AFRL-SR-AR-TR-10-0330

REPORT DOCUMENTATION PAGE

1. REPORT DATE (*DD-MM-YYYY*)	2. REPORT TYPE	3. DATES COVERED (*From - To*)
31-03-2009	Final Technical	01 APR 2005 - 31 AUG 2008

4. TITLE AND SUBTITLE
Development of analysis tools for certification of flight control laws

5a. CONTRACT NUMBER

5b. GRANT NUMBER
FA9550-05-1-0266

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
Dr. Andrew Packard (UC Berkely)
Dr. Pete Seiler (Honeywell Laboratories)
Dr. Gary Balas (Univ. of Minnesota)

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Department of Mechanical Engineering
University of California
5116 Etchverry Hall, Berkely, CA 94720-1740

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Air Force Office of Scientific Research
875 N. Randolph St, Suite 325
Arlington, VA 22203

10. SPONSOR/MONITOR'S ACRONYM(S)
AFOSR

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Distribution A

13. SUPPLEMENTARY NOTES

14. ABSTRACT
We have improved our ability to analyze uncertain system dynamics, including nonaffine parametric uncertainty as well as unmodeled dynamics. This entailed polytopic covering methods for graphs of vector-valued polynomial functions, and local small-gain theorems. We use simulation as a key step in aiding the nonconvex search for proofs (Lyapunov functions) and proof certificates (multipliers). Some aspects of the calculation are trivially parallelizable, and we have employed a 9-machine cluster to speed-up the analysis of uncertain systems. We also began more detailed study of systems with marginally stable linearizations (ie., adaptive systems). Finally, we made precise our claim that these techniques represent a quantitative and definitive improvement over linearized analysis.

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
U	U	U	UU		Dr. Andrew Packard
					19b. TELEPHONE NUMBER (*Include area code*)

DEVELOPMENT OF ANALYSIS TOOLS FOR CERTIFICATION OF FLIGHT CONTROL LAWS

FA9550-05-1-0266

Andy Packard
Mechanical Eng., Univ. of California
5116 Etcheverry Hall, Berkeley, CA 94720-1740

Pete Seiler
Honeywell Laboratories - Minneapolis
3660 Technology Drive, Minneapolis, MN 55418

Gary Balas
Aerospace Eng. and Mechanics, Univ. of Minnesota
107 Akerman Hall, Minneapolis, MN 55455

Introduction

This is the final report, covering the period May 1, 2005 through August 31, 2008. In this report, we discuss our progress on nonlinear systems analysis, based on Lyapunov and dissipation inequalities, using sum-of-squares (SOS) decompositions to verify set containments.

We have improved our ability to analyze uncertain system dynamics, including nonaffine parametric uncertainty as well as unmodeled dynamics. This entailed polytopic covering methods for graphs of vector-valued polynomial functions, and local small-gain theorems. We use simulation as a key step in aiding the nonconvex search for proofs (Lyapunov functions) and proof certificates (multipliers). Some aspects of the calculation are trivially parallelizable, and we have employed a 9-machine cluster to speed-up the analysis of uncertain systems. We also began more detailed study of systems with marginally stable linearizations (ie., adaptive systems). Finally, we made precise our claim that these techniques represent a quantitative and definitive improvement over linearized analysis.

Notation: $\mathbb{R}[x]$ represents the set of polynomials in x with real coefficients. For $\pi \in \mathbb{R}[x]$, $\partial(\pi)$ denotes the degree of π. The subset $\Sigma[x] := \{\pi_1^2 + \cdots + \pi_m^2 \ : \ \pi_1, \cdots, \pi_m \in \mathbb{R}[x]\}$ is the set of SOS polynomials.

Uncertain Systems Analysis

Uncertainty in the vector field includes non-affine dependence on a parameter vector δ lying in a polytope $\boldsymbol{\Delta}$, namely

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} \delta_i f_i(x(t)) + \sum_{j=1}^{m_{pu}} g_j(\delta) f_{m+j}(x(t)), \qquad \delta \in \boldsymbol{\Delta}$$

and unmodeled dynamics,

$$\begin{aligned}\dot{x}(t) &= f_0(x(t), w(t)) + \textstyle\sum_{i=1}^{m} \delta_i f_i(x(t), w(t)) + \sum_{j=1}^{m_{pu}} g_j(\delta) f_{m+j}(x(t), w(t)) \\ z &= h(x) \\ w &= \Phi(z)\end{aligned}$$

Here Φ represents unmodeled dynamics, for example, finite-dimensional, linear, time-invariant operators with specified upper bound on induced $\mathcal{L}_2$ norm, for example $\|\Phi\|_{\mathcal{L}_2 \to \mathcal{L}_2} < 1$.

The tools we have developed to address this problem are

1. region-of-attraction analysis for systems with affine parameter uncertainty using a single Lyapunov function
2. local induced $\mathcal{L}_2 \to \mathcal{L}_2$ gain analysis for systems with affine parameter uncertainty, using a single polynomial storage function
3. covering, with a polytope, the graph of vector-valued polynomial function over a polytopic domain,
4. informal branch-and-bound
5. local small gain theorems.

Next, we illustrate the calculations that are possible with these methods.

Controlled short period aircraft dynamics, parametric uncertainty
We apply the robust ROA analysis for uncertain controlled short period aircraft dynamics (see the project website for the parameters used in the model)

$$\dot{x}_p = \begin{bmatrix} c_{01}(x_p) + \delta_1 c_{11}(x_p) + \delta_1^2 q_{31}(x_p) \\ q_{02}(x_p) + \delta_1 \ell_{12}^T x_p + \delta_2 q_{22}(x_p) \\ x_1 \end{bmatrix} + \begin{bmatrix} \ell_b^T x_p + b_{11} + b_{12}\delta_1 \\ b_{21} + b_{22}\delta_2 \\ 0 \end{bmatrix} u,$$

where $x_p = [x_1 \; x_2 \; x_3]^T$, x_1, x_2, and x_3 denote the pitch rate, the angle of attack, and the pitch angle, respectively, c_{01} and c_{11} are cubic polynomials, q_{02}, q_{22}, and q_{31} are quadratic polynomials, ℓ_{12} and ℓ_b are vectors in $\mathcal{R}^3$, b_{11}, b_{12}, b_{21}, and $b_{22} \in \mathcal{R}$, and u, the elevator deflection, is the control input. Variations in the center of gravity in the longitudinal direction are modeled by $\delta_1 \in [0.99, 2.05]$ and variations in the mass are modeled $\delta_2 \in [-0.1, 0.1]$. Note that the parametric uncertainty includes one nonaffine term (ie., δ_1^2). The control input is determined by $\dot{x}_4 = -0.864 y_1 + -0.321 y_2$ and $u = 2x_4$, where x_4 is the controller state and the plant output $y = [x_1 \; x_3]^T$. Define $x := \begin{bmatrix} x_p^T & x_4 \end{bmatrix}^T$ and the shape factor $p(x) := x^T x$. We applied a branch-and-bound type procedure with $\partial(V) = 2$ and $\partial(V) = 4$ on a 9-processor computer cluster: after the first *B&B* iteration, the cell with the smallest lower bound is subdivided into 3 subcells and cells with 2-nd, 3-rd, and 4-th smallest lower bounds are sub-divided into 2 subcells. Fig. 1 shows the lower bounds and upper bounds. Note that quadratic Lyapunov functions (several, as different Lyapunov functions are employed

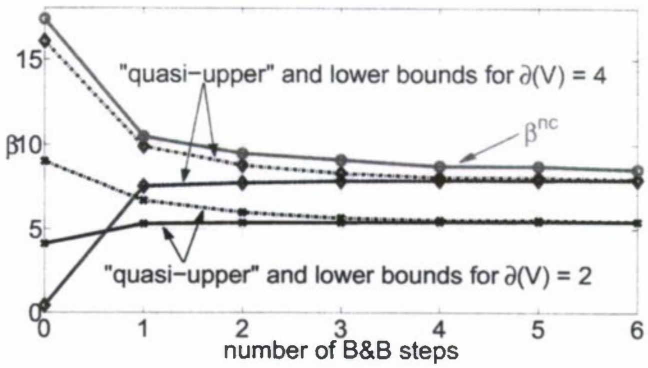

Figure 1: Lower bounds for β^*_Δ with $\partial(V) = 2$ (solid black with "×") and $\partial(V) = 4$ (solid blue curve with "⋄") and β^{nc} (solid red with "∘") computed at the centers of the cells generated by the *B&B Algorithm* for the $\partial(V) = 4$ run. Dashed curves are for (computed values of) $\beta_{\{\delta\}}$ where δ is the center of the cell with the smallest lower bound at the corresponding step of the *B&B* refinement procedure for $\partial(V) = 2$ (dashed black with "×") and $\partial(V) = 4$ (dashed blue with "⋄").

in different cells across the parameter space) certify that all initial conditions $x_0 \in \mathbf{R}^4$ satisfying $x_0^T x_0 \leq 5.4$ are in the region-of-attraction. Likewise, a collection of quartic Lyapunov functions certify that all initial conditions $x_0 \in \mathbf{R}^4$ satisfying $x_0^T x_0 \leq 7.8$ are in the region-of-attraction. The smallest value of p attained on divergent trajectories, β^{nc}, is 8.6 and obtained for $(\delta_1, \delta_2) = (2.039, -0.099)$ and the initial condition $(0.17, 2.65, -0.10, 1.24)$.

Aircraft dynamics, parametric uncertainty and unmodeled dynamics

Next, consider the same system with additional unmodeled dynamics at the plant input, as shown in Figure 2.

