6)$$

The subintegral expressions in (85.4) are finite at any θ and φ, where $\cos\beta$, $\sin\alpha$, $\cos\alpha$ are known functions of coordinate φ, determined on the basis of (79.5). Going through the circumference, loads $M_{1(0)}$, $M_{2(0)}$ undergo discontinuity

$$\left.\begin{aligned}[M_{1(0)}]&=[r_{1(0)}]=m_0,\\ [M_{2(0)}]&=[r_{2(0)}]=\mu m_0.\end{aligned}\right\} \tag{85.7}$$

Let us compute using (85.4), the bending moments in the pole $\theta=0$. Since at $\theta=0$, $\psi=0$, $\sin^2\alpha=\sin^2\varphi$, $\cos\beta=1$, $\sin\beta=0$, then integration in (85.4) is done directly and we obtain

$$M_{1(0)}(0)=M_{2(0)}(0)=\pi\frac{m_0\sin\theta_1}{2}(1+\mu)\left[\operatorname{ctg}\theta_1\zeta_4(\theta_1)+\frac{1}{2}\zeta_4'(\theta_1)\right]. \tag{85.8}$$

Since at θ_1 we can approximately set

$$\operatorname{ctg}\theta_1\approx\frac{1}{\theta_1},\quad \zeta_4(\theta_1)=-\frac{1}{\pi}+\frac{2\gamma^2\theta_1^2}{16},\quad \zeta_4'(\theta_1)=\frac{4\gamma^2\theta_1}{16},$$

then in this instance (85.8) can be written in the following manner:

$$M_{1(0)}(0)=M_{2(0)}(0)=\pi\frac{m_0\sin\theta_1}{2}(1+\mu)\left[-\frac{1}{\pi\theta_1}+\frac{\theta_1\gamma^2}{4}\right]. \tag{85.9}$$

Let us also give the formulas for amplitudes of the bending moments when the shell is loaded along parallel θ_1 by moments of intensity $m_1\cos\varphi$:

$$\left.\begin{aligned}M_{1(1)}(\theta)=&\frac{m_1\sin\theta_1}{2}\int_0^\pi\Big\{(1+\mu)\frac{\cos\beta}{\sin\psi}\left[\cos\psi\zeta_4(\psi)+\frac{1}{\pi}\right]+\\ &+[1-(1-\mu)\sin^2\alpha]\cos\beta\zeta_4'(\psi)-\\ &-2(1-\mu)\sin\alpha\cos\alpha\frac{\sin\beta}{\sin\psi}\left[\zeta_4(\psi)+\frac{1}{\pi}\right]\Big\}\cos\varphi\,d\varphi+r_{1(1)}(\theta),\\ M_{2(1)}(\theta)=&\frac{m_1\sin\theta_1}{2}\int_0^\pi\Big\{(1+\mu)\frac{\cos\beta}{\sin\psi}\left[\cos\psi\zeta_4(\psi)+\frac{1}{\pi}\right]+\\ &+[\mu+(1-\mu)\sin^2\alpha]\cos\beta\zeta_4'(\psi)+\\ &+2(1-\mu)\sin\alpha\cos\alpha\frac{\sin\beta}{\sin\psi}\left[\zeta_4(\psi)+\frac{1}{\pi}\right]\Big\}\cos\varphi\,d\varphi+r_{2(1)}(\theta).\end{aligned}\right| \tag{85.10}$$

where $r_{1(1)}(\theta)$, $r_{2(1)}(\theta)$ differ from $r_{1(0)}(\theta)$, $r_{2(0)}(\theta)$ only by the presence in the subintegral expression of the factor $\cos\varphi$. Calculations give

$$\int_0^\pi \frac{\cos\beta\cos\varphi}{\sin\psi}\,d\varphi = -\frac{\pi}{2}\frac{(1-\cos\theta)(1-\cos\theta_1)}{\sin\theta\sin^2\theta_1} + \begin{cases} -\dfrac{\pi}{2}\dfrac{(1+\cos\theta)(1-\cos\theta_1)}{\sin\theta\sin^2\theta_1}, & \theta>\theta_1, \\ \dfrac{\pi}{2}\dfrac{(1+\cos\theta_1)(1-\cos\theta)}{\sin\theta\sin^2\theta_1}, & \theta<\theta_1. \end{cases}$$

$$\int_0^\pi \frac{\sin\alpha\cos\alpha\sin\beta}{\sin\psi}\cos\varphi\,d\varphi = \begin{cases} \dfrac{\pi}{2}\dfrac{\sin\theta_1}{\sin^2\theta}, & \theta>\theta_1, \\ 0, & \theta<\theta_1. \end{cases}$$

$$\left.\begin{aligned} r_{1(1)}(\theta) &= \begin{cases} \dfrac{m_1\sin\theta_1}{2\sin\theta}\left[(1+\mu)\dfrac{1-\cos\theta_1}{\sin^2\theta_1}+(1-\mu)\dfrac{\sin\theta_1}{\sin\theta}\right], & \theta>\theta_1, \\ -\dfrac{m_1\sin\theta_1}{2\sin\theta}(1+\mu)\dfrac{(1-\cos\theta)\cos\theta_1}{\sin^2\theta_1}, & \theta<\theta_1, \end{cases} \\ r_{2(1)}(\theta) &= \begin{cases} \dfrac{m_1\sin\theta_1}{2\sin\theta}\left[(1+\mu)\dfrac{1-\cos\theta_1}{\sin^2\theta_1}-(1-\mu)\dfrac{\sin\theta_1}{\sin\theta}\right], & \theta>\theta_1, \\ -\dfrac{m_1\sin\theta_1}{2\sin\theta}(1+\mu)\dfrac{(1-\cos\theta)\cos\theta_1}{\sin^2\theta_1}, & \theta<\theta_1. \end{cases} \end{aligned}\right\} \quad (85.11)$$

Going through section θ_1 the moments undergo jumps of continuity equal to

$$[M_{1(1)}(\theta_1)] = r_{1(1)}(\theta_1^+) - r_{1(1)}(\theta_1^-) = m_1,$$
$$[M_{1(2)}(\theta_1)] = r_{2(1)}(\theta_1^+) - r_{2(1)}(\theta_1^-) = \mu m_1$$

In section $\theta=0$

$$M_{1(1)}(0) = M_{2(1)}(0) = 0,$$

however, subtending the circumference of the load θ_1 to pole $\theta=0$ and carrying out in the second line of (85.11) passage to the limit in such a way that $(\pi R\sin\theta_1 m_1)_{\theta_1\to 0} = M$, we find that in the case when a concentrated moment M is applied in the pole, the following equalities hold:

$$\underset{\theta_1\to 0}{M_{1(1)}}(0) = \underset{\theta_1\to 0}{M_{2(1)}}(0) = \left[-\frac{M(1+\mu)}{4\pi R\theta_1}\right]_{\theta_1\to 0}.$$

CHAPTER VI

A TORUS-SHAPED SHELL

§ 86. Solving the Equation of the Problem About the Equilibrium of a Circular Torus-Shaped Shell During Axisymmetric and Bending Loads

Torus-shaped shell (Fig. 39) differs from the previous cylindrical, conic and spherical shells by the fact that its geometry is characterized by two essential parameters: ratio of the radius of the generatrix of the circumference to the thickness of the shell $\frac{a}{h}$ and by the ratio of the radii $\frac{d}{a}$. In accordance with this the basic resolvant solving equation for a torus-shaped shell during axisymmetric and bending loads will also contain two parameters:

$$4\gamma^4 = 12(1-\mu^2)\frac{a^2}{h^2}, \tag{86.1}$$

$$\lambda = \frac{d}{a}. \tag{86.2}$$

The first of them characterizes the relative shell thickness, and the second essentially characterizes the middle surface: at $\lambda=0$ the surface is spherical, at $\lambda\to\infty$ it becomes a circular cylindrical surface.

The principal radii of curvature and the radius of the parallel circle of the middle surface of a torus-shaped shell are equal to

$$\left.\begin{aligned} R_1 = a, \quad R_2 = a\frac{\lambda+\sin\theta}{\sin\theta}, \\ \nu = a(\lambda+\sin\theta). \end{aligned}\right\} \tag{86.3}$$

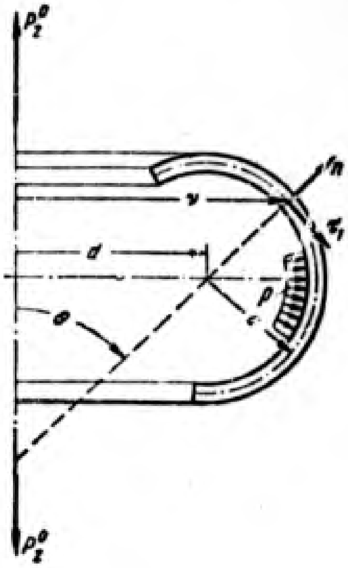


Fig. 39. Torus-shaped shell.

The basic resolvant equation of axisymmetric stress (12.13) for a torus-shaped shell takes the form

$$\frac{d^2\sigma}{d\theta^2}+\frac{\cos\theta}{\lambda+\sin\theta}\frac{d\sigma}{d\theta}+2i\gamma^2\sigma\left[\frac{\sin\theta}{\lambda+\sin\theta}+i\frac{\cos^2\theta}{2\gamma^2(\lambda+\sin\theta)^2}\right]=$$
$$=-4\gamma^4\left[\frac{\Phi_2}{a(\lambda+\sin\theta)}+\frac{i}{2\gamma^2}\frac{1}{a(\lambda+\sin\theta)}\left(\mu\frac{d\Phi_1}{d\theta}+\frac{\cos\theta}{\lambda+\sin\theta}\Phi_1\right)\right]. \qquad (86.4)$$

$\Phi_1(\theta)$, $\Phi_2(\theta)$ — functions of external load, determined by equations (11.11) The latter, taking into account (86.3), can be rewritten in the following manner:

$$\left.\begin{aligned}\Phi_1(\theta)&=-\cos\theta a^2\int_{\theta_0}^{\theta}q_r(\lambda+\sin\theta)\,d\theta+\\&\qquad+\sin\theta\left[\frac{P_z^0}{2\pi}+a^2\int_{\theta_0}^{\theta}q_z(\lambda+\sin\theta)\,d\theta\right],\\\Phi_2(\theta)&=-\sin\theta a^2\int_{\theta_0}^{\theta}q_r(\lambda+\sin\theta)\,d\theta-\\&\qquad-\cos\theta\left[\frac{P_z^0}{2\pi}+a^2\int_{\theta_0}^{\theta}q_z(\lambda+\sin\theta)\,d\theta\right].\end{aligned}\right\} \qquad (86.5)$$

Forces and bending moments in the shell are determined through function σ by formulas (12.11), where in them it is necessary to set

$b=a$, $\alpha=1$ (a shell of constant thickness is considered) and to take into account formulas (86.3).

The character of the solution of the basic resolvant equation depends on the relationship between parameters (86.1), (86.2) and on the amount of parameter λ. When the shell is thin-walled and the ratio

$$\frac{2\gamma^2}{\lambda} \gg 1.$$

the solution of the uniform resolvant equation can be constructed by the asymptotic method, where it will be useful both in the vicinity of $\theta=0(\pi)$, and also far from it (§ 89). At $\lambda \gg 1$ the solution and the working formulas are considerably simplified, since terms of order $\frac{1}{\lambda}$ in comparison with unity can be neglected. For sufficiently smooth loads the solution of the resolvant equation with the right side non-zero, at $\frac{2\gamma^2}{\lambda} \gg 1$ also can be constructed by the asymptotic method. This solution has a comparatively simple form and on its basis can be obtained convenient working formulas for determination of displacements and stresses in different constructions containing torus-shaped shell: in the end walls (§§ 91, 92), in tubular and lens compensators (§§ 90, 93). When the right side of the resolvant equation is a periodic function of coordinate θ, the particular solution corresponding to this right side can be constructed in the form of a trigonometric series. The coefficients of this series are determined from an unbounded system of algebraic equations. For a simple right side (for example, of the form of $\cos\theta$) the solution of this system is expressed through unbounded chain fractions, which are calculated more easily the greater the relative shell thickness. On the basis of this solution simple working formulas also are obtained for a tubular compensator compressible by an axial force (§ 87) and bending moment (§ 88). In § 94 a thin torus-shaped shell $\left(\frac{2\gamma^2}{\lambda} \gg 1\right)$, intersecting the axis of revolution $(\lambda < 1)$ is considered.

Returning to equation (86.4), we introduced instead of σ new function

$$\sigma_0 = -\bar{\sigma}, \qquad (86.6)$$

where $\bar{\sigma}=\Psi_0+2i\gamma^2V_0$. Having replaced in (86.4), i by $-i$ and σ by $\bar{\sigma}$, taking into account (86.6), we obtain

$$\frac{d^2\sigma_0}{d\theta^2}+\frac{\cos\theta}{\lambda+\sin\theta}\frac{d\sigma_0}{d\theta}-2\gamma^2 i\sigma_0\frac{\sin\theta}{\lambda+\sin\theta}-\sigma_0\frac{\cos^2\theta}{(\lambda+\sin\theta)^2}=$$
$$=4\gamma^4\frac{\Phi_2}{a(\lambda+\sin\theta)}-2\gamma^2 i\frac{1}{a(\lambda+\sin\theta)}\left(\mu\frac{d\Phi_1}{d\theta}+\frac{\cos\theta}{\lambda+\sin\theta}\Phi_1\right). \qquad (86.7)$$

Let us examine a load of the following particular form:

$$\left.\begin{array}{l} q_n=p \quad (q_\theta=p\sin\theta,\ q_z=p\cos\theta), \\ q_1=0, \quad P_z^0\neq 0. \end{array}\right\} \qquad (86.8)$$

in order to simplify the left and right sides of (86.7), we will again replace the basic variable:

$$\sigma_0=\frac{\sigma_1}{(\lambda+\sin\theta)}+2i\gamma^2\frac{\Phi_2}{a\sin\theta}+2i\gamma^2Ca\frac{\cos\theta}{(\lambda+\sin\theta)\sin\theta}. \qquad (86.9)$$

$$C=\frac{\lambda}{2}\left[\frac{P_z^0}{\pi a^3}+p\left(\lambda^2-\frac{v_0^2}{a^2}\right)\right]. \qquad (86.10)$$

here v_0 — radius of the parallel circle of edge section θ_0.

To determine σ_1 we obtain the equation

$$(\lambda+\sin\theta)\frac{d^2\sigma_1}{d\theta^2}-\cos\theta\frac{d\sigma_1}{d\theta}-2i\gamma^2\sigma_1\sin\theta+\sigma_1\sin\theta=$$
$$=-4\gamma^4Ca\cos\theta-2i\gamma^2a\left[\frac{p\lambda}{2}+C(1+\mu)\right]\cos\theta$$

or, dropping $\frac{1}{2\gamma^2}$ in comparison with unity

$$(\lambda+\sin\theta)\frac{d^2\sigma_1}{d\theta^2}-\cos\theta\frac{d\sigma_1}{d\theta}-2i\gamma^2\sigma_1\sin\theta=-4\gamma^4Ca\cos\theta. \qquad (86.11)$$

We hold (86.11) to be the basic resolvant equation of the axisymmetric problem. Forces, moments and displacements are expressed through σ_1 in the following manner:

$$\left.\begin{aligned}
T_1 &= -\frac{1}{2\gamma^2}\frac{\operatorname{Im}\sigma_1\cos\theta}{(\lambda+\sin\theta)^2} + Ca\frac{\lambda\sin\theta+1}{\lambda(\lambda+\sin\theta)^2} + \frac{pa}{2}\frac{2\lambda+\sin\theta}{\lambda+\sin\theta},\\
N_1 &= -\frac{1}{2\gamma^2}\frac{\operatorname{Im}\sigma_1\sin\theta}{(\lambda+\sin\theta)^2} - Ca\frac{\cos\theta}{(\lambda+\sin\theta)^2},\\
H_e &= -\frac{1}{2\gamma^2}\frac{\operatorname{Im}\sigma_1}{(\lambda+\sin\theta)^2} + Ca\frac{\cos\theta}{\lambda(\lambda+\sin\theta)^2} + \\
&\qquad + \frac{pa}{2}\frac{(2\lambda+\sin\theta)\cos\theta}{\lambda+\sin\theta},\\
T_2 &= -\frac{1}{2\gamma^2}\left[\frac{1}{\lambda+\sin\theta}\operatorname{Im}\frac{d\sigma_1}{d\theta} - \frac{\cos\theta\operatorname{Im}\sigma_1}{(\lambda+\sin\theta)^2}\right] + \\
&\qquad + \frac{pa}{2} - Ca\frac{\lambda\sin\theta+1}{\lambda(\lambda+\sin\theta)^2},
\end{aligned}\right\} \qquad (86.12)$$

$$\left.\begin{aligned}
M_1 &= \frac{a}{4\gamma^4}\left[\frac{1}{\lambda+\sin\theta}\operatorname{Re}\frac{d\sigma_1}{d\theta} - \frac{(1-\mu)\cos\theta}{(\lambda+\sin\theta)^2}\operatorname{Re}\sigma_1\right],\\
M_2 &= \frac{a}{4\gamma^4}\left[\frac{(1-\mu)\cos\theta}{(\lambda+\sin\theta)^2}\operatorname{Re}\sigma_1 + \frac{\mu}{\lambda+\sin\theta}\operatorname{Re}\frac{d\sigma_1}{d\theta}\right],
\end{aligned}\right\} \qquad (86.13)$$

$$\left.\begin{aligned}
&Eh\vartheta_1 = -\frac{\operatorname{Re}\sigma_1}{\lambda+\sin\theta}, \quad \Delta_e = \frac{\nu}{Eh}(T_2-\mu T_1),\\
&\Delta_z = \int_{\theta_0}^{\theta}(-ae_1\sin\theta + a\vartheta_1\cos\theta)\,d\theta + K.
\end{aligned}\right\} \qquad (86.14)$$

Particular solution (86.11) for large values of parameter $2\gamma^2$ in comparison with unity, if we exclude from analysis the area of variable θ, where $\sin\theta\approx 0$, can be obtained by the usual method of dividing the right side by the coefficient of σ_1:

$$\tilde{\sigma}_1 = -i2\gamma^2 Ca\operatorname{ctg}\theta. \qquad (86.15)$$

To it corresponds the zero-moment stressed state

$$\left.\begin{aligned}
&\tilde{T}_1 = \frac{Ca}{\lambda(\lambda+\sin\theta)\sin\theta} + \frac{pa}{2}\frac{2\lambda+\sin\theta}{\lambda+\sin\theta},\\
&\tilde{T}_2 = -\frac{Ca}{\lambda\sin^2\theta} + \frac{pa}{2}, \quad \tilde{N}_1 = \tilde{M}_1 = \tilde{M}_2 = 0.
\end{aligned}\right\} \qquad (86.16)$$

It is easy to see that at $\theta=0, \pi$ forces increase without limit and the zero-moment solution is unsuitable. For a closed shell under only the action of uniform pressure, $C=0$ and forces remain bounded everywhere:

$$\tilde{T}_1 = \frac{pa}{2}\frac{2\lambda+\sin\theta}{\lambda+\sin\theta}, \quad \tilde{T}_2 = \frac{pa}{2}. \qquad (86.17)$$

When shell rotates with angular velocity ω around axis OZ, the components of the distributed load are equal to

$$q_\theta = \rho h\omega^2 v, \quad q_z = 0. \tag{86.18}$$

ρ — mass density.

In accordance with (86.5) we have

$$\Phi_1(\theta) = -\cos\theta\rho h\omega^2 a^3 \int_{\theta_0}^{\theta} (\lambda + \sin\theta)^2 d\theta,$$
$$\Phi_2(\theta) = -\sin\theta\rho h\omega^2 a^3 \int_{\theta_0}^{\theta} (\lambda + \sin\theta)^2 d\theta.$$

Making the replacement

$$\sigma_0 = \frac{\sigma_1}{\lambda + \sin\theta} + 2i\gamma^2 \frac{\Phi_2}{a\sin\theta},$$

instead of equation (86.7) we will obtain an equation which is distinguished from (86.11) by the right side, which now will have the form

$$F(\theta) = 2i\gamma^2(3+\mu)\rho h\omega^2 a^2(\lambda + \sin\theta)^3\cos\theta -$$
$$- 2i\gamma^2\mu\rho h\omega^2 a^2(\lambda + \sin\theta)\sin\theta \int_{\theta_0}^{\theta} (\lambda + \sin\theta)^2 d\theta.$$

The second term in this expression can be discarded inasmuch as the particular solution corresponding to this term, being obtained by dividing it by the coefficient of σ_1, at all θ is a small quantity in comparison with the quantity $2i\gamma^2\frac{\Phi_2}{a\sin\theta}$. The problem is reduced to the solution of equation (86.11) with the right side $F(\theta) = 2i\gamma^2(3+\mu)\rho h\omega^2 a^2(\lambda + \sin\theta)^3\cos\theta$. Forces and moments are determined according to formulas (86.12), (86.13). In this case it is necessary to set in them $C = p = 0$ and to add to the calculated values of force of the zero-moment state

$$\tilde{T}_1 = 0, \quad \tilde{N}_1 = 0, \quad \tilde{T}_2 = \rho h\omega^2 v^2.$$

Going to consideration of the deformation of a torus under a bending load, we write equation (16.22), transforming it allowing for (86.1), (86.3). We obtain

$$\frac{d^2\sigma}{d\theta^2}+\frac{\cos\theta}{\lambda+\sin\theta}\frac{d\sigma}{d\theta}+\sigma\left[2i\gamma^2\frac{\sin\theta}{\lambda+\sin\theta}-\frac{4}{(\lambda+\sin\theta)^2}\right]= \\ =4\gamma^4\left[\Phi_3(\theta)+\frac{ia}{2\gamma^2}\Phi_4(\theta)\right]. \qquad (86.19)$$

where, in accordance with (16.10), (16.11), $\Phi_3(\theta)$, $\Phi_4(\theta)$ are known functions of the load. In the absence of distributed load they have the form

$$\left.\begin{aligned}\Phi_3(\theta)&=-\frac{P_1}{\pi\nu}\sin\theta+\frac{\cos\theta}{\nu^2}\left[M_1\frac{1}{\pi}-\frac{P_1a}{\pi}(\cos\theta-\cos\theta_0)\right],\\ \Phi_4(\theta)&=\frac{\mu}{a}\frac{df_1}{d\theta}+\frac{\cos\theta}{a(\lambda+\sin\theta)}f_1,\end{aligned}\right\} \qquad (86.20)$$

where

$$f_1(\theta)=-\frac{P_1}{\pi a}\frac{\cos\theta}{\lambda+\sin\theta}- \\ -\frac{\sin\theta}{(\lambda+\sin\theta)^2}\left[\frac{M_1}{\pi a^2}-\frac{P_1}{\pi a}(\cos\theta-\cos\theta_0)\right]. \qquad (86.21)$$

Dropping in the right side of (86.19) small terms and introducing new unknown function

$$\sigma_2=\bar{\sigma}, \qquad (86.22)$$

instead of (86.19) we obtain

$$\frac{d^2\sigma_2}{d\theta^2}+\frac{\cos\theta}{\lambda+\sin\theta}\frac{d\sigma_2}{d\theta}+ \\ +\sigma_2\left[-2i\gamma^2\frac{\sin\theta}{\lambda+\sin\theta}-\frac{4\cos^2\theta}{(\lambda+\sin\theta)^2}\right]=4\gamma^4\Phi_3(\theta). \qquad (86.23)$$

Let us make one replacement of the variable to simplify the coefficient of σ_2. Assuming

$$\sigma_2 = \frac{\sigma_3}{(\lambda + \sin\theta)^2}. \qquad (86.24)$$

we have

$$(\lambda + \sin\theta)\frac{d^2\sigma_3}{d\theta^2} - 3\cos\theta\frac{d\sigma_3}{d\theta} - \\ - 2i\gamma^2\sigma_3\sin\theta = 4\gamma^4\Phi_3(\theta)(\lambda + \sin\theta)^3. \qquad (86.25)$$

Equation (86.25) we select as the basic resolvant equation of the problem. The expressions for forces and moments through function σ_3 are obtained on the basis of formulas (16.20), assuming in them $b=a$, $\alpha=1$ and taking into account transformations (86.22), (86.24):

$$\left.\begin{aligned} t_1 &= -\frac{1}{2\gamma^2}\frac{\cos\theta \operatorname{Im}\sigma_3}{(\lambda+\sin\theta)^3} - \frac{1}{4\gamma^4}\frac{\sin\theta}{\lambda+\sin\theta}\operatorname{Re}\frac{d\sigma_3}{d\theta} + \\ &\quad + f_0(\theta) + f_1(\theta), \\ t_2 &= -\frac{1}{2\gamma^2}\left[\frac{1}{(\lambda+\sin\theta)^2}\operatorname{Im}\frac{d\sigma_3}{d\theta} - \frac{\cos\theta \operatorname{Im}\sigma_3}{(\lambda+\sin\theta)^3}\right] - \\ &\quad - \frac{\cos\theta}{a(\lambda+\sin\theta)} f_2(\theta), \\ s_{(1)} &= -\frac{1}{2\gamma^2}\frac{\operatorname{Im}\sigma_3}{(\lambda+\sin\theta)^3} - \frac{1}{a(\lambda+\sin\theta)} f_2(\theta), \end{aligned}\right\} \qquad (86.26)$$

$$\left.\begin{aligned} m_1 &= \frac{a}{4\gamma^4}\frac{1}{(\lambda+\sin\theta)^2}\left[\operatorname{Re}\frac{d\sigma_3}{d\theta} - \frac{(1-\mu)}{(\lambda+\sin\theta)}\cos\theta \operatorname{Re}\sigma_3\right], \\ m_2 &= \frac{a}{4\gamma^4}\frac{1}{(\lambda+\sin\theta)^2}\left[\mu \operatorname{Re}\frac{d\sigma_3}{d\theta} + \frac{(1-\mu)}{(\lambda+\sin\theta)}\cos\theta \operatorname{Re}\sigma_3\right], \\ h_{(1)} &= -\frac{a}{4\gamma^4}\frac{(1-\mu)\operatorname{Re}\sigma_3}{(\lambda+\sin\theta)^3}. \end{aligned}\right\} \qquad (86.27)$$

Here

$$\left.\begin{aligned} f_0(\theta) &= -\frac{a\cos\theta}{\lambda+\sin\theta}\int_{\theta_0}^{\theta}(q_{1(1)}\cos\theta + q_{n(1)}\sin\theta)(\lambda+\sin\theta)\,d\theta + \\ &\quad + \frac{a\sin\theta}{(\lambda+\sin\theta)^2}\int_{\theta_0}^{\theta}(q_{n(1)}\cos\theta - q_{1(1)}\sin\theta)(\lambda+\sin\theta)^2\,d\theta - \\ &\quad - \frac{a\sin\theta}{(\lambda+\sin\theta)^2}\int_{\theta_0}^{\theta}\sin\theta\left[\int_{\theta_0}^{\theta}(q_{1(1)}\cos\theta - q_{2(1)} + \right. \\ &\quad \left. + q_{n(1)}\sin\theta)(\lambda+\sin\theta)\,d\theta\right]d\theta, \\ f_2(\theta) &= a^2\int_{\theta_0}^{\theta} q_{2(1)}(\lambda+\sin\theta)\,d\theta. \end{aligned}\right\} \qquad (86.28)$$

In the absence of distributed loads $f_0 = f_2 = 0$ and the amplitude of radial force h_e is connected with $s_{(1)}$ by the relationship resulting from (15.21):

$$h_e = -\frac{P_1}{\pi a} \frac{1}{\lambda + \sin\theta} + s_{(1)}. \quad (86.29)$$

By division of the right side of (86.25) by the coefficient of σ_3 we obtain the zero-moment solution

$$\tilde{\sigma}_3 = l 2\gamma^2 \frac{\Phi_3(\theta)}{\sin\theta} (\lambda + \sin\theta)^3. \quad (86.30)$$

In the absence of distributed loads it has the form

$$\sigma_3 = l 2\gamma^2 \Big[\frac{M_1}{\pi a^2} \operatorname{ctg}\theta (\lambda + \sin\theta) - \frac{P_1}{\pi a} (\lambda + \sin\theta)^2 - \\ - \frac{P_1}{\pi a} \operatorname{ctg}\theta (\lambda + \sin\theta)(\cos\theta - \cos\theta_0)\Big]. \quad (86.31)$$

To it correspond forces of the zero-moment state

$$\left.\begin{aligned} \tilde{t}_1 &= -\frac{M_1}{\pi a^2} \frac{1}{\sin\theta (\lambda + \sin\theta)^2} + \frac{P_1}{\pi a} \frac{\cos\theta - \cos\theta_0}{\sin\theta (\lambda + \sin\theta)^2}, \\ \tilde{t}_2 &= \frac{M_1 - P_1 a (\cos\theta - \cos\theta_0)}{\pi a^2 \sin^2\theta (\lambda + \sin\theta)}, \\ \tilde{s}_{(1)} &= \frac{P_1}{\pi a (\lambda + \sin\theta)} - \frac{M_1 - P_1 a (\cos\theta - \cos\theta_0)}{\pi a^2 (\lambda + \sin\theta)^2} \operatorname{ctg}\theta. \end{aligned}\right\} \quad (86.32)$$

$$\tilde{h}_e = -\frac{M_1 - P_1 a (\cos\theta - \cos\theta_0)}{\pi a^2 (\lambda + \sin\theta)^2} \operatorname{ctg}\theta. \quad (86.33)$$

§ 87. Periodic Particular Solution of Equation (86.11). Axial Extension of a Tubular Compensator

As it was shown in the previous section, the zero-moment solution (86.15) cannot serve as the particular solution of resolvant equation (86.11), since it turns into infinity in points $\theta = 0, \pi$. It is possible, however, to build the particular solution of this equation which possesses the feature of periodicity and is finite everywhere. We look for it in the form of a series [114]

$$\sigma_1 = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \ldots$$
$$\ldots + b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + \ldots \qquad (87.1)$$