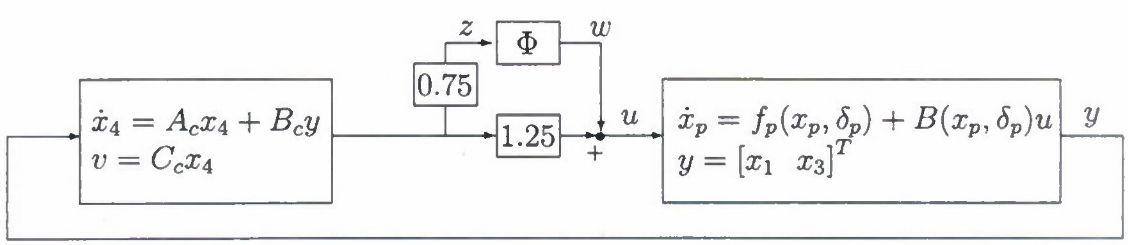

Figure 2: Controlled short period aircraft dynamics with unmodeled dynamics ($\delta_p := (\delta_1, \delta_2)$).

The assumption is that Φ is any stable, linear time-invariant (this could be relaxed) operator, with induced $\mathcal{L}_2$ norm less than 1. We again repeat the analysis using both quadratic and quartic storage functions, which locally certify bounds on the gain (with Φ removed) from w to z (recall that this system is not globally stable) in the presence of the parametric uncertainty. These local $\mathcal{L}_2$ gains are used in conjunction with a local small-gain theorem to yield results such as:

- (using quadratic storage functions) For all (finite-dimensional, linear, time-invariant)

Φ, satisfying $\|\Phi\|_{\mathcal{L}_2 \to \mathcal{L}_2} \leq 1$, assuming that the initial condition of Φ is 0, then all plant/controller initial conditions $x_0 \in \mathbf{R}^4$ satisfying $x_0^T x_0 \leq 2.4$, are in the robust region-of-attraction.

- (using quartic storage functions) For all (finite-dimensional, linear, time-invariant) Φ, satisfying $\|\Phi\|_{\mathcal{L}_2 \to \mathcal{L}_2} \leq 1$, assuming that the initial condition of Φ is 0, then all plant/controller initial conditions $x_0 \in \mathbf{R}^4$ satisfying $x_0^T x_0 \leq 4.1$, are in the robust region-of-attraction.

In conclusion, we have established provable and certifiable inner estimates of the region-of-attraction of a (nominally) 4-state nonlinear system with both parametric and unmodeled dynamics uncertainty. The informal use of branch-and-bound in the parametric uncertainty space was handled efficiently using a small-scale parallel cluster of 9 machines.

Viewing our approach as quantitative extension of linearized analysis

Practical nonlinear analysis often couples extensive nonlinear simulation with extensive linearized analysis (such as stability, stability margins, Bode plots of linearized I/O maps, etc). Here we show that common linearized analysis techniques can be rigorously quantified using the SOS approaches. The next lemma is key to these derivations.

Lemma: Let $z(x)$ be a vector of all monomials of degree 2 with no repetition. Let $Q = Q^T \succ 0$. There exists a positive definite matrix $H = H^T$ such that $x^T x x^T Q x = z(x)^T H z(x)$.

SOS region-of-attraction analysis: For the autonomous system $\dot{x} = f(x)$, and a positive-definite function p, if there exist positive-definite, radially unbounded function l_1, positive-definite function l_2, SOS polynomials s_1, s_2 and s_3, a polynomial function V, and positive constants γ and β such that $V(0) = 0$ and

$$V - l_1 \in \Sigma[x], \tag{1a}$$
$$-\left[(\beta - p)s_1 + (V - \gamma)\right] \in \Sigma[x], \tag{1b}$$
$$-\left[(\gamma - V)s_2 + \nabla V f s_3 + l_2\right] \in \Sigma[x], \tag{1c}$$

then $\{x : p(x) \leq \beta\} \subseteq \{x : V(x) \leq \gamma\} =: \Omega$, and for all $x(0) \in \Omega$, the solution satisfies $x(t) \in \Omega \ \forall t$ and $\lim_{t \to \infty} x(t) = 0$.

Now, consider the system $\dot{x}(t) = Ax(t) + f_{23}(x(t))$ where A is Hurwitz, and f_{23} is a polynomial with quadratic and cubic terms. For any quadratic, positive-definite l_1, l_2 and p, the equations (1) are feasible using quadratic V, constant s_1, s_3 and quadratic s_2 (suboptimal, feasable values are easily determined from A and f_{23}). *Consequently, if the local stability of an equilibrium point of a system with a cubic vector field is decidable using linearized analysis, then the SOS region-of-attraction analysis will always yield a quantitative, certified, inner estimate of the region of attraction.*

SOS $\mathcal{L}_2$ gain analysis: For the driven system $\dot{x} = f(x, w), z = h(x)$, if there exist positive-definite, radially unbounded function l_1, SOS polynomial s_1, a polynomial function V, and positive constants γ and R such that $V(0) = 0$ and

$$V - l_1 \in \Sigma[x], \tag{2a}$$

$$-\left[(R^2 - V)s_1 + \nabla V f - w^T w + \tfrac{1}{\gamma^2} h^T h\right] \in \Sigma[x], \tag{2b}$$

then for all w, with $\|w\|_{2,T} \leq R$, the solution from $x(0) = 0$ satisfies $V(x(t)) \leq R^2$ and $\|z\|_{2,T} \leq \gamma \|w\|_{2,T}$.

Now consider the system $\dot{x} = Ax + f_2(x) + f_3(x) + [B + g_1(x)]\, w$ and $z = Cx + h_2(x)$ where g_1 is purely linear, f_2 and h_2 are purely quadratic, and f_3 is purely cubic.

Suppose the linearization (A, B and C) has A Hurwitz, and $\|C(sI - A)^{-1}B\|_\infty < \gamma$. For any quadratic, positive-definite l_1, there exist $R > 0$ such that the equations (2) are feasible (possibly after scaling the state coordinates, $x \leftarrow \alpha x$, by a computable scalar $\alpha > 0$) using quadratic V, and quadratic s_1 (suboptimal, feasable values are easily determined from A, f_2, f_3, etc.). *Consequently, if the local input/output gain around an equilibrium point of a system with a cubic vector field is bounded using linearized analysis, then the SOS $\mathcal{L}_2$ gain analysis will always yield a quantitative, certified, ball of disturbances such that the same gain bound holds for the nonlinear system.*

Acknowledgment/Disclaimer
This work was sponsored (in part) by the Air Force Office of Scientific Research, USAF, under grant/contract number FA9550-05-1-0266. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Air Force Office of Scientific Research or the U.S. Government.

Personnel Supported During Duration of Grant
Weehong Tan (UC Berkeley, 04/2005-01/2006, externally funded, DSO National Laboratories, Singapore, but implicitly supported through the supervisory role of Packard, PhD in January 2006)
Ufuk Topcu (UC Berkeley, 08/2005-07/2008, PhD in July 2008)
Tim Wheeler (UC Berkeley, 08/2005-12/05, 5/06-8/06)

Publications (items 1-4, and 6-7 appear, in order, at the end of this report)

A. Archival Journals

1. Ufuk Topcu, Andrew Packard, Peter Seiler and Gary Balas, "Robust Region of Attraction Estimation," provisionally accepted, *IEEE Transactions on Automatic Control*, 2009.

2. Ufuk Topcu and Andrew Packard, "Local Stability Analysis for Uncertain Nonlinear Systems," to appear *IEEE Transactions on Automatic Control*, 2009

3. Ufuk Topcu, Andrew Packard and Peter Seiler, "Local Stability Analysis Using Simulations and Sum-of-Squares Programming," *Automatica*, vol. 44, pp. 2669-2675, 2008.

4. Weehong Tan and Andrew Packard, "Stability Region Analysis using polynomial and composite polynomial Lyapunov functions and Sum of Squares Programming," *IEEE Transactions on Automatic Control*, vol. 53, no. 2, pp 565-571, February 2008.

B. Conference Proceedings

5. Ufuk Topcu, Andrew Packard, Peter Seiler and Gary J. Balas, "Stability region estimation for systems with unmodeled dynamics," to appear *2009 European Control Conference*, September 2009.

6. Ufuk Topcu and Andrew Packard, "Local Robust Performance Analysis for Nonlinear Dynamical Systems," to appear *2009 American Control Conference*, June 2009.

7. Ufuk Topcu and Andrew Packard, "Linearized Analysis versus Optimization-based Nonlinear Analysis for Nonlinear Systems," to appear *2009 American Control Conference*, June 2009.

8. Weehong Tan, Ufuk Topcu, Peter Seiler, Gary Balas and Andrew Packard, "Simulation-aided Reachability and Local Gain Analysis for Nonlinear Dynamical Systems," *2008 IEEE Conference on Decision and Control*, pp. 4097-4102, doi 10.1109/CDC.2008.4739425, December 2008.

9. Ufuk Topcu, Andrew Packard, Peter Seiler and Gary Balas, "Local Stability Analysis For Uncertain Nonlinear Systems Using A Branch-and-Bound Algorithm," *2008 American Control Conference*, pp. 3428-3433 doi 10.1109/ACC.2008.4587023, June 2008.

10. Ufuk Topcu and Andrew Packard, "Stability region analysis for uncertain nonlinear systems," *2007 IEEE Conference on Decision and Control*, pp. 1693-1698, doi 10.1109/CDC.2007.4434914, December 2007.

11. Ufuk Topcu, Andrew Packard, Peter Seiler and Timothy Wheeler, "Stability Region Analysis Using Simulations and Sum-of-Squares Programming," *2007 American Control Conference*, pp. 6009-6014, doi 10.1109/ACC.2007.4283013, July 2007.

12. Weehong Tan, Tim Wheeler and Andrew Packard, "Local gain analysis of nonlinear systems," *2006 American Control Conference*, pp. 92-96, doi 10.1109/ACC.2006.1655336, June 2006.

13. Weehong Tan and Andrew Packard, "Stability region analysis using sum of squares programming," *2006 American Control Conference*, pp. 2297-2302, doi 10.1109/ACC.2006.1656562, June 2006.

Honors and Awards Received
IEEE Fellow (Packard)

AFRL Point of Contact
Siva Banda (`Siva.Banda@wpafb.af.mil`) and David Doman (`David.Doman@wpafb.af.mil`).

Transitions
None yet. On Tuesday, June 9, 2009, we are hosting a 1-day workshop (*Quantitative Local Analysis of Nonlinear Systems*) at the 2009 American Control Conference. The workshop participants will learn the theory, as well as work with the software tools we have developed in the course of this AFOSR grant.