Substituting (87.1) into equation (86.11) and combining terms containing trigonometric functions of angles of identical multiplicity, we obtain relationships for determination of the coefficients

$$\left.\begin{aligned} &-n^2 a_n \lambda + \frac{1}{2}[-2i\gamma^2 - (n+1)(n+2)]\, b_{n+1} - \\ &\qquad -\frac{1}{2}[-2i\gamma^2 - (n-1)(n-2)]\, b_{n-1} = \beta_n, \\ &-n^2 \lambda b_n + \frac{1}{2}[-2i\gamma^2 - (n-1)(n-2)]\, a_{n-1} - \\ &\qquad -\frac{1}{2}[-2i\gamma^2 - (n+1)(n+2)]\, a_{n+1} = 0, \end{aligned}\right\} \qquad (87.2)$$

where

$$\beta_n = -4\gamma^4 C a, \quad n = 1,$$
$$\beta_n = 0, \qquad n > 1, \quad n = 0.$$

The first four equations of this unbounded system have the form

$$\left.\begin{aligned} &-\lambda a_1 - \frac{1}{2}(2i\gamma^2 + 2\cdot 3)\, b_2 = -4\gamma^4 C a, \\ &-\lambda b_1 - \frac{1}{4}(2i\gamma^2 - 0\cdot 1)\, a_3 + \frac{1}{2}(2i\gamma^2 + 2\cdot 3)\, a_2 = 0, \\ &-2^2\lambda a_2 - \frac{1}{2}(2i\gamma^2 + 3\cdot 4)\, b_3 + \frac{1}{2}(2i\gamma^2 + 1\cdot 0)\, b_1 = 0, \\ &-2^2\lambda b_2 - \frac{1}{2}(2i\gamma^2 + 1\cdot 0)\, a_1 + \frac{1}{2}(2i\gamma^2 + 3\cdot 4)\, a_3 = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \qquad (87.3)$$

It is simple to see that the derived system of equations breaks into two, where the first contains coefficients of a — an odd number and b — an even, and the second is the reverse: a — with even numbers and b — with odd. The second system has, moreover, null right sides. This means that we can set

$$a_{2k} = b_{2k+1} = 0, \quad k = 0, 1, 2, \ldots$$

The solution of the first system can be obtained in the form of continued fractions

$$\left.\begin{aligned}
&\frac{4\gamma^2 Ca}{a_1}=\frac{\lambda}{\gamma^2}+\cfrac{1-i\frac{2\cdot 3}{2\gamma^2}}{\frac{2^2\lambda}{\gamma^2}+\cfrac{\left(1-i\frac{3\cdot 4}{2\gamma^2}\right)\left(1-i\frac{1\cdot 2}{2\gamma^2}\right)}{\frac{3^2\lambda}{\gamma^2}+\cfrac{\left(1-i\frac{4\cdot 5}{2\gamma^2}\right)\left(1-i\frac{2\cdot 3}{2\gamma^2}\right)}{\frac{4^2\lambda}{\gamma^2}+\dots}}},\\
&-i\frac{a_n}{b_{n-1}}=\cfrac{1-i\frac{(n-1)(n-2)}{2\gamma^2}}{\frac{n^2\lambda}{\gamma^2}+\cfrac{\left[1-i\frac{n(n-1)}{2\gamma^2}\right]\left[1-i\frac{(n+1)(n+2)}{2\gamma^2}\right]}{\frac{(n+1)^2\lambda}{\gamma^2}+\cfrac{\left[1-i\frac{(n+1)n}{2\gamma^2}\right]\left[1-i\frac{(n+2)(n+3)}{2\gamma^2}\right]}{\frac{(n+2)^2\lambda}{\gamma^2}+\dots}}},\\
&i\frac{b_{n+1}}{a_n}=\\
&=\cfrac{1-i\frac{n(n-1)}{2\gamma^2}}{\frac{(n+1)^2\lambda}{\gamma^2}+\cfrac{\left[1-i\frac{(n+1)n}{2\gamma^2}\right]\left[1-i\frac{(n+2)(n+3)}{2\gamma^2}\right]}{\frac{(n+2)^2\lambda}{\gamma^2}+\cfrac{\left[1-i\frac{(n+2)(n+1)}{2\gamma^2}\right]\left[1-i\frac{(n+3)(n+4)}{2\gamma^2}\right]}{\frac{(n+3)^2\lambda}{\gamma^2}+\dots}}},\\
&n=1,\ 3,\ 5,\ \dots
\end{aligned}\right\}\qquad (87.4)$$

In this way, the particular solution of equation (86.11) has the form

$$\sigma_1=a_1\cos\theta+a_3\cos 3\theta+\dots+b_2\sin 2\theta+b_4\sin 4\theta+\dots \qquad (87.5)$$

Arrangement of continued fractions (87.4) is such that they converge even better than greater the ratio $\frac{\lambda}{\gamma^2}$. For a very thin shell $(2\gamma^2\gg 1)$, but such that $\frac{\lambda}{\gamma^2}\gg 1$, the particular solution of equation (86.11) can be approximately presented in the form

$$\sigma_1\approx\frac{4\gamma^4 Ca}{\lambda}\cos\theta. \qquad (87.6)$$

Really, at $2\gamma^2\to\infty$ and simultaneously $\frac{\lambda}{\gamma^2}\to\infty$ from (87.4) we obtain

$$a_1\to\frac{4\gamma^4 Ca}{\lambda},\quad a_n\to 0,\quad b_{n-1}\to 0.$$

Let us note that (87.6) is the particular solution of the equation

$$\lambda \frac{d^2\sigma_1}{d\theta^2} = -4\gamma^4 C\alpha \cos\theta. \tag{87.7}$$

which is obtained from (86.11), if we leave in its left side only the term containing λ. Solution (87.7), just as (87.5), satisfies boundary conditions of the following form:

$$\sigma_1\left(\pm\frac{\pi}{2}\right) = 0. \tag{87.8}$$

These conditions correspond to the fact that on the edges $\theta = \pm\frac{\pi}{2}$ both angle of rotation ϑ_1, and shearing force turn into zero. Let us designate relative axial displacement of shell edges $\theta = \pm\frac{\pi}{2}$ under edge conditions (87.8) using the approximate formula

$$\Delta_z = \int\limits_{+\frac{\pi}{2}}^{-\frac{\pi}{2}} a\vartheta_1 \cos\theta\, d\theta \approx -\int\limits_{+\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{a}{Eh\lambda} \operatorname{Re}\sigma_1 \cos\theta\, d\theta. \tag{87.9}$$

On the basis of (87.6) and (86.10) we obtain

$$\Delta_z = \frac{12(1-\mu^2)a^3}{Eh^3}\left[\frac{\pi}{2}\frac{P_z^0}{2\pi a\lambda} + pa\frac{\pi}{2}\right]. \tag{87.10}$$

At $p=0$ this expression by only the factor $(1-\mu^2)$ differs from the amount of displacement of a curved beam of unit width cut out from the shell [141]. Toward the end of the beam is applied vertical force $P_z^0/2\pi a\lambda$.

The periodic particular solution (87.4), (87.5) can be used to determine the displacement of the edge sections and the stressed state of a tubular compensator [139]. The compensator is cut out in section $\theta = -\frac{\pi}{2}$ of a torus-shaped shell, the edges of which $\left(\theta = -\frac{\pi}{2},\ \theta = \frac{3\pi}{2}\right)$ are joined to the tube. We can approximately set that the tube possesses infinite rigidity relative to the angle of rotation and zero rigidity in the radial direction. This means that in sections $\theta=-\frac{\pi}{2}$, $\theta=\frac{3\pi}{2}$ the angle of rotation and shearing force are equal to zero. Because of the symmetry of construction the same conditions exist in section $\theta=\frac{\pi}{2}$. In this way, the compensator can be approximately calculated as a torus-shaped shell under edge conditions (87.8). Let us examine the elongation of a compensator by axial forces P_z^0 (Fig. 40).

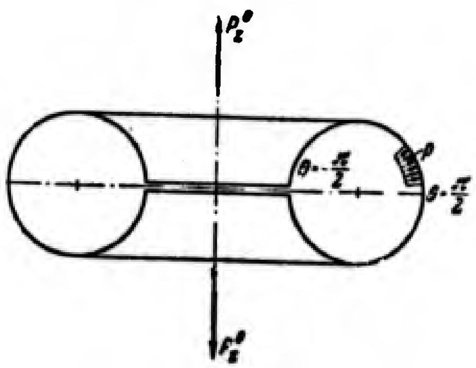


Fig. 40. Tubular compensator stretched by axial forces.

Determining axial displacement in general, when parameters λ, $2\gamma^2$ can be any amount, using equations (87.9) and (87.5) we obtain

$$\Delta_z=\frac{a}{Eh\lambda}\frac{\pi}{2}\operatorname{Re}a_1. \qquad (87.11)$$

At $p=0$ using formula (86.10), we have

$$C=\frac{\lambda}{2}\frac{P_z^0}{\pi a^2}. \qquad (87.12)$$

Substituting into (87.11) the quantity a_1, in accordance with (87.4), (87.12) we find axial displacement of the edge of the compensator relative to plane of symmetry $\theta=\frac{\pi}{2}$

$$\Delta_z=\frac{P_z^0\gamma^2}{Eh}\operatorname{Re}\cfrac{1}{\cfrac{\lambda}{\gamma^2}+\cfrac{\left(1-i\frac{2\cdot 3}{2\gamma^2}\right)}{\cfrac{2^2\lambda}{\gamma^2}+\cfrac{\left(1-i\frac{3\cdot 4}{2\gamma^2}\right)\left(1-i\frac{1\cdot 2}{2\gamma^2}\right)}{\cfrac{3^2\lambda}{\gamma^2}+\cdots}}}. \qquad (87.13)$$

Substituting solution (87.5) into the first formula of (86.13), we determine the bending moment

$$M_1(\theta)=\frac{a}{4\gamma^4}\frac{1}{\lambda+\sin\theta}\times$$
$$\times\operatorname{Re}\Big[-a_1\sin\theta-3a_3\sin 3\theta+\ldots+2b_2\cos 2\theta+4b_4\cos 4\theta+\ldots$$
$$\ldots-\frac{(1-\mu)\cos\theta}{\lambda+\sin\theta}(a_1\cos\theta+a_3\cos 3\theta+\ldots+b_2\sin 2\theta+b_4\sin 4\theta+\ldots)\Big]. \qquad (87.14)$$

At $\lambda\gg 1$ expression (87.14) can be simplified and brought to the form

$$M_1(\theta)=$$
$$=\frac{a}{4\gamma^4\lambda}\operatorname{Re}(-a_1\sin\theta-3a_3\sin 3\theta-\ldots+2b_2\cos 2\theta+4b_4\cos 4\theta+\ldots). \qquad (87.15)$$

At $\theta=\frac{\pi}{4}$ on the basis of (87.15) we find that the bending moment in this section is equal to

$$M_1\left(\frac{\pi}{4}\right)=\frac{a}{4\gamma^4\lambda}\frac{1}{\sqrt{2}}\operatorname{Re}(-a_1-3a_3-\ldots). \qquad (87.16)$$

If the relationships of dimensions ar such that during the calculation of the first continued fractions in (87.15) we can be limited to two steps, making subsequent calculations also with the corresponding correctness, i.e., assuming

$$\left.\begin{aligned}
&\frac{4\gamma^2 Ca}{a_1}=\frac{\lambda}{\gamma^2}+\cfrac{1-i\frac{2\cdot 3}{2\gamma^2}}{\frac{2^2\lambda}{\gamma^2}+\cfrac{\left(1-i\frac{3\cdot 4}{2\gamma^2}\right)\left(1-i\frac{1\cdot 2}{2\gamma^2}\right)}{\frac{3^2\lambda}{\gamma^2}}},\\
&i\frac{b_2}{a_1}=\cfrac{1}{\frac{2^2\lambda}{\gamma^2}+\cfrac{\left(1-i\frac{2\cdot 1}{2\gamma^2}\right)\left(1-i\frac{3\cdot 4}{2\gamma^2}\right)}{\frac{3^2\lambda}{\gamma^2}}}, \quad -i\frac{a_3}{b_2}=\cfrac{1-i\frac{2\cdot 1}{2\gamma^2}}{\frac{3^2\lambda}{\gamma^2}},\\
&a_5=b_4=\ldots=0,
\end{aligned}\right\} \qquad (87.17)$$

then it is easy to see that in this instance (at $\lambda\gg 1$

$$a_3\approx a_1\frac{\gamma^4}{3^2\cdot 2^2\cdot\lambda^2}.$$

for calculation of the bending moment in section $\theta=\frac{\pi}{4}$ we derive the approximate formula

$$M_1\left(\frac{\pi}{4}\right)=-\frac{a}{4\gamma^4\lambda}\frac{1}{\sqrt{2}}\left(1+\frac{\gamma^4}{2^2\cdot 3\lambda^2}\right)\operatorname{Re} a_1.$$

Meridian stresses from this moment at $\rho=0$ are equal to

$$\sigma_1\left(\frac{\pi}{4}\right)=\mp\frac{3P_z^0}{\pi h^2\gamma^2}\left(1+\frac{\gamma^4}{2^2\cdot 3\lambda^2}\right)\frac{1}{\sqrt{2}}\operatorname{Re}\cfrac{1}{\cfrac{\lambda}{\gamma^2}+\cfrac{1-i\frac{2\cdot 3}{2\gamma^2}}{\cfrac{2^2\lambda}{\gamma^2}+\cfrac{\left(1-i\frac{3\cdot 4}{2\gamma^2}\right)\left(1-i\frac{2\cdot 1}{2\gamma^2}\right)}{\frac{3^2\lambda}{\gamma^2}}}}. \tag{87.18}$$

Using (87.13) and (87.18) we compute axial displacement and stress in section $\theta=\frac{\pi}{4}$' in a tubular compensator stretched by forces P_z^0 and having the dimensions:

$d=22{,}8$ *см*, $a=2{,}8$ *см*, $h=0{,}2$ *см*,

$\lambda=8.14$, $2\gamma^2=46{,}3$, $\frac{\lambda}{\gamma^2}=0{,}353$,

$E=2{,}1\cdot 10^6$ kg/cm^2 $\mu=0{,}3$.

Calculating the amount of first continuous fraction (87.17) to within the third significant digit, we obtain

$$\frac{4\gamma^2 Ca}{a_1}=0{,}924-i0{,}0465.$$

At $P_z^0=935$ kg

$$|\Delta_z|\approx\frac{P_z^0\cdot 46{,}3}{2\cdot 2{,}1\cdot 10^6\cdot 0{,}2}\,\frac{1}{0{,}924}=\frac{P_z^0}{16\,800}=0{,}595\cdot 10^{-4}\ \text{см}.$$

The axial displacement of the whole compensator is equal to

$$|\Delta_z|=\frac{P_z^0}{8400}=1{,}19\cdot 10^{-4}\ \text{см}.$$

Stresses in filiaments of the external surface in section $\theta = \frac{\pi}{4}$ at $P_z^0 = 935$ kg are equal to

$$\sigma_1\left(\frac{\pi}{4}\right) = -\frac{6 \cdot 935 \cdot 0{,}707}{0{,}2^2 \cdot 3{,}14 \cdot 46{,}3}\left(1 + \frac{1}{12 \cdot 0{,}353^2}\right)\frac{1}{0{,}924} = -1220 \text{ kg/cm}^2.$$

§ 88. Periodic Particular Solution of Equation (86.25)

The particular solution of basic resolvant equation (86.25) at a bending load of the form $P_1 = 0,\ M_1 \neq 0$ also can be looked for in the form of (87.5). The right term of equation (86.25) in this instance has the form

$$A_1 \cos\theta + B_2 \sin 2\theta, \tag{88.1}$$

where

$$A_1 = \frac{M_1}{\pi a^2} 4\gamma^4 \lambda, \quad B_2 = \frac{M_1}{\pi a^2} 4\gamma^4 \cdot \frac{1}{2}. \qquad (88.1)\ [sic]$$

To determine the coefficients of series (87.5) now we obtain the system:

$$\left.\begin{array}{l}
-1^2 a_1\lambda - \frac{1}{2} b_2 (2l\gamma^2 + 2 \cdot 5) = A_1, \\
-2^2 b_2\lambda - \frac{1}{2} a_1 (2l\gamma^2 - 1 \cdot 2) + \frac{1}{2} a_3 (2l\gamma^2 + 3 \cdot 6) = B_2, \\
-3^2 a_3\lambda + \frac{1}{2} b_2 (2l\gamma^2 - 2 \cdot 1) - \frac{1}{2} b_4 (2l\gamma^2 + 4 \cdot 7) = 0, \\
-4^2 b_4\lambda - \frac{1}{2} a_3 (2l\gamma^2 - 3 \cdot 0) + \frac{1}{2} a_5 (2l\gamma^2 + 5 \cdot 8) = 0, \\
-5^2 a_5\lambda + \frac{1}{2} b_4 (2l\gamma^2 + 4 \cdot 1) - \frac{1}{2} b_6 (2l\gamma^2 + 6 \cdot 9) = 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}\right\} \tag{88.2}$$

From (88.2) we find recurrent formulas for the coefficients

$$
\left.
\begin{aligned}
&\frac{b_n}{a_{n-1}}\,i = \\
&= \cfrac{1 - i(n-1)(n-4)\frac{1}{2\gamma^2}}{\cfrac{n^2\lambda}{\gamma^2} + \cfrac{\left[1 - i(n+1)(n+4)\frac{1}{2\gamma^2}\right]\left[1 - in(n-3)\frac{1}{2\gamma^2}\right]}{\cfrac{(n+1)^2\lambda}{\gamma^2} + \cfrac{\left[1 - i(n+2)(n+5)\frac{1}{2\gamma^2}\right]\left[1 - i(n+1)(n-2)\frac{1}{2\gamma^2}\right]}{\cfrac{(n+2)^2\lambda}{\gamma^2} + \ldots}}}, \\
&-\frac{a_{n+1}}{b_n}\,i = \\
&= \cfrac{1 - in(n-3)\frac{1}{2\gamma^2}}{\cfrac{(n+1)^2\lambda}{\gamma^2} + \cfrac{\left[1 - i(n+2)(n+5)\frac{1}{2\gamma^2}\right]\left[1 - i(n+1)(n-2)\frac{1}{2\gamma^2}\right]}{\cfrac{(n+2)^2\lambda}{\gamma^2} + \ldots}}, \\
&\qquad n = 2, 4, 6, \ldots
\end{aligned}
\right\}
\tag{88.3}
$$

For example, at $n=2$ we have

$$
-\frac{a_3}{b_2}\,i = \cfrac{1 + i2\cdot 1\frac{1}{2\gamma^2}}{\cfrac{3^2\lambda}{\gamma^2} + \cfrac{\left(1 - i\frac{4\cdot 6}{2\gamma^2}\right)\left(1 + i\frac{3\cdot 0}{2\gamma^2}\right)}{\cfrac{4^2\lambda}{\gamma^2} + \ldots}}. \tag{88.4}
$$

Substituting (88.4) into the second equation of (88.2), we obtain instead of it one equation containing two unknown coefficients: b_2 and a_1. After this from the first and transformed second equation of (88.2) we find the numerical values of a_1, b_2, and then, using (88.3), compute the subsequent coefficients. At sufficiently large λ we can approximately set

$$
B_2 \approx 0.
$$

Then, to determine a_1, we obtain the continued fraction:

$$
-\frac{A_1}{\gamma^2 a_1} = \frac{\lambda}{\gamma^2} + \cfrac{\left(1 - i\frac{2\cdot 5}{2\gamma^2}\right)\left(1 + i\frac{1\cdot 2}{2\gamma^2}\right)}{\cfrac{2^2\lambda}{\gamma^2} + \cfrac{\left(1 - i\frac{3\cdot 6}{2\gamma^2}\right)\left(1 + i\frac{2\cdot 1}{2\gamma^2}\right)}{\cfrac{3^2\lambda}{\gamma^2} + \cfrac{\left(1 - i\frac{4\cdot 7}{2\gamma^2}\right)\left(1 + i\frac{3\cdot 0}{2\gamma^2}\right)}{\cfrac{4^2\lambda}{\gamma^2} + \ldots}}}. \tag{88.5}
$$

The solution satisfies the conditions

$$\sigma_3\left(\pm\frac{\pi}{2}\right)=0 \tag{88.6}$$

or, because of substitutions (86.22), (86.24) and relationships (16.17), (16.12), the conditions

$$\Psi\left(\pm\frac{\pi}{2}\right)=\nu\left(\pm\frac{\pi}{2}\right)=0. \tag{88.7}$$

Let us determine angular displacement of section $\theta=-\frac{\pi}{2}$ of the shell relative to the middle section $\theta=+\frac{\pi}{2}$ under the action of an external bending moment M_1

$$\omega_y=-\frac{\Delta_{z(1)}\left(-\frac{\pi}{2}\right)}{\nu\left(-\frac{\pi}{2}\right)}\approx-\frac{\Delta_{z(1)}\left(-\frac{\pi}{2}\right)}{a\lambda}. \tag{88.8}$$

By formula (18.2) we have

$$\frac{\Delta_{z(1)}}{\nu}=\Psi-\int_{\frac{\pi}{2}}^{\theta}a\varkappa_{1(1)}\,d\theta+D_1.$$

Assuming $\Delta_{z(1)}\left(\frac{\pi}{2}\right)=0$ and taking into account the first condition of (88.7), we find that $D_1=0$ and

$$\frac{\Delta_{z(1)}\left(-\frac{\pi}{2}\right)}{a\lambda}=\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}}a\varkappa_{1(1)}\,d\theta.$$

Since

$$\varkappa_{1(1)}=\frac{1}{Eh^3}(m_1-\mu m_2)=\frac{1}{Eha}\operatorname{Re}\left\{\frac{d}{d\theta}\left[\frac{\sigma_3}{(\lambda+\sin\theta)^2}\right]+\frac{\cos\theta\sigma_3}{(\lambda+\sin\theta)^3}\right\},$$

then, taking into account the edge conditions for σ_3, finally we obtain

$$\omega_y = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\operatorname{Re} \sigma_3 \cos\theta}{Eh(\lambda + \sin\theta)^3}\, d\theta \approx \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\operatorname{Re} \sigma_3 \cos\theta}{Eh\lambda^3}\, d\theta = -\frac{\pi}{2Eh\lambda^3} \operatorname{Re} a_1 .$$

Substituting the value of $\operatorname{Re} a_1$ into this formula we find

$$\omega_y = \frac{2\gamma^2}{Eh\lambda^2 a^2} M_1 \operatorname{Re} \frac{1}{\frac{\lambda}{\gamma^2} + \frac{\left(1 - i\frac{2\cdot 5}{2\gamma^2}\right)\left(1 + i\frac{1\cdot 2}{2\gamma^2}\right)}{2^2\lambda/\gamma^2 + \ldots}} . \qquad (88.9)$$

To determine meridian bending stress in section $\theta = \frac{\pi}{4}$ at large λ when during calculation of solution (87.5) we can be limited to the first three terms of the series, we can obtain a formula analogous to (87.18),

$$\sigma_1\left(\frac{\pi}{4}\right) = \pm \frac{6}{h^2} \frac{M_1\sqrt{2}}{\pi a 2\gamma^2}\left(1 + \frac{\gamma^4}{2^2\cdot 3\cdot \lambda^2}\right) \times$$
$$\times \operatorname{Re} \frac{1}{\frac{\lambda}{\gamma^2} + \frac{\left(1 - i\frac{2\cdot 5}{2\gamma^2}\right)\left(1 + i\frac{1\cdot 2}{2\gamma^2}\right)}{\frac{2^2\lambda}{\gamma^2} + \ldots}} . \qquad (88.10)$$

Formulas (88.9), (88.10) can be used to approximately calculate the rigidity and stresses in a tubular compensator which is subjected to bending by external moments M.

§ 89. Solution of Basic Equations (86.11), (86.25) for the Case $\frac{2\gamma^2}{\lambda} \gg 1$

Equations (86.11) and (86.25), describing the deformation of a torus-shaped shell during axisymmetric and bending loads, differ from one another only in the constant factor in the coefficient of the first derivative and by the form of the right side. The solutions of the corresponding uniform equations have the character of edge effects, about which we spoke in § 19, only in that area of change in θ, in which the values of $\sin\theta$ are strongly nonzero. At $\theta = 0, \pi$ the terms containing large parameter $2\gamma^2$ in these equations vanish. In accordance with this function $\psi(\theta)$ in equations (19.13), (19.15) in these points assumes an unbounded large value and cannot be dropped in comparison with the first term, containing the parameter $2\gamma^2$. It is necessary to build such solutions of uniform equations (86.11), (86.25) which, remaining limited in the vicinity of points $\theta=0, \pi$, with sufficient distance from them would assume the character of edge effects.

There are a great many works [113], [132], [167], ..., which consider equations of the form

$$\frac{d}{d\theta}\left[p(\theta)\frac{d\sigma}{d\theta}\right]+\left[\frac{1}{\varepsilon}q(\theta)+r(\theta)\right]\sigma=f(\theta). \qquad (89.1)$$

$p(\theta)$, $r(\theta)$ — actual functions which do not possess singularities on the section of change in θ $[a, b]$, where $p(\theta)$ does not turn into zero in any point of the section, $q(\theta)$ has a simple zero in point $\theta=0$. Parameter ε can have both real and imaginary values and

$$\left|\frac{1}{\varepsilon}\right|\gg 1. \qquad (89.2)$$

Equations (86.11), (86.25), which are of interest to us, easily can be brought to the form of (89.1). The large parameter in the involved case has the value

$$\frac{1}{\varepsilon}=-\frac{2i\gamma^2}{\lambda}=-i\mu_1^3, \quad \mu_1^3=\frac{2\gamma^2}{\lambda}. \qquad (89.3)$$

Method [132], which will be stated below is suitable for calculation of shells whose geometric dimensions satisfy the requirement

$$\frac{2\gamma^2}{\lambda}=\sqrt{12(1-\mu^2)}\frac{a}{h\lambda}\gg 1. \qquad (89.4)$$

The concrete notation of equations (86.11) and (86.25) in the form of (89.1) gives for the axisymmetric case

$$p(\theta)=\frac{1}{1+a\sin\theta},\quad q(\theta)=\frac{\sin\theta}{(1+a\sin\theta)^2},\quad r(\theta)=0, \tag{89.5}$$

$$a=\frac{1}{\lambda}<1,\quad f(\theta)=\frac{F_1 a}{(1+a\sin\theta)^2}, \tag{89.6}$$

for the case of a bending load

$$\left.\begin{aligned} &p(\theta)=\frac{1}{(1+a\sin\theta)^3},\quad q(\theta)=\frac{\sin\theta}{(1+a\sin\theta)^4},\\ &r(\theta)=0,\quad f(\theta)=\frac{F_3 a}{(1+a\sin\theta)^4}.\end{aligned}\right\} \tag{89.7}$$

Here F_1 and F_3 — the right parts of (86.11) and (86.25).

Let us examine the interval $0\leqslant\theta\leqslant\frac{\pi}{2}$. By replacement of the dependent and independent variables

$$\left.\begin{aligned} &\sigma=\eta w,\quad w=\frac{1}{\sqrt{pu'}},\quad u'=\frac{dv}{d\theta},\\ &u=u_0(\theta)+\varepsilon u_1(\theta)+\varepsilon^2 u_2(\theta)+\ldots\end{aligned}\right\} \tag{89.8}$$

equation (89.1) is brought to the form

$$\varepsilon\frac{d^2\eta}{du^2}+A(\theta)\eta=g(u), \tag{89.9}$$

where

$$\left.\begin{aligned} &A(\theta)=\frac{1}{pu'^2 w}\left[(q+\varepsilon r)w+\varepsilon pu'^2\frac{d^2w}{du^2}+\varepsilon(pu''+p'u')\frac{dw}{du}\right],\\ &g(u)=\frac{\varepsilon f(\theta)}{pu'^2 w}=\frac{f\varepsilon w}{u'}.\end{aligned}\right\} \tag{89.10}$$

Assuming $A(\theta)=u$ and equating terms with identical powers of ε, we find the relationships for determination of u_0, u_1, etc.

$$\left.\begin{aligned} &u_0=\frac{q(\theta)}{p(\theta)u_0'^2},\\ &\frac{d}{d\theta}\left(u_0^{\frac{1}{2}}u_1\right)=\frac{1}{2u_0^{1/2}}v[rv+(pv')'],\qquad v=\frac{1}{\sqrt{pu_0'}}.\end{aligned}\right\} \tag{89.11}$$

From the first relationship of (89.11) follows

$$u_0=\left(\frac{3}{2}\int_0^u \sqrt{\frac{q(x)}{p(x)}}\,dx\right)^{\frac{2}{3}}. \qquad (89.12)$$

Equation (89.9) now can be rewritten in the form

$$\varepsilon\frac{d^2\eta}{du^2}+u\eta=g(u). \qquad (89.13)$$

By one more replacement of the independent variable

$$u=\varepsilon^{\frac{1}{3}}t \qquad (89.14)$$

we convert (89.13) into the equation

$$\frac{d^2\eta}{dt^2}+t\eta=g_1(t), \quad g_1(t)=\varepsilon^{-\frac{1}{3}}g(u). \qquad (89.15)$$

The uniform equation corresponding to (89.15) is the known Airy equation

$$\eta''+t\eta=0,$$

which with the aid of elementary substitutions is converted into a Bessel equation of the order $\frac{1}{3}$. The general solution can be represented in the form

$$\eta=C_1h_1(t)+C_2h_2(t). \qquad (89.16)$$

Functions $h_1(t)$, $h_2(t)$ are expressed through Hankel functions in the following manner:

$$h_1(t)=\left(\frac{2}{3}t^{3/2}\right)^{\frac{1}{3}}H^{(1)}_{\frac{1}{3}}\left(\frac{2}{3}t^{3/2}\right), \quad h_2(t)=\left(\frac{2}{3}t^{3/2}\right)^{\frac{1}{3}}H^{(2)}_{\frac{1}{3}}\left(\frac{2}{3}t^{3/2}\right).$$

ere are tables of functions h_1, h_2 for imaginary values of the rgument $t=x+iy$ with the interval of change x, y through 0, 1 [173].

In the appendix is given a copy of these tables for $t=iy$ (Table 4).

Functions $h_1(t)$, $h_2(t)$ are representable by infinite power series of he following form:

$$\left.\begin{aligned}
h_1(t) &= g(t) + i\frac{\sqrt{3}}{3}[g(t) - 2f(t)],\\
h_2(t) &= g(t) - i\frac{\sqrt{3}}{3}[g(t) - 2f(t)],\\
f(t) &= \frac{2^{\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)}\left[1 + \sum_{m=1}^{\infty}(-1)^m\frac{(3m-2)(3m-5)\ldots 4\cdot 1}{(3m)!}t^{3m}\right],\\
g(t) &= \frac{2^{2/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}\left[t + \sum_{m=1}^{\infty}(-1)^m\frac{(3m-1)(3m-4)\ldots 5\cdot 2}{(3m+1)!}t^{3m+1}\right].
\end{aligned}\right\} \quad (89.17)$$

$$\left.\begin{aligned}
h_1(0) &= -\frac{2}{\sqrt{3}}i\frac{2^{1/3}}{\Gamma\left(\frac{2}{3}\right)} = -i1.0744,\\
h_2(0) &= \frac{2}{\sqrt{3}}i\frac{2^{1/3}}{\Gamma\left(\frac{2}{3}\right)} = +i1.0744,\\
h_1'(0) &= \frac{2^{2/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}\left(1 + i\frac{\sqrt{3}}{3}\right), \quad h_2'(0) = \frac{2^{2/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}\left(1 - i\frac{\sqrt{3}}{3}\right),\\
W &= h_1h_2' - h_2h_1' = -\frac{4}{\sqrt{3}}i\frac{2^{2/3}}{3^{1/3}}\frac{1}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)} = -2i\beta^2,\\
\beta &= 0.853667.
\end{aligned}\right\} \quad (89.18)$$

t is not difficult to see that functions $h_1(t)$, $h_2(t)$ possess the ollowing features:

$$\left.\begin{aligned}
h_1(\bar{t}) &= \overline{h_2(t)}, \quad h_2(\bar{t}) = \overline{h_1(t)},\\
h_1'(\bar{t}) &= \overline{h_2'(t)}, \quad h_2'(\bar{t}) = \overline{h_1'(t)}.
\end{aligned}\right\} \quad (89.19)$$

or values of t large in absolute value, there exist the asymptotic epresentations

$$
\left.
\begin{aligned}
&h_1(t) \sim \beta t^{-\frac{1}{4}}\left[1+\sum_{m=1}^{\infty}(-i)^m c_m t^{-\frac{3m}{2}}\right]\exp\left(\frac{2}{3}it^{\frac{3}{2}}-\frac{5\pi i}{12}\right), \\
&\qquad\qquad -\frac{2\pi}{3}<\arg t<\frac{4\pi}{3}, \\
&h_2(t) \sim \beta t^{-\frac{1}{4}}\left[1+\sum_{m=1}^{\infty}(i)^m c_m t^{-\frac{3m}{2}}\right]\exp\left(-\frac{2}{3}it^{\frac{3}{2}}+\frac{5\pi i}{12}\right), \\
&\qquad\qquad -\frac{4\pi}{3}<\arg t<\frac{2\pi}{3}, \\
&c_m = \frac{(9-4)(81-4)\ldots[9(2m-1)^2-4]}{2^{4m}\cdot 3^m\cdot m!}.
\end{aligned}
\right\}
\qquad (89.20)
$$

$$
\left.
\begin{aligned}
&h_1'(t) \sim \beta t^{\frac{1}{4}}\exp\left(\frac{2}{3}it^{\frac{3}{2}}+\frac{\pi i}{12}\right), \quad -\frac{2\pi}{3}<\arg t<\frac{4\pi}{3}, \\
&h_2'(t) \sim \beta t^{\frac{1}{4}}\exp\left(-\frac{2}{3}it^{\frac{3}{2}}-\frac{\pi i}{12}\right), \quad -\frac{4\pi}{3}<\arg t<\frac{2\pi}{3}.
\end{aligned}
\right\}
\qquad (89.21)
$$

From (89.20) it follows that the constructed solution of the uniform equation, if parameter $\frac{1}{\varepsilon}$ is large in absolute value, possesses the features of edge effect. Really, taking into account (89.3), (89.5), (89.7), (89.12), (89.14) and keeping in expansion (89.8) only the term containing ε in the zero degree, we obtain

$$
\exp\left[\pm\left(\frac{2}{3}it^{3/2}-\frac{5\pi i}{12}\right)\right]=
$$
$$
=\exp\left\{\pm\left[-(1+i)\gamma\int_0^{\theta}\sqrt{\frac{\sin x}{\lambda+\sin x}}\,dx-\frac{5\pi i}{12}\right]\right\}.
$$

Note that the solution keeps the features of edge effect for any complex ε. The exception is only the case when ε is equal to a real number.

The particular solution of heterogeneous equation (89.15) is looked for by the method of variation of arbitrary constants. In this case one must take into account that $h_1(t)$ is that solution of the uniform equation which increases along the positive direction of a beam $t=\rho u$ $\left(\rho=\varepsilon^{-\frac{1}{3}}\right)$, and $h_2(t)$ — is the solution which decreases along the same direction so that $h_1(-\infty\rho)=0$, $h_2(+\infty\rho)=0$. Then the particular solution can be represented in the form

$$
\eta=\frac{1}{W}\left[h_2(t)\int_{-\infty}^{t}g_1(\tau)h_1(\tau)\,d\tau+h_1(t)\int_{t}^{\infty}g_1(\tau)h_2(\tau)\,d\tau\right]. \qquad (89.22)
$$

Remembering that $t = u\rho$, $g_1(t) = \rho g(u)$, we rewrite (89.22) in the following manner:

$$\eta = \frac{\rho^2}{W}\left[h_2(\rho u)\int_{-\infty}^{u} g(\xi)\,h_1(\rho\xi)\,d\xi + h_1(\rho u)\int_{u}^{\infty} g(\xi)\,h_2(\rho\xi)\,d\xi\right]. \qquad (89.23)$$

Substituting into (89.23) the expression for $g(\xi)$ in the form of the series

$$g(\xi) = \sum_{n=0}^{\infty} \frac{(\xi-u)^n}{n!}\, g^{(n)}(u) \qquad (89.24)$$

and going again to integration over variable $\tau = \rho\xi$, we obtain

$$\eta = \rho e_0(t)\, g(u) + e_1(t)\, g'(u) + \dots + \frac{1}{\rho^{n-1}}\, e_n(t)\, g^{(n)}(u) + \dots. \qquad (89.25)$$

where

$$\left.\begin{aligned} e_0(t) &= \frac{1}{W}\left[h_2(t)\int_{-\infty}^{t} h_1(\tau)\,d\tau + h_1(t)\int_{t}^{\infty} h_2(\tau)\,d\tau\right], \\ e_n(t) &= \frac{1}{Wn!}\left[h_2(t)\int_{-\infty}^{t} (\tau-t)^n h_1(\tau)\,d\tau + h_1(t)\int_{t}^{\infty} (\tau-t)^n h_2(\tau)\,d\tau\right]. \end{aligned}\right\} \qquad (89.26)$$

With the aid of integration by parts we can establish that between these functions exist the recurrent relationships

$$(n+3)\,e_{n+3}(t) + t e_{n+2}(t) + e_n(t) = 0. \qquad (89.27)$$

where $e_1(t) = 1 - t e_0(t)$, and function $e_0(t)$ satisfies the equation

$$e_0'' + t e_0 = 1. \qquad (89.28)$$

This directly follows from (89.25) if we set in this expression $g(u) = 1$.

Function $e_0(t)$ is connected with the known Lommel function of the second kind and order 0, 1/3 by such a relationship:

$$e_0(t)=\left(\frac{2}{3}\right)^{2/3}\left(\frac{2}{3}t^{3/2}\right)^{\frac{1}{3}}S_{0,\frac{1}{3}}\left(\frac{2}{3}t^{3/2}\right). \qquad (89.29)$$

Let us remember that the Lommel function of order μ, ν is a function wich satisfies the equation [79]

$$x^2y'' + xy' + (x^2 - \nu^2)y = x^{\mu+1} \qquad (89.30)$$

so that function $S_{0,1/3}$ (just as the function of the first kind $s_{0,1/3}$) is the solution of equation (89.30) at $\nu=\frac{1}{3}$, $\mu=0$.

When ν is fractional the function of the first kind $s_{\mu,\nu}$ has the expression

$$s_{\mu,\nu}=\frac{\pi}{2\sin\pi\nu}\left[I_\nu(x)\int_0^x \xi^\mu I_{-\nu}(\xi)\,d\xi - I_{-\nu}(x)\int_0^x \xi^\mu I_\nu(\xi)\,d\xi\right].$$

Furthermore, it is represented by a power series

$$s_{\mu,\nu}(x)=x^{\mu-1}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{x}{2}\right)^{2m+2}\Gamma\left(\frac{\mu+\nu}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{\mu+\nu}{2}+m+\frac{3}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}+m+\frac{3}{2}\right)},$$

if $(\mu+\nu)$ or $(\mu-\nu)$ are not equal to an odd negative number.

Function of the second kind $S_{\mu,\nu}$ is connected with the function of the first kind by the relationship

$$S_{\mu,\nu}(x)=s_{\mu,\nu}(x)+\frac{2^{\mu-1}\Gamma\left(\frac{\mu+\nu}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}+\frac{1}{2}\right)}{\sin\nu\pi}\times$$
$$\times\left[\cos\left(\frac{\mu-\nu}{2}\pi\right)I_{-\nu}(x)-\cos\left(\frac{\mu+\nu}{2}\pi\right)I_\nu(x)\right]$$

(ν is fractional).

Using the written out expressions and (89.29), we obtain

$$e_0(0)=\frac{1}{3^{2/3}}\Gamma\left(\frac{1}{3}\right)=1{,}287899,\quad e_0'(0)=-3^{-1/3}\Gamma\left(\frac{2}{3}\right)=-0{,}938893.$$

It is simple to write out a direct representation of function $e_0(t)$ in the form of a series in powers of t

$$\begin{aligned}e_0(t)=e_0(0)\left(1-\frac{1\cdot t^3}{3!}+\frac{1\cdot 4}{6!}t^6-\frac{1\cdot 4\cdot 7}{9!}t^9+\ldots\right)+\\ +e_0'(0)t\left(1-\frac{2}{4!}t^3+\frac{2\cdot 5}{7!}t^6-\frac{2\cdot 5\cdot 8}{10!}t^9+\ldots\right)+\\ +t^2\left(\frac{1}{2!}-\frac{3}{5!}t^3+\frac{3\cdot 6}{8!}t^6-\frac{3\cdot 6\cdot 9}{11!}t^9+\ldots\right).\end{aligned}\tag{89.31}$$

On the condition that

$$\frac{\pi}{3}<|\arg t|<\frac{2\pi}{3}$$

for $e_0(t)$ there exists the integral representation

$$e_0(t)=\int_0^\infty e^{-tx-\frac{x^3}{3}}dx.\tag{89.32}$$

For large $|t|$ from (89.32) we obtain the asymptotic expansion

$$e_0(t)\sim\frac{1}{t}+\sum_{k=1}^{\infty}\frac{(-1)^k(3k-1)!}{(k-1)!\,3^{k-1}t^{3k+1}}.\tag{89.33}$$

Using expression (89.32) and the formula for the Fourier cosine transformation, applicable to the function $f(x)=e^{-x^3/3}$

$$e^{-\frac{x^3}{3}}=\frac{2}{\pi}\int_0^\infty\cos\alpha x\,d\alpha\int_0^\infty e^{-\frac{t^3}{3}}\cos\alpha t\,dt,\tag{89.34}$$

it is easy to calculate what $\int_0^\infty \operatorname{Re} e_0(iy)\,dy$ is equal to. First note that

$$\operatorname{Re} e_0(iy) = \int_0^\infty e^{-\frac{x^3}{3}} \cos yx\, dx.$$

Making x approach zero in the left and right parts of (89.34), we have

$$1 = \frac{2}{\pi}\int_0^\infty da \int_0^\infty e^{-\frac{t^3}{3}} \cos at\, dt;$$

hence it follows that

$$\int_0^\infty \operatorname{Re} e_0(iy)\,dy = \frac{\pi}{2}. \qquad (89.35)$$

Thus, the particular solution of equation (89.13) (formula (89.25)) is obtaired, which is a series in negative powers of large parameter $\rho = \varepsilon^{-1/3}$. For practical purposes frequently it is sufficient to be limited to one term of the series

$$\eta \approx \varepsilon^{-\frac{1}{3}} e_0(t)\, \zeta(u).$$

In accordance with this in creating the fundamental solutions of the uniform equation in representation (89.8) also one ought to discard small terms, assuming

$$u \approx u_0(\theta).$$

Being limited to the indicated zero approximation, we write the general solutions of the basic resolvant equations (86.11) and (86.25) which describe axisymmetric deformation and deformation under bending load:

$$
\left.
\begin{aligned}
&\sigma_1 = A_1 w_1 h_1(t) + A_2 w_1 h_2(t) + 2\gamma^2 C\alpha\mu_1 \frac{\cos\theta}{\sin\theta} u_0 e_0(t),\\
&\frac{d\sigma_1}{d\theta} = A_1\left[i\mu_1 w_1 \frac{du_0}{d\theta} h_1'(t) + \underline{h_1(t)\frac{dw_1}{d\theta}}\right] +\\
&\qquad + A_2\left[i\mu_1 w_1 \frac{du_0}{d\theta} h_2'(t) + \underline{h_2(t)\frac{dw_1}{d\theta}}\right] +\\
&\qquad\qquad + 2\gamma^2 C\alpha\mu_1 \cdot i\mu_1 \operatorname{ctg}\theta\, u_0 e_0'(t)\frac{du_0}{dt}.
\end{aligned}
\right\} \qquad (89.36)
$$

Here

$$
t = i\mu_1 u_0, \quad w_1 = \left(\frac{u_0}{\sin\theta}\right)^{\frac{1}{4}} (1 + \alpha\sin\theta)^{3/4},
$$

$$
0 \leqslant \theta \leqslant \frac{\pi}{2}, \quad u_0 = \left(\frac{3}{2}\int_0^\theta \sqrt{\frac{\sin x}{1 + \alpha\sin x}}\, dx\right)^{2/3}.
$$

$$
\left.
\begin{aligned}
&\sigma_3 = A_1 w_3 h_1(t) + A_2 w_3 h_2(t) -\\
&\qquad - \mu_1 2\gamma^2 \Phi_3(\theta)(\lambda + \sin\theta)^3 \frac{u_0}{\sin\theta} e_0(t),\\
&\frac{d\sigma_3}{d\theta} = A_1\left[i\mu_1 w_3 \frac{du_0}{d\theta} h_1'(t) + \underline{h_1(t)\frac{dw_3}{d\theta}}\right] +\\
&\qquad + A_2\left[i\mu_1 w_3 \frac{du_0}{d\theta} h_2'(t) + \underline{h_2(t)\frac{dw_3}{d\theta}}\right] -\\
&\qquad\qquad - i\mu_1^2 2\gamma^2 \Phi_3(\theta)(\lambda + \sin\theta)^3 \frac{u_0}{\sin\theta}\frac{du_0}{d\theta} e_0'(t).
\end{aligned}
\right\} \qquad (89.37)
$$

where

$$
w_3 = \left(\frac{u_0}{\sin\theta}\right)^{1/4} (1 + \alpha\sin\theta)^{7/4}.
$$

For small α there exist the approximate equalities

$$
\left.
\begin{aligned}
&\left(\frac{u_0}{\sin\theta}\right)^{1/4} \approx 1, \quad w_1 \approx w_3 \approx 1,\\
&\frac{du_0}{d\theta} \approx 1, \quad \frac{dw_1}{d\theta} \approx \frac{dw_3}{d\theta} \approx 0.
\end{aligned}
\right\} \qquad (89.38)
$$

Taking into account that the particular solution was constructed with an accuracy at which quantities of the order $1/\mu_1$ were dropped in comparison with unity, in the uniform solution we also can discard analogous terms. Therefore in formulas (89.36), (89.37) we can drop the underlined terms.

For large t $e_0(t)\approx\frac{1}{t}$, and the particular solutions

$$\left.\begin{aligned}\sigma_1&=2\gamma^2Ca\mu_1\frac{\cos\theta}{\sin\theta}u_0e_0(t),\\ \sigma_3&=-\mu_12\gamma^2\Phi_3(\theta)(\lambda+\sin\theta)^3\frac{u_0}{\sin\theta}e_0(t)\end{aligned}\right\}\qquad(89.39)$$

coincide with the zero-moment solutions

$$\dot{\sigma}_1=-i2\gamma^2Ca\frac{\cos\theta}{\sin\theta},\quad \tilde{\sigma}_3=i2\gamma^2\Phi_3(\theta)(\lambda+\sin\theta)^3\frac{1}{\sin\theta}.$$

On the other hand, at $\theta=0$ the right parts of (89.39), unlike the zero-moment solutions, remain bounded, since

$$\lim_{\theta\to0}\frac{u_0}{\sin\theta}=1.$$

Formulas (89.36), (89.37) represent the desired solution in the section $0\leqslant\theta\leqslant\frac{\pi}{2}$. In order to build a solution valid at $-\frac{\pi}{2}\leqslant\theta\leqslant0$, we represent equation (86.11) in the form

$$\frac{d}{d\theta}\left[\frac{1}{1+c\sin\theta}\frac{d\sigma_1}{d\theta}\right]-\frac{2i\gamma^2}{\lambda}\frac{\sin\theta\sigma_1}{(1+a\sin\theta)^2}=\frac{F_1a}{(1+a\sin\theta)^2}\qquad(89.40)$$

$$(F_1=-4\gamma^4Ca\cos\theta)$$

and replace the argument

$$\theta=-\theta_1.$$

then equation (89.40) will pass into equation

$$\frac{d}{d\theta_1}\left[\frac{1}{1-a\sin\theta_1}\frac{d\sigma_1}{d\theta_1}\right]+\frac{2i\gamma^2}{\lambda}\frac{\sin\theta_1\sigma_1}{(1-a\sin\theta_1)^2}=\frac{F_1a}{(1-a\sin\theta_1)^2}.\qquad(89.41)$$

This equation also belongs to the type of equations (89.1), where

$$\frac{1}{\varepsilon}=\frac{2i\gamma^2}{\lambda},\qquad(89.42)$$

$$p(\theta_1)=\frac{1}{1-\alpha\sin\theta_1},\quad q(\theta_1)=\frac{1}{(1-\alpha\sin\theta_1)^2},\quad f(\theta_1)=\frac{F_1\alpha}{(1-\alpha\sin\theta_1)^2}. \tag{89.43}$$

Large parameter (89.42) is obtained from (89.3) by the replacement of i by $-i$, and formulas (89.43) differ from formulas (89.5) only by the sign on α in the denominators. In accordance with this the desired solution is obtained from (89.36) (the same applies even to (89.37)) by replacing

$$\left.\begin{aligned}
&u_0 \text{ by } v_0=\left[\frac{3}{2}\int_0^{\theta_1}\sqrt{\frac{\sin x}{1-\alpha\sin x}}\,dx\right]^{2/3},\\
&t \text{ by } t_1=-i\mu_1 v_0(\theta_1),\\
&w_1 \text{ by } w_1=\left(\frac{v_0}{\sin\theta_1}\right)^{\frac{1}{4}}(1-\alpha\sin\theta_1)^{3/4},\\
&w_3 \text{ by } w_3=\left(\frac{v_0}{\sin\theta_1}\right)^{\frac{1}{4}}(1-\alpha\sin\theta_1)^{7/4}.
\end{aligned}\right\} \tag{89.44}$$

Making the indicated replacing, we obtain expressions for σ_1, σ_3 and their derivatives, valid at $-\frac{\pi}{2}\leqslant\theta\leqslant 0$:

$$\left.\begin{aligned}
&\sigma_1=B_1w_1h_1(t_1)+B_2w_1h_2(t_1)+2\gamma^2C\alpha\mu_1\frac{\cos\theta_1}{\sin\theta_1}v_0e_0(t_1),\\
&\frac{d\sigma_1}{d\theta}=-\frac{d\sigma_1}{d\theta_1}=i\mu_1w_1B_1h_1'(t_1)v_0'(\theta_1)+i\mu_1w_1B_2h_2'(t_1)v_0'(\theta_1)+\\
&\qquad+i\mu_1^2 2\gamma^2C\alpha\frac{\cos\theta_1}{\sin\theta_1}v_0e_0'(t_1)\frac{dv_0}{d\theta_1}.
\end{aligned}\right\} \tag{89.45}$$

$$\left.\begin{aligned}
&\sigma_3=B_1w_3h_1(t_1)+B_2w_3h_2(t_1)-\\
&\qquad-\mu_1 2\gamma^2\Phi_3(\theta)(\lambda+\sin\theta)^3\frac{v_0}{\sin\theta_1}e_0(t_1),\\
&\frac{d\sigma_3}{d\theta}=-\frac{d\sigma_3}{d\theta_1}=i\mu_1w_3B_1h_1'(t_1)\frac{dv_0}{d\theta_1}+i\mu_1w_3B_2h_2'(t_1)\frac{dv_0}{d\theta_1}-\\
&\qquad-i\mu_1^2 2\gamma^2\Phi_3(\theta)(\lambda+\sin\theta)^3\frac{v_0}{\sin\theta_1}e_0'(t_1)\frac{dv_0}{d\theta_1}.
\end{aligned}\right\} \tag{89.46}$$

Since at $\theta=0$, or, which the same, at $\theta_1=0$, the values of w_1, w_3, t and t_1, $\frac{du_0}{d\theta}$ and $\frac{dv_0}{d\theta_1}$ in both cases coincide, then it is easy to see that at

$$B_1=A_1,\quad B_2=A_2 \tag{89.47}$$

solution (89.45) is an analytic extension of solution (89.36). In exactly the same manner (89.46) is an analytic extension of (89.37) from the domain $0 < \theta < \frac{\pi}{2}$ into the domain $-\frac{\pi}{2} < \theta < 0$.

Let us imagine a torus-shaped shell bounded by the edges θ_0, θ_k, where

$$-\frac{\pi}{2} \leqslant \theta_0 < 0, \quad \frac{\pi}{2} \geqslant \theta_k > 0.$$

Remembering the features of functions $h_1(t)$, $h_2(t)$, expressed by formulas (89.19), and asymptotic representations of these functions, existing for large values of the argument, we come to the conclusion that function

$$H_1(\theta) = \begin{cases} h_1(t_1) = \overline{h_2(i\mu_1 v_0)}, & \theta_0 \leqslant \theta \leqslant 0, \\ h_1(t) = h_1(i\mu_1 u_0), & \theta_k \geqslant \theta \geqslant 0, \end{cases}$$

in absolute value decreases in proportion to the distance from edge θ_0 to edge θ_k, and the function

$$H_2(\theta) = \begin{cases} h_2(t_1) = \overline{h_1(i\mu_1 v_0)}, & \theta_0 \leqslant \theta \leqslant 0, \\ h_2(t) = h_2(i\mu_1 u_0), & \theta_k \geqslant \theta \geqslant 0, \end{cases}$$

in absolute value increases when θ from θ_0 to θ_k. In this way, that part of solutions (89.36), (89.37) and (89.45), (89.46), (89.47), which contains constant A_1, describes the stressed state of edge θ_0, and the term containing A_2, describes the stressed state of edge θ_k.

In the appendix are given tables for calculation of the integrals $\int_0^\theta \sqrt{\frac{\sin x}{1 \pm a \sin x}}\,dx$ at different values of $a = \frac{1}{\lambda}$, and also tables of function $e_0(t)$ and $e_0'(t)$.

§ 90. Elongation and Bend of a Tubular Compensator

In an axisymmetrically loaded shell, bounded by sections $\theta_0 = -\frac{\pi}{2}$, $\theta_k = +\frac{\pi}{2}$ under the edge conditions

$$\vartheta_1^0 = H_e^0 = 0, \quad \vartheta_1^k = H_e^k = 0 \tag{90.1}$$

edge effects are absent. Constants A_1, A_2 in solutions (89.36), (89.45) are equal to zero. The stressed state is described by the particular solution

$$\sigma_1 = \begin{cases} 2\gamma^2 Ca\mu_1 \dfrac{\cos\theta}{\sin\theta} u_0 e_0(t), & 0 \leqslant \theta, \\ 2\gamma^2 Ca\mu_1 \dfrac{\cos\theta_1}{\sin\theta_1} v_0 e_0(t_1), & \theta \leqslant 0, \end{cases} \qquad \theta_1 = -\theta, \tag{90.2}$$

$$\frac{d\sigma_1}{d\theta} = \begin{cases} i2\gamma^2 Ca\mu_1^2 \operatorname{ctg}\theta\, u_0 \dfrac{du_0}{d\theta} e_0'(t), & 0 \leqslant \theta, \\ i2\gamma^2 Ca\mu_1^2 \operatorname{ctg}\theta_1 v_0 \dfrac{dv_0}{d\theta} e_0'(t_1), & \theta \leqslant 0, \end{cases} \qquad C = \frac{\lambda}{2}\left[\frac{P_z^0}{\pi a^2} + p(2\lambda - 1)\right], \tag{90.3}$$

since $v_0 = a(\lambda - 1)$.

Let us examine a tubular compensator stretched by force P_z^0, in the presence of internal pressure p. From considerations of symmetry in section $\theta = +\frac{\pi}{2}$ conditions (90.1) should be held. We consider that they exist also in section $\theta = -\frac{\pi}{2}$.

The bending moment in instantaneous section θ we compute using the approximation formula

$$M_1 = \frac{a}{4\gamma^4} \frac{1}{\lambda} \operatorname{Re} \frac{d\sigma_1}{d\theta}. \tag{90.4}$$

Substituting into (90.4) the expression for $\frac{d\sigma_1}{d\theta}$ at $\theta > 0$, we obtain

$$M_1 = -\frac{1}{2\mu_1\lambda} \operatorname{ctg}\theta\, u_0 \frac{du_0}{d\theta}\left[\frac{P_z^0}{\pi} + pa^2(2\lambda - 1)\right] \operatorname{Im} e_0'(t).$$

The imaginary part of function $e_0'(t)$ has a maximum for the value $t=t^*=$ $=1.225i$:

$$\max \operatorname{Im} e_0'(t) = 0.753.$$

To the value of t^* corresponds to angular coordinate $\theta^* = \frac{1.225}{\mu_1}$. Since parameter μ_1 is great in comparison with unity, this means that θ^* is small and we can take $\operatorname{ctg}\theta^* u_0(\theta_*)\frac{du_0}{d\theta^*} \approx 1$.

Taking into account everything we said, we obtain

$$M_{1\,max} \approx -\frac{1}{2\mu_1\lambda}\left[\frac{P_z^0}{\pi} + pa^2(2\lambda - 1)\right]0.753.$$

Maximum bending stresses are equal to

$$\sigma_{1\,max} = \mp(\pm)\frac{3\mu_1^2}{\sqrt{12(1-\mu^2)}}\frac{a}{h}\left[\frac{P_z^0}{\pi a^2} + p(2\lambda - 1)\right]0.753. \qquad (90.5)$$

At $p=0$ formula (90.5) can be represented in the form

$$\sigma_1 = \mp(\pm)2.99(1-\mu^2)^{-\frac{1}{6}}\left(\frac{a}{h}\right)^{\frac{2}{3}}\lambda^{\frac{1}{3}}\frac{P_z^0}{2\pi ah}, \qquad (90.6)$$

where $\frac{P_z^0}{2\pi ah}$ — tensile stress in a cylindrical tube of radius a with wall thickness h. In formulas (90.5), (90.6) the upper sign refers to stresses in the filaments of the external surface, and the lower sign refers to stresses in filaments of the internal surface. The signs in parentheses should be taken during the calculation of stresses in section $\theta = -\theta^*$.