New Discoveries
Nonaffine parameter uncertainty in region-of-attraction and gain analysis. Special minimum-volume polytopic coverings of graphs of polynomial functions. Local small-gain theorems, and application to region-of-attraction analysis with unmodeled dynamics. Superior performance on all examples tested thusfar.

Robust Region-of-Attraction Estimation

Ufuk Topcu, Andrew Packard, Peter Seiler, and Gary Balas

Abstract

We propose a method to compute invariant subsets of the region-of-attraction for the asymptotically stable equilibrium points of polynomial dynamical systems with bounded parametric uncertainty. Parameter-independent Lyapunov functions are used to characterize invariant subsets of the robust region-of-attraction. A branch-and-bound type refinement procedure is implemented to reduce the conservatism. We demonstrate the method on an example from the literature and uncertain controlled short period aircraft dynamics.

I. INTRODUCTION

We consider the problem of computing invariant subsets of the region-of-attraction (ROA) for systems with polynomial vector fields and bounded parametric uncertainty. Since computing the exact ROA, even for systems with known dynamics, is hard, research has focused on determining Lyapunov functions whose sublevel sets characterize invariant subsets of the ROA [8], [9], [19]. Recent advances in polynomial optimization based on sum-of-squares (SOS) relaxations [12] are utilized to determine invariant subsets of the ROA for systems with known polynomial and/or rational dynamics solving optimization problems with matrix inequality constraints [21], [15], [7], [14], [17]. The literature on ROA analysis for systems with uncertain dynamics includes a generalization of Zubov's method [4] and an iterative algorithm that asymptotically gives the robust ROA for systems with time-varying perturbations [11]. Systems with parametric uncertainties are considered in [5], [13], [18]. The focus in [5] is on computing the largest sublevel set of a given Lyapunov function that can be certified to be an invariant subset of the ROA. References [13], [5] propose parameter-dependent Lyapunov functions which lead to

U. Topcu (utopcu@jagger.me.berkeley.edu) and A. Packard are with the Department of Mechanical Engineering, The University of California, Berkeley; P. Seiler is with the Honeywell Labs; G. Balas is with the Department of Aerospace Engineering and Mechanics, The University of Minnesota.

potentially less conservative estimate of the ROA compared to parameter-independent Lyapunov functions at the expense of increased computational complexity.

This paper follows [16], using bilinear sum-of-squares optimization to determine invariant subsets of the robust ROA. The differences lie in the allowed uncertain parameter dependence and the class of Lyapunov functions. The approach in [16] employs parameter-independent Lyapunov functions for systems whose vector field depends affinely on uncertain parameters known to lie in a given polytope. This is reminiscent of *quadratic stability analysis* [3], where a single quadratic Lyapunov function certifies the stability of an entire family of uncertain linear systems, usually described by a polytope of linear vector fields. Of course, in both cases, using a common Lyapunov function tends to yield conservative results. Additionally, the restriction to polytopes of vector fields is undesirable. This paper partially alleviates both of these limitations. First, vector fields are allowed to depend affinely on *polynomial* functions of the uncertain parameters, and we develop methods to cover these with a polytope of vector fields (so that [16] applies). Additionally, we propose a branch-and-bound type refinement procedure [10] to partition the uncertainty set and compute a different parameter-independent Lyapunov function for each cell, hence implicitly using piecewise constant (across uncertainty space), yet parameter-dependent Lyapunov functions. In fact, in robustness analysis involving time-invariant unknown parameters, it is common, [2], [22], to combine easily-computable sufficient conditions with branch-and-bound strategies, often yielding improved analysis results.

An alternate for the conservativeness of parameter-independent Lyapunov functions is using polynomially parameter-dependent Lyapunov functions as proposed in [5], [13]. Although SOS optimization can be used with parameter-dependent Lyapunov functions, the ensuing optimization problem is challenging because uncertain parameters are treated as additional independent variables in the SOS conditions, which can greatly affect the size of the semidefinite programs. Moreover, choosing a suitable and effective polynomially parameter-dependent basis for the Lyapunov function is not intuitive.

Finally, we remark that the methodology based on the branch-and-bound algorithm, applied to robust region-of-attraction analysis here, is generally applicable to local reachability and gain analysis of systems with parametric uncertainty.

Notation: $\mathbb{R}[x]$ represents the set of polynomials in x with real coefficients. For $\pi \in \mathbb{R}[x]$, $\partial(\pi)$ denotes the degree of π. The subset $\Sigma[x] := \{\pi_1^2 + \cdots + \pi_m^2 \ : \ \pi_1, \cdots, \pi_m \in \mathbb{R}[x]\}$ is the set of SOS polynomials. For $\eta \in \mathcal{R}$ and $g : \mathcal{R}^n \to \mathcal{R}$, the η-sublevel set $\Omega_{g,\eta}$ of g is defined as $\Omega_{g,\eta} := \{x \in \mathcal{R}^n : g(x) \leq \eta\}$. In several places, a relationship between an algebraic condition on real variables and state properties of a dynamical system is claimed, often using the same symbol for a particular real variable in the algebraic statement as well as the state of the dynamical system. This could be a source of confusion, so care on the reader's part is required.

II. ESTIMATION OF THE ROBUST ROA OF SYSTEMS WITH PARAMETRIC UNCERTAINTY

Consider the system governed by

$$\dot{x}(t) = f(x(t), \delta), \tag{1}$$

where $\delta \in \boldsymbol{\Delta} \subset \mathcal{R}^m$ is the vector of unknown parameters and $\boldsymbol{\Delta}$ is a known bounded polytope. For each $\delta \in \boldsymbol{\Delta}$, $f(\cdot, \delta) : \mathcal{R}^n \to \mathcal{R}^n$ is locally Lipschitz and satisfies $f(0, \delta) = 0$. The robust ROA is the intersection of the ROAs for all systems governed by (1), i.e., $\bigcap_{\delta \in \boldsymbol{\Delta}} \{\mathrm{x}_0 \in \mathcal{R}^n : \lim_{t \to \infty} \varphi(t; \mathrm{x}_0, \delta) = 0\}$, where $\varphi(t; \mathrm{x}_0, \delta)$ denotes the solution of (1) at time t with initial condition x_0 and fixed parameter value $\delta \in \boldsymbol{\Delta}$. Trivial extensions of results found in classic textbooks [20] show that sublevel sets of appropriate Lyapunov functions are invariant subsets of the robust ROA. For a subset $D \subset \mathcal{R}^m$ and continuously differentiable function V, define $\mathcal{M}_{D,V} := \bigcap_{\delta \in D} \{x \in \mathcal{R}^n : \nabla V(x) f(x, \delta) < 0\}$.

Proposition 2.1: If there exist $\gamma > 0$ and a continuously differentiable $V : \mathcal{R}^n \to \mathcal{R}$ such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ for all } x \neq 0, \tag{2}$$

$$\Omega_{V,\gamma} \text{ is bounded, and} \tag{3}$$

$$\Omega_{V,\gamma} \backslash \{0\} \subset \mathcal{M}_{\boldsymbol{\Delta},V}, \tag{4}$$

hold, then for all $\mathrm{x}_0 \in \Omega_{V,\gamma}$ and for all $\delta \in \boldsymbol{\Delta}$, $\varphi(t; \mathrm{x}_0, \delta)$ exists, satisfies $\varphi(t; \mathrm{x}_0, \delta) \in \Omega_{V,\gamma}$ for all $t \geq 0$, and $\lim_{t \to \infty} \varphi(t; \mathrm{x}_0, \delta) = 0$, i.e., $\Omega_{V,\gamma}$ is an invariant subset of the robust ROA. $\triangleleft$

We now restrict our attention to a special case, where the dependence of f on δ is affine, to obtain conditions equivalent to (4) for this special case and suitable for numerical verification (a generalization to polynomial dependence on δ is treated in section III). Assume that the vector field in (1) is in the form

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} \delta_i f_i(x(t)), \tag{5}$$

where $f_0, f_1, \ldots, f_m : \mathcal{R}^n \to \mathcal{R}^n$ are known locally Lipschitz functions and satisfy $f_i(0) = 0$ for $i = 0, 1, \ldots, m$, and $\delta \in \Delta$. Further, denote the set of vertices (extreme points) of Δ by $\mathcal{E}_\Delta$. Then, the following follows from the affine dependence of the vector field on δ, [16].

Proposition 2.2: For the vector field in (5) and and continuously differentiable function V, the equality $\mathcal{M}_{\Delta,V} = \mathcal{M}_{\mathcal{E}_\Delta,V}$ holds. $\lhd$

Consequently, for any continuously differentiable function V satisfying (2), (3), and

$$\Omega_{V,\gamma} \backslash \{0\} \subset \mathcal{M}_{\mathcal{E}_\Delta,V}, \tag{6}$$

the sublevel set $\Omega_{V,\gamma}$ is an invariant subset of the robust ROA. In order to enlarge the computed invariant subset of the robust ROA by choice of V, we introduce a fixed, positive definite, convex function p, called the *analysis shape factor* and maximize β while imposing the constraints (2)-(3), (6), and $\Omega_{p,\beta} := \{x \in \mathcal{R}^n \ : \ p(x) \leq \beta\} \subseteq \Omega_{V,\gamma}$. This is written as an optimization problem,

$$\beta_\Delta^{opt}(\mathcal{V}) := \max_{V \in \mathcal{V}, \beta > 0, \gamma > 0} \beta \ \text{ subject to (2), (3), (6), and } \Omega_{p,\beta} \subseteq \Omega_{V,\gamma}. \tag{7}$$

Here, $\mathcal{V}$ denotes the set of candidate Lyapunov functions over which the maximum is computed, for example all continuously differentiable functions. In practice, p is problem-dependent, and chosen by the analyst. Since the form of the certified inner estimate of the robust region-of-attraction is a sublevel set of p, the sublevel sets of p should be well-understood (for in high-dimensions they cannot be visualized), and should reflect directionality/scaling information that the analyst is interested in learning with regard to the robust region-of-attraction. In order to relax the problem in (7) to a SOS programming problem, we require $f_0, f_1, \ldots, f_m$ and p to be polynomials and restrict V to be a polynomial in x of fixed degree. Further, we use generalizations of the S-procedure [16] to obtain sufficient conditions for the set containment constraints in (7) and the well-known SOS sufficient condition for polynomial nonnegativity [12]: if $\pi \in \Sigma[x]$, then π is nonnegative.