Calculating circumferential forces using the corresponding formulas (86.12), on the basis of solution (90.2), (90.3) we obtain

$$T_2 = -\frac{1}{\lambda+\sin\theta}Ca\mu_1^2\operatorname{ctg}\theta u_0\frac{du_0}{d\theta}\operatorname{Re}e_0'(t) + \frac{pa}{2} - Ca\frac{\lambda\sin\theta + 1}{\lambda(\lambda+\sin\theta)^2} \approx$$
$$\approx -\frac{a\mu_1^2}{2}\operatorname{ctg}\theta u_0\frac{du_0}{d\theta}\operatorname{Re}e_0'(t)\left[\frac{P_z^0}{\pi a^2} + p(2\lambda - 1)\right].$$

they have a maximum value at $\theta=0$. Circumferential tensile stresses in this section are equal to

$$\sigma_2(0)=\frac{T_2(0)}{h}=-\frac{a\mu_1^2}{2}\operatorname{Re} e_0'(0)\left[\frac{P_z^0}{\pi a^2}+p(2\lambda-1)\right]. \qquad (90.7)$$

where $\operatorname{Re} e_0'(0)=-0.939$. At $p=0$ we have

$$\sigma_2(0)=2.15(1-\mu^2)^{1/4}\left(\frac{a}{h}\right)^{3/2}\lambda^{1/4}\frac{P_z^0}{2\pi dh}. \qquad (90.8)$$

Let us determine the axial displacement of the compensator under the action of stretching forces P_z^0 in the presence of interior pressure p. The displacement of section $\theta=\text{const}$ relative to fixed section $\theta=\frac{\pi}{2}$ is equal to

$$\Delta_z(0)\approx\int_{\pi/2}^{\theta} a\theta_1\cos\theta\,d\theta.$$

Relative displacement of edges $\theta=-\frac{\pi}{2}$, $\theta=\frac{3\pi}{2}$ is calculated by the formula

$$\Delta_z=\frac{2a}{Eh}\int_{-\pi/2}^{+\pi/2}\frac{\operatorname{Re}\sigma_1\cos\theta}{\lambda+\sin\theta}\,d\theta\approx$$

$$\approx\frac{4\gamma^2a^2}{Eh}\left[\frac{P_z^0}{\pi a^2}+p(2\lambda-1)\right]\int_0^{\pi/2}\frac{\cos^2\theta}{\sin\theta}u_0\operatorname{Re} e_0(i\mu_1u_0)\mu_1\,d\theta,$$

since $\operatorname{Re} e_0(i\mu_1u_0)=\operatorname{Re} e_0(-i\mu_1u_0)$, $u_0\approx v_0$. For large values of parameter μ_1 the integral in the right side of this equation is approximately calculated with the aid of (89.35)

$$\int_0^{\pi/2}\frac{\cos^2\theta}{\sin\theta}u_0\operatorname{Re} e_0(i\mu_1u_0)\,d(\mu_1\theta)\approx\int_0^{\infty}\operatorname{Re} e_0(iy)\,dy=\frac{\pi}{2}.$$

Relative axial displacement of the edges of the compensator is

$$\Delta_z = \sqrt{12(1-\mu^2)}\,\frac{a}{Eh^2}\left[P_z^{?} + p\pi a^2(2\lambda - 1)\right]. \tag{90.9}$$

Under bending load $M \neq 0$, $P = 0$ the particular solution of equation (86.25), possessing the form

$$\sigma_3 = \begin{cases} -\mu_1 2\gamma^2 \Phi_3(\theta)(\lambda + \sin\theta)^3 \dfrac{u_0}{\sin\theta} e_0(t), & \theta > 0, \\ -\mu_1 2\gamma^2 \Phi_3(\theta)(\lambda + \sin\theta)^3 \dfrac{v_0}{\sin\theta_1} e_0(t_1), & \theta < 0, \end{cases}$$

$$\frac{d\sigma_3}{d\theta} = \begin{cases} -i\mu_1^2 2\gamma^2 \Phi_3(\theta)(\lambda + \sin\theta)^3 \dfrac{u_0}{\sin\theta} \dfrac{du_0}{d\theta} e_0'(t), & \theta > 0, \\ -i\mu_1^2 2\gamma^2 \Phi_3(\theta)(\lambda + \sin\theta)^3 \dfrac{v_0}{\sin\theta_1} \dfrac{dv_0}{d\theta_1} e_0'(t_1), & \theta < 0, \end{cases}$$

satisfies the conditions $\sigma_3\left(\pm\frac{\pi}{2}\right) = 0$, since in this instance $\Phi_3(0) = \frac{M}{\pi}\frac{\cos\theta}{\nu^2}$. Using this solution to determine angular displacement of half the tubular compensator under the action of bending moment M, we obtain

$$\omega_y = \int\limits_{\pi/2}^{-\pi/2} \frac{\mathrm{Re}\,\sigma_3 \cos\theta}{Eh\lambda^3}\,d\theta \approx 2\gamma^2 \frac{M}{\pi a^2}\frac{1}{Eh\lambda^2}\cdot 2\int\limits_0^{\pi/2} \cos^2\theta \frac{u_0}{\sin\theta}\,\mathrm{Re}\,e_0(t)\,\mu_1\,d\theta.$$

Calculating the integral just as was done above, finally we have

$$\omega_y = \sqrt{12(1-\mu^2)}\,\frac{M}{aEh^2\lambda^2}. \tag{90.10}$$

The angular displacement of the edge sections relative to one another is

$$2\omega_y = \frac{4\sqrt{3(1-\mu^2)}}{aEh^2\lambda^2} M. \tag{90.11}$$

Maximum bending stresses occur in planes $\varphi = 0, \pi$ at $\theta = 1.225\mu_1$ and are calculated using the formula

$$\sigma_{1\,max} = \pm\frac{6m_{1\,max}}{h^2},$$

where

$$m_{1\,max} = \frac{M}{\pi a \lambda^2 \mu_1} 0{,}753.$$

In this way, finally

$$\sigma_{1\,max} = \pm 0{,}955(1-\mu^2)^{-1/4}\left(\frac{a}{h}\right)^{1/2}\lambda^{-1/2}\frac{M}{d^2 h}. \qquad (90.12)$$

The formulas for calculation of the compensator given in this section are simpler and handier than the corresponding formulas obtained in §§ 87, 88. However, one ought to have in mind that the applicability of these formulas is limited by an essential requirement: $\mu_1 \gg 1$.

§ 91. Stressed State of a Quarter-Torus

Using the asymptotic solution introduced in § 89, we will examine the stressed state of a quarter of a torus-shaped shell (Fig. 41), loaded by uniform internal pressure P_z^0, by axial force p and by edge forces and moments distributed along the parallel circle (θ_0) and (θ_k) with constant intensity $H_e^0, M_1^0, H_e^k, M_1^k$ [31]. We will hold that the parameter μ_1 is so big that during determination of the arbitrary constants in solution (89.36), the mutual influence of the edges can be neglected. Then during determination of the stressed state in the vicinity of edge $\theta_0 = 0$ we can set $A_2 = 0$, i.e., write the solution in the form

$$\left.\begin{aligned} \sigma_1 &= A_1 w_1 h_1(t) + 2\gamma^2 C a \mu_1 \frac{\cos\theta}{\sin\theta} u_0 e_0(t), \\ \frac{d\sigma_1}{d\theta} &= A_1 i \mu_1 w_1 \frac{du_0}{d\theta} h_1'(t) + 2\gamma^2 i \mu_1^2 C a \operatorname{ctg}\theta\, u_0 \frac{du_0}{d\theta} e_0'(t). \end{aligned}\right\} \qquad (91.1)$$

Separating the real and imaginary parts, we have

$$\left.\begin{aligned} \operatorname{Re}\sigma_1 &= w_1 [\operatorname{Re} A_1 \operatorname{Re} h_1(t) - \operatorname{Im} A_1 \operatorname{Im} h_1(t)] + \\ &\qquad + 2\gamma^2 C a \mu_1 \frac{\cos\theta}{\sin\theta} u_0 \operatorname{Re} e_0(t), \\ \operatorname{Im}\sigma_1 &= w_1 [\operatorname{Re} A_1 \operatorname{Im} h_1(t) + \operatorname{Im} A_1 \operatorname{Re} h_1(t)] + \\ &\qquad + 2\gamma^2 C a \mu_1 \frac{\cos\theta}{\sin\theta} u_0 \operatorname{Im} e_0(t). \end{aligned}\right| \qquad (91.2)$$

$$\operatorname{Re}\frac{d\sigma_1}{d\theta}=-\mu_1 w_1\frac{du_0}{d\theta}\left[\operatorname{Re}A_1\operatorname{Im}h_1'(\xi)+\operatorname{Im}A_1\operatorname{Re}h_1'(\xi)\right]-$$
$$-2\gamma^2\mu_1^2 Ca\operatorname{ctg}\theta u_0\frac{du_0}{d\theta}\operatorname{Im}e_0'(\xi).$$
$$\operatorname{Im}\frac{d\sigma_1}{d\theta}=\mu_1 w_1\frac{du_0}{d\theta}\left[\operatorname{Re}A_1\operatorname{Re}h_1'(\xi)-\operatorname{Im}A_1\operatorname{Im}h_1'(\xi)\right]+$$
$$+2\gamma^2\mu_1^2 Ca\operatorname{ctg}\theta u_0\frac{du_0}{d\theta}\operatorname{Re}e_0'(\xi).$$

(91.2)
(Cont'd)

Using formulas (86.12), (86.13) from the edge conditions at $\theta=0$: $H_e=H_e^0$, $M_1=M_1^0$, we derive the equations for determination of the real and imaginary parts of A_1:

$$-\frac{1}{2\gamma^2\lambda^3}\operatorname{Re}A_1\operatorname{Im}h_1(0)=H_e^0-\frac{Ca}{\lambda^3}-pa,$$
$$-\frac{a\mu_1}{4\gamma^4\lambda}\left[\operatorname{Re}A_1\operatorname{Im}h_1'(0)+\operatorname{Im}A_1\operatorname{Re}h_1'(0)\right]=M_1^0. \qquad (91.3)$$

In this case we accept $\frac{du_0}{d\theta}\approx 1$, $a\approx 0$.

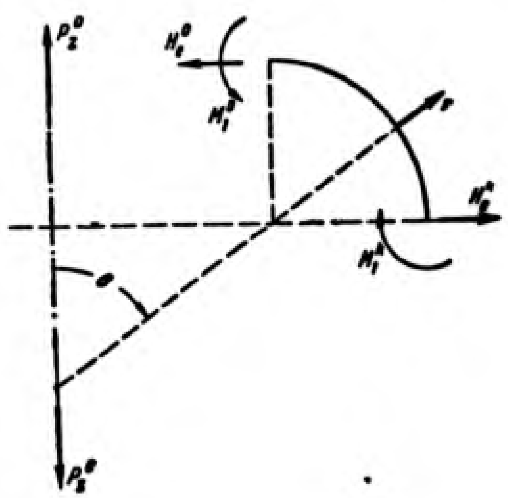


Fig. 41. A quarter section of a torus-shaped shell loaded by forces and moments evenly distributed on the edge.

Solving (91.3), we find

$$\operatorname{Re}A_1=-\frac{2\gamma^2\lambda^3}{\operatorname{Im}h_1(0)}\left(H_e^0-\frac{Ca}{\lambda^3}-pa\right),$$
$$\operatorname{Im}A_1=-\frac{4\gamma^4\lambda}{a\mu_1}\frac{M_1^0}{\operatorname{Re}h_1'(0)}+$$
$$+2\gamma^2\lambda^2\left(H_e^0-\frac{Ca}{\lambda^3}-pa\right)\frac{\operatorname{Im}h_1'(0)}{\operatorname{Im}h_1(0)\operatorname{Re}h_1'(0)}. \qquad (91.4)$$

The bending moment in any section θ (in the vicinity of edge $\theta=0$) is equal to

$$\begin{aligned} M_1 = M_1^0 \frac{w_1\lambda}{\lambda+\sin\theta}\frac{\operatorname{Re} h_1'(t)}{\operatorname{Re} h_1'(0)} + \\ + \left(H_e^0 - \frac{Ca}{\lambda^3} - pa\right)\frac{\mu_1 w_1 \lambda^2 a}{2\gamma^2(\lambda+\sin\theta)}\frac{\operatorname{Re} h_1'(0)\operatorname{Im} h_1'(t) - \operatorname{Re} h_1'(t)\operatorname{Im} h_1'(0)}{\operatorname{Re} h_1'(0)\operatorname{Im} h_1(0)} - \\ - \frac{\mu_1^2 a^2}{2\gamma^2(\lambda+\sin\theta)} C \operatorname{ctg}\theta u_v \operatorname{Im} e_v'(t). \end{aligned} \qquad (91.5)$$

Calculating on the basis of (86.14), (91.1), (91.4) the angle of rotation and radial displacement at edge $\theta=0$, we obtain

$$\left.\begin{aligned} \vartheta_1^0 = -\frac{4\gamma^4}{Eha\mu_1}\frac{\operatorname{Im} h_1(0)}{\operatorname{Re} h_1'(0)} M_1^0 + \frac{2\gamma^2\lambda}{Eh}\frac{\operatorname{Im} h_1'(0)}{\operatorname{Re} h_1'(0)} H_e^0 - \\ - \frac{2\gamma^2\mu_1}{Eh}\frac{P_z^0}{2\pi a}\operatorname{Re} e_0(0) - \frac{2\gamma^2\lambda pa}{Eh}\frac{\operatorname{Im} h_1'(0)}{\operatorname{Re} h_1'(0)}, \\ \Delta_e^0 = \frac{a\lambda}{Eh}(T_2^0 - \mu T_1^0) = \\ = \frac{a\lambda^2\mu_1}{Eh}\frac{[\operatorname{Re} h_1'(0)]^2 + [\operatorname{Im} h_1'(0)]^2}{\operatorname{Im} h_1(0)\operatorname{Re} h_1'(0)}(H_e^0 - pa) - \\ - \frac{2\gamma^2\lambda}{Eh}\frac{\operatorname{Im} h_1'(0)}{\operatorname{Re} h_1'(0)} M_1^0 - \frac{P_z^0 d}{2\pi a Eh}\mu_1^2 \operatorname{Re} e_0'(0). \end{aligned}\right\} \qquad (91.6)$$

During calculations it has been taken into consideration that in the involved case

$$v_0 = a\lambda, \quad C = \frac{\lambda P_z^0}{2\pi a^2} \qquad (91.7)$$

furthermore, quantities of order $\frac{1}{\mu_1}$ in comparison with unity were dropped.

Let us designate relative axial displacement of the edges θ_0, θ_k, assuming that the edge $\theta_k = \frac{\pi}{2}$ is free from forces

$$M_1^k = H_e^k = 0. \qquad (91.8)$$

In this instance the edge effect of edge θ_k is absent, constant A_2 is accurately equal to zero and axial displacement is determined on the basis of (91.1) by the formula

$$\Delta_z(\theta) = -\frac{a}{Eh}\int_0^{\theta}\frac{\operatorname{Re}\sigma_1\cos\theta}{\lambda+\sin\theta}\,d\theta + D. \tag{91.9}$$

Constant of integration D is found from the condition $\Delta_z\left(\frac{\pi}{2}\right)=0$, then

$$\Delta_z(\theta) = \frac{a}{Eh}\int_{\theta}^{\frac{\pi}{2}}\frac{\operatorname{Re}\sigma_1\cos\theta}{\lambda+\sin\theta}\,d\theta. \tag{91.10}$$

Substituting into the right part of (91.10) the expression for $\operatorname{Re}\sigma_1$, according to (91.2) and (91.4), and setting

$$\frac{1}{\lambda+\sin\theta}\approx\frac{1}{\lambda},\quad \theta=0,$$

we find

$$\Delta_z(0) = -\frac{2\gamma^2\lambda a}{Eh\mu_1}(H_c^0 - pa)\left[\frac{I_1}{\operatorname{Im}h_1(0)} + \frac{I_2\operatorname{Im}h_1'(0)}{\operatorname{Im}h_1(0)\operatorname{Re}h_1'(0)}\right] + \\ + \frac{4\gamma^4}{Eh\mu_1^2}\frac{I_2}{\operatorname{Re}h_1'(0)}M_1^0 + 2\gamma^2 Ca^2\frac{\pi}{2}\frac{1}{Eh\lambda}. \tag{91.11}$$

where the designations

$$\left.\begin{aligned} \mu_1\int_0^{\pi/2}\operatorname{Re}h_1(t)\cos\theta\,d\theta &= I_1,\\ \mu_1\int_0^{\pi/2}\operatorname{Im}h_1(t)\cos\theta\,d\theta &= I_2 \end{aligned}\right\} \tag{91.12}$$

have been introduced, and, furthermore, it has been taken into account that

$$\mu_1 \int_0^{\pi/2} \operatorname{Re} e_0(t) \frac{\cos^2\theta}{\sin\theta} u_0 \, d\theta \approx \int_0^{\pi/2} \operatorname{Re} e_0(iy) \, dy = \frac{\pi}{2}.$$

Integrals (91.12) can be calculated with respect to any of the quadrature formulas, approximately setting $\cos\theta \approx 1$ and replacing the upper limit of integration by ∞, since when θ is small $\cos\theta$ is really close to unity, and for large θ the absolute value of t is great (on the strength of the fact that parameter μ_1 is big) and the functions themselves $\operatorname{Re} h_1(t)$, $\operatorname{Im} h_1(t)$ insignificantly differ from zero. In this way

$$\mu_1 \int_0^{\pi/2} \operatorname{Re} h_1(t) \cos\theta \, d\theta \approx \int_0^{\infty} \operatorname{Re} h_1(iy) \, dy$$

and, consequently,

$$I_1 \approx \int_0^{\infty} \operatorname{Re} h_1(iy) \, dy.$$

similarly

$$I_2 \approx \int_0^{\infty} \operatorname{Im} h_1(iy) \, dy.$$

Calculation using the formula for trapezoids with the aid of Table 4 of the appendix gives

$$I_1 = -0.504; \quad I_2 = -0.873.$$

Formulas (91.6), (91.11) are simple to rewrite with numerical coefficients:

$$\left.\begin{aligned} \vartheta_1^0 &= \frac{4\gamma^4}{Eha\mu_1} 1.585 M_1^0 + \frac{2\gamma^2\lambda}{Eh} 0.578 (H_e^0 - pa) - \frac{2\gamma^2\mu_1}{Eh} \frac{P_z^0}{2\pi a} 1.288, \\ \Delta_e^0 &= -\frac{a\lambda^2\mu_1}{Eh} 0.841 (H_e^0 - pa) - \frac{2\gamma^2\lambda}{Eh} 0.578 M_1^0 + \frac{P_z^0 d\mu_1^2}{2\pi a Eh} 0.939. \end{aligned}\right\} \quad (91.13)$$

$$\Delta_s^0 = -\frac{2\gamma^3\lambda a}{Eh\mu_1}(H_e^0 - pa)\,0{,}938 - \frac{4\gamma^4}{Eh\mu_1^2}\,1{,}288 M_1^0 + 2\gamma^2 \frac{P_s^0}{4Eh}. \tag{91.14}$$

§ 92. Coupling a Quarter-Torus with Two Long Cylindrical Shells. The End Walls

Let us examine the coupling of a quarter-torus with two cylindrical shells (Fig. 42) [142]. We formulate first the conditions of coupling the torus with a cylinder of less radius. The positive directions of the forces, moments and displacements in the torus and the cylinder are shown in Fig. 43. At the coupling place the conditions

$$\left.\begin{array}{ll} \vartheta_{1ц} = \vartheta_{1т}, & w_ц = \Delta_{eт}, \\ M_{1ц} = M_{1т}, & N_{1ц} = H_{eт} \end{array}\right\} \tag{92.1}$$

should be held. Between the quantities ϑ_1, w and N_1, M_1 on the edge of the cylindrical shell exist the relationships (formulas (27.21))

$$\left.\begin{array}{r} \vartheta_{1ц} = \dfrac{2\gamma_ц^2}{Eh_ц} N_{1ц} - \dfrac{4\gamma_ц^3}{rEh_ц} M_{1ц}, \\ w_ц - \dfrac{1}{Eh_ц}\left(pr^2 - \mu\dfrac{P_s^0}{2\pi}\right) = \dfrac{2\gamma_ц r}{Eh_ц} N_1^0 - \dfrac{2\gamma_ц^2}{Eh_ц} M_1^0. \end{array}\right\}$$

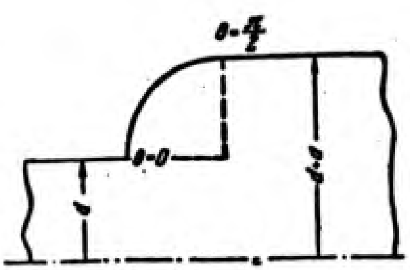


Fig. 42.

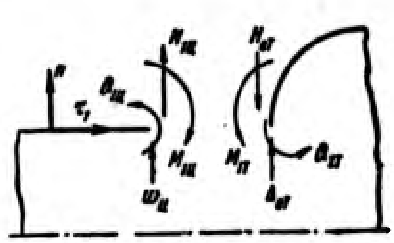


Fig. 43.

Fig. 42. Coupling a quarter-torus with two cylindrical shells.

Fig. 43. Positive directions of forces and moments in a torus and in a cylinder.

Subsequently we assume that the thicknesses of the walls of the shells are identical: $h_n = h_\tau = h$. Substituting the expressions for angle of rotation and radial displacement in the torus and in the cylinder, expressed by formulas (92.2), (91.6) through H_e and M_1, into the first two conditions of (92.1) and taking into consideration that by the remaining two conditions (92.1) $M_{1n} = M_{1\tau} = M_1^0$, $N_{1n} = H_{e\tau} = H_e^0$, we obtain two equations for determination of the moment and radial force at the place of the connection:

$$\left.\begin{aligned}
&H_e^0\left[1 - \frac{\operatorname{Im} h_1'(0)}{\operatorname{Re} h_1'(0)}\right] - \frac{\sqrt{2}\mu_1^{1/2}}{a} M_1^0\left[1 - \frac{\mu_1^{1/2}}{\sqrt{2}}\frac{\operatorname{Im} h_1(0)}{\operatorname{Re} h_1'(0)}\right] = \\
&\qquad = -\mu_1 \operatorname{Re} e_0(0)\frac{P_z^0}{2\pi d}, \\
&H_e^0\left\{1 - \frac{1}{\sqrt{2}\mu_1^{1/2}}\frac{[\operatorname{Re} h_1'(0)]^2 + [\operatorname{Im} h_1'(0)]^2}{\operatorname{Im} h_1(0)\operatorname{Re} h_1'(0)}\right\} - \\
&\qquad - \frac{\mu_1^{1/2}}{a\sqrt{2}}\left[1 - \frac{\operatorname{Im} h_1'(0)}{\operatorname{Re} h_1'(0)}\right] M_1^0 = -\frac{P_z^0}{2\pi d}\frac{\mu_1^{1/2}}{\sqrt{2}} \operatorname{Re} e_0'(0).
\end{aligned}\right\}$$

During the composition of equations (92.3) it was taken into account that

$$4\gamma_n^4 = 4\gamma_\tau^4\lambda^2$$

quantities of the order of $\frac{1}{\mu_1}$ in comparison with unity also are dropped. If we discard also quantities of order $\frac{1}{\mu_1^{1/2}}$ in comparison with unity, then system (92.3) is still more simplified and the solution assumes the form

$$\left.\begin{aligned}
M_1^0 &= -\frac{1}{\mu_1}\frac{\operatorname{Re} e_0(0)\operatorname{Re} h_1'(0)}{\operatorname{Im} h_1(0)}\frac{P_z^0}{2\pi\lambda}, \\
H_e^0 &= -\mu_1^{1/2}\frac{P_z^0}{2\pi d}\left\{\frac{\operatorname{Re} e_0(0)}{\sqrt{2}}\frac{[\operatorname{Re} h_1'(0) - \operatorname{Im} h_1'(0)]}{\operatorname{Im} h_1(0)} + \frac{\operatorname{Re} e_0'(0)}{\sqrt{2}}\right\}.
\end{aligned}\right\} \qquad (92.4)$$

Substituting (92.4) into formula (91.14), it is easy to see that the first term in the right part of (91.14) will be proportional to the quantity $2\gamma^2\frac{P_z^0}{2\pi E h}\mu_1^{1/2}$, and the second term to a quantity greater by a factor of $\mu_1^{1/2}$. In this way, with the accepted correctness of calculations during the determination of Δ_z^0 (and also during the determination of Δ_r'', ϑ_1^0) instead of (92.4) we can set

$$\left.\begin{aligned} M_1^0 &= -\frac{1}{\mu_1}\frac{\operatorname{Re} e_0(0)\operatorname{Re} h_1'(0)}{\operatorname{Im} h_1(0)}\frac{P_z^0}{2\pi\lambda}, \\ H_e &= 0. \end{aligned}\right\} \qquad (92.5)$$

With the aid of the first formula of (91.6) it is simple to be convinced that such a bending moment has under the edge conditions

$$\vartheta_1^0 = 0, \quad H_e^0 = 0. \qquad (92.6)$$

Considering the coupling of a torus with a cylinder of greater radius $\left(\theta = \frac{\pi}{2},\ r = a(\lambda + 1)\right)$, it is possible to arrive at the conclusion that also in this instance the conditions of the coupling of the torus with the cylinder can be approximately replaced by edge conditions of the form

$$\vartheta_1^k = 0, \quad H_r^k = 0. \qquad (92.7)$$

Note that under the conditions (92.7) the edge effect of edge $\theta_k = \frac{\pi}{2}$ is absent. The stressed state in the vicinity of this edge is described by the particular solution which for large μ_1 practically coincides with the zero-moment solution. In this way, the end wall shown in Fig. 44a can be approximately calculated as a quarter-torus under edge conditions (92.6), (92.7).

Let us determine the maximum bending moment (it is equal to $M_1(0) = M_1^0$) and the axial displacement of the end wall (Fig. 44a). The axial force acting in section $\theta = 0$, is equal to

$$P_z^0 = p\pi(d^2 - \nu_*^2) = p\pi a^2\left(\lambda^2 - \frac{\nu_*^2}{a^2}\right), \qquad (92.8)$$

where ν_* — radius of the thickening, while $\nu_* < \nu_0$.

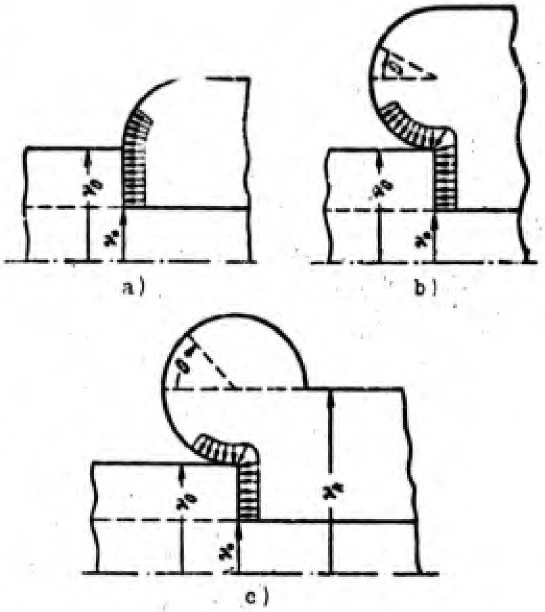


Fig. 44. End walls, containing: a) a quarter-torus, b) a half-torus and c) three quarters of a torus.

Substituting (92.5) into (91.11) and ignoring small quantities, allowing for (91.7) we obtain

$$\Delta_z^0 = \frac{2\gamma^2}{Eh}\frac{P_z^0}{4}\left[1 - \frac{2}{\pi}\frac{l_2 \operatorname{Re} e_0(0)}{\operatorname{Im} h_1(0)}\right]. \tag{92.9}$$

or

$$\Delta_z^0 = \frac{p\pi a^3}{2Eh^2}\sqrt{3(1-\mu^2)}\left(\lambda^2 - \frac{\nu_*^2}{a^2}\right)\left(1 - \frac{2,1}{\pi}\right). \tag{92.10}$$

$$M_1^0 = \frac{0{,}406}{\mu_1}\frac{pa^2\left(\lambda^2 - \frac{\nu_*^2}{a^2}\right)}{\lambda}. \tag{92.11}$$

Maximum bending stresses are computed by the formula

$$|\sigma_{1\max}| = \frac{6M_1^0}{h^2} = \frac{3\mu_1^2}{\sqrt{12(1-\mu^2)}}\frac{pa}{h}\left(\lambda^2 - \frac{\nu_*^2}{a^2}\right)0{,}813. \tag{92.12}$$

Carrying out the same approximation approach for calculation of the end wall, containing a half-torus (Fig. 44b), we obtain the formulas

$$P_z^0 = p\pi(\nu_0^2 - \nu_*^2).$$

$$C = \frac{\lambda}{2}\left[\frac{p(\nu_0^2-\nu_*^2)}{a^2} + p\left(\lambda^2 - \frac{\nu_0^2}{a^2}\right)\right] = \frac{\lambda}{2}\, p\left(\lambda^2 - \frac{\nu_*^2}{a^2}\right). \qquad (92.13)\ (92.14)$$

$$\Delta_z = \frac{p\pi a^3}{Eh^2}\sqrt{3(1-\mu^2)}\left(\lambda^2 - \frac{\nu_*^2}{a^2}\right). \qquad (92.15)$$

Maximum bending stresses occur at $\theta = \pm\theta_*\left(\theta_* = \frac{1{,}225}{\mu_1}\right)$ and are equal to

$$|\sigma_{1\max}| = \frac{3\mu_1^2}{\sqrt{12(1-\mu^2)}}\frac{pa}{h}\left(\lambda^2 - \frac{\nu_*^2}{a^2}\right)0{,}753. \qquad (92.16)$$

To determine the axial displacement of the wall containing the torus with an aperture angle of $\frac{3\pi}{2}$ (Fig. 44c), it is necessary to combine displacement of the section containing the half-torus (formula (92.15)), and the displacement of the quarter-torus (length $\frac{\pi}{2} \leqslant \theta \leqslant \pi$). During calculation of the latter we must assume that on section ν_k $(\nu_k = a\lambda)$ is applied an axial stretching force $P_z^0 = p\pi(\nu_k^2 - \nu_*^2)$. In this way, on this part of the torus

$$C = \frac{\lambda}{2}\left[\frac{P_z^0}{\pi a^2} + \left(\lambda^2 - \frac{\nu_k^2}{a^2}\right)p\right] = \frac{\lambda}{2}\, p\left(\lambda^2 - \frac{\nu_*^2}{a^2}\right)$$

the axial displacement of this part is determined again using formula (92.10). The total displacement of the wall is equal to

$$\Delta_z = \frac{p\pi a^3}{Eh^2}\sqrt{3(1-\mu^2)}\left(\lambda^2 - \frac{\nu_*^2}{a^2}\right)\left(\frac{3}{2} - \frac{1{,}05}{\pi}\right). \qquad (92.17)$$

Bending stresses in section $\theta = \pm\theta_*$ in this wall are determined according to formula (92.16), and stresses in section $\theta = \pi$ (at the place of coupling with the cylinder) using formula (92.12). Note that practically the coupling of the torus with the cylinder is carried out smoothly and the stress peak in this place flattens itself As the working stresses therefore we use the stresses in sections $\theta = \pm\theta_*$.

Let us compare the rigidity and the stressed state of end walls of different form at an identical shell thickness h and constant radii ν_0 and ν_k (Fig. 45). Wall 1 is formed by a torus-shaped shell with the parameters

$$a_{(1)}=\frac{\nu_k-\nu_0}{2},\quad d_{(1)}=\frac{\nu_k+\nu_0}{2},$$
$$\mu_1^{(1)}=\left(\sqrt{12(1-\mu^2)}\frac{a_1}{h\lambda_1}\right)^{1/2}=\left[\sqrt{12(1-\mu^2)}\frac{(\nu_k-\nu_0)^2}{2h(\nu_k+\nu_0)}\right]^{1/2}.$$
$$\lambda_{(1)}=\frac{\nu_k+\nu_0}{\nu_k-\nu_0},\quad d_{(1)}=a_{(1)}\lambda_{(1)}.$$

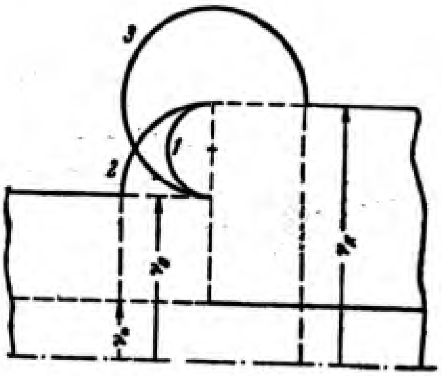


Fig. 45. End walls of different forms, which couple two cylinders of preassigned radius.