Let $\mathcal{V}_{poly} \subseteq \mathcal{V}$, $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ be prescribed finite-dimensional subsets of $\mathbb{R}[x]$, and denote $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$. For a polytopic subset D of Δ and positive definite polynomials l_1 and l_2

(typically $l_i(x) = \epsilon_i x^T x$ with small scalars ϵ_i), define $\beta_D(\mathcal{V}_{poly}, \mathcal{S})$ as

$$\beta_D(\mathcal{V}_{poly}, \mathcal{S}) := \max_{V \in \mathcal{V}_{poly}, \beta, \gamma, s_1 \in \mathcal{S}_1, s_{2\delta} \in \mathcal{S}_2, s_{3\delta} \in \mathcal{S}_3} \beta \text{ subject to}$$

$$s_1 \in \Sigma[x], s_{2\delta} \in \Sigma[x], s_{3\delta} \in \Sigma[x], \quad \text{for all } \delta \in \mathcal{E}_D, \tag{8a}$$

$$\beta > 0, \ \gamma > 0, \ V(0) = 0, \ V \in \mathcal{V}_{poly}, \quad V - l_1 \in \Sigma[x], \tag{8b}$$

$$-[(\beta - p)s_1 + (V - \gamma)] \in \Sigma[x], \text{ and} \tag{8c}$$

$$-[(\gamma - V)s_{2\delta} + \nabla V(f_0 + \textstyle\sum_{i=1}^m \delta_i f_i)s_{3\delta} + l_2] \in \Sigma[x], \quad \text{for all } \delta \in \mathcal{E}_D. \tag{8d}$$

The feasibility of the constraints in (8) is sufficient for the feasibility of the constraints in (7). Therefore, $\beta_{\Delta}(\mathcal{V}_{poly}, \mathcal{S}) \leq \beta_{\Delta}^{opt}(\mathcal{V})$.

The optimization in (8) is naturally converted to a bilinear semidefinite program (SDP), with 3 "types" of decision variables: the free parameters in V, the free parameters in the s polynomials, and the free parameters introduced by the SOS constraints. The SDP is bilinear in the free parameters in V and multipliers s, as evidenced by the product terms (e.g. $Vs_{2\delta}$, $\nabla V f s_{3\delta}$, etc). We have made significant pragmatic progress in obtaining high-quality solutions to (8), using simulation to first derive a convex outer-bound on the set of feasible V parameters [17], followed by coordinatewise optimization over V and $(s_1, s_{2\delta}, s_{3\delta})$. Nevertheless, the nonconvexity is not to be taken lightly, and <u>any</u> numerical attempt to compute $\beta_D(\mathcal{V}_{poly}, \mathcal{S})$ must itself treated as a lower bound.

Finally, note that, if l_1 and l_2 have positive definite quadratic part, then the feasibility of (8) implies the robust stability of the uncertain linearized dynamics using common quadratic Lyapunov function. For systems with cubic vector fields, the feasibility of (8) is also necessary by the following theorem whose proof is in the Appendix.

Theorem 2.1: Let $f_0, \ldots, f_m$ be cubic polynomials in x satisfying $f_0(0) = \ldots = f_m(0) = 0$, $P \succ 0$, $L_1 \succ 0$, $L_2 \succ 0$, $p(x) = x^T P x$, $l_1(x) = x^T L_1 x$, and $l_2(x) = x^T L_2 x$. For $\delta \in \Delta$, let A_δ be such that $A_\delta x$ is the linear (in x) part of $f_0(x) + \sum_{i=1}^m \delta_i f_i(x)$. If there exists $Q \succ 0$ satisfying $A_\delta^T Q + Q A_\delta \prec 0$ for all $\delta \in \mathcal{E}_{\Delta}$, then the constraints in (8) are feasible. $\triangleleft$

III. POLYNOMIAL PARAMETRIC UNCERTAINTY

We extend the development in section II to systems with polynomial parametric uncertainty. Specifically, we consider the system

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} \delta_i f_i(x(t)) + \sum_{j=1}^{m_{pu}} g_j(\delta) f_{m+j}(x(t)), \tag{9}$$

where $f_0, f_1, \ldots, f_m, f_{m+1} \ldots, f_{m+m_{pu}} : \mathcal{R}^n \to \mathcal{R}^n$ are vector valued polynomial functions satisfying $f_0(0) = \ldots = f_{m+m_{pu}}(0) = 0$, and $g_1, \ldots, g_{m_{pu}} \in \mathbb{R}[\delta]$ are scalar valued polynomial functions, and δ takes values in a bounded polytope $\mathbf{\Delta}$. Note that $g_1, \ldots, g_{m_{pu}}$ are bounded since they are polynomials with the bounded domain $\mathbf{\Delta}$. We begin with $m_{pu} = 1$ (for simplicity) and then generalize for $m_{pu} \geq 1$.

Replacing $g_1(\delta)$ by an artificial parameter ϕ, the dynamics in (9) can be written as

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} \delta_i f_i(x(t)) + \phi f_{m+1}(x(t)). \tag{10}$$

Our approach is based on covering the graph of g, $\{(\zeta, g(\zeta)) \in \mathcal{R}^{m+1} \ : \ \zeta \in \mathbf{\Delta}\}$, by a bounded polytope $\Gamma \subset \mathcal{R}^{m+1}$. Then, the dependence of the vector field in (10) on the parameters (δ, ϕ) is affine and (δ, ϕ) takes values in the bounded polytope Γ. Therefore, results from section II are applicable for the system in (10) by replacing $\mathbf{\Delta}$ by Γ.

A polytope Γ covering the graph of g can be obtained by bounding g from above and below by affine functions $a_u^T\delta + b_u$ and $a_l^T\delta + b_l$ over the set $\mathbf{\Delta}$, namely $\Gamma(a_l, a_u, b_l, b_u) := \{(\zeta, \psi) \in \mathcal{R}^{m+1} : \zeta \in \mathbf{\Delta},\ a_l^T\zeta + b_l \leq \psi \leq a_u^T\zeta + b_u\}$. The volume of Γ is a linear function of a_l, a_u, b_l, and b_u, $\texttt{Volume}(\Gamma(a_l, a_u, b_l, b_u)) = (a_u - a_l)^T \int_{\mathbf{\Delta}} \zeta d\zeta + (b_u - b_l)\int_{\mathbf{\Delta}} d\zeta$. The polytope with smallest volume among such covering polytopes can be characterized via

$$\begin{aligned} &\min_{a_l, a_u, b_l, b_u} \ \texttt{Volume}(\Gamma(a_l, a_u, b_l, b_u)) \text{ subject to} \\ &g(\delta) - (a_l^T\delta + b_l) \geq 0 \quad \text{and} \quad g(\delta) - (a_u^T\delta + b_u) \leq 0, \quad \forall \delta \in \mathbf{\Delta}. \end{aligned} \tag{11}$$

Using the generalized S-procedure [16], an upper bound for this minimal volume can be computed by a linear SOS optimization problem. To this end, let affine functions h_i, $i = 1, \ldots, N$, provide an inequality description for $\mathbf{\Delta}$, i.e., $\mathbf{\Delta} = \{\zeta \in \mathcal{R}^m \ : \ h_i(\zeta) \geq 0,\ i = 1, \ldots, N\}$.

Proposition 3.1: The value of the optimization problem

$$\min_{a_l, a_u, b_l, b_u, \sigma_{ui} \in \mathcal{S}_{ui}, \sigma_{li} \in \mathcal{S}_{li}} \ \texttt{Volume}(\Gamma(a_l, a_u, b_l, b_u)) \text{ subject to } \ \sigma_{ui} \in \Sigma[\delta],\ \sigma_{li} \in \Sigma[\delta]\ , i = 1, \ldots, N,$$

$$-g(\delta) + (a_u^T\delta + b_u) - \textstyle\sum_{i=1}^{N} \sigma_{ui}(\delta) h_i(\delta) \in \Sigma[\delta], \tag{12a}$$

$$g(\delta) - (a_l^T\delta + b_l) - \textstyle\sum_{i=1}^{N} \sigma_{li}(\delta) h_i(\delta) \in \Sigma[\delta] \tag{12b}$$

is an upper bound for (11). Here $\mathcal{S}$'s are finite dimensional subsets of $\mathbb{R}[\delta]$. ◁

Remarks 3.1: Note that $\texttt{Volume}(\Gamma(a_l, a_u, b_l, b_u)) = \texttt{Volume}(\Gamma(0, a_u, 0, b_u)) - \texttt{Volume}(\Gamma(0, a_l, 0, b_l))$, and therefore the optimizing values of the variables a_l, a_u, b_l, and b_u in Proposition 3.1 can equivalently be computed by two smaller optimization problems. $\triangleleft$

In case $m_{pu} \geq 1$, affine upper and lower bounds for $g_1, \ldots, g_{m_{pu}}$ can be used to construct a polytope (with $2^{m+m_{pu}}$ vertices) covering the graph of $(g_1, \ldots, g_{m_{pu}})$, [1].

Proposition 3.2: For $j = 1, \ldots, m_{pu}$, let $a_{lj}^T\delta + b_{lj}$ and $a_{uj}^T\delta + b_{uj}$ be affine functions bounding g_j over $\mathbf{\Delta}$ from below and above, respectively. Then, the polytope Γ with the vertex set $\mathcal{E}_\Gamma := \bigcup_{\zeta \in \mathcal{E}_{\mathbf{\Delta}}} \{(\zeta, \psi_1, \ldots, \psi_{m_{pu}}) \in \mathcal{R}^{m+m_{pu}} \ : \ \psi_j = a_{\alpha j}^T\zeta + b_{\alpha j}, \ \alpha \in \{l, u\}, \ j = 1, \ldots, m_{pu}\}$ contains the graph of $(g_1, \ldots, g_{m_{pu}})$. $\triangleleft$

This gives one specific procedure to cover the graph of a vector-valued multivariate polynomial by a convex polytope. Further research that advances graph covering strategies and quantifies the trade-off between the number of vertices and the volume of the covering polytope would be relevant and applicable to the robust ROA problem. Finally, Proposition 3.2 can be used with bounded non-polynomial g_j's as long as affine upper and lower bounds are provided.