The axial displacement of the wall and the maximum bending stresses in it are equal to

$$\left.\begin{aligned}\Delta_z^{(1)}&=\frac{p\pi\sqrt{3(1-\mu^2)}}{Eh^2}\,\frac{(\nu_k-\nu_0)}{2}\left[\frac{(\nu_k+\nu_0)^2}{4}-\nu_0^2\right],\\ \sigma_{max}^{(1)}&=\frac{3p}{h^{3/2}\left(\sqrt{12(1-\mu^2)}\right)^{1/2}}\,\frac{1}{a_{(1)}^{1/2}\lambda_{(1)}^{2/3}}\left(d_{(1)}^2-\nu_0^2\right)0{,}753.\end{aligned}\right\}\qquad(92.18)$$

Wall 2 contains a torus which possesses the dimensions

$$a_{(2)}=\nu_k-\nu_0,\quad d_{(2)}=\nu_0,$$
$$\lambda_{(2)}=\frac{\nu_0}{\nu_k-\nu_0},\quad \mu_1^{(2)}=\left(\sqrt{12(1-\mu^2)}\frac{a_{(2)}}{h\lambda_{(2)}}\right)^{1/2}.$$

according to (92.10), (92.12),

$$\left.\begin{aligned} \Delta_z^{(2)} &= \frac{p\pi}{Eh^2}\sqrt{3(1-\mu^2)}\,\frac{(\nu_k-\nu_0)}{2}(\nu_0^2-\nu_*^2)\left(1-\frac{2{,}1}{\pi}\right), \\ \sigma_{max}^{(2)} &= \frac{3p}{h^{4/3}\left(\sqrt{12(1-\mu^2)}\right)^{1/3}}\,\frac{1}{a_{(2)}^{1/3}\lambda_{(2)}^{2/3}}(d_{(2)}^2-\nu_*^2)\,0{,}813. \end{aligned}\right\} \qquad (92.19)$$

Wall 3, containing a torus with aperture angle equal to $\frac{3\pi}{2}$, has the dimensions

$$a_{(3)} = \nu_k - \nu_0, \quad d_{(3)} = \nu_k, \quad \lambda_{(3)} = \frac{\nu_k}{\nu_k - \nu_0},$$
$$\mu_1^{(3)} = \left(\sqrt{12(1-\mu^2)}\,\frac{a_{(3)}}{h\lambda_{(3)}}\right)^{1/3},$$

the axial displacement and maximum stresses in it are equal to

$$\left.\begin{aligned} \Delta_z^{(3)} &= \frac{p\pi}{Eh^2}\sqrt{3(1-\mu^2)}(\nu_k-\nu_0)(\nu_k^2-\nu_*^2)\left(\frac{3}{2}-\frac{1{,}05}{\pi}\right), \\ \sigma_{max}^{(3)} &= \frac{3p}{h^{4/3}\left(\sqrt{12(1-\mu^2)}\right)^{1/3}}\,\frac{1}{a_{(3)}^{1/3}\lambda_{(3)}^{2/3}}(d_{(2)}^2-\nu_*^2)\cdot 0{,}813. \end{aligned}\right\} \qquad (92.20)$$

Let us make up the ratio of axial displacements

$$\Delta_z^{(1)} : \Delta_z^{(2)} : \Delta_z^{(3)} =$$
$$= \left[\frac{(\nu_k+\nu_0)^2}{4} - \nu_*^2\right] : \left[(\nu_0^2-\nu_*^2)\left(1-\frac{2{,}1}{\pi}\right)\right] : \left[(\nu_k^2-\nu_*^2)\left(3-\frac{2{,}1}{\pi}\right)\right].$$

This ratio shows that of the examined three walls the most advantageo with respect to rigidity (i.e., giving the least axial displacement) is wall 2. Really, we minimize the values of first and third terms of the ratio, replacing in them ν_k by ν_0, then we get the ratio

$$1 : \left(1-\frac{2{,}1}{\pi}\right) : \left(3-\frac{2{,}1}{\pi}\right),$$

from which it is clear that the displacement of wall 2 is less than any of the minimized displacements of walls 1 and 3.

Comparing the ratio of maximum stresses, we have

$$\sigma^{(1)} : \sigma^{(2)} : \sigma^{(3)} = \\ = \left[\frac{1}{a_{(1)}^{1/2}\lambda_{(1)}^{2/3}}(d_{(1)}^2 - \nu_e^2)\frac{0{,}753}{0{,}813}\right] : \left[\frac{1}{a_{(2)}^{1/2}\lambda_{(2)}^{2/3}}(d_{(2)}^2 - \nu_e^2)\right] : \left[\frac{1}{a_{(3)}^{1/2}\lambda_{(3)}^{2/3}}(d_{(3)}^2 - \nu_e^2)\right].$$

Since

$$a_{(2)} = a_{(3)}, \quad a_{(3)} = 2a_{(1)},$$
$$\frac{\lambda_{(3)}}{\lambda_{(2)}} = \frac{\nu_k}{\nu_0}, \quad \frac{\lambda_{(3)}}{\lambda_{(1)}} = \frac{\nu_k}{\nu_k + \nu_e},$$

this ratio can be rewritten thus:

$$\sigma^{(1)} : \sigma^{(2)} : \sigma^{(3)} = \\ = \left[2^{1/2}\left(\frac{\nu_k}{\nu_k + \nu_e}\right)^{2/3}(d_{(1)}^2 - \nu_e^2)\,0{,}926\right] : \left[\left(\frac{\nu_1}{\nu_0}\right)^{2/3}(d_{(2)}^2 - \nu_e^2)\right] : (d_{(3)}^2 - \nu_e^2).$$

In every concrete case it is simple to calculate the ratio of maximum stresses in the walls of the involved types. For example, at

$$\nu_k = 96\ см, \quad \nu_0 = 47{,}5\ см, \quad \nu_e = 26{,}5\ см$$

the ratio proves to be the following:

$$4000 : 2480 : 8500 = 1{,}61 : 1 : 3{,}43.$$

Wall 2 is in this instance the most advantageous with respect to stress.

§ 93. Extension of the Lens Compensator

Let us examine the lens compensator, the half-lens of which consists of two sections of a torus-shaped shell [140] (Fig. 46).

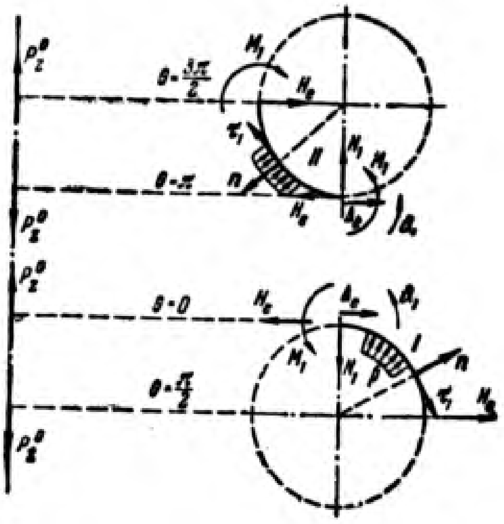


Fig. 46. The composition of edge conditions and coupling conditions of two sections of a torus-shaped shell, forming the half-lens of a compensator.

Length I of the shell is characterized by a change in angular coordinate θ within limits from 0 to $\frac{\pi}{2}$. in section II θ changes from π to $\frac{3\pi}{2}$. The directions of the normal and tangential vectors (n, τ) and positive directions of forces and moments are shown in the same place (Fig. 46). Formulas (86.12)-(86.14) and resolvant equation (86.11) are adequate for a description of the stressed state of the shell in section II. It is necessary only to take into account that of the loads – pressure and axial force – have in this section directions which are opposite those which were accepted earlier in § 86. In this way, instead of (86.11)-(86.14) we have

$$(\lambda+\sin\theta)\frac{d^2\sigma_1^{II}}{d\theta^2}-\cos\theta\frac{d\sigma_1^{II}}{d\theta}-2i\gamma^2\sigma_1^{II}\sin\theta=4\gamma^4Ca\cos\theta. \tag{93.1}$$

$$\left.\begin{aligned}
T_1^{II}&=-\frac{1}{2\gamma^2}\frac{\operatorname{Im}\sigma_1^{II}\cos\theta}{(\lambda+\sin\theta)^2}-Ca\frac{\lambda\sin\theta+1}{\lambda(\lambda+\sin\theta)^2}-\frac{pa}{2}\frac{2\lambda+\sin\theta}{\lambda+\sin\theta},\\
T_2^{II}&=-\frac{1}{2\gamma^2}\left[\frac{1}{\lambda+\sin\theta}\operatorname{Im}\frac{d\sigma_1^{II}}{d\theta}-\frac{\cos\theta\operatorname{Im}\sigma_1^{II}}{(\lambda+\sin\theta)^2}\right]-\\
&\qquad-\frac{pa}{2}+Ca\frac{\lambda\sin\theta+1}{\lambda(\lambda+\sin\theta)^2},\\
N_1^{II}&=-\frac{1}{2\gamma^2}\frac{\operatorname{Im}\sigma_1^{II}\sin\theta}{(\lambda+\sin\theta)^2}+Ca\frac{\cos\theta}{(\lambda+\sin\theta)^2},\\
H_e^{II}&=N_1^{II}\sin\theta+T_1^{II}\cos\theta.
\end{aligned}\right\} \tag{93.2}$$

$$M_1^{II}=\frac{a}{4\gamma^4}\left[\frac{1}{\lambda+\sin\theta}\operatorname{Re}\frac{d\sigma_1^{II}}{d\theta}-\frac{(1-\mu)\cos\theta}{(\lambda+\sin\theta)^2}\operatorname{Re}\sigma_1^{II}\right]. \tag{93.3}$$

$$\left.\begin{aligned}
&Eh\vartheta_1^{II}=-\frac{\operatorname{Re}\sigma_1^{II}}{\lambda+\sin\theta},\quad Eh\Delta_e^{II}=\nu(T_2^{II}-\mu T_1^{II}).\\
&\Delta_z^{II}\approx\int_{\theta_0}^{\theta}a\vartheta_1^{II}\cos\theta\,d\theta+K_1.
\end{aligned}\right\} \tag{93.4}$$

In the involved case $v_0 = a(\lambda - 1)$, therefore here

$$C = \frac{\lambda}{2}\left[\frac{P_z^0}{\pi a^2} + p(2\lambda - 1)\right]. \qquad (93.5)$$

We transform equation (93.1) using the substitution

$$\theta = \pi + \theta_1; \qquad (93.6)$$

we obtain

$$(\lambda - \sin\theta_1)\frac{d^2\sigma_1^{II}}{d\theta_1^2} + \cos\theta_1 \frac{d\sigma_1^{II}}{d\theta_1} + 2i\gamma^2\sigma_1^{II}\sin\theta_1 = -4\gamma^4 Ca\cos\theta_1,$$

or, which is the same,

$$\frac{d}{d\theta_1}\left[\frac{1}{1 - a\sin\theta_1}\frac{d\sigma_1^{II}}{d\theta_1}\right] + 2i\gamma^2\sigma_1^{II}\frac{a\sin\theta_1}{(1 - a\sin\theta_1)^2} = -\frac{4\gamma^4 Ca\cos\theta_1 a}{(1 - a\sin\theta_1)^2}.$$

This equation coincides with equation (89.41), and the solution has the form

$$\left.\begin{aligned} \sigma_1^{II} &= B_1 w_1(\theta_1) h_1(t_1) + B_2 w_1(\theta_1) h_2(t_1) + \\ &\quad + 2\gamma^2 Ca\mu_1 \operatorname{ctg}\theta_1 v_0(\theta_1) e_0(t_1), \\ \frac{d\sigma_1^{II}}{d\theta} &= \frac{d\sigma_1^{II}}{d\theta_1} = -[i\mu_1 w_1 v_0'(\theta_1) B_1 h_1'(t_1) + i\mu_1 w_1 v_0'(\theta_1) B_2 h_2'(t_1) + \\ &\quad + i\mu_1^2 2\gamma^2 Ca \operatorname{ctg}\theta_1 v_0(\theta_1) v_0'(\theta_1) e_0'(t_1)] \end{aligned}\right\} \qquad (93.7)$$

while

$$t_1 = -i\mu_1 v_0(\theta_1), \quad \theta_1 = \theta - \pi. \qquad (93.8)$$

The solution

$$h_1[-i\mu_1 v_0(\theta_1)] = \overline{h_2[i\mu_1 v_0(\theta_1)]}.$$

having as a factor the constant B_1, describes the stressed state in the neighborhood of edge $\theta_1=\frac{\pi}{2}\left(\theta=\frac{3\pi}{2}\right)$, since it decreases in absolute value from edge $\theta_1=\frac{\pi}{2}$ to edge $\theta_1=0$. The solution $h_2(t_1)=\overline{h_1[i\mu_1v_0(\theta_1)]}$ describes the stressed state in the vicinity of edge $\theta_1=0$ $(\theta=\pi)$.

On section 1 formulas (89.36) are effective. Complex constants of integration B_1, B_2 and A_1, A_2, in (93.7) and (89.36), are determined from the edge conditions and coupling conditions. From considerations of symmetry in section $\theta=\frac{\pi}{2}$ we set equal to zero the angle of rotatio and radial force

$$\vartheta_1^{\text{I}}\left(\frac{\pi}{2}\right)=H_e^{\text{I}}\left(\frac{\pi}{2}\right)=0. \tag{93.9}$$

For simplicity of the solution the same conditions are put on edge $\theta=\frac{3\pi}{2}$:

$$\vartheta_1^{\text{II}}\left(\frac{3\pi}{2}\right)=H_e^{\text{II}}\left(\frac{3\pi}{2}\right)=0. \tag{93.10}$$

Let us note that these conditions are accurate if we imagine that we consider one of the interior lenses of compensator consisting of several lenses. However, for the extreme lens, combinable with a cylindrical tube, the inaccuracy connected with the acceptance of such conditions is small.

Coupling conditions, as it is easy to see from the illustration (Fig. 46), should have the following form:

$$\left.\begin{array}{ll} H_e^{\text{I}}(0)=-H_e^{\text{II}}(\pi), & M_1^{\text{I}}(0)=-M_1^{\text{II}}(\pi), \\ \Delta_e^{\text{I}}(0)=\Delta_e^{\text{II}}(\pi), & \vartheta_1^{\text{I}}(0)=\vartheta_1^{\text{II}}(\pi). \end{array}\right\} \tag{93.11}$$

Let us note that since $H_e^{\text{I}}(0)=T_1^{\text{I}}(0)$, and $H_e^{\text{II}}(\pi)=-T_1^{\text{II}}(\pi)$, the first conditio of (93.11) is equivalent to the condition $T_1^{\text{I}}(0)=T_1^{\text{II}}(\pi)$. Taking this into account, we replace the first two conditions of (93.11) by the following:

$$\left.\begin{aligned} T_1^{I}(0) &= T_1^{II}(\pi), \\ T_2^{I}(0) &= T_2^{II}(\pi). \end{aligned}\right\} \tag{93.12}$$

Using formulas (93.2)-(93.4) and (86.12)-(86.14) we write out the quantities necessary for composition of the edge conditions and coupling conditions:

$$\left.\begin{aligned} \vartheta_1^{I}\left(\frac{\pi}{2}\right) &= -\frac{1}{Eh(\lambda+1)}\,\mathrm{Re}\,\sigma_1^{I}\left(\frac{\pi}{2}\right), \\ H_e^{I}\left(\frac{\pi}{2}\right) &= -\frac{1}{2\gamma^2(\lambda+1)^2}\,\mathrm{Im}\,\sigma_1^{I}\left(\frac{\pi}{2}\right). \end{aligned}\right\} \tag{93.13}$$

$$\left.\begin{aligned} \vartheta_1^{II}\left(\frac{3\pi}{2}\right) &= -\frac{1}{Eh(\lambda-1)}\,\mathrm{Re}\,\sigma_1^{II}\left(\frac{3\pi}{2}\right), \\ H_e^{II}\left(\frac{3\pi}{2}\right) &= -\frac{1}{2\gamma^2(\lambda-1)^2}\,\mathrm{Im}\,\sigma_1^{II}\left(\frac{3\pi}{2}\right). \end{aligned}\right\} \tag{93.14}$$

$$\left.\begin{aligned} T_1^{I}(0) &= -\frac{1}{2\gamma^2\lambda^2}\,\mathrm{Im}\,\sigma_1^{I}(0)+\frac{Ca}{\lambda^3}+pa, \\ T_2^{I}(0) &= -\frac{1}{2\gamma^2\lambda}\left(\mathrm{Im}\,\frac{d\sigma_1^{I}}{d\theta}\right)_{\theta=0}+\frac{pa}{2}-\frac{Ca}{\lambda^3}+\frac{1}{2\gamma^2\lambda^2}\,\mathrm{Im}\,\sigma_1^{I}(0), \\ M_1^{I}(0) &= \frac{a}{4\gamma^4\lambda}\left[\mathrm{Re}\left(\frac{d\sigma_1^{I}}{d\theta}\right)_{\theta=0}-\frac{(1-\mu)}{\lambda}\,\mathrm{Re}\,\sigma_1^{I}(0)\right], \\ \lambda Eh\vartheta_1^{I}(0) &= -\mathrm{Re}\,\sigma_1^{I}(0). \end{aligned}\right\} \tag{93.15}$$

$$\left.\begin{aligned} T_1^{II}(\pi) &= \frac{1}{2\gamma^2\lambda^2}\,\mathrm{Im}\,\sigma_1^{II}(\pi)-\frac{Ca}{\lambda^3}-pa, \\ T_2^{II}(\pi) &= -\frac{1}{2\gamma^2\lambda}\left(\mathrm{Im}\,\frac{d\sigma_1^{II}}{d\theta}\right)_{\theta=\pi}-\frac{pa}{2}+\frac{Ca}{\lambda^3}-\frac{1}{2\gamma^2\lambda^2}\,\mathrm{Im}\,\sigma_1^{II}(\pi), \\ M_1^{II}(\pi) &= \frac{a}{4\gamma^4\lambda}\left[\left(\mathrm{Re}\,\frac{d\sigma_1^{II}}{d\theta}\right)_{\theta=\pi}+\frac{(1-\mu)}{\lambda}\,\mathrm{Re}\,\sigma_1^{II}(\pi)\right], \\ \lambda Eh\vartheta_1^{II}(\pi) &= -\mathrm{Re}\,\sigma_1^{II}(\pi). \end{aligned}\right\} \tag{93.16}$$

With the aid of formulas (93.13)-(93.16) edge conditions (93.9), (93.10), coupling condition (93.12) and the remaining two conditions of (93.11) are brought to the form

$$\sigma_1^{I}\left(\frac{\pi}{2}\right)=0, \quad \sigma_1^{II}\left(\frac{3\pi}{2}\right)=0. \tag{93.17}$$

$$\overline{\sigma_1^{II}(\pi)} = \sigma_1^{I}(0) - i\left(\frac{2Ca}{\lambda}\,2\gamma^2 + 2pa\,2\gamma^2\lambda^2\right). \tag{93.18}$$

$$\overline{\left(\frac{d\sigma_1^{II}}{d\theta}\right)_{\theta=\pi}} = -\left(\frac{d\sigma_1^{I}}{d\theta}\right)_{\theta=0} + i\,3pa\,2\gamma^2\lambda. \tag{93.19}$$

In deriving (93.19) Poisson coefficient μ was accepted as zero. From the form of formulas (93.16) it follows that this is equivalent to neglecting terms of the order of $\frac{1}{\mu_1}$ in comparison with unity, i.e., it will agree withthe accepted accuracy of calculations. Formulas (93.17)-(93.19) serve for determination of unknown constants A_1, A_2, B_1, B_2. We determine the constants ignoring the mutual influence of edges. Edge conditions (93.17) are satisfied if we set

$$A_2 = B_1 = 0. \tag{93.20}$$

Taking into account (93.20), we have

$$\left.\begin{aligned} \sigma_1^{\mathrm{I}}(0) &= A_1 h_1(0) + 2\gamma^2 Ca\mu_1 e_0(0), \\ \left(\frac{d\sigma_1^{\mathrm{I}}}{d\theta}\right)_{\theta=0} &= A_1 i\mu_1 h_1'(0) + 2\gamma^2 iCa\mu_1^2 e_0'(0), \\ \sigma_1^{\mathrm{II}}(\pi) &= B_2 h_2(0) + 2\gamma^2 Ca\mu_1 e_0(0), \\ \left(\frac{d\sigma_1^{\mathrm{II}}}{d\theta}\right)_{\theta=\pi} &= -i\mu_1 B_2 h_2'(0) - i\mu_1^2 2\gamma^2 Cae_0'(0). \end{aligned}\right\} \tag{93.21}$$

Substituting expressions (93.21) into conditions (93.18), (93.19), we obtain a system of equations for determination of th constants A_1, B_2:

$$\left.\begin{aligned} &\overline{B_2}\,\overline{h_2(0)} = A_1 h_1(0) - i\,2\gamma^2\left(\frac{2Ca}{\lambda} + \lambda^2 2pa\right), \\ &\overline{B_2}\,\overline{h_2'(0)} + \mu_1 2\gamma^2 Ca\overline{e_0'(0)} = \\ &\qquad = -A_1 h_1'(0) - 2\gamma^2\mu_1 Cae_0'(0) + \frac{3pa}{\mu_1} 2\gamma^2\lambda. \end{aligned}\right\} \tag{93.22}$$

Remembering the properties of functions $h_1(\xi), h_2(\xi), e_0(\xi)$, we have

$$\overline{h_2(0)} = h_1(0), \quad \overline{h_2'(0)} = h_1'(0),$$
$$\overline{e_0'(0)} = e_0'(0),$$

and the solution of system (93.22) has the form

$$A_1 = -2\gamma^2\mu_1 Ca \frac{e_0'(0)}{h_1'(0)} + \underline{\frac{l}{h_1(0)} 2\gamma^2\left(\frac{Ca}{\lambda} + pa\lambda^2\right)} + \underline{\frac{3pa}{2\mu_1} 2\gamma^2\lambda \frac{1}{h_1'(0)}},$$
$$B_2 = \bar{A}_1 + \underline{\frac{l}{h_2(0)} 2\gamma^2\left(\frac{2Ca}{\lambda} + 2pa\lambda^2\right)}. \tag{93.23}$$

Dropping in the right side of (93.23) the underlined small terms, we obtain finally

$$A_1 = -2\gamma^2\mu_1 Ca \frac{e_0'(0)}{h_1'(0)},$$
$$B_2 = -2\gamma^2\mu_1 Ca \frac{\overline{e_0'(0)}}{\overline{\overline{h_1'(0)}}}. \tag{93.24}$$

Substituting (93.24) into expressions for $\sigma_1^{\text{I}}(\theta)$, $\sigma_1^{\text{II}}(\theta)$, we have

$$\sigma_1^{\text{I}}(\theta) = -2\gamma^2\mu_1 Ca \frac{e_0'(0)}{h_1'(0)} h_1(t) + 2\gamma^2 Ca\mu_1 \frac{\cos\theta}{\sin\theta} u_0(\theta)\, e_0(t),$$
$$\sigma_1^{\text{II}}(\theta) = -2\gamma^2\mu_1 Ca \frac{\overline{e_0'(0)}}{\overline{\overline{h_1'(0)}}} h_2(t_1) + 2\gamma^2 Ca\mu_1 \frac{\cos\theta_1}{\sin\theta_1} v_0(\theta_1)\, e_0(t_1),$$
$$t_1 = -l\mu_1 v_0(\theta_1), \quad \theta_1 = \theta - \pi. \tag{93.25}$$

Now, when the solution is constructed, it is simple to calculate all forces, moments and displacements in any section of the shell. In this case for the purpose of obtaining the simplest equations we will set $\frac{1}{\lambda} \approx 0$ and assume that

$$u_0(\theta) \approx \left(\frac{3}{2}\int_0^{\theta} \sqrt{\sin x}\, dx\right)^{2/3},$$
$$v_0(\theta_1) \approx \left(\frac{3}{2}\int_0^{\theta_1} \sqrt{\sin x}\, dx\right)^{2/3},$$
$$u_0'(\theta) \approx v_0'(\theta_1) \approx 1. \tag{93.26}$$

The axial displacement of segment $\theta = 0$ with respect to segment $\theta = \frac{\pi}{2}$, in the plane of symmetry of the compensator, we find from the formula

$$\Delta_z^{I} = -\frac{a}{Eh\lambda}\int\limits_{\pi/2}^{0} \mathrm{Re}\,\sigma_1^{I}\cos\theta\,d\theta.$$

The axial shift of segment $\theta = \frac{3\pi}{2}$ relative to segment $\theta = \pi$ is

$$\Delta_z^{II} = -\frac{a}{Eh\lambda}\int\limits_{\pi}^{3\pi/2} \mathrm{Re}\,\sigma_1^{II}\cos\theta\,d\theta.$$

Combining them, we obtain the shift of the half-lens under the action of axial force P_z^0 and pressure p

$$\begin{aligned}\Delta &= \frac{a}{Eh\lambda}\left[\int\limits_0^{\pi/2} \mathrm{Re}\,\sigma_1^{I}\cos\theta\,d\theta + \int\limits_0^{\pi/2} \mathrm{Re}\,\sigma_1^{II}\cos\theta_1\,d\theta_1\right] = \\ &= \frac{2Ca^2}{Eh\lambda}\,2\gamma^2\left[\int\limits_0^{\pi/2} u(0)\frac{\cos^2\theta}{\sin\theta}\,\mathrm{Re}\,e_0(t)\,\mu_1\,d\theta - \right. \\ &\left. - \int\limits_0^{\pi/2} \mathrm{Re}\left[\frac{e_0'(0)}{h_1'(0)}h_1(t)\right]\cos\theta\,\mu_1\,d\theta\right]. \end{aligned} \qquad (93.27)$$

The first term in the right side of (93.27) was approximately calculated earlier and is $\frac{\pi}{2}$. In this way,

$$\Delta = \frac{2Ca^2}{Eh\lambda}\,2\gamma^2\,\frac{\pi}{2}\,(1-\delta), \qquad (93.28)$$

where

$$\delta = \frac{2}{\pi}\int\limits_0^{\pi/2} \mathrm{Re}\left[\frac{e_0'(0)}{h_1'(0)}h_1(t)\right]\cos\theta\,\mu_1\,d\theta. \qquad (93.29)$$

The dependence of $(1-\delta)$ on parameter μ_1 is shown in Fig. 47. For large μ_1 the quantity δ is simply calculated in the following manner:

$$\delta \approx \frac{2}{\pi}\int\limits_0^{\infty} \mathrm{Re}\left[\frac{e_0'(0)}{h_1'(0)}h_1(t)\right]dt = -\frac{2}{\pi}(1{,}036 I_1 + 0{,}600 I_2) = 0{,}665,$$

$$(1-\delta) = 0{,}335.$$

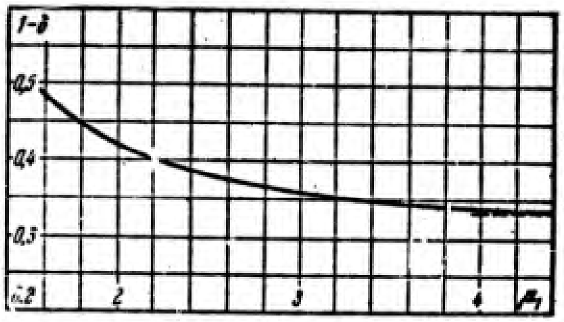


Fig. 47. Dependence $(1-\delta)$ from parameter μ_1.

The relative deviation of the edges of the whole compensator, consisting of two half-tenses, is equal to

$$2\Delta = \frac{\sqrt{12(1-\mu^2)}\,a}{Eh^2}\left[P_z^0 + p\pi a^2(2\lambda-1)\right](1-\delta). \qquad (93.30)$$

Calculating the derivatives $\frac{d\sigma_1^{I}}{d\theta}$, $\frac{d\sigma_1^{II}}{d\theta}$, and from them bending moments M_1^{I}, M_1^{II} and bending stresses, we obtain

$$\sigma_1^{I} = \pm\frac{6M_1^{I}}{h^2} = \mp\frac{6\mu_1^2 a}{2h\sqrt{12(1-\mu^2)}}\left[\frac{P_z^0}{\pi a^2} + p(2\lambda-1)\right]k(\theta), \qquad (93.31)$$

$$\sigma_1^{II} = \mp\frac{6M_1^{II}}{h^2} = \pm\frac{6\mu_1^2 a}{2h\sqrt{12(1-\mu^2)}}\left[\frac{P_z^0}{\pi a^2} + p(2\lambda-1)\right]k(\theta_1), \qquad (93.32)$$

where

$$k(\theta) = u_0(\theta)\,\mathrm{ctg}\,\theta\,\mathrm{Im}\,e_0'(t) + 1.04\,\mathrm{Im}\,h_1'(t) - 0.600\,\mathrm{Re}\,h_1'(t),$$
$$t = i\mu_1 u_0(\theta),$$
$$k(\theta_1) = u_0(\theta_1)\,\mathrm{ctg}\,\theta_1\,\mathrm{Im}\,e_0'(t) + 1.04\,\mathrm{Im}\,h_1'(t) - 0.600\,\mathrm{Re}\,h_1'(t),$$
$$t = i\mu_1 u_0(\theta_1).$$

In writing formulas (93.31), (93.32) it has been taken into account that on part II of the half lens the bending moment is considered positive, if it causes elongation of the interior filaments. Therefore stresses were calculated here using the formula $\sigma_1 = \mp\frac{6M_1^{II}}{h^2}$, in order that in both formulas the upper sign referred to stresses in the external filaments of the half-lens and the lower to stresses in interior filaments.

Setting $u_0(\theta)\operatorname{ctg}\theta\approx 1$, we determine the maximum of the $k(\theta)$ expression It exists approximately for the same value of $\mu_1 u_0(\theta_*)=1.225$ at which function $\operatorname{Im}e_0'(t)$ takes the greatest value, while $k(\theta_*)=0.430$. The maximum bending stress from axial force $(p=0)$ is

$$\sigma_{1\,max}=\frac{6\mu_1^2}{\sqrt{12(1-\mu^2)}}\,\frac{P_z^0}{2\pi ah}0.430. \tag{93.33}$$

Let us note that formulas (93.31), (93.32) are adequate for calculating stress only when $C\neq 0$. Let us examine the case when $C=0$. From formula (93.30) it follows that in this case the plates of the compensator do not separate in the axial direction $(\Delta_z=0)$ and the distance force acting on the extreme sections of the compensator, because of the presence of internal pressure is equal to

$$P_z^0=-p(2\lambda-1). \tag{93.34}$$

Thus, in order to make the axial displacement of the half-lens equal to zero, on the compensator, it is necessary to apply two compressive forces in the amount $p(2\lambda-1)$. In this case stresses from pressure in the half-lenses of the compensator will differ from zero. To determine them we keep in formulas (93.23) the underlined terms, sinc the basic terms in this instance cancel out. As a result we will have

$$\left.\begin{aligned}A_1&=\frac{i}{h_1(0)}\,pa\,2\gamma^2\lambda^2,\\ B_2&=\frac{i}{h_1(0)}\,pa\,2\gamma^2\lambda^2.\end{aligned}\right\} \tag{93.35}$$

The formula for calculation of flexural stresses in this instance assumes the form

$$\sigma_1=\pm\frac{pa}{h}\,\frac{6\mu_1}{1.074\sqrt{12(1-\mu^2)}}\operatorname{Im}h_1'(t). \tag{93.36}$$

while

$t = l\mu_1 u_0(\theta)$ for part I, $t = l\mu_1 u_0(\theta_1)$ for part II.

The maximum stresses from pressure in the compensator at $\Delta_z = 0$ are in segment $\theta = 0$ and are equal to

$$\sigma_{1\,max} = \frac{3\mu_1}{\sqrt{3(1-\mu^2)}} \frac{pa}{h} 0.365. \qquad (93.37)$$

We will make the calculation of a compensator which possesses the dimensions:

$$a = 4.8 \text{ cm}, \quad \lambda = 8.35,$$
$$h = 0.6 \text{ cm}, \quad \mu_1 = 1.47.$$

The force necessary for elongation of the compensator by $2\Delta = 0.1$ cm in the absence of internal pressure is determined with the aid of formula (93.30) and Fig. 47.

The results of experimental analysis of this compensator are presented in [122].

During the action of force $P_z^0 = 3650$ kg, computable using formula (93.30), the amount of axial shift is 0.04 cm, and experiment gives a value of 0.039 cm.

Figure 8 gives the graphs of meridian flexural stress in external filaments of a compensator during elongation by $2\Delta_z = 0.1$ cm ($p = 0$) (curve 1) and with internal pressure $p = 20$ kg/cm^2 (axial shift $\Delta_z = 0$) (curve 2). The dotted line plots experimental curves, and the continuous line the calculated curves.

The comparison of the calculated and experimental data indicates that the approximate formulas based on asymptotic solution of resolvant equations are adequate for rough practical calculations even when parameter μ_1 insignificantly exceeds one.

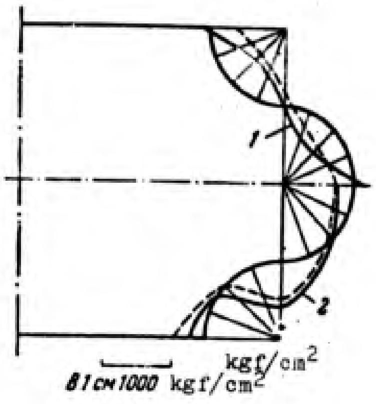


Fig. 48. Meridia. flexural stress in external filaments of a compensator during elongation by 0.1 cm (curve 1) and with internal pressure $p=20$ kg/cm^2 (curve 2).

At the end of § 90 of this chapter on the basis of the asymptotic solution a tubular compensator was examined. Let us compare tubular and lens compensators which possess an identical radius of the torus a and identical radius of the tube (Fig. 49a, b).

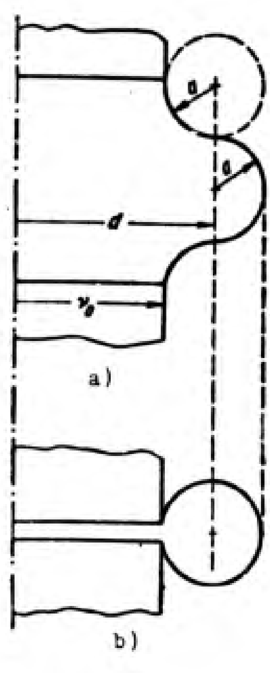


Fig. 49. Lens (a) and tubular (b) compensators, having identical radius of the torus and radius of the tube.

Comparing formulas (93.30) and (90.9), we see that at an assigned force P_z^0 $(p=0)$ the shift of the lens compensator will be $(1-\delta)=0.335$ of the shift of a tubular compensator. Assuming that both compensators must pick up the same temperature elongation or shortening of the tube Δ, we find that in this case $P_\tau^0 = P_z^0 \cdot 0.335$. From comparison of formulas (90.5) and (93.33) at $p=0$ it follows that for equal Δ the flexural stresses in the lens compensator exceed stresses in a tubular compensator, namely:

$$\frac{\sigma_{1л}}{\sigma_{1т}} = \frac{0{,}430}{0{,}753}\frac{P^0_л}{P^0_т} = 1{,}7.$$

§ 94. Torus-Shaped Shell with Parameters $\frac{2\gamma^2}{\lambda} \gg 1$, $\lambda < 1$ (Fig. 50)

In the previous sections we considered torus-shaped shells with geometric dimensions satisfying the conditions $\frac{2\gamma^2}{\lambda} \gg 1$, $\lambda > 1$, where both these conditions essentially were used: the first in creation of an asymptotic approach to the solution of the basic equation, the second in deriving the working formulas in different concrete cases. In applications, however, we find shells with other relationships of the parameters.

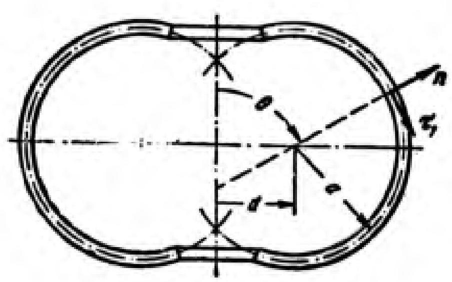


Fig. 50. Torus-shaped shell with parameter of the torus $\lambda = \frac{d}{a} < 1$.

We will stop at $\frac{2\gamma^2}{\lambda} \gg 1$, $\lambda < 1$ [141]. For an example we will consider axisymmetric deformation. In the area of change of θ, not containing the vicinity of point $\theta_* = -\arcsin\lambda$, the solution is constructed the same way as was described in § 89, and is presented by formulas (89.36) for $\theta > 0$ and formulas (89.45) for $\theta < 0$. The value of the integrals

$$\int_0^\theta \sqrt{\frac{\sin x}{\lambda + \sin x}}\,dx \quad \text{and} \quad \int_0^\theta \sqrt{\frac{\sin x}{\lambda - \sin x}}\,dx \tag{94.1}$$

at $\lambda < 1$ are given in Tables 8, 9 of the appendix. Near $\theta_* = -\arcsin\lambda$ the torus is a shell of the conic class. Leaving equation (86.7) instead of substitution (86.9) we make the following:

$$\sigma_0 = \frac{\sigma_1}{\lambda + \sin\theta} + 2i\gamma^2\frac{\Phi_2}{a\sin\theta}, \quad \theta = -\theta_1. \tag{94.2}$$

Then for determination of σ_1 we obtain the equation

$$(\lambda - \sin\theta_1)\frac{d^2\sigma_1}{d\theta_1^2} + \cos\theta_1 \frac{d\sigma_1}{d\theta_1} + 2l\gamma^2\sigma_1 \sin\theta_1 = \psi(\theta_1). \qquad (94.3)$$

where

$$\psi(\theta_1) = -\frac{2l\gamma^2}{a}\Big[-\frac{P_z^0}{2\pi}\frac{\cos\theta_1}{\sin\theta_1} + \frac{P_z^0}{2\pi}\frac{\cos\theta_1}{\sin^2\theta_1}(\lambda - \sin\theta_1) + (\lambda - \sin\theta_1)^2\left(\frac{P_z^0}{2\pi}\frac{2\cos\theta_1}{\sin^3\theta_1} - \frac{3}{2}pa^2\frac{\cos\theta_1}{\sin\theta_1}\right) + (\lambda - \sin\theta_1)^3\frac{5}{2}pa^2\frac{\cos\theta_1}{\sin^2\theta_1} + (\lambda - \sin\theta_1)^4 pa^2\frac{\cos\theta_1}{\sin^3\theta_1}\Big]. \qquad (94.4)$$

In the left and right sides of the obtained equation we drop terms of the order of $1/2\gamma^2$ in comparison with unity. We introduce new variables x and τ, connected to θ_1 and σ_1 by the relationships

$$x = \int_{\theta_1}^{\theta_{1*}} \sqrt{\frac{\sin\varphi}{\lambda - \sin\varphi}}\, d\varphi, \quad \sigma_1 = \tau\frac{(\lambda - \sin\theta_1)^{1/4}}{(\sin\theta_1)^{1/4}}. \qquad (94.5)$$

In this case we find that τ should satisfy the equation

$$\frac{d^2\tau}{dx^2} + \tau\Big\{2l\gamma^2 + \frac{3}{4} + \frac{1}{4}\frac{\lambda - \sin\theta_1}{\sin\theta_1} + \cos^2\theta_1\Big[-\frac{15}{16}\frac{1}{\sin\theta_1(\lambda - \sin\theta_1)} + \frac{1}{8}\frac{1}{\sin^2\theta_1} + \frac{5}{16}\frac{\lambda - \sin\theta_1}{\sin^3\theta_1}\Big]\Big\} = \frac{\psi(\theta_1)}{(\sin\theta_1)^{1/4}\cdot(\lambda - \sin\theta_1)^{3/4}}. \qquad (94.6)$$

Improper integral (94.5) converges, since, with the exception of $\lambda = 1$, $\theta_{1*} = \frac{\pi}{2}$, when φ is close to θ_{1*}, the condition

$$\sqrt{\frac{\sin\varphi}{\lambda - \sin\varphi}} = \sqrt{\frac{\sin\varphi}{2\cos\frac{\theta_{1*}+\varphi}{2}\sin\frac{\theta_{1*}-\varphi}{2}}} < \sqrt{\frac{\sin\theta_{1*}}{\cos\theta_{1*}}}\frac{1}{\left[2\sin\frac{\theta_{1*}-\varphi}{2}\right]^{1/2}} \approx \sqrt{\frac{\sin\theta_{1*}}{\cos\theta_{1*}}}\frac{1}{(\theta_{1*}-\varphi)^{1/2}} \quad (\varphi < \theta_{1*}). \qquad (94.7)$$

is executed, where $\sqrt{\operatorname{tg}\theta_{1*}}$ is a bound quantity.

Since when θ_1 is close to θ_{1*} we can write

$$\lambda - \sin\theta_1 = \sin\theta_{1*} - \sin\theta_1 \approx (\theta_{1*} - \theta_1)\cos\theta_{1*},$$
$$x \approx 2(\theta_{1*} - \theta_1)^{1/2}\sqrt{\operatorname{tg}\theta_{1*}},$$

then, in this case

$$\lambda - \sin\theta_1 \approx \frac{1}{4}\frac{\cos^2\theta_{1*}}{\sin\theta_{1*}}x^2. \tag{94.8}$$

Taking into account (94.8), we will rewrite equation (94.6) in the form

$$\left.\begin{aligned} &\frac{d^2\tau}{dx^2} + \tau\left[2i\gamma^2 - \frac{15}{4x^2} + \ldots\right] = A_0 x^{-1/2}(1 + a_1x^2 + \ldots), \\ &A_0 = \frac{2i\gamma^2}{a}\frac{P_z^0}{2\pi}\frac{\cos\theta_1}{(\sin\theta_1)^{1/4}}\left(\frac{1}{4}\frac{\cos^2\theta_{1*}}{\sin\theta_{1*}}\right)^{-1/4},\end{aligned}\right\} \tag{94.9}$$

where the dots designate terms containing higher positive powers (x^2). In the coefficient of τ the term containing large parameter $2\gamma^2$ is kept, and the main singularity is separated, characterizing the amount of increase of the function when $\theta_1 \to \theta_{1*}$. Let us note that the term $-\frac{15}{4x^2}$ when $\theta_1 \ll \theta_{1*}$ strongly differs from the corresponding term $-\frac{15}{16}\frac{\cos^2\theta_1}{\sin\theta_1(\lambda - \sin\theta_1)}$ in equation (94.6), however, inasmuch as in this case it is small in comparison with $2i\gamma^2$, this difference has no value. The equation

$$\frac{d^2y}{dx^2} + y\left(2i\gamma^2 - \frac{15}{4x^2}\right) = 0 \tag{94.10}$$

we call the "standard" with respect to uniform equation (94.6), understanding by this that the solution of equation (94.10) asymptotically approaches the solution of equation (94.6) with an increase of parameter $2\gamma^2$. Equation (94.10) was considered in § 46 during the calculation of a conic shell. Its solution has the form

$$y = x^{1/2}\left[C_1 I_2(\gamma x\sqrt{2i}) + C_2 H_2^{(1)}(\gamma x\sqrt{2i})\right].$$

In accordance with this we will write out the approximate solution of equation (94.9) in the following manner:

$$\tau \approx \tau_0 + \tau_1. \qquad (94.11)$$

where

$$\tau_0 = (\lambda - \sin\theta_1)^{1/4}\left[C_1 I_2(\gamma x\sqrt{2i}) + C_2 H_2^{(1)}(\gamma x\sqrt{2i})\right], \quad \tau_1 = \frac{1}{2i\gamma^2} A_0 x^{-3/2}.$$

when x is small

$$\tau_1 \approx \frac{P_z^0}{2\pi a}\frac{\cos\theta_1}{(\sin\theta_1)^{7/4}}(\lambda - \sin\theta_1)^{-3/4}.$$

Particular solution τ_1 is sufficiently accurate when x is close to zero; for larger x accurate knowledge of this solution is not necessary, since, as compared to the second term in the right side of (94.2), it will be a small quantity (of the order of $\frac{1}{2\gamma^2}$ in comparison with unity). Really, writing out the general solution of the initial equation (86.7) on the basis of (94.11), (94.5), (94.2), we obtain

$$\sigma_0 = \frac{P_z^0}{2\pi a}\frac{\cos\theta_1}{\sin^2\theta_1}\frac{1}{\lambda - \sin\theta_1} + 2i\gamma^2\frac{\Phi_2}{a\sin\theta} + \frac{1}{(\sin\theta_1)^{1/4}}\left[C_1 I_2(\gamma x\sqrt{2i}) + C_2 H_2^{(1)}(\gamma x\sqrt{2i})\right]. \qquad (94.12)$$

The first term in the right side corresponds to the particular solution and has essential value only when $(\lambda - \sin\theta_1) \to 0$. The second term, corresponding to the usual zero-moment solution, in this case remains bound. For a shell closed in the top $(\theta_0 = \theta_{1*}, x_0 = 0)$ in the absence of axial force it is necessary to set $P_z^0 = 0$, $C_2 = 0$. If $P_z^0 \neq 0$, then C_2 is calculated from the condition that σ_0 is finite at $x = 0$. Such a condition is realized because $H_2^{(1)}(\gamma x\sqrt{2i})$ contains singularity

$$-\frac{2}{\pi\gamma^2 x^2} = -\frac{1}{2\pi\gamma^2}\frac{\cos^2\theta_{1*}}{\sin\theta_{1*}}\frac{1}{\lambda - \sin\theta_1}.$$

Satisfying the requirement that $\sigma_0(0)$ is bounded, we have

$$C_2 = \frac{P_z^0 \gamma^2}{a} \frac{1}{(\sin\theta_{1*})^{5/4}\cos\theta_{1*}}. \qquad (94.13)$$

As an example let us examine a closed shell when there is no concentrated force in the top. From conditions of symmetry in segment $\theta = \frac{\pi}{2}$ we place the conditions

$$\vartheta_1 = 0, \quad H_e = 0.$$

Ignoring the mutual influence of the edges $\theta = -\beta$ and $\theta = \frac{\pi}{2}$, on the basis of these conditions in solution (89.36) we set $A_2 = 0$ and for the area of change of θ, not containing the top, we write out the solution of equation (86.11) in the form

$$\left.\begin{aligned}
\sigma_1 &= A_1 w_1 h_1(t) + 2\gamma^2 C a\mu_1 \frac{\cos\theta}{\sin\theta} u_0 e_0(t),\\
w_1 &= \left(\frac{u_0}{\sin\theta}\right)^{1/2}(1 + a\sin\theta)^{3/4}, \quad t = i\mu_1 u(\theta), \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2},\\
\sigma_1 &= A_1 w_1 h_1(t_1) + 2\gamma^2 C a\mu_1 \frac{\cos\theta_1}{\sin\theta_1} v_0 e_0(t_1),\\
w_1 &= \left(\frac{v_0}{\sin\theta_1}\right)^{1/2}(1 - a\sin\theta_1)^{3/4}, \quad t_1 = -i\mu_1 v_0(\theta_1),\\
\theta_1 &= -\theta, \quad \theta_* \ll \theta < 0.
\end{aligned}\right\} \qquad (94.14)$$

According to (86.9) the following solution of the initial equation (86.7) corresponds to it:

$$\left.\begin{aligned}
\sigma_0 &= \frac{1}{(\lambda + \sin\theta)}\left[A_1 w_1 h_1(i\mu_1 u_0) + 2\gamma^2 C a\mu_1 \frac{\cos\theta}{\sin\theta} u_0 e_0(t)\right] +\\
&\quad + \frac{2i\gamma^2\Phi_2}{a\sin\theta} + 2i\gamma^2 C a \frac{\cos\theta}{(\lambda + \sin\theta)\sin\theta}, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2},\\
\sigma_0 &= \frac{1}{(\lambda - \sin\theta_1)}\left[A_1 w_1 h_1(-i\mu_1 v_0) + 2\gamma^2 C a\mu_1 \frac{\cos\theta_1}{\sin\theta_1} v_0 e_0(-i\mu_1 v_0)\right] +\\
&\quad + \frac{2i\gamma^2\Phi_2}{a\sin\theta} + 2i\gamma^2 C a \frac{\cos\theta}{(\lambda + \sin\theta)\sin\theta}, \qquad \theta_* \ll \theta < 0.
\end{aligned}\right\} \qquad (94.15)$$

where $P_z^0 = 0$, $v_0 = 0$ and

$$C = \frac{1}{2}\left[\frac{P_z^0}{\pi a^2} + p\left(\lambda^2 - \frac{v_0^2}{a^2}\right)\right] = \frac{p\lambda^3}{2}. \qquad (94.16)$$

In a certain segment $\theta_1 = -\theta = \beta$, rather far from point $\theta_1 = \theta = 0$ when $2\gamma^2 \gg 1$, setting $e_0(t) \approx \frac{1}{t}$, the expression for $\sigma_0(-\beta)$ can be written in the form

$$\sigma_0(-\beta) = \frac{A_1}{\lambda - \sin\theta_1} w_1 h_1(-i\mu_1 v_0) - \frac{2i\gamma^2 \Phi_2(-\beta)}{a \sin\beta}.$$

Using solution (94.12), adequate in the neighborhood of $\theta_{1*} = \arcsin\lambda$, not containing point $\theta_1 = 0$, we write

$$\sigma_0(-\beta) = \frac{1}{(\sin\theta_1)^{1/4}} C_1 I_2(\gamma x \sqrt{2i}) - 2i\gamma^2 \frac{\Phi_2(-\beta)}{a \sin\beta}.$$

Both solutions coincide in the given point together with all its derivatives if we set $A_1 = C_1 = 0$. In this way the conic part of the shell containing top $\theta = -\arcsin\lambda$, is located practically in the zero-moment state. The essential bending stresses and the large catenary stresses, considerably exceeding zero-moment, appear in the neighborhood of point $\theta = 0$.

Figures 51, 52 show the distribution of forces and moments in a shell with parameters $2\gamma^2 = 165$, $\lambda = 0.9$, $C = \frac{p\lambda^3}{2}$. In this case curves 1, 2 and 3 in Fig. 51 represent T_1/pa, T_2/pa, N_1/pa, and curves 1, 2 Fig. 52 the distribution of $2\gamma^2 M_1/pa^2$, $2\gamma^2 M_2/pa^2$.

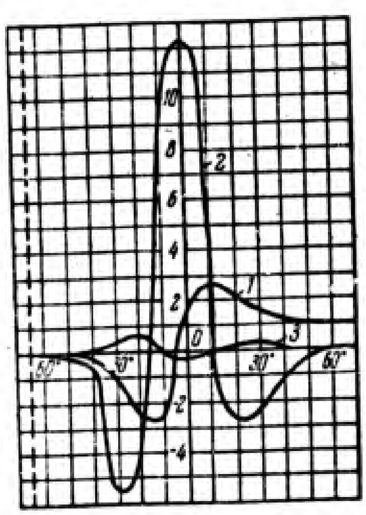


Fig. 51. Distribution of forces T_1/pa, T_2/pa, N_1/pa (curves 1, 2, 3 respectively) along the meridian of the torus with parameter $\lambda = 0.9$.

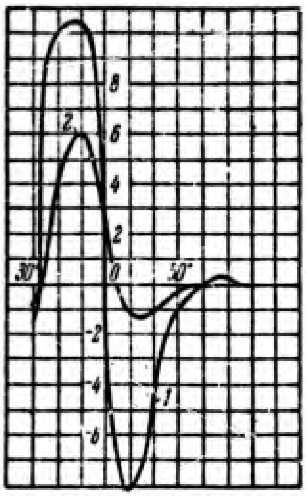


Fig. 52. Distribution of bending moments $2\gamma^2 M_1/pa^2$, $2\gamma^2 M_2/pa^2$ (curves 1, 2) along the meridian of the torus with parameter $\lambda = 0.9$.

It is interesting to note that in this case in a closed torus-shaped shell ($\lambda < 1$) under the action of internal pressure essential flexural stresses unlike the closed torus at $\lambda > 1$, were obtained, which under the action of such a load is found practically in the zero-moment state. It is natural to expect that with a decrease of λ to zero the flexural stresses will decrease, since at $\lambda = 0$ the torus will become a sphere.

CHAPTER VII

INTERNAL STRESSES IN SHELLS OF REVOLUTION

§ 95. Formulation of the Problem

The problem about determination of the stressed state which appears in an elastic body under the action of applied external forces is the ordinary problem of the theory of elasticity. The total system of equations of the ordinary problem consists of equations which connect stresses and the components of deformations expressed through displacements, equations of equilibrium and boundary conditions (the case of purely static boundary conditions is considered). The equations of equilibrium or boundary conditions (and perhaps also others) contain nonzero right parts, which depend on the assigned external loads. The components of deformation satisfy uniform differential equations - the equations of compatibility which are identities relative to the components of the displacements. On solving the problem under stresses, the system of equations consisting of equations of statics and equations of compatibility, also written in stresses, is heterogeneous either because of the equations of equilibrium, or because of the boundary conditions.

It is possible to imagine yet another problem, the total system of equations of which will consist of uniform equations of statics and uniform static boundary conditions, but heterogeneous differential equations relative to the components of deformation, connected with the components of stress by the same Hooke's law. In essence, these differential relationships between the components of deformation now should be called equations of "incompatibility" of deformations.

This type of problem is called the dislocation problem. An illustration of such a problem is the problem of the determination of internal stresses in a body free from external forces during nonlinear temperature distribution in the space occupied by the body. In §§ 20-23 the problem of determination of temperature stresses in a free shell of revolution was considered. The components of internal forces and moments are connected in this instance by uniform equations of equilibrium, and they should satisfy uniform boundary conditions. The components of deformation consist of two members: the first members are connected with forces and moments by the usual Hooke's law (we call them conditionally the "elastic" components of deformations), the second members (let us call them the dislocation members) are the temperature elongations which would take place in the individual shell elements, are not connected with each other. The total components of deformation should, of course, satisfy the equations of compatibility, which now turn into nonuniform differential equations relative to the "elastic" components. The right part of these equations define the incompatibility of the deformations induced by the assigned temperature distribution.

The dislocation components of deformation in general can be formed by the usual formulas expressing deformations through displacements with the aid of a certain potential function, which possesses formally the structure of a vector of displacement. However, unlike the actual displacement, the potential function is a multiple-valued vectorial function.

For the shells of revolution in which it appears an internal axisymmetric stressed state or a state which varies according to the law of cos ϕ (sin ϕ), let us formulate such a dislocation problem.

There exist uniform equations of statics. They admit the first two integrals, in which the constants of integration are equal to zero, since external forces are absent. There exist also uniform equations of compatibility. However, the constants of integration in the first two integrals of the equations of compatibility, unlike the usual problem, now are not equal to zero. Internal forces and moments are connected with the "elastic" components of deformation, which are in the left part of the equations of continuity (or their first integrals) by the usual relationships of elasticity. The edges of the shell θ_0, θ_1 are free from external forces. The enumerated equations for a system which is heterogeneous because of the nonzero constants of integration in the first integrals of the equations of continuity. As will be indicated below (§§ 96-98), such heterogeneity indicates the incompatibility of deformations calculated through a potential function of the following form [31]:

$$U^{\lambda} = [U_0 + \Omega \times r] \frac{\varphi}{2\pi}, \tag{95.1}$$

U_0 . Ω – constant vectors, which possess in the system of orthogonal axes XYZ the components

$$u_{0x},\ u_{0y},\ u_{0z},$$
$$\omega_x,\ \omega_y,\ \omega_z.$$

function U^{λ} is multiple-valued: on going around any closed contour $(\varphi_0 \cdot \varphi_0 + 2\pi)$ it grows by the amount contained in the right part of (95.1) in parentheses. The stressed state corresponding to potential function (95.1) appears in an initially unstressed shell if in it we make a cut along the meridian $\phi = 0$ (or on some other curve, intersecting both edges θ = const), shift the edges of the section and turn one relative to the other by the amount U^0 . Ω and then again combine. This procedure is realizable only for such shells whose meridian does not intersect the axis of revolution, for example, for shells in the form of a strip (spherical, conic and others), cylindrical and toroidal shells, while a toridal shell can be both closed and with respect to coordinate θ [118], [170].

Let us write out, using Table § 15, the components of vectorial function U^{λ} in movable axes τ_1, τ_2, n

$$\left.\begin{aligned}
u^{\lambda} &= [(u_{0x}\cos\varphi + u_{0y}\sin\varphi)\cos\theta - u_{0z}\sin\theta + \\
&\quad + (\omega_y\cos\varphi - \omega_x\sin\varphi)(Z\cos\theta + v\sin\theta)]\frac{\varphi}{2\pi}, \\
v^{\lambda} &= [-u_{0x}\sin\varphi + u_{0y}\cos\varphi + v\omega_z - \\
&\quad - (\omega_y\sin\varphi + \omega_x\cos\varphi)Z]\frac{\varphi}{2\pi}, \\
w^{\lambda} &= [(u_{0x}\cos\varphi + u_{0y}\sin\varphi)\sin\theta + \cos\theta u_{0z} - \\
&\quad - (\omega_y\cos\varphi - \omega_x\sin\varphi)(v\cos\theta - Z\sin\theta)]\frac{\varphi}{2\pi}.
\end{aligned}\right\} \tag{95.2}$$

Calculating on the basis of (95.2) and (3.19) the component of deformation, we will differentiate the axisymmetric and nonaxisymmetric cases.

1. Axiaymmetric case:

a)

$$\left.\begin{aligned} &\gamma^{s} = -\frac{1}{2\pi\nu} u_{0z} \sin\theta, \quad \tau^{s} = \frac{1}{2\pi\nu^{2}} u_{0z} \cos^{2}\theta, \\ &\varepsilon_1^{s} = \varepsilon_2^{s} = \varkappa_1^{s} = \varkappa_2^{s} = 0. \end{aligned}\right\} \tag{95.3}$$

b)

$$\left.\begin{aligned} &\varepsilon_2^{s} = +\frac{1}{2\pi} \omega_z, \quad \varkappa_2^{s} = +\frac{1}{2\pi\nu} \omega_z \sin\theta, \\ &\varepsilon_1^{s} = \gamma^{s} = \varkappa_1^{s} = \tau_1^{s} = 0. \end{aligned}\right\} \tag{95.4}$$

2. Nonaxisymmetric case:

a)

$$\left.\begin{aligned} &\varepsilon_2^{s} = \frac{1}{2\pi\nu} (u_{0z} - Z\omega_x) \cos\varphi, \\ &\gamma^{s} = \frac{1}{2\pi\nu} [u_{0y} \cos\theta - \omega_x (Z\cos\theta + \nu\sin\theta)] \sin\varphi, \\ &\varkappa_2^{s} = \frac{1}{2\pi\nu^{2}} [-u_{0y} \sin\theta - \omega_x (2\nu\cos\theta - Z\sin\theta)] \cos\varphi, \\ &\tau^{s} = \frac{1}{2\pi\nu^{2}} [u_{0y} \sin\theta\cos\theta - \omega_x \sin\theta (\nu\sin\theta + Z\cos\theta)] \sin\varphi, \\ &\varepsilon_1^{s} = \varkappa_1^{s} = 0. \end{aligned}\right\} \tag{95.5}$$

b)

$$\left.\begin{aligned} &\varepsilon_2^{s} = \frac{1}{2\pi\nu} (-u_{0x} - Z\omega_y) \sin\varphi, \\ &\gamma^{s} = \frac{1}{2\pi\nu} [u_{0x} \cos\theta + \omega_y (Z\cos\theta + \nu\sin\theta)] \cos\varphi, \\ &\varkappa_2^{s} = \frac{1}{2\pi\nu^{2}} [u_{0x} \sin\theta - \omega_y (\nu\cos\theta - Z\sin\theta)] \sin\varphi, \\ &\tau^{s} = \frac{1}{2\pi\nu^{2}} [u_{0x} \sin\theta\cos\theta + \omega_y \sin\theta (\nu\sin\theta + Z\cos\theta)] \cos\varphi, \\ &\varepsilon_1^{s} = \varkappa_1^{s} = 0. \end{aligned}\right\} \tag{95.6}$$

§ 96. The First Case of an Axisymmetric Stressed State

Let us consider an axisymmetric case 1 a). Deformations (95.3) atisfy differential equations of compatibility (3.30). The second quation of (3.30), in the considered case can be written in the orm of

$$\frac{d}{d\theta}\left[\nu^{2}\left(-\tau + \frac{\gamma}{2R_{2}}\right)\right] + \frac{d}{d\theta}\left(\nu\sin\theta\frac{\gamma}{2}\right) = 0. \tag{96.1}$$

or

$$\nu^2\left(-\tau+\frac{\gamma}{2R_2}\right)+\nu\sin\theta\frac{\gamma}{2}=A_1. \qquad (96.2)$$

Substituting (95.3) into equation (96.2), we are convinced that constant A_1, which is the measure of the incompatibility of dislocati deformations, is equal to

$$A_1=-\frac{u_{z0}}{2\pi}. \qquad (96.3)$$

The incompatibility of dislocation deformations is the reason for the beginning of "elastic: deformations, connected with forces and moments by the elasticity relationships. The total components of deformation should satisfy all the equations of compatibility, including equation (96.2) at $A_1 = 0$. Hence we obtain

$$\nu^2\left(-\tau^e+\frac{\gamma^e}{2R_2}\right)+\nu\sin\theta\frac{\gamma^e}{2}=\frac{u_{z0}}{2\pi}, \qquad (96.4)$$

here the "e" marks "elastic" components of deformation.