IV. BRANCH-AND-BOUND TYPE REFINEMENT IN THE PARAMETER SPACE

The optimization problem in (8), when applied with $D = \mathbf{\Delta}$, provides a method for computing invariant subsets of the robust ROA characterized by a single Lyapunov function. Therefore, results by (8) may be conservative: the certified invariant subset may be too small relative to the robust ROA. On the other hand, a less conservative estimate of the robust ROA can be obtained by solving (8) for each $\delta \in \mathbf{\Delta}$ with $D = \{\delta\}$. For a subset $D \subseteq \mathbf{\Delta}$, define

$$\beta_D^*(\mathcal{V}_{poly}, \mathcal{S}) := \min_{\delta \in D} \ \beta_{\{\delta\}}(\mathcal{V}_{poly}, \mathcal{S}). \tag{13}$$

Then, $\beta_{\mathbf{\Delta}}(\mathcal{V}_{poly}, \mathcal{S}) \leq \beta_{\mathbf{\Delta}}^*(\mathcal{V}_{poly}, \mathcal{S})$. However, computing $\beta_{\mathbf{\Delta}}^*(\mathcal{V}_{poly}, \mathcal{S})$ requires solving an optimization problem for each $\delta \in \mathbf{\Delta}$, and consequently is impractical. Next, we propose an informal "branch-and-bound" type procedure for computing lower and upper bounds for $\beta_{\mathbf{\Delta}}^*(\mathcal{V}_{poly}, \mathcal{S})$, i.e., localizing the value of $\beta_{\mathbf{\Delta}}^*(\mathcal{V}_{poly}, \mathcal{S})$. The method is based on computing a different Lyapunov function for each cell of a finite partition, $\mathcal{D}$, of $\mathbf{\Delta}$.

Branch-and-bound (B&B) is an algorithmic method for global optimization based on two steps: first the search region is partitioned into a union of smaller regions , or *cells* (branching) and then upper and lower bounds for the objective function restricted to each cell are computed (bounding) [10]. These steps are repeated, refining the partition each repetition (e.g. subdividing the cell with the worst lower bound). If the upper and lower bounds are such that their difference converges to zero uniformly as the size of the cell goes to zero, then the B&B algorithm converges to a global optimum. Without such specific guarantees, steps are simply repeated until the gap between the upper and lower bounds gets suitably small or a maximum number of steps is reached. Additionally, for our problem, we take into account the polytopic covering described in section III, and recompute this covering whenever any cell is subdivided.

The lower and upper bounds are defined over any polytope $D \in \Delta$. Certainly $\beta_D(\mathcal{V}_{poly}, \mathcal{S})$ is a lower bound for $\beta_D^*(\mathcal{V}_{poly}, \mathcal{S})$. Upper bounds for $\beta_D^*(\mathcal{V}_{poly}, \mathcal{S})$ can be obtained via divergent trajectories and infeasibility of certain necessary conditions for the constraints in (8). Let $\delta \in D$ and $\beta^{nc}(\delta)$ be the minimum value of p attained on all non-convergent trajectories of (5), with $\beta^{nc}(\delta) := \infty$ if there is no non-convergent trajectory. Since every trajectory entering an invariant subset of the robust ROA has to converge to the origin, $\Omega_{p,\beta^{nc}(\delta)}$ cannot be a subset of the robust ROA; hence, for any $\mathcal{V}_{poly}$ and $\mathcal{S}$, $\beta_D^*(\mathcal{V}_{poly}, \mathcal{S}) < \beta^{nc}(\delta)$. Unfortunately, $\beta^{nc}(\delta)$, as defined, is impossible to compute. But, any non-convergent trajectory yields an upper bound on $\beta^{nc}(\delta)$, and consequently on $\beta_D^*(\mathcal{V}_{poly}, \mathcal{S})$. In order to establish another upper bound, let $\beta > 0$ and $\delta \in D$ be fixed. If there exists $V \in \mathcal{V}$ certifying that $\Omega_{p,\beta}$ is in the robust ROA through the constraints in (7), then V has to be (i) positive for all nonzero $x \in \mathcal{R}^n$, (ii) less than or equal to 1 (without loss of generality) and decreasing along every trajectory of (5) (for this fixed δ) starting in $\Omega_{p,\beta}$. Therefore, if no $V \in \mathcal{V}$ satisfies properties (i) and (ii), then there is no $V \in \mathcal{V}$ certifying that $\Omega_{p,\beta}$ is in the robust ROA via (7). The minimum such value, denoted $\beta^{lp}(\delta)$, is an upper bound on $\beta_D^*(\mathcal{V}_{poly}, \mathcal{S})$. In the case V is parameterized as $V(x) = \alpha^T z(x)$ with z a vector of basis functions and α a vector of real scalar decision variables, constraints on V along trajectories are affine constraints on α; consequently, an upper bound on $\beta^{lp}(\delta)$ can be determined by simulation and linear programming (see [17] for more details). As in all *B&B* algorithms, the minimum (over the subsets that make up the partition of Δ) of these upper and lower bounds are upper and lower bounds for $\beta_\Delta^*(\mathcal{V}_{poly}, \mathcal{S})$.

V. IMPLEMENTATION ISSUES

The optimization problem in (8) provides a recipe to compute invariant subsets of the robust ROA. However, the number of constraints in (8) and consequently the number of decision variables increase exponentially with $m + m_{pu}$ because (8d) contains a SOS constraint for each vertex value of the uncertainty polytope. The increase in the problem size may render (8) computationally challenging for even modest values of $m + m_{pu}$. This difficulty can be partially alleviated by accepting suboptimal solutions for (8) in a sequential manner [16]. To this end, let D be a polytopic subset of Δ and D_{sample} be a finite sample in D:

- Solve (8) with D_{sample}, $\mathcal{V}_{poly}$, and $\mathcal{S}$ and call the optimizing Lyapunov function V_{sample}.
- For each $\delta \in \mathcal{E}_D$, compute

$$\begin{aligned} \gamma_\delta := \max_{0<\gamma, s_{2\delta}\in\mathcal{S}_2, s_{3\delta}\in\mathcal{S}_3} \quad & \gamma \quad \text{subject to } s_{2\delta} \in \Sigma[x], \text{ and } s_{3\delta} \in \Sigma[x], \\ & -[(\gamma - V_{sample})s_{2\delta} + \nabla V_{sample}(f_0 + \textstyle\sum_{i=1}^m \delta_i f_i))s_{3\delta} + l_2] \in \Sigma[x], \end{aligned} \tag{14}$$

  and define $\gamma^{subopt} := \min\{\gamma_\delta \ : \ \delta \in \mathcal{E}_D\}$. At this point, $\Omega_{V_{sample},\gamma^{subopt}}$ is an invariant subset of the robust ROA.
- Determine the largest sublevel set $\Omega_{p,\beta_D^{subopt}(\mathcal{V}_{poly},\mathcal{S})}$ of p contained in $\Omega_{V_{sample},\gamma^{subopt}}$ by solving

$$\begin{aligned} \beta_D^{subopt}(\mathcal{V}_{poly}, \mathcal{S}) := \max_{s_1\in\mathcal{S}_1,\beta} \quad & \beta \quad \text{subject to } s_1 \in \Sigma[x] \\ -[(\beta - p)s_1 + V_{sample} - \gamma^{subopt}] \in \Sigma[x]. & \end{aligned} \tag{15}$$

While this sequential procedure sacrifices optimality (i.e., for a given polytopic subset $D \subseteq \Delta$, $\beta_D^{subopt}(\mathcal{V}_{poly}, \mathcal{S}) \leq \beta_D(\mathcal{V}_{poly}, \mathcal{S})$), it provides practical advantages: For a fixed Lyapunov function candidate V_{sample}, constraints in (8d) (which contain one SOS constraint for each vertex value of D) decouple. Therefore, it is possible to determine largest value of γ such that $\Omega_{V_{sample},\gamma} \subset \{x \in \mathcal{R}^n : \nabla V_{sample}(x)f(x,\delta) < 0\}$ for every $\delta \in \mathcal{E}_D$ by solving (14) independently for each $\delta \in \mathcal{E}_D$.

Remarks 5.1: Let $D \subseteq \Delta$ and $D_{sample} \subset D$ be a singleton. Then, the value $\beta_{D_{sample}}(\mathcal{V}_{poly}, \mathcal{S}) - \beta_D^{subopt}(\mathcal{V}_{poly}, \mathcal{S})$ is always non-negative and can be interpreted as a measure of potential improvement in the lower bound for $\beta_\Delta^*(\mathcal{V}_{poly}, \mathcal{S})$ from further sub-dividing D in the *B&B* refinement procedure. Therefore, it may be used as a stopping criterion in an informal *B&B* algorithm. However, we re-emphasize that $\beta_{D_{sample}}(\mathcal{V}_{poly}, \mathcal{S})$ is computed solving a non-convex optimization problem, so that its use as an upper bound is ad hoc and referred to as a "quasi-upper" bound (for example see Figure 2). ◁

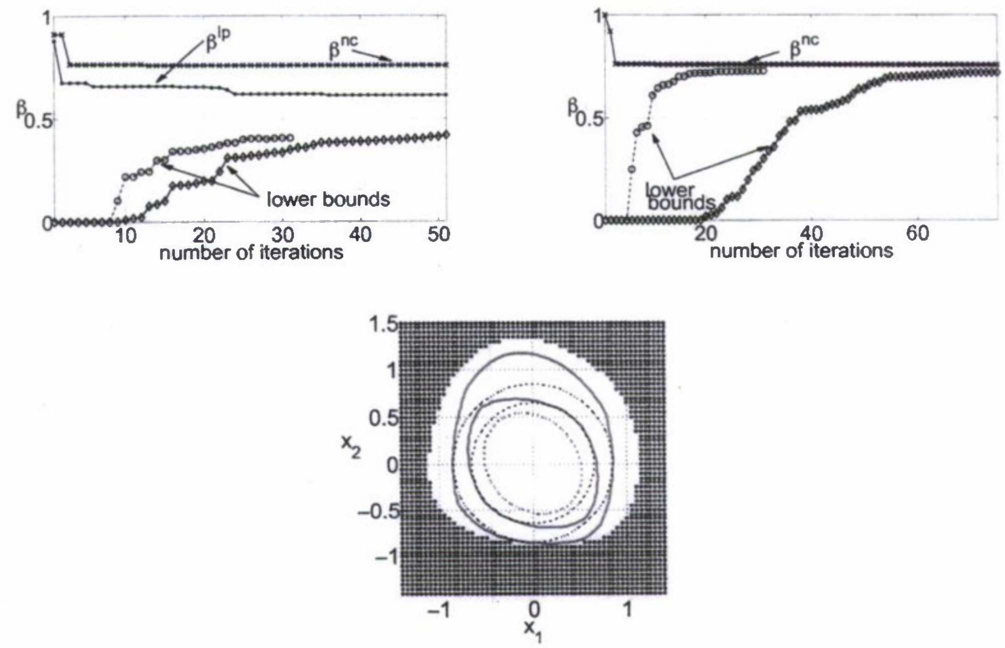

Fig. 1. Top figures: Bounds for $\beta^*_{[0,1]}$ vs. number of $B\&B$ iterations with $\partial(V) = 2$ (left) and $\partial(V) = 4$ (right). Curves with "o" are for the lower bounds obtained by directly solving (8) with D taken as the vertices of the corresponding cell and curves with "◇" are for the lower bounds obtained by applying the sequential procedure from section V by taking D_{sample} as the center of the corresponding cell. Bottom figure: Intersections of sublevel sets of V's certified to be in the robust ROA with $\partial(V) = 2$ (inner red, solid curve) and $\partial(V) = 4$ (outer red, solid curve), Sublevel sets of p certified to be in the robust ROA $\partial(V) = 2$ (inner black, dashed curve) and $\partial(V) = 4$ (outer black, dashed curve), estimate of the robust ROA reported in [6] (blue, dotted curve). Gray dots are the initial conditions of trajectories which do not converge to the origin for some $\delta \in [0, 1]$.

VI. Examples

For the following examples, we implemented the sequential procedure from section V using the method from [17] in the first step with $l_1(x) = l_2(x) = 10^{-6}x^Tx$ and $p(x) = x^Tx$.

A. An example from the literature

Consider the system, [6], governed by

$$\dot{x} = \begin{bmatrix} -x_1 \\ 3x_1 - 2x_2 \end{bmatrix} + \begin{bmatrix} -6x_2 + x_2^2 + x_1^3 \\ -10x_1 + 6x_2 + x_1x_2 \end{bmatrix} \delta + \begin{bmatrix} 4x_2 - x_2^2 \\ 12x_1 - 4x_2 \end{bmatrix} \delta^2,$$

with $\delta \in [0, 1]$. We applied the refinement procedure with the initial partition $\{[0, 1]\}$ for $\partial(V) = 2$ and $\partial(V) = 4$. Upper and lower bounds for $\beta^*_{[0,1]}$ (top left for $\partial(V) = 2$ and top right for $\partial(V) = 4$) and certified invariant subsets of the robust ROA are shown in Fig. 1. In both cases, the first iteration (a parameter independent Lyapunov function for Δ, [16]) and even a few more do not yield a certified region.

B. Controlled short period aircraft dynamics

We apply the robust ROA analysis for uncertain controlled short period aircraft dynamics (see the Appendix for the parameters used in the model)

$$\dot{x}_p = \begin{bmatrix} c_{01}(x_p) + \delta_1 c_{11}(x_p) + \delta_1^2 q_{31}(x_p) \\ q_{02}(x_p) + \delta_1 \ell_{12}^T x_p + \delta_2 q_{22}(x_p) \\ x_1 \end{bmatrix} + \begin{bmatrix} \ell_b^T x_p + b_{11} + b_{12}\delta_1 \\ b_{21} + b_{22}\delta_2 \\ 0 \end{bmatrix} u,$$

where $x_p = [x_1 \ x_2 \ x_3]^T$, x_1, x_2, and x_3 denote the pitch rate, the angle of attack, and the pitch angle, respectively, c_{01} and c_{11} are cubic polynomials, q_{02}, q_{22}, and q_{31} are quadratic polynomials, ℓ_{12} and ℓ_b are vectors in $\mathcal{R}^3$, b_{11}, b_{12}, b_{21}, and $b_{22} \in \mathcal{R}$, and u, the elevator deflection, is the control input. Variations in the center of gravity in the longitudinal direction are modeled by $\delta_1 \in [0.99, 2.05]$ and variations in the mass are modeled $\delta_2 \in [-0.1, 0.1]$.. The control input is determined by $\dot{x}_4 = -0.864y_1 + -0.321y_2$ and $u = 2x_4$, where x_4 is the controller state and the plant output $y = [x_1 \ x_3]^T$. Define $x := \begin{bmatrix} x_p^T & x_4 \end{bmatrix}^T$. We applied the branch-and-bound type refinement procedure with $\partial(V) = 2$ and $\partial(V) = 4$ using the sequential implementation on a computer cluster with 9 processors: after the first $B\&B$ iteration, the cell with the smallest lower bound is subdivided into 3 subcells and cells with 2-nd, 3-rd, and 4-th smallest lower bounds are sub-divided into 2 subcells. Fig. 2 shows the lower bounds and upper bounds. Smallest value of p attained on divergent trajectories, β^{nc}, is 8.60 and obtained for $(\delta_1, \delta_2) = (2.039, -0.099)$ and the initial condition $(0.17, 2.65, -0.10, 1.24)$.

C. Controlled short period aircraft dynamics with first-order unmodeled dynamics

Consider the closed loop system in Figure 3 where uncertain first-order dynamics are introduced between the controller output (v) and the plant input (u) from section VI-B

$$u(s) = 1.25 + G(s; \delta_3, \delta_4)v(s) = \left[1.25 + 0.75\delta_3 \frac{s - \delta_4}{s + \delta_4}\right] v(s). \tag{16}$$

Here, $\delta_3 \in [-1, 1]$ and $\delta_4 \in [10^{-2}, 10^2]$ are uncertain parameters and $G(s; \delta_3, \delta_4)$ is introduced to examine the effect of unmodeled dynamics on the ROA. Let $\dot{x}_5 = -\delta_4 x_5 - \delta_4 v$ and $u =$

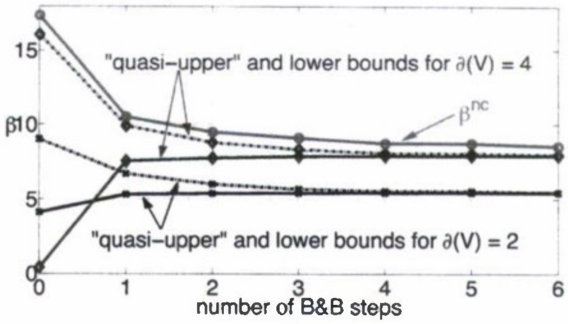

Fig. 2. Lower bounds for β^*_{Δ} with $\partial(V) = 2$ (solid black with "×") and $\partial(V) = 4$ (solid blue curve with "◇") and β^{nc} (solid red with "∘") computed at the centers of the cells generated by the *B&B Algorithm* for the $\partial(V) = 4$ run. Dashed curves are for (computed values of) $\beta_{\{\delta\}}$ where δ is the center of the cell with the smallest lower bound at the corresponding step of the *B&B* refinement procedure for $\partial(V) = 2$ (dashed black with "×") and $\partial(V) = 4$ (dashed blue with "◇").

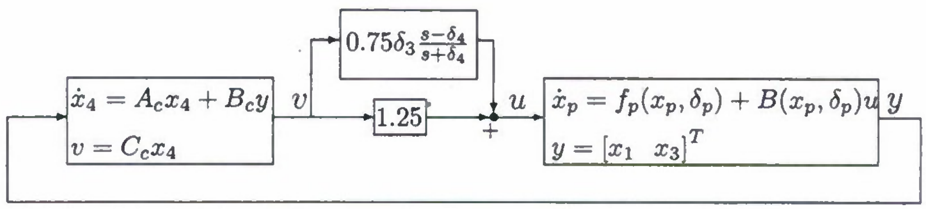

Fig. 3. Closed-loop system with the uncertain first-order dynamics between the controller and the plant ($\delta_p = (\delta_1, \delta_2)$).

$1.5\delta_3 x_5 + (1.25 + 0.75\delta_3)v$ be a realization of G and $x = \left[x_p^T \; x_4 \; x_5\right]^T$ denote the state of the closed loop dynamics. We applied the *B&B* refinement with $\partial(V) = 2$ for two cases: (i) $(\delta_1, \delta_2) = (1.52, 0)$, (ii) δ_1 and δ_2 are treated as uncertain parameters (as in section VI-B) where the resultant vector field is affine in δ_1, δ_2, δ_3, δ_4, $\delta_1\delta_3$, $\delta_2\delta_3$, and δ_1^2 and the covering polytopes are in $\mathcal{R}^7$ with 128 vertices. For $p(x) = x^T x$, $\Omega_{p,4.90}$ is shown to be in the robust ROA for case (i) whereas $\Omega_{p,2.80}$ is certified to be in the robust ROA for case (ii).