The second equation of equilibrium (4.22) allows the first integral, where the constant of integration in it is equal to zero, since external intensities are absent,

$$\nu S+2H\sin\theta=0. \qquad (96.5)$$

Adding to (96.4), (96.5) the relationships

$$H=D(1-\mu)\tau^e,\quad S=B\frac{1-\mu}{2}\gamma^e, \qquad (96.6)$$

we obtain the total system of equations for determination of unknown quantities τ^e, γ^e, S and H. Substituting (96.6) into (96.5), we find

$$\gamma^e=-\frac{1}{3}\frac{h^2}{\nu}\sin\theta\tau^e.$$

Taking into account this relationship, from (96.4) we have

$$\tau^e\approx\frac{u_{z0}}{2\pi\nu^2}. \qquad (96.7)$$

Internal force and twisting moment are equal to

$$S = -\frac{E}{\pi(1+\mu)} \frac{h^3}{12\nu^3} \sin\theta\, u_{0z}. \qquad (96.8)$$

$$H = \frac{Eh}{\pi 24(1+\mu)} \frac{h^3}{\nu^3} u_{0z}. \qquad (96.9)$$

The greatest tangential stress appear from the twisting moment

$$|\tau_{12}| \approx |\tau_{21}| \approx \frac{6H}{h^2} = \frac{E}{\pi 4(1+\mu)} \frac{h^2}{\nu^2} \frac{u_{0z}}{h}. \qquad (96.10)$$

§ 97. The Second Case of an Axisymmetric Stressed State. Meissner Equations

We will consider axisymmetric case 1 b) (§ 95). Substituting deformations (95.4) into equations of continuity (11.5) and (11.7), we find that (11.7) is satisfied identically, and constant C_1 in the right part of (11.5), characterizing the incompatibility of dislocation deformations, is equal to

$$C_1 = +\frac{\omega_z}{2\pi}. \qquad (97.1)$$

Deformations noted by "*e*" should satisfy the equations

$$\left.\begin{aligned} &\nu\varkappa_2^e \sin\theta + \frac{\cos\theta}{R_1}\left[\frac{d}{d\theta}(\nu\varepsilon_2^e) - \varepsilon_1^e R_1 \cos\theta\right] = -\frac{\omega_z}{2\pi}, \\ &\frac{d}{d\theta}\left\{\nu\cos\theta\varkappa_2^e - \frac{\sin\theta}{R_1}\left[\frac{d}{d\theta}(\nu\varepsilon_2^e) - \varepsilon_1^e R_1\cos\theta\right]\right\} - \varkappa_1^e R_1 = 0. \end{aligned}\right\} \qquad (97.2)$$

Equations (97.2) express fact that total deformations $(\varepsilon_1^e + \varepsilon_1^s)$, $(\varepsilon_2^e + \varepsilon_2^s)$ etc., satisfy equations (11.5), (11.7) at $C_1 = 0$.

Adding to (97.2) uniform equations of statics, which are obtained from (11.1), (11.3) at $q_v = q_z = 0$, $P_z^0 = 0$,

$$\left.\begin{aligned} &\frac{d}{d\theta}[\nu(T_1\cos\theta + N_1\sin\theta)] - T_2 R_1 = 0, \\ &T_1\sin\theta - N_1\cos\theta = 0, \end{aligned}\right\} \qquad (97.3)$$

$$\nu R_1 N_1 = \frac{d}{d\theta}(\nu M_1) - M_2 R_1 \cos\theta \qquad (97.4)$$

and relationships of elasticity

$$\left.\begin{aligned} \varepsilon_1^e &= \frac{1}{Eh}(T_1 - \mu T_2), & \varepsilon_2^e &= \frac{1}{Eh}(T_2 - \mu T_1), \\ \varkappa_1^e &= \frac{12}{Eh^3}(M_1 - \mu M_2), & \varkappa_2^e &= \frac{12}{Eh^3}(M_2 - \mu M_1), \end{aligned}\right\} \qquad (97.5)$$

we obtain the total system of equations for determination of all unknowns. Exactly as for the case of the usual axisymmetric problem, we introduce function of stresses V and the function of displacements Ψ. With the aid of representations

$$\left.\begin{aligned} &\nu T_1 = V\cos\theta, \quad T_2 = \frac{1}{R_1}\frac{dV}{d\theta}, \\ &\nu N_1 = V\sin\theta, \end{aligned}\right\} \qquad (97.6)$$

$$\left.\begin{aligned} \varkappa_1^e &= -\frac{1}{R_1}\frac{d\Psi}{d\theta} + \frac{\omega_z}{2\pi R_1 \sin^2\theta}, \\ \varkappa_2^e &= -\frac{\cos\theta}{\nu}\Psi - \frac{\omega_z}{2\pi\nu\sin\theta}, \end{aligned}\right\} \qquad (97.7)$$

$$\frac{1}{R_1}\frac{d}{d\theta}(\nu\varepsilon_2^e) - \varepsilon_1^e\cos\theta = \Psi\sin\theta \qquad (97.8)$$

equation of statics (97.3) and equation (97.2) are identically satisfied.

Expressing ε_1^e, ε_2^e and M_1, M_2 through functions V and Ψ with the aid of Hooke's law and formulas (97.6), (97.7), we obtain

$$\left.\begin{aligned} \varepsilon_1^e &= \frac{1}{Eh}\left(\frac{V\cos\theta}{\nu} - \frac{\mu}{R_1}\frac{dV}{d\theta}\right), \\ \varepsilon_2^e &= \frac{1}{Eh}\left(\frac{1}{R_1}\frac{dV}{d\theta} - \mu\frac{V\cos\theta}{\nu}\right), \end{aligned}\right\} \qquad (97.9)$$

$$\left.\begin{aligned} M_1 &= -D\left(\frac{1}{R_1}\frac{d\Psi}{d\theta} + \frac{\mu\cos\theta}{\nu}\Psi\right) + D\frac{\omega_z}{2\pi}\left(\frac{1}{R_1\sin^2\theta} - \frac{\mu}{\nu\sin\theta}\right), \\ M_2 &= -D\left(\frac{\cos\theta}{\nu}\Psi + \frac{\mu}{R_1}\frac{d\Psi}{d\theta}\right) + D\frac{\omega_z}{2\pi}\left(\frac{\mu}{R_1\sin^2\theta} - \frac{1}{\nu\sin\theta}\right). \end{aligned}\right\} \qquad (97.10)$$

Substituting (97.9) into (97.8) and (97.10) into (97.4), taking into account in this case (97.6), we obtain two equations for the determination of unknown functions V and Ψ, or the associated functions

$$V_0 = \frac{V}{b}, \quad \Psi_0 = Eh\,\Psi. \tag{97.11}$$

$$\left.\begin{aligned}
&\frac{d^2\Psi_0}{d\theta^2} + \left(\frac{R_1\cos\theta}{\nu} - \frac{1}{R_1}\frac{dR_1}{d\theta}\right)\frac{d\Psi_0}{d\theta} + \\
&+\Psi_0\left(-\frac{\mu R_1\sin\theta}{\nu} - \frac{R_1^2\cos^2\theta}{\nu^2}\right) + 4\gamma^4\frac{R_1^2}{\nu b}\sin\theta\, V_0 = \\
&\quad = +\frac{Eh\omega_z R_1}{2\pi}\left\{\frac{d}{d\theta}\left[\frac{1}{\sin^2\theta}\left(\frac{1}{R_1} - \frac{\mu}{R_2}\right)\right] + \right. \\
&\qquad \left. + \frac{R_1\cos\theta}{\nu}\frac{(1-\mu)}{\sin^2\theta}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right\}. \\
&\frac{d^2V_0}{d\theta^2} + \left(\frac{R_1\cos\theta}{\nu} - \frac{1}{R_1}\frac{dR_1}{d\theta}\right)\frac{dV_0}{d\theta} + \\
&\quad +V_0\left(\frac{\mu R_1\sin\theta}{\nu} - \frac{R_1^2\cos^2\theta}{\nu^2}\right) - \frac{R_1^2\sin\theta}{\nu b}\Psi_0 = 0.
\end{aligned}\right\} \tag{97.12}$$

It is easy to see that the left parts of (97.12), as was to be expected, wholly coincide with the left parts of equations (12.6). The right parts of (97.12) are slowly varying functions, which do not possess singularities if sin θ in the considered interval does not turn into zero. With these conditions, setting

$$\sigma = \Psi_0 - 2i\gamma^2 V_0,$$

it is simple to find the approximate particular solution of system (97.12). It has the form

$$\left.\begin{aligned}
&\tilde{\sigma} = -i\frac{E\omega_z hR_1}{2\pi\cdot 2\gamma^2}\left\{\frac{d}{d\theta}\left[\frac{1}{\sin^2\theta}\left(\frac{1}{R_1} - \frac{\mu}{R_2}\right)\right] + \right. \\
&\qquad \left. + \frac{(1-\mu)R_1\cos\theta}{\nu\sin^2\theta}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right\}. \\
&\tilde{V}_0 = -\frac{1}{2\gamma^2}\,\mathrm{Im}\,\sigma, \quad \tilde{\Psi}_0 = \mathrm{Re}\,\tilde{\sigma} = 0.
\end{aligned}\right\} \tag{97.13}$$

To solution (97.13) corresponds the purely moment stressed state of the shell

$$\left.\begin{aligned}
&\tilde{M}_1 = +\frac{Eh\omega_z b^2}{2\pi\cdot 4\gamma^4\sin^2\theta}\left(\frac{1}{R_1} - \frac{\mu}{R_2}\right), \\
&\tilde{M}_2 = +\frac{Eh\omega_z b^2}{2\pi\cdot 4\gamma^4\sin^2\theta}\left(\frac{\mu}{R_1} - \frac{1}{R_2}\right),
\end{aligned}\right\} \tag{97.14}$$

since the forces $\tilde{T}_1$, $\tilde{T}_2$, computable by this solution, have an order of magnitude of $\frac{bEh}{4\gamma^4}\frac{\omega_z h}{\nu}$ and stress from the forces $\tilde{T}_1$, $\tilde{T}_2$ are small in comparison with stresses from $\tilde{M}_1$, $\tilde{M}_2$ as h/b is small in comparison with unity. In this way, with sufficient correctness it is possible to set $\tilde{T}_1 = \tilde{T}_2 = 0$, or

$$\tilde{V}_\varphi = 0. \tag{97.15}$$

Let us note that the same solution can be constructed by setting in (97.2) $\varepsilon_2^e = \varepsilon_1^e = 0$. The simplified equations (97.2) are satisfied using representations (97.7) at $\Psi = 0$.

The constructed solution does not satisfy conditions on the edges θ = const, which should be free from streses. The first formula of (97.14) in this case gives us

$$\left.\begin{aligned} \tilde{M}_1^0 &= +\frac{Eh\omega_x b^2}{2\pi \cdot 4\gamma^4 \sin^2\theta_0}\left(\frac{1}{R_1} - \frac{\mu}{R_2}\right)_{\theta=\theta_0}, \\ \tilde{M}_1^1 &= +\frac{Eh\omega_x b^2}{2\pi \cdot 4\gamma^4 \sin^2\theta_1}\left(\frac{1}{R_1} - \frac{\mu}{R_2}\right)_{\theta=\theta_1}. \end{aligned}\right\} \tag{97.16}$$

In order to satisfy the edge conditions, for the obtained solution it is necessary to supplement solutions of the edge effect or type, which satisfy the following requirements:

$$\left.\begin{aligned} M_1^0 &= -\tilde{M}_1^0, \quad H_\varphi^0 = 0, \\ M_1^1 &= -\tilde{M}_1^1, \quad H_\varphi^1 = 0. \end{aligned}\right\} \tag{97.17}$$

§ 98. The Stressed State Proportional to cos φ (sin φ)

Let us consider the event 2 a) (§ 95). The amplitude of the total components of deformation $(\varepsilon_{1(1)}^e + \varepsilon_{1(1)}^s)$, $(\varepsilon_{2(1)}^e + \varepsilon_{2(1)}^s)$, ... should satisfy equations of continuity (15.24), (15.25), (15.33) at $C_3 = C_4 = 0$. Taking into account formulas (95.5), we obtain the following differential equations which connect the "elastic" components:

$$\left.\begin{aligned} &\nu(\varkappa_{2(1)}^e \cos\theta + \tau_{(1)}^e) - \frac{\nu\sin\theta}{R_1}\frac{d\varepsilon_{2(1)}^e}{d\theta} - \\ &\qquad - (\varepsilon_{2(1)}^e - \varepsilon_{1(1)}^e)\sin\theta\cos\theta = +\frac{\omega_x}{\pi}, \\ &-\frac{\nu\cos\theta}{R_1}\frac{d\varepsilon_{2(1)}^e}{d\theta} + \gamma_{(1)}^e\cos\theta + \varepsilon_{2(1)}^e\sin^2\theta + \varepsilon_{1(1)}^e\cos^2\theta - \\ &\qquad - \varkappa_{2(1)}^e\nu\sin\theta = -\frac{1}{\pi\nu}(u_{0y} - Z\omega_x), \\ &-R_1\varkappa_{1(1)}^e - \nu\frac{d\tau_{(1)}^e}{d\theta} - 2R_1\cos\theta\,\tau_{(1)}^e + \gamma_{(1)}^e\cos\theta + \\ &\qquad + \frac{d\gamma_{(1)}^e}{d\theta}\sin\theta + \gamma_{(1)}^e\frac{R_1\sin\theta\cos\theta}{\nu} + \frac{R_1\sin\theta}{\nu}\varepsilon_{1(1)}^e = 0. \end{aligned}\right\} \tag{98.1}$$

Excluding the domain $\theta = 0$, as the particular solution of system (98.1) we take the purely flexural solution. Setting

$$\tilde{\gamma}^e_{(1)} = \tilde{\varepsilon}^e_{2(1)} = \tilde{\varepsilon}^e_{1(1)} = 0, \tag{98.2}$$

we obtain

$$\left.\begin{aligned} \tilde{\varkappa}^e_{1(1)} &= -\frac{1}{\pi\nu R_1 \sin^2\theta}(u_{0y} - Z\omega_x), \\ \tilde{\varkappa}^e_{2(1)} &= \frac{1}{\pi\nu^2 \sin\theta}(u_{0y} - Z\omega_x), \\ \tilde{\tau}^e_{(1)} &= \frac{\omega_x}{\pi\nu} - \frac{\operatorname{ctg}\theta}{\pi\nu^2}(u_{0y} - Z\omega_x). \end{aligned}\right\} \tag{98.3}$$

To solution (98.3) corresponds the purely moment stressed state

$$\left.\begin{aligned} \tilde{m}_1 &= -\frac{D}{\pi\nu}\frac{1}{\sin^2\theta}\left(\frac{1}{R_1} - \frac{\mu}{R_2}\right)(u_{0y} - Z\omega_x), \\ \tilde{m}_2 &= -\frac{D}{\pi\nu}\frac{1}{\sin^2\theta}\left(\frac{\mu}{R_1} - \frac{1}{R_2}\right)(u_{0y} - Z\omega_x), \\ \tilde{h}_{(1)} &= -\frac{D}{\pi}(1-\mu)\left[\frac{\operatorname{ctg}\theta}{\nu^2}(u_{0y} - Z\omega_x) - \frac{\omega_x}{\nu}\right], \\ \tilde{t}_1 &= \tilde{t}_2 = \tilde{n}_1 = 0. \end{aligned}\right\} \tag{98.4}$$

Near the edges of the shell to state (98.4) it is necessary to add the stressed state of edge effect type which satisfy the edge conditions

$$\left.\begin{aligned} m_1^0 &= -\tilde{m}_1^0, \quad h_e^0 = 0, \\ m_1^1 &= -\tilde{m}_1^1, \quad h_e^1 = 0. \end{aligned}\right\} \tag{98.5}$$

§ 99. Internal Stresses in Cylindrical, Conic and Spherical Shells

Let us write out the equations for calculation of internal stresses in shells of a concrete shape. For a cylindrical shell formulas (97.14), corresponding to the purely moment axisymmetric stressed state, assume the form

$$\left.\begin{aligned} \tilde{M}_1 &= -\frac{\mu EhR}{2\pi}\frac{\omega_z}{4\gamma^4}, \\ \tilde{M}_2 &= -\frac{EhR}{2\pi}\frac{\omega_z}{4\gamma^4} \quad \left(4\gamma^4 = \frac{12(1-\mu^2)R^2}{h^2}\right). \end{aligned}\right\} \tag{99.1}$$

Setting $M_1^0 = -\tilde{M}_1, M_1^L = -\tilde{M}_1, N_1^0 = N_1^L = 0$, by formulas (27.15), (27.16) we determine the forces and moments of the edge effect

$$\left.\begin{aligned} M_1 &= -\tilde{M}_1 [\varphi(\gamma\xi)+\varphi(\gamma\xi_1)], \\ T_2 &= \tilde{M}_1 \frac{2\gamma^2}{R} [\psi(\gamma\xi)+\psi(\gamma\xi_1)], \\ M_2 &= \mu M_1 \qquad \left(\xi_1 = \frac{L}{R}-\xi\right). \end{aligned}\right| \tag{99.2}$$

Summarizing (99.1) and (99.2), we have

$$\left.\begin{aligned} M_1 &= -\frac{\mu EhR\omega_z}{2\pi\cdot 4\gamma^4}[1-\varphi(\gamma\xi)-\varphi(\gamma\xi_1)], \\ M_2 &= -\frac{EhR\omega_z}{2\pi\cdot 4\gamma^4}[1-\mu^2\varphi(\gamma\xi)-\mu^2\varphi(\gamma\xi_1)], \\ T_2 &= -\frac{\mu Eh\omega_z}{2\pi\cdot 2\gamma^2}[\psi(\gamma\xi)+\psi(\gamma\xi_1)]. \end{aligned}\right| \tag{99.3}$$

From consideration of (99.3) it is clear that the stress from bending moment M_2, taking place far from the edges of the shell, are the working stresses

$$\sigma_2 = \pm\frac{6\tilde{M}_2}{h^2} = \mp\frac{Eh\omega_z}{4\pi R(1-\mu^2)}. \tag{99.4}$$

For the dislocation which is characterized by parameters u_{0y}, ω_x, we have the purely moment stressed state, proportional to cos ϕ (sin ϕ). The amplitudes of the moments, according to (98.4), are equal to

$$\left.\begin{aligned} \tilde{m}_1 &= \frac{\mu Eh}{\pi\cdot 4\gamma^4}(u_{0y}+s\omega_x), \quad \tilde{m}_2 = \frac{Eh}{\pi\cdot 4\gamma^4}(u_{0y}+s\omega_x), \\ \tilde{h}_{(1)} &= +\frac{EhR}{\pi\cdot 4\gamma^4}\omega_x. \end{aligned}\right\} \tag{99.5}$$

When $\omega_x=0$, $u_{0y}\neq 0$, just as in the case above, the greatest stresses are obtained from the circumferential moment and exist far from the edges of the shell in segments $\varphi=0, \pi$. In absolute value they are equal to

$$|\sigma_2| = \frac{E}{2\pi(1-\mu^2)}\frac{h}{R}\frac{|u_{0y}|}{R}.$$

At $u_{0y}=0$, $\omega_x\neq 0$, determining the moments of the edge effect by formulas (34.23), setting $n_1^L=0$, $m_1^L=-\tilde{m}_1^L$, we obtain near edge $s=L$ the following expression for the amplitude of the moment

$$m_2 = +\frac{EhR}{\pi\cdot 4\gamma^4}\omega_x\,\{\xi-\mu^2 l\varphi[\gamma(l-\xi)]\}.$$

Assume $\xi=l$, $\varphi=0, \pi$ the stresses corresponding to this moment in absolute value are equal to

$$|\sigma_2| = \frac{El}{2\pi}\frac{h}{R}|\omega_x|.$$

As we see from the considered illustration, edge effect has a secondary value. Therefore for conic and spherical shells we will be limited to the determination of stresses by the purely moment state. Formulas (97.14), (98.4), allowing for the designations of § 44 for a conic shell assume the form

$$\tilde{M}_2 = -\frac{D}{2\pi}\frac{\omega_x}{\nu\cos\beta}, \quad \tilde{m}_2 = \frac{D}{\pi\nu^2\cos\beta}[u_{0y} - (\nu_0 - \nu)\operatorname{ctg}\beta\omega_x],$$
$$|\sigma_2| = \left|\frac{6\tilde{M}_2}{h^2}\right| = \frac{E}{4\pi(1-\mu^2)}\frac{h}{\nu\cos\beta}|\omega_x|,$$
$$|\sigma_2| = \left|\frac{6m_2}{h^2}\right| = \frac{Eh}{2\pi(1-\mu^2)\nu^2\cos\beta}|u_{0y} - (\nu_0 - \nu)\omega_x\operatorname{ctg}\beta|. \quad (99.6)$$

For a spherical shell we have

$$\left.\begin{aligned} \tilde{M}_1 &= -\tilde{M}_2 = +\frac{D(1-\mu)}{2\pi R\sin^2\theta}\omega_x, \\ \tilde{m}_1 &= -\tilde{m}_2 = -\frac{D(1-\mu)}{\pi R^2\sin^3\theta}[u_{0y} - R(\cos\theta - \cos\theta_1)\omega_x], \\ \tilde{h}_{(1)} &= -\frac{D(1-\mu)}{\pi R^2}\left\{\frac{\cos\theta}{\sin^3\theta}[u_{0y} - R(\cos\theta - \cos\theta_1)\omega_x] - \frac{\omega_x R}{\sin\theta}\right\}. \end{aligned}\right\} \quad (99.7)$$

$$|\sigma_1| = |\sigma_2| = \left|\frac{6\tilde{M}_1}{h^2}\right| = \frac{Eh}{4\pi R(1+\mu)}\frac{|\omega_x|}{\sin^2\theta}.$$
$$|\sigma_1| = |\sigma_2| = \left|\frac{6\tilde{m}_1}{h^2}\right| = \frac{Eh}{2\pi(1+\mu)R^2\sin^3\theta}|u_{0y} - R(\cos\theta - \cos\theta_1)\omega_x|.$$

APPENDIX

Table 1. Values of functions $\varphi, \psi, \theta, \zeta$: $\theta(x) = e^{-x}\cos x$, $\zeta(x) = e^{-x}\sin x$, $\varphi(x) = \theta(x) + \zeta(x)$, $\psi(x) = \theta(x) - \zeta(x)$

x	φ	ψ	θ	ζ
0	1,0000	1,0000	1,0000	0
0,1	0,9907	0,8100	0,9003	0,0903
0,2	0,9651	0,6398	0,8024	0,1627
0,3	0,9267	0,4888	0,7077	0,2189
0,4	0,8784	0,3564	0,6174	0,2610
0,5	0,8231	0,2415	0,5323	0,2908
0,6	0,7628	0,1431	0,4530	0,3099
0,7	0,6997	0,0599	0,3798	0,3199
0,8	0,6354	−0,0093	0,3131	0,3223
0,9	0,5712	−0,0657	0,2527	0,3185
1,0	0,5083	−0,1108	0,1988	0,3096
1,1	0,4467	−0,1457	0,1510	0,2967
1,2	0,3899	−0,1716	0,1091	0,2807
1,3	0,3355	−0,1807	0,0729	0,2626
1,4	0,2849	−0,2011	0,0419	0,2430
1,5	0,2384	−0,2068	0,0158	0,2226
1,6	0,1959	−0,2077	−0,0059	0,2018
1,7	0,1576	−0,2047	−0,0235	0,1812
1,8	0,1234	−0,1985	−0,0376	0,1610
1,9	0,0932	−0,1899	−0,0484	0,1415
2,0	0,0667	−0,1794	−0,0563	0,1230
2,1	0,0439	−0,1673	−0,0618	0,1057
2,2	0,0244	−0,1548	−0,0652	0,0895
2,3	0,0060	−0,1416	−0,0668	0,0748
2,4	−0,0056	−0,1282	−0,0669	0,0613
2,5	−0,0166	−0,1149	−0,0658	0,0492
2,6	−0,0254	−0,1019	−0,0636	0,0383
2,7	−0,0320	−0,0895	−0,0608	0,0287
2,8	−0,0369	−0,0777	−0,0573	0,0204
2,9	−0,0403	−0,0666	−0,0534	0,0132
3,0	−0,0423	−0,0563	−0,0493	0,0071
3,1	−0,0431	−0,0469	−0,0450	0,0019
3,2	−0,0431	−0,0383	−0,0407	−0,0024
3,3	−0,0422	−0,0306	−0,0364	−0,0058
3,4	−0,0408	−0,0237	−0,0323	−0,0085
3,5	−0,0389	−0,0177	−0,0283	−0,0106
3,6	−0,0366	−0,0124	−0,0245	−0,0121
3,7	−0,0341	−0,0079	−0,0210	−0,0131
3,8	−0,0314	−0,0040	−0,0177	−0,0137
3,9	−0,0286	−0,0008	−0,0147	−0,0140
4,0	−0,0258	0,0019	−0,0120	−0,0139
4,1	−0,0204	0,0057	−0,0074	−0,0131
4,2	−0,0204	0,0057	−0,0074	−0,0131
4,3	−0,0179	0,0070	−0,0054	−0,0125
4,4	−0,0155	0,0079	−0,0038	−0,0117
4,5	−0,0132	0,0085	−0,0023	−0,0108
4,6	−0,0111	0,0089	−0,0011	−0,0100
4,7	−0,0092	0,0090	0,0001	−0,0091
4,8	−0,0075	0,0089	0,0007	−0,0082
4,9	−0,0059	0,0087	0,0014	−0,0073
5,0	−0,0046	0,0084	0,0019	−0,0065
5,1	−0,0033	0,0080	0,0023	−0,0057
5,2	−0,0023	0,0075	0,0026	−0,0049
5,3	−0,0014	0,0069	0,0023	−0,0042
5.4	−0,0006	0,0064	0,0029	−0,0035
5,5	0,0000	0,0058	0,0029	−0,0029
5,6	0,0005	0,0052	0,0029	−0,0029
5,7	0,0010	0,0046	0,0028	−0,0018
5,8	0,0013	0,0041	0,0027	−0,0014
5,9	0,0015	0,0036	0,0026	−0,0010
6,0	0,0013	0,0031	0,0024	−0,0007

Table 2. Values of functions χ_1, χ_2, χ_3: $\chi_1(x) = \frac{\text{ch}\, x + \cos x}{\text{sh}\, x + \sin x}$, $\chi_2(x) = \frac{\text{sh}\, x - \sin x}{\text{sh}\, x + \sin x}$, $\chi_3(x) = \frac{\text{ch}\, x - \cos x}{\text{sh}\, x + \sin x}$

x	χ_1	χ_2	χ_3	x	χ_1	χ_2	χ_3
0,2	5,000	0,0068	0,100	1,8	0,735	0,5050	0,855
0,4	2,502	0,0268	0,200	2,0	0,738	[illegible]	0,925
0,6	1,674	0,0601	0,300	2,5	0,802	0,8220	1,045
0,8	1,267	0,1065	0,400	3,0	0,893	0,9770	1,090
1,0	1,033	0,1670	0,500	3,5	0,966	1,0500	1,085
1,2	0,890	0,2370	0,596	4,0	1,005	1,0580	1,050
1,4	0,803	0,3170	0,689	4,5	1,017	1,0400	1,027
1,6	0,755	0,4080	0,775	5,0	1,017	1,0300	1,008

Table 3. Values of functions $\psi_1, \psi_2, \psi_3, \psi_4$ and their derivatives (formulas (46.9)).

x	$\psi_1(x)$	$\psi_2(x)$	$\frac{d\psi_1(x)}{dx}$	$\frac{d\psi_2(x)}{dx}$
0,00	+1,0000	0,0000	—0,0000	0,0000
0,20	+1,0000	—0,0100	—0,0005	—0,1000
0,40	+0,9996	—0,0400	—0,0040	—0,2000
0,60	+0,9980	—0,0950	—0,0135	—0,3000
0,80	+0,9936	—0,1599	—0,0320	—0,3991
1,00	+0,9844	—0,2500	—0,0624	—0,4974
1,20	+0,9676	—0,3587	—0,1078	—0,5935
1,40	+0,9401	—0,4867	—0,1709	—0,6860
1,60	+0,8979	—0,6327	—0,2545	—0,7727
1,80	+0,8367	—0,7853	—0,3612	—0,8509
2,00	+0,7517	—0,9723	—0,4931	—0,9170
2,20	+0,6377	—1,1610	—0,6520	—0,9661
2,40	+0,4890	—1,3575	—0,8392	—0,9944
2,60	+0,3001	—1,5569	—1,0552	—0,9943
2,80	+0,0651	—1,7529	—1,2993	—0,9589
3,00	—0,2214	—2,0228	—1,7141	—0,8223
3,20	—0,5644	—2,1016	—1,8636	—0,7499
3,40	—0,9680	—2,2334	—2,1755	—0,5577
3,60	—1,4353	—2,3199	—2,4983	—0,2936
3,80	—1,9674	—2,3454	—2,8221	+0,0526
4,00	—2,5634	—2,2927	—3,1346	+0,4912
4,20	—3,2195	—2,1422	—3,4199	+1,0318
4,40	—3,9283	—1,6726	—3,6587	+1,6833
4,60	—4,6784	—1,4610	—3,8280	+2,4520
4,80	—5,4531	—0,8837	—3,9006	+3,3422
5,00	—6,2301	—0,1160	—3,8454	+4,3542
5,20	—6,9803	+0,8658	—3,6270	+6,4835
5,40	—7,6674	+2,0845	—3,2063	+6,7198
5,60	—8,2466	+3,5597	—2,5409	+8,0453
5,80	—8,7937	+5,3068	—1,5856	+9,4332
6,00	—8,8523	+7,3347	—0,2931	+10,3462
x	$\psi_3(x)$	$\psi_4(x)$	$\frac{d\psi_3(x)}{dx}$	$\frac{d\psi_4(x)}{dx}$
0,00	+0,5000		0,0000	
0,20	+0,4826	—1,1034	—0,1419	+3,1340
0,40	+0,4480	—0,6765	—0,1970	+1,4974
0,60	+0,4058	—0,4412	—0,2216	+0,9273
0,80	+0,3606	—0,2883	—0,2286	+0,6286
1,00	+0,3151	—0,1825	—0,2243	+0,4422
1,20	+0,2713	—0,1076	—0,2129	+0,3149
1,40	+0,2302	—0,0542	—0,1971	+0,2235
1,60	+0,1926	—0,0166	—0,1788	+0,1560
1,80	+0,1558	+0,0094	—0,1594	+0,1056
2,00	+0,1289	+0,0265	—0,1399	+0,0679
2,20	+0,1026	+0,0371	—0,1210	+0,0397
2,40	+0,0804	+0,0429	—0,1032	+0,0189