VII. CONCLUSIONS

This paper considers the problem of finding certified, inner-estimates of the region-of-attraction for a certain class of uncertain nonlinear systems. At its core, the solution approach combines Lyapunov analysis, S-procedure relaxations, and SOS/SDP optimization. Four factors contribute to the problem complexity: number of state variables; degree of vector field; number of uncertain parameters; dependence of vector field on uncertain parameters. The challenges associated with state dimension and vector field degree (often large optimization problems) appear somewhat common across solution techniques. By contrast, the issues which arise from uncertainty are attacked using a variety of diverse techniques.

We address the difficulties due to parameter uncertainty through parallelization, partitioning the parameter space, solving a large number of (uncoupled) sub-problems. While the Lyapunov function for each sub-problem is independent of the uncertain parameter, the net result yields a parameter-dependent (piecewise-constant in the parameter) Lyapunov function. This is an alternative to more direct approaches which use explicitly parameter-dependent Lyapunov functions, e.g. [13], [5], and a single optimization (with additional indeterminate and decision variables, used to represent the uncertain parameters and capture their constraints) to solve the problem.

Of course, the question of how fine the parameter space partition must be before the proposed method yields a certified robust ROA is still largely open, so it is impossible to say that one approach is superior/inferior to another. Similarly, we do not claim that the proposed strategy is practical for all instances of systems modeled by (9). Indeed, large numbers of uncertain parameters, entering the dynamics in complex ways might require an untenable level of parameter space partitioning to yield a positive result. Nevertheless, we have illustrated the approach on several academic, but nontrivial, examples, including a 5-state, 4-parameter model with non-affine parameter dependence. Moreover, for cubic (in state) vector fields, we have a (weak) positive result which follows from Theorem 2.1: for any specific partition of the parameter space, if over each cell, the linearized uncertain dynamics are quadratically stable, then the certification conditions (8) are guaranteed to be feasible (with analytically derived choices for the decision variables). Among other things, this implies that the uncertain linearization could provide insight into the level of parameter space division needed for robust region-of-attraction certification.

VIII. ACKNOWLEDGMENTS

This work was sponsored by the Air Force Office of Scientific Research, USAF, under grant/contract number FA9550-05-1-0266. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the AFOSR or the U.S. Government.

REFERENCES

[1] F. Amato, F. Garofalo, and L. Glielmo. Polytopic coverings and robust stability analysis via Lyapunov quadratic forms. In *Variable Structure and Lyapunov Control*, pages 269–288. Springer-Verlag, 1994.

[2] V. Balakrishnan, S. Boyd, and S. Balemi. Branch and bound algorithm for computing the minimum stability degree of parameter-dependent linear systems. *Int. J. of Robust and Nonlinear Control*, 1(4):295–317, 1991.

[3] B.R. Barmish. Necessary and sufficient conditions for quadratic stabilizability of an uncertain system. *Journal of Optimization Theory and Applications*, 46(4):399–408, 1985.

[4] F. Camilli, L. Grune, and F. Wirth. A generalization of Zubov's method to perturbed systems. *SIAM Journal on Control and Optimization*, 40(2):496–515, 2001.

[5] G. Chesi. Estimating the domain of attraction of uncertain poynomial systems. *Automatica*, 40:1981–1986, 2004.

[6] G. Chesi. On the estimation of the domain of attraction for uncertain polynomial systems via lmis. In *Proc. Conf. on Decision and Control*, pages 881–886, Bahamas, 2004.

[7] G. Chesi, A. Garulli, A. Tesi, and A. Vicino. LMI-based computation of optimal quadratic Lyapunov functions for odd polynomial systems. *Int. J. Robust Nonlinear Control*, 15:35–49, 2005.

[8] E. J. Davison and E. M. Kurak. A computational method for determining quadratic Lyapunov functions for nonlinear systems. *Automatica*, 7:627–636, 1971.

[9] R. Genesio, M. Tartaglia, and A. Vicino. On the estimation of asymptotic stability regions: State of the art and new proposals. *IEEE Transaction on Automatic Control*, 30(8):747–755, 1985.

[10] E.L. Lawler and D.E.Wood. Branch-and-bound methods: a survey. *Operations Research*, 14(4):679–719, 1966.

[11] A. Paice and F. Wirth. Robustness analysis of domains of attraction of nonlinear systems. In *Proceedings of the Mathematical Theory of Networks and Systems*, pages 353–356, 1998.

[12] P. Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Mathematical Programming Series B*, 96(2):293–320, 2003.

[13] W. Tan. *Nonlinear control analysis and synthesis using sum-of-squares programming*. Ph.D. dissertation, UC, Berkeley, May 2006.

[14] W. Tan and A. Packard. Stability region analysis using polynomial and composite polynomial Lyapunov functions and sum-of-squares programming. *IEEE Transactions on Automatic Control*, 53(2):565–571, 2008.

[15] B. Tibken and Y. Fan. Computing the domain of attraction for polynomial systems via BMI optimization methods. In *Proc. American Control Conf.*, pages 117–122, Minneapolis, MN, 2006.

[16] U. Topcu and A. Packard. Local stability analysis for uncertain nonlinear systems. 2008. To appear in the IEEE Transactions on Automatic Control.

[17] U. Topcu, A. Packard, and P. Seiler. Local stability analysis using simulations and sum-of-squares programming. *Automatica*, 44:2669 – 2675, 2008.

[18] A. Trofino. Robust stability and domain of attraction of uncertain nonlinear systems. In *Proc. American Control Conf.*, pages 3707–3711, Chicago, IL, 2000.

[19] A. Vannelli and M. Vidyasagar. Maximal Lyapunov functions and domains of attraction for autonomous nonlinear systems. *Automatica*, 21(1):69–80, 1985.

[20] M. Vidyasagar. *Nonlinear Systems Analysis*. Prentice Hall, 2^{nd} edition, 1993.

[21] T-C. Wang, S. Lall, and M. West. Polynomial level-set methods for nonlinear dynamical systems analysis. In *Proc. Allerton Conf. on Communication, Control, and Computing*, Allerton, IL, 2005.

[22] P. M. Young, M. P. Newlin, and J. C. Doyle. μ analysis with real parametric uncertainty. In *Proc. Conf. on Decision and Control*, pages 1251–1256, 1991.

IX. APPENDIX

Let $z(x)$ be a vector of all monomials of degree 2 with no repetition and n_z be its length.

Lemma 9.1: Let $Q \in \mathcal{R}^n$ and $Q = Q^T \succeq 0$. Then there exists a positive definite matrix $H \in \mathcal{R}^{n_z \times n_z}$ such that $x^T x x^T Q x = z(x)^T H z(x)$. $\triangleleft$

Proof: Let $L_i \in \mathcal{R}^{n \times n_z}$ be such that $L_i z(x) = x_i x$, then $\left(x^T x\right) x^T Q x = \sum_{i=1}^{n} (x_i x)^T Q (x_i x) = \sum_{i=1}^{n} z(x) L_i^T Q L_i z(x) = z(x)^T H z(x)$ Note that $L = \left[L_1^T \dots L_{n_z}^T\right]^T$ has full column rank since every entry of $z(x)$ is $x_j x_k$ for some $1 \leq j,k \leq n$; consequently, $H = L^T (I_n \otimes Q) L \succ 0$. ■

Proof of Theorem 2.1: Let $\tilde{Q} \succ 0$ satisfy $A_\delta \tilde{Q} + \tilde{Q} A_\delta \preceq -2L_2$, for all $\delta \in \mathcal{E}_{\Delta}$, and $\tilde{Q} \succeq L_1$ (such $\tilde{Q}$ can be obtained by properly scaling Q). Let $\epsilon = \lambda_{min}(L_2)$, $V(x) := x^T \tilde{Q} x$, and H be a positive definite Gram matrix for $(x^T x) V(x)$ (which exists by Lemma 9.1). Let $M_{2\delta} \in \mathcal{R}^{n \times n_z}$, and the symmetric matrix $M_{3\delta} \in \mathcal{R}^{n_z \times n_z}$ be such that $x^T M_{2\delta} z(x)$, and $z(x)^T M_{3\delta} z(x)$ are cubic and quartic (in x) parts of $\nabla V(f_0(x) + \sum_{i=1}^{m} \delta_i f_i(x))$, respectively. Define $s_1(x) = \lambda_{max}(\tilde{Q})/\lambda_{min}(P)$, $s_{2\delta}(x) = \alpha_{2\delta} x^T x$ with $\alpha_{2\delta} = \lambda_{max}\left(M_{3\delta}^+ + \frac{1}{2\epsilon} M_{2\delta}^T M_{2\delta}\right)/\lambda_{min}(H)$ (where for a symmetric matrix Λ, Λ^+ denotes the projection on the positive semidefinite cone), $s_{3\delta}(x) = 1$, $\gamma = \min\left\{\epsilon/(2\alpha_{2\delta}) \ : \ \delta \in \mathcal{E}_{\Delta}\right\}$, and $\beta = \gamma/(2s_1)$. Then, $V - l_1$ and $-\left[(\beta - p)s_1 + (V - \gamma)\right]$ are SOS since they are positive semidefinite quadratic polynomials. For $\delta \in \mathcal{E}_{\Delta}$, $b_\delta(x) = -\left[(\gamma - V)s_{2\delta} + \frac{\partial V}{\partial x} f_\delta s_{3\delta} + l_2\right] = \left[x^T \ \ z(x)^T\right] B_\delta \left[x^T \ \ z(x)^T\right]^T$, where

$$B_\delta = \begin{bmatrix} -\gamma\alpha_{2\delta} I - L_2 - (A_\delta^T \tilde{Q} + \tilde{Q} A_\delta) & -M_{2\delta}/2 \\ -M_{2\delta}^T/2 & \alpha_{2\delta} H - M_{3\delta} \end{bmatrix} \succeq \begin{bmatrix} \frac{\epsilon}{2} I & -M_{2\delta}/2 \\ -M_{2\delta}^T/2 & \alpha_{2\delta} H - M_{3\delta} \end{bmatrix}. \tag{17}$$

Note that $\alpha_{2\delta} H = \frac{\lambda_{max}\left(M_{3\delta}^+ + \frac{1}{2\epsilon} M_{2\delta}^T M_{2\delta}\right)}{\lambda_{min}(H)} H \succeq \lambda_{max}\left(M_{3\delta}^+ + \frac{1}{2\epsilon} M_{2\delta}^T M_{2\delta}\right) I \succeq \lambda_{max}\left(M_{3\delta} + \frac{1}{2\epsilon} M_{2\delta}^T M_{2\delta}\right) I \succeq M_{3\delta} + \frac{1}{2\epsilon} M_{2\delta}^T M_{2\delta}$. Consequently, B_δ is positive semidefinite by the Schur complement formula applied to the far left term in (17) and $b_\delta \in \Sigma[x]$. ■

Parameters for the uncertain controlled short period aircraft dynamics: $c_{01}(x_p) = -0.24366x_2^3 + 0.082272x_1x_2 + 0.30492x_2^2 + 0.015426x_2x_3 - 3.1883x_1 - 2.7258x_2 - 0.59781x_3$; $\ell_b = [0 \ \ -0.041136 \ \ 0]^T$; $b_{11} = 1.594150$; $q_{02}(x_p) = -0.054444x_2^2 + 0.10889x_2x_3 - 0.054444x_3^2 + 0.91136x_1 - 0.64516x_2 - 0.016621x_3$; $b_{21} = 0.0443215$; $c_{11}(x_p) = 0.30765x_2^3 + 0.099232x_2^2 + 0.12404x_1 + 0.90912x_2 + 0.023258x_3$; $b_{12} = -0.06202$; $\ell_{12} = [0 \ \ 0.00045754 \ \ 0]^T$; $q_{22}(x_p) = -0.054444x_2^2 + 0.10889x_2x_3 - 0.054444x_3^2 - 0.6445x_2 - 0.016621x_3$; $b_{22} = 0.044321$.

Local Stability Analysis for Uncertain Nonlinear Systems

Ufuk Topcu and Andrew Packard

Abstract—**We propose a method to compute provably invariant subsets of the region-of-attraction for the asymptotically stable equilibrium points of uncertain nonlinear dynamical systems. We consider polynomial dynamics with perturbations that either obey local polynomial bounds or are described by uncertain parameters multiplying polynomial terms in the vector field. This uncertainty description is motivated by both incapabilities in modeling, as well as bilinearity and dimension of the sum-of-squares programming problems whose solutions provide invariant subsets of the region-of-attraction. We demonstrate the method on three examples from the literature and a controlled short period aircraft dynamics example.**

I. INTRODUCTION

We consider the problem of computing invariant subsets of the region-of-attraction (ROA) for uncertain systems with polynomial nominal vector fields and local polynomial uncertainty description. Since computing the exact ROA, even for systems with known dynamics, is hard, researchers have focused on finding Lyapunov functions whose sublevel sets provide invariant subsets of the ROA [1], [2], [3], [4], [5]. Recent advances in polynomial optimization based on sum-of-squares (SOS) relaxations [6], [7] are utilized to determine invariant subsets of the ROA for systems with known polynomial and/or rational dynamics solving optimization problems with matrix inequality constraints [8], [9], [10], [11], [12], [13]. Ref. [14] provides a generalization of Zubov's method to uncertain systems and [15] investigates robustness of the ROA under time-varying perturbations and proposes an iterative algorithm that asymptotically gives the robust ROA. Parametric uncertainties are considered in [16], [17], [18]. The focus in [16] is on computing the largest sublevel set of a *given* Lyapunov function that can be certified to be an invariant subset of the ROA. [17], [18] propose parameter-dependent Lyapunov functions which lead to potentially less conservative results at the expense of increased computational complexity.

Similar to other problems in local analysis of dynamical systems based on Lyapunov arguments and SOS relaxations [9], [11], [12], [17], [13], [19], our formulation leads to optimization problems with bilinear matrix inequality (BMI) constraints. BMIs are nonconvex and bilinear SDPs (those with BMI constraints) are generally harder than linear SDPs. Consequently, approaches for solving SDPs with BMIs are limited to local search schemes [20], [21], [22], [23]).

The uncertainty description detailed in section III contains two types of uncertainty: uncertain components in the vector field that obey local polynomial bounds and/or uncertain parameters appearing affinely and multiplying polynomial terms. Using this description, we develop an SDP with BMIs to compute robustly invariant subsets of the ROA. The number of BMIs (and consequently the number of variables) in this problem increases exponentially with the sum of the number of components of the vector field containing uncertainty with polynomial bounds and the number of uncertain parameters. One way to deal with this difficulty is first to compute a Lyapunov function for a particular system (imposing extra robustness constraints) and then determine the largest sublevel set in which the computed Lyapunov function serves as a local stability certificate for the whole family of systems. Once a Lyapunov function is determined for the system in the first step, second step involves solving smaller decoupled linear SDPs. Therefore, this two step procedure is well suited for parallel computation leading to relatively efficient numerical implementation. Moreover, recently developed methods [13], [24], which use simulation to aid in the nonconvex search for Lyapunov functions, extend easily to the robust ROA analysis using simulation data for finitely many systems from the family of possible systems (e.g. systems corresponding to the vertices of the uncertainty polytope when the uncertainty can be described by a polytope). In the examples in this paper, we implement this generalization of the simulation based ROA analysis method from [13], [24].

Authors are with the Department of Mechanical Engineering, The University of California, Berkeley, 94720-1740, USA utopcu, pack@jagger.me.berkeley.edu

The rest of the paper is organized as follows: Section II reviews results on computing the ROA for systems with known polynomial dynamics. Section III is devoted to the discussion of the motivation for this work and the setup for the uncertain system analysis. In section IV provides a generalization of the results from section II to the case of dynamics with uncertainty. The methodology is demonstrated with three small examples from the literature and a five-state example in section V.

Notation: For $x \in \mathcal{R}^n$, $x \succeq 0$ means that $x_k \geq 0$ for $k = 1, \cdots, n$. For $Q = Q^T \in \mathcal{R}^{n\times n}$, $Q \succeq 0$ ($Q \succ 0$) means that $x^TQx \geq 0$ (≤ 0) for all $x \in \mathcal{R}^n$. $\mathbb{R}[x]$ represents the set of polynomials in x with real coefficients. The subset $\Sigma[x] := \{\pi \in \mathbb{R}[x] \ : \ \pi = \pi_1^2 + \pi_2^2 + \cdots + \pi_m^2,\ \pi_1, \cdots, \pi_m \in \mathbb{R}[x]\}$ of $\mathbb{R}[x]$ is the set of SOS polynomials. For $\pi \in \mathbb{R}[x]$, $\partial(\pi)$ denotes the degree of π. For subsets $\mathcal{X}_1$ and $\mathcal{X}_2$ of a vector space $\mathcal{X}$, $\mathcal{X}_1 + \mathcal{X}_2 := \{x_1 + x_2 \ : \ x_1 \in \mathcal{X}_1,\ x_2 \in \mathcal{X}_2\}$. In several places, a relationship between an algebraic condition on some real variables and state properties of a dynamical system is claimed, and same symbol for a particular real variable in the algebraic statement as well as the state of the dynamical system is used. This could be a source of confusion, so care on the reader's part is required. $\triangleleft$

II. COMPUTATION OF INVARIANT SUBSETS OF REGION-OF-ATTRACTION

In this section, we give a characterization of invariant subsets of ROA using Lyapunov functions and formulate a bilinear optimization problem for computing these functions when they are restricted to be polynomial. These results will be modified to compute invariant subsets of the ROA for systems with uncertainty in section IV. Now, consider the system governed by

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the state vector and $f : \mathcal{R}^n \to \mathcal{R}^n$ is such that $f(0) = 0$, i.e., the origin is an equilibrium point of (1) and f is locally Lipschitz on $\mathcal{R}^n$. Let $\varphi(t; \mathrm{x}_0)$ denote the solution to (1) with the initial condition $x(0) = \mathrm{x}_0$. If the origin is asymptotically stable but not globally attractive, one often wants to know which trajectories converge to the origin as time approaches ∞. This gives rise to the following definition of the *region-of-attraction*:

Definition 2.1: The region-of-attraction R_0 of the origin for the system (1) is

$$R_0 := \left\{\mathrm{x}_0 \in \mathcal{R}^n \ : \ \lim_{t\to\infty} \varphi(t; \mathrm{x}_0) = 0\right\}.$$

$\triangleleft$

For $\eta > 0$ and a function $V : \mathcal{R}^n \to \mathcal{R}$, define the η-sublevel set of V as

$$\Omega_{V,\eta} := \{x \in \mathcal{R}^n \ : \ V(x) \leq \eta\}.$$

For simplicity, $\Omega_{V,1}$ is denoted by Ω_V. Lemma 2.2 provides a characterization of invariant subsets of the ROA in terms of sublevel sets of appropriate Lyapunov functions.

Lemma 2.2: If there exists a continuously differentiable function

$V : \mathcal{R}^n \to \mathcal{R}$ such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ for all } x \neq 0, \tag{2}$$

$$\Omega_V \text{ is bounded, and} \tag{3}$$

$$\Omega_V \setminus \{0\} \subset \{x \in \mathcal{R}^n \; : \; \nabla V(x) f(x) < 0\}, \tag{4}$$

then for all $\mathbf{x}_0 \in \Omega_V$, the solution of (1) exists, satisfies $\varphi(t; \mathbf{x}_0) \in \Omega_V$ for all $t \geq 0$, and $\lim_{t \to \infty} \varphi(t; \mathbf{x}_0) = 0$, i.e., Ω_V is an invariant subset of R_0. ◁

Lemma 2.2 is proven in [11], [12] using a similar result from [25]. If the dynamical system has an exponentially stable linearization, one can impose a stricter condition replacing (4), for $\mu \geq 0$, by

$$\Omega_V \setminus \{0\} \subset \{x \in \mathcal{R}^n \; : \; \nabla V(x) f(x) < -\mu V(x)\} \tag{5}$$

With nonzero μ, (5) not only assures that trajectories starting in Ω_V stay in Ω_V and converge to the origin but also imposes a bound on the rate of exponential decay of V certifying the convergence and provides an implicit threshold for the level of a disturbance that could drive the system out of Ω_V. Therefore