Table 3. (cont'd.)

x	$v_3(x)$	$v_4(x)$	$\frac{dv_3(x)}{dx}$	$\frac{dv_4(x)}{dx}$
2,60	+0,0314	+0,0446	—0,0868	+0,0039
2,80	+0,0455	+0,0447	—0,0719	+0,0065
3,00	+0,0326	+0,0427	—0,0586	—0,0137
3,20	+0,0220	+0,0394	—0,0459	—0,0137
3.40	+0,0137	+0,0356	—0,0369	—0,0204
3,60	+0,0072	+0,0314	—0,0284	—0,0213
3,80	+0,0022	+0,0260	—0,0212	—0,0210
4,00	- 0,0014	+0,0230	—0,0152	—0,0200
4,20	—0,0039	+0,0192	—0,0104	—0,0185
4,40	—0,0056	+0,0156	—0,0065	—0,0268
4,60	—0,0066	+0,0125	—0,0035	—0,0148
4,80	—0,0071	+0,0097	—0,0012	—0,0129
5,00	—0,0071	+0,0073	+0,0005	—0,0109
5,20	—0,0069	+0,0053	+0,0017	—0,0091
5,40	—0,0065	+0,0037	+0,0025	—0,0075
5,60	—0,0059	+0,0023	+0,0030	—0,0060
5,80	—0,0053	+0,0012	+0,0033	—0,0047
6,00	—0,0046	+0,0004	+0,0033	—0,0036

Table 4. Values of Airy functions $h_1(iy), h_2(iy)$ and their derivatives $\frac{dh_1}{dz}, \frac{dh_2}{dz}$ at $z=iy$

y	Re h_1	Im h_1	Re h_1'	Im h_1'
0	—0,000000	—1,074000	0,678300	0,391600
0,1	—0,039000	—1,007000	0,678200	0,386500
0,2	—0,077000	—0,938800	0,677300	0,371900
0,3	—0,113100	—0,871100	0,674900	0,349400
0,4	—0,146600	—0,803900	0,670300	0,320100
0,5	—0,177000	—0,737200	0,663000	0,285500
0,6	—0,203600	—0,671400	0,652400	0,246800
0,7	—0,226200	—0,606800	0,638400	0,205300
0,8	—0,244600	—0,543800	0,620700	0,162300
0,9	—0,258700	—0,482800	0,599300	0,118700
1,0	—0,268400	—0,424100	0,574200	0,075700
1,1	—0,273800	—0,368100	0,545700	0,034200
1,2	—0,275300	—0,315000	0,514100	—0,005000
1,3	—0,272900	—0,265300	0,479800	—0,041200
1,4	—0,267200	—0,219200	0,443300	—0,073900
1,5	—0,258300	—0,176700	0,405200	- 0,102500
1,6	—0,246800	—0,138100	0,366000	-- 0,126800
1,7	—0,233100	—0,103500	0,326400	- 0,146600
1,8	—0,217800	—0,072850	0,287000	—0,162000
1,9	—0,200800	—0,046100	0,248300	- 0,172900
2,0	—0,183200	—0,023160	0,210800	-- 0,179600
2,1	—0,165100	—0,003880	0,175100	- 0,182300
2,2	—0,146800	+0,011940	0,141600	- 0,181300
2,3	—0,128900	+0,024530	0,110600	- 0,177200
2,4	—0,111500	+0,034160	0,082410	- 0,170200
2,5	—0,094930	0,041110	0,057160	—0,160900
2,6	—0,079330	0,045690	0,034970	—0,149800
2,7	—0,065020	0,048210	0,015870	—0,137300
2,8	—0,051950	0,048970	—0,000170	—0,123900
2,9	—0,040250	0,048280	—0,013260	—0,110000
3,0	—0,029950	0,046410	—0,023580	—0,096050
3,1	—0,021030	0,043650	—0,031310	—0,082290
3,2	—0,013470	0,040230	—0,036700	—0,069070
3,3	—0,007192	0,036380	- 0,040020	—0,056620
3,4	—0,002114	0,032280	—0,041540	—0,045120
3,5	0,001867	0,028120	—0,041550	—0,034700
3,6	0,004865	0,024010	—0,040330	—0,025460
3,7	0,006999	0,020080	—0,038140	—0,017420
3,8	0,008389	0,016410	—0,035230	—0,010590
3,9	0,009155	0,013050	—0,031830	—0,004929
4,0	0,009412	0,010050	—0,028150	—0,000379
4,1	0,009265	0,007429	—0,024360	0,003146
4,2	0,008813	0,005182	—0,020600	0,005748
4,3	0,008143	0,003305	—0,016990	0,007537

Table 4. (cont'd.)

y	$\mathrm{Re}\, h_1$	$\mathrm{Im}\, h_1$	$\mathrm{Re}\, h_1'$	$\mathrm{Im}\, h_1'$
4,4	0,007329	0,001776	—0,013620	0,008629
4,5	0,006436	0,000571	—0,010550	0,009139
4,6	0,005517	—0,000346	—0,007836	0,009179
4,7	0,004613	—0,001009	—0,005482	0,008854
4,8	0,003755	—0,001454	—0,003498	0,008261
4,9	0,002967	—0,001720	—0,001872	0,007484
5,0	0,002261	—0,001840	—0,000582	0,006597
5,1	0,001649	—0,001847	0,000401	0,005663
5,2	0,001129	—0,001769	0,001112	0,004729
5,3	0,000702	—0,001632	0,001588	0,003834
5,4	0,000360	—0,001453	0,001868	0,003006
5,5	0,000098	—0,001263	0,001989	0,002264
5,6	—0,000096	—0,001064	0,001987	0,001618
5,7	—0,000229	—0,000870	0,001892	0,001072
5,8	—0,000313	—0,000688	0,001734	0,000625
5,9	—0,000358	—0,000524	0,001536	0,000272
6,0	—0,000371	—0,000381	0,001318	0,000004

y	$\mathrm{Re}\, h_2$	$\mathrm{Im}\, h_2$	$\mathrm{Re}\, h_2'$	$\mathrm{Im}\, h_2'$
0	0,0000	1,0740	0,6783	—0,3916
0,1	0,0390	1,1420	0,6784	—0,3860
0,2	0,0768	1,2100	0,6793	—0,3683
0,3	0,1122	1,2780	0,6817	—0,3372
0,4	0,1437	1,3470	0,6862	—0,2912
0,5	0,1699	1,4150	0,6933	—0,2290
0,6	0,1889	1,4850	0,7033	—0,1492
0,7	0,1991	1,5560	0,7159	—0,0503
0,8	0,1983	1,6290	0,7309	0,0692
0,9	0,1845	1,7020	0,7473	0,2108
1,0	0,1553	1,7780	0,7635	0,3762
1,1	0,1084	1,8550	0,7775	0,5570
1,2	0,0410	1,9330	0,7862	0,7849
1,3	—0,0495	2,0120	0,7859	1,0320
1,4	—0,1663	2,0900	0,7715	1,3090
1,5	—0,3123	2,1660	0,7371	1,6170
1,6	—0,4908	2,2370	0,6752	1,9580
1,7	—0,7051	2,2990	0,5768	2,3330
1,8	—0,9585	2,3500	0,4317	2,7400
1,9	—1,2540	2,3840	0,2274	3,1780
2,0	—1,5950	2,3930	—0,0499	3,6440
2,1	—1,9840	2,3710	—0,4162	4,1330
2,2	—2,4220	2,3070	—0,8893	4,6370
2,3	—2,9110	2,1890	—1,4890	5,1440
2,4	—3,4510	2,0040	—2,2360	5,6380
2,5	—4,0380	1,7360	—3,1530	6,0980
2,6	—4,6680	1,3670	—4,2620	6,4950
2,7	—5,3340	0,8764	—5,5870	6,7950
2,8	—6,0220	0,2416	—7,1490	6,9520
2,9	—6,7170	—0,5619	—8,9650	6,9100
3,0	—7,3950	—1,5600	—11,0500	6,6010
3,1	—8,0260	—2,7810	—13,4000	5,9440
3,2	—8,5690	—4,2500	—16,0200	4,8420
3,3	—8,9760	—5,9920	—18,8700	3,1840
3,4	—9,1830	—8,0310	—21,9200	0,8420
3,5	—9,1170	—10,3800	—25,0900	—2,3270
3,6	—8,6850	—13,0500	—28,2600	—6,4780
3,7	—7,7830	—16,0300	—31,2800	—11,7800
3,8	—6,2860	—19,2900	—33,9400	—18,4000
3,9	—4,0540	—22,7900	—35,9500	—26,4900
4,0	—0,9330	—26,4500	—36,9700	—36,2200
4,1	3,2480	—30,1400	—36,5300	—47,6800
4,2	8,6640	—33,6900	—34,1000	—60,9400
4,3	15,4900	—36,8700	—29,0200	—75,9600
4,4	23,9100	—39,3800	—20,5000	—92,5700
4,5	34,0500	—40,8300	—7,6600	—110,5000
4,6	46,0300	—40,7400	10,4900	—129,1600
4,7	59,8700	—38,5200	35,0500	—147,6000
4,8	75,5100	—33,4700	67,1500	—164,8000
4,9	92,7400	—24,8000	107,9000	—179,1000
5,0	111,1600	—11,5700	158,3000	—188,3000
5,1	130,1000	7,2200	219,3000	—189,7000

Table 4. (cont'd.)

y	Re h_2	Im h_2	Re h_2'	Im h_2'
5,2	148,7000	32,6500	291,1000	—179,7000
5,3	165,6000	65,8100	373,8000	—154,2000
5,4	178,9000	107,7000	466,1000	—103,1000
5,5	186,3000	159,3000	566,0000	—35,8000
5,6	184,9000	221,0000	669,5000	69,4000
5,7	171,1000	293,1000	770,7000	214,2000
5,8	140,5000	374,8000	861,2000	405,9000
5,9	98,2000	464,6000	929,2000	651,1000
6,0	8,3000	559,4000	959,4000	955,7000

Table 5. Values of functions $e_0(iy)$ and $e_0'(iy)$.

y	Re $e_0(iy)$	Im $e_0(iy)$	Re $e_0'(iy)$	Im $e_0'(iy)$
0,00	1,288	—0,000	—0,939	0,000
0,05	1,287	—0,047	—0,937	0,050
0,10	1,283	—0,094	—0,932	0,100
0,15	1,277	—0,140	—0,924	0,149
0,20	1,268	—0,186	—0,913	0,197
0,25	1,257	—0,231	—0,899	0,245
0,30	1,244	—0,276	—0,882	0,292
0,35	1,228	—0,320	—0,862	0,337
0,40	1,210	—0,362	—0,839	0,380
0,45	1,190	—0,403	—0,813	0,422
0,50	1,168	—0,443	—0,785	0,462
0,55	1,144	—0,482	—0,755	0,500
0,60	1,118	—0,519	—0,723	0,535
0,65	1,090	—0,554	—0,688	0,568
0,70	1,061	—0,588	—0,652	0,599
0,75	1,030	—0,619	—0,614	0,627
0,80	0,998	—0,649	—0,574	0,652
0,85	0,965	—0,677	—0,534	0,675
0,90	0,931	—0,702	—0,493	0,695
0,95	0,896	—0,726	—0,450	0,712
1,00	0,860	—0,747	—0,407	0,726
1,05	0,823	—0,767	—0,364	0,737
1,10	0,786	—0,784	—0,321	0,745
1,15	0,749	—0,799	—0,278	0,750
1,20	0,711	—0,812	—0,235	0,753
1,25	0,673	—0,822	—0,193	0,753
1,30	0,636	—0,830	—0,151	0,750
1,35	0,599	—0,836	—0,110	0,745
1,40	0,562	—0,841	—0,070	0,737
1,45	0,525	—0,844	—0,031	0,727
1,50	0,489	—0,846	0,007	0,715
1,55	0,454	—0,845	0,043	0,700
1,60	0,419	—0,842	0,078	0,683
1,65	0,385	—0,837	0,111	0,664
1,70	0,352	—0,831	0,142	0,644
1,75	0,320	—0,823	0,171	0,623
1,80	0,290	—0,814	0,198	0,601
1,85	0,261	—0,804	0,222	0,578
1,90	0,232	—0,793	0,244	0,554
1,95	0,204	—0,780	0,264	0,529
2,00	0,178	—0,766	0,282	0,503
2,05	0,153	—0,751	0,298	0,476
2,10	0,130	—0,736	0,313	0,449
2,15	0,108	—0,720	0,326	0,421
2,20	0,088	—0,703	0,337	0,394
2,25	0,069	—0,686	0,346	0,366
2,30	0,051	—0,669	0,353	0,339
2,35	0,035	—0,651	0,358	0,312
2,40	0,020	—0,633	0,361	0,286
2,45	0,006	—0,615	0,362	0,260

Table 5. (cont'd.)

y	Re e_0 (l/y)	Im c_0 (l/y)	Re c_0' (l/y)	Im c_0' (l/y)
2,50	—0,006	—0,597	0,362	0,235
2,55	—0,017	—0,579	0,361	0,211
2,60	—0,027	—0,561	0,358	0,188
2,65	—0,036	—0,543	0,354	0,165
2,70	—0,044	—0,525	0,349	0,144
2,75	—0,050	—0,508	0,342	0,123
2,80	—0,056	—0,491	0,335	0,104
2,85	—0,061	—0,475	0,327	0,086
2,90	—0,065	—0,459	0,318	0,069
2,95	—0,068	—0,443	0,308	0,053
3,00	—0,070	—0,426	0,298	0,038
3,05	—0,071	—0,413	0,287	0,024
3,10	—0,072	—0,399	0,276	0,011
3,15	—0,073	—0,386	0,264	0,000
3,20	—0,073	—0,373	0,253	—0,010
3,25	—0,072	—0,360	0,241	—0,019
3,30	—0,071	—0,348	0,230	—0,027
3,35	—0,069	—0,337	0,218	—0,034
3,40	—0,067	—0,327	0,206	—0,040
3,45	– 0,[illegible]65	—0,317	0,195	—0,015
3,50	—0,063	—0,307	0,184	—0,050
3,55	—0,060	—0,298	0,173	—0,053
3,60	—0,058	—0,290	0,163	—0,056
3,65	—0,056	—0,282	0,152	—0,058
3,70	—0,053	—0,275	0,142	—0,059
3,75	—0,050	—0,268	0,132	—0,059
3,80	—0,047	—0,261	0,123	—0,059
3,85	—0,044	—0,255	0,114	—0,059
3,90	—0,041	—0,250	0,106	—0,058
3,95	—0,038	—0,246	0,098	—0,057
4,00	—0,035	—0,242	0,090	—0,056
4,05	—0,032	—0,238	0,083	—0,054
4,10	—0,029	—0,234	0,077	—0,052
4,15	—0,027	—0,230	0,071	—0,050
4,20	—0,024	—0,226	0,066	—0,048
4,25	—0,022	—0,223	0,061	—0,046
4,30	—0,020	—0,220	0,057	—0,043
4,35	—0,018	—0,218	0,053	—0,040
4,40	—0,016	—0,215	0,049	—0,038
4,45	—0,014	—0,213	0,046	—0,035
4,50	—0,012	—0,210	0,043	—0,032
4,55	—0,011	—0,208	0,040	—0,030
4,60	—0,009	—0,206	0,038	—0,027
4,65	—0,008	—0,205	0,036	—0,025
4,70	—0,007	—0,203	0,034	—0,022
4,75	—0,006	—0,201	0,033	—0,020
4,80	—0,005	—0,200	0,031	—0,018
4,85	—0,004	—0,198	0,030	—0,016
4,90	—0,003	—0,197	0,029	—0,014
4,95	—0,003	—0,195	0,029	—0,012
5,00	—0,002	—0,191	0,028	—0,010

Table 6. Values of $I_1(\theta) = \int_0^{\theta} \frac{\sqrt{\sin t}}{\sqrt{1+\lambda^{-1}\sin t}}\,dt$

$\theta°$	$1/\lambda = 0$	$1/\lambda = 0,1$	$1/\lambda = 0,2$	$1/\lambda = 0,3$	$1/\lambda = 0,4$	$1/\lambda = 0,5$
0	0	0	0	0	0	0
5	0,0172	0,0171	0,0170	0,0170	0,0170	0,0170
10	0,0486	0,0483	0,0481	0,0478	0,0476	0,0473
15	0,0892	0,0885	0,0878	0,0872	0,0866	0,0859
20	0,1369	0,1355	0,1342	0,1329	0,1316	0,1305
25	0,1908	0,1884	0,1861	0,1839	0,1818	0,1798
30	0,2591	0,2463	0,2428	0,2394	0,2362	0,2332
35	0,3141	0,3087	0,3036	0,2989	0,2943	0,2900
40	0,3822	0,3748	0,3679	0,3614	0,3553	0,3497
45	0,4538	0,4441	0,4352	0,4268	0,4190	0,4116
50	0,5287	0,5164	0,5050	0,4945	0,4847	0,4756
55	0,6065	0,5913	0,5773	0,5644	0,5524	0,5415
60	0,6866	0,6682	0,6515	0,6360	0,6218	0,6086
65	0,7688	0,7470	0,7274	0,7092	0,6924	0,6771
70	0,8527	0,8273	0,8042	0,7834	0,7640	0,7464
75	0,9378	0,9086	0,8823	0,8584	0,8364	0,8168
80	1,0240	0,9909	0,9611	0,9341	0,9095	0,8873
85	1,1109	1,0737	1,0404	1,0103	0,9830	0,9584
90	1,1981	1,1570	1,1202	1,0870	1,0568	1,0294

Table 7. Values of $I_2(\theta_1) = \int_0^{\theta_1} \frac{\sqrt{\sin t}}{\sqrt{1-\lambda^{-1}\sin t}}\,dt$

θ_1^0	$1/\lambda = 0,1$	$1/\lambda = 0,2$	$1/\lambda = 0,3$	$1/\lambda = 0,4$	$1/\lambda = 0,5$
0	0	0	0	0	0
5	0,0172	0,0173	0,0173	0,0174	0,0174
10	0,0488	0,0490	0,0493	0,0496	0,0499
15	0,0898	0,0907	0,0915	0,0921	0,0929
20	0,1383	0,1398	0,1415	0,1430	0,1446
25	0,1932	0,1959	0,1987	0,2016	0,2046
30	0,2540	0,2581	0,2625	0,2672	0,2722
35	0,3199	0,3259	0,3325	0,3395	0,3471
40	0,3901	0,3985	0,4077	0,4177	0,4286
45	0,4640	0,4756	0,4880	0,5016	0,5167
50	0,5420	0,5567	0,5728	0,5908	0,6111
55	0,6231	0,6414	0,6619	0,6849	0,7111
60	0,7068	0,7293	0,7546	0,7833	0,8163
65	0,7930	0,8200	0,8505	0,8856	0,9264
70	0,8809	0,9128	0,9492	0,9912	1,0407
75	0,9705	1,0075	1,0499	1,0993	1,1580
80	1,0613	1,1034	1,1526	1,2096	1,2782
85	1,1527	1,2004	1,2558	1,3210	1,3997
90	1,2448	1,2982	1,3603	1,4338	1,5224

Table 8. Values of $I_3(\theta) = \int_0^{\theta} \frac{\sqrt{\sin t}}{\sqrt{\lambda+\sin t}}\,dt$

$\theta°$	$\lambda = 0,9$	$\lambda = 0,8$	$\lambda = 0,7$	$\lambda = 0,6$	$\lambda = 0,5$	$\theta°$	$\lambda = 0,9$	$\lambda = 0,8$	$\lambda = 0,7$	$\lambda = 0,6$	$\lambda = 0,5$
0	0	0	0	0	0	48	0,430	0,446	0,468	0,488	0,516
4	0,011	0,012	0,013	0,014	0,015	52	0,478	0,495	0,518	0,540	0,570
8	0,034	0,036	0,038	0,041	0,044	56	0,525	0,546	0,569	0,594	0,625
12	0,062	0,065	0,069	0,074	0,080	60	0,574	0,595	0,621	0,647	0,680
16	0,094	0,099	0,105	0,111	0,119	64	0,623	0,645	0,672	0,701	0,736
20	0,130	0,136	0,143	0,152	0,163	68	0,674	0,698	0,725	0,756	0,791
24	0,168	0,175	0,185	0,195	0,208	72	0,722	0,750	0,778	0,810	0,847
28	0,208	0,217	0,228	0,241	0,256	76	0,775	0,800	0,830	0,865	0,905
32	0,249	0,261	0,273	0,288	0,306	80	0,825	0,852	0,835	0,920	0,962
36	0,292	0,306	0,320	0,337	0,357	84	0,875	0,905	0,938	0,975	1,020
40	0,338	0,352	0,368	0,386	0,409	88	0,925	0,956	0,992	1,030	1,080
44	0,384	0,399	0,416	0,437	0,462	90	0,950	0,985	1,020	1,060	1,106

Table 9. Values of $I_\lambda(\theta_1) = \int\limits_0^{\theta_1} \frac{\sqrt{\sin t}}{\sqrt{\lambda - \sin t}}\, dt$

θ_1^0	$\lambda = 0{,}9$ $\theta_1^* = 1{,}120$	$\lambda = 0{,}8$ $\theta_1^* = 0{,}928$	$\lambda = 0{,}7$ $\theta_1^* = 0{,}776$	$\lambda = 0{,}6$ $\theta_1^* = 0{,}644$	$\lambda = 0{,}5$ $\theta_1^* = 0{,}524$
0	0	0	0	0	0
4	0,012	0,013	0,014	0,015	0,017
8	0,037	0,040	0,043	0,047	0,053
12	0,071	0,077	0,083	0,092	0,104
16	0,114	0,123	0,135	0,150	0,172
20	0,165	0,178	0,197	0,222	0,262
24	0,224	0,245	0,272	0,313	0,385
28	0,292	0,322	0,362	0,429	0,582
32	0,370	0,413	0,474	0,590	
36	0,460	0,520	0,616	0,890	
40	0,564	0,647	0,810		
44	0,683	0,810	1,233		
48	0,822	1,020			
52	0,993	1,395			
56	1,202				
60	1,493				
64	2,317				
θ_1^*	2,48	1,720	1,410	1,110	0,86

LIST OF DESIGNATIONS

OX, OY, OZ — the axes in a Cartesian rectangular system of coordinates

i, j, k — unit vectors, directed along axes OX, OY, OZ

r — radius vector of a certain point M on a middle surface

X, Y, Z — coordinates a point in a Cartesian rectangular system of coordinates

α, β — curvilinear surface coordinates

θ, φ — angular geographical coordinates on a surface of rotation

v — radius of a parallel circle

τ_1, τ_2 — unit vectors of the tangent to a meridian and parallel circle

n — unit vector of the normal to a surface

R_1, R_2 — radii of principal curvatures of a surface of revolution

e — unit vector of a normal to a parallel circle

t — unit vector of a tangent to a certain curve on a

d_1s, d_2s — elements of length of an arc of meridian and parallel

$d\sigma$ — element of length of an arc of a certain curve on the surface

$\frac{1}{R_t}$ — curvature of a certain curve on the surface

$\frac{1}{R_\alpha}, \frac{1}{R_\beta}, \frac{1}{R_{\alpha\beta}}$ — curvature and twisting of a surface referred to arbitrary curvilinear coordinates

t_α, t_β — unit vectors of tangents to coordinate lines α, β,

U — vector of the displacement of a point of a middle surface during deformation

u, v, w — projections of a vector of displacement U onto axes τ_1, τ_2, n.

Δ_r, Δ_z – radial and axial component of displacement

r^* – radius vector of a point of a middle surface after deformation

n^* – vector of normal to deformed surface

τ_1^*, τ_2^* – vectors of tangents to coordinate lines after deformation

d_1s^*, d_2s^* – elements of lengths of arcs of coordinate lines after deformation

$\frac{1}{R_1^*}, \frac{1}{R_2^*}, \frac{1}{R_{12}^*}$ – curvature and twisting of surface after deformation

$-\vartheta_1, \vartheta_2$ – angles of rotation of normal to surface around axes τ_2, τ_1 during deformation

γ_1, γ_2 – angles between vectors τ_1, τ_1^* and τ_2, τ_2^* respectively

$\varepsilon_1, \varepsilon_2, \gamma, \varkappa_1, \varkappa_2, \tau$ – components of deformation of middle surface of shell

Ω – vector of rotation of an element of a continuous medium during deformation

$\vartheta_2, -\vartheta_1, \delta$ – projections of vector Ω onto axes τ_1, τ_2, n

h – thickness of shell

R – radius vector of a point of a shell

ζ – distance from middle surface, read on the side of a normal

T_1, T_2, N – vectors of tangents to coordinate lines and normals on an equidistant surface

$d\Sigma$ – element of the area of an equidistant surface

dU – volume element

$U^{(\zeta)}$ – vector of the displacement of a point lying on the surface ζ = const during deformation

$u^{(\zeta)}, v^{(\zeta)}, w^{(\zeta)}$ – projections of vector $U^{(\zeta)}$ onto axes τ_1, τ_2, n,

R^* – radius vector of a point of a shell after deformation

T_1^*, T_2^*, N^* – vectors of tangents and normal on an equidistant surface after deformation

e_1, e_2, ω – relative elongations and shear on an equidistant surface during deformation

σ_1, σ_2 – normal stress on areas perpendicular to vectors τ_1 and τ_2,

$\tau_{12}, \tau_{13}, \tau_{23}$ – tangential stress in these areas

k_1, k_2 – vectors of stress on areas perpendicular to vectors τ_1, τ_2,

K_1, K_2, M_1, M_2 – vectors of total forces and moments applied to the edges of a chosen element of a shell

$T_1, S_{12}, Q_1, M_1, H_{12}$, – internal forces and moments applied a unit length of an arc of the parallel circle of the middle surface.

$T_2, S_{21}, Q_2, M_2, H_{21}$ — internal forces and moments referred to a unit length of an arc of the meridian of the middle surface

S, H — applied tangential force and twisting moment

N_1, N_2 — applied shearing forces, vector of volume forces referred to a unit of volume

H_e — radial force

F — vector of volume forces referred to a unit of volume

p^+, p^- — vectors of surface forces referred to a unit of area (external and internal respectively)

E, L — applied vectors of forces and moments of external forces, referred to a unit of area of the middle surface

$F_1, F_2, F_n, p_1^+, p_2^+, p_n^+, p_1^-, p_2^-, p_n^-, E_1 \approx q_1, E_2 \approx q_2, E_n \approx q_n$

L_1, L_2, L_n — projections of vectors F, p^+, p^-, E, L onto axes τ_1, τ_2, n,

B — tensile rigidity

D — cylindrical or flexural rigidity

E — Young's modulus

μ — Poisson's coefficient

A, B, C — stress functions

$T_{1(0)}, T_{1(k)}, T_1^{(k)}$ $(k = 1, 2, 3, \ldots)$ — coefficients of the expansion of meridian force T_1 in a trigonometric series in coordinate ϕ

$T_{2(0)}, T_{2(k)}, T_2^{(k)}, \ldots, H^{(0)}, H_{(k)}, H^{(k)}, \varepsilon_{1(0)}, \varepsilon_{1(k)}, \varepsilon_1^{(k)}, \ldots, \tau^{(0)}, \tau_{(k)}, \tau^{(k)},$
$\vartheta_{1(0)}, \vartheta_{1(k)}, \vartheta_1^{(k)}, \vartheta_2^{(0)}, \vartheta_{2(k)}, \vartheta_2^{(k)}, u_{(0)}, u_{(k)}, u^{(k)}, \ldots, w_{(0)}, w_{(k)}, w^{(k)}, q_{1(0)},$
$q_{1(k)}, q_1^{(k)}, \ldots, q_{n(0)}, q_{n(k)}, q_n^{(k)}$ — analogous meaning

$P_x, P_y, P_z, M_x, M_y, M_z$ — projections of principal vector and principal moment of external loads applied to one edge of a shell

ϑ_1, V — Meissner functions in the axisymmetric problem

Ψ, V — Meissner-type functions in the problem of deformation of a shell under a bending load

σ — with different indexes is used for a complex combination from Meissner functions and Meissner-type functions

$4\gamma^4 = \frac{12(1-\mu^2)b^2}{h^2}$ — parameter characterizing relative thickness of of shell, b — certain characteristic geometric dimension

μ_1 — basic parameter of asymptotic integrating of the equation of deformation of a toroidal shell

a — radius of the forming circumference of a torus

d — radius of circumference of the centers of the forming circles of a toroidal surface

$t(\theta, \varphi)$ — temperature as a function of coordinates

t^m — average wall temperature

Δt — temperature drop with shell thickness

$\varepsilon_1^t, \varepsilon_2^t, \varkappa_1^t, \varkappa_2^t$ — components of temperature deformation

$t_{(0)}^m, t_{(k)}^m, t^{m(k)}; \Delta t_{(0)}, \Delta t_{(k)}, \Delta t^{(k)}$ — coefficients of the expansion of function t^m and Δt, in a trigonometric series in coordinate

$\varepsilon_1^{(e)}, \varepsilon_2^{(e)}, \ldots, \varkappa_2^{(e)}, \tau^{(e)}$ — components of "elastic" deformation

$\varepsilon_1^{(g)}, \varepsilon_2^{(g)}, \ldots, \varkappa_2^{(g)}, \tau^{(g)}$ — component of dislocation deformation.

Note. The lisc does not include auxiliary designations or the designation of special functions (Bessel, Legendre and others). Both the basic so and auxiliary designations are explained in the text of the book.

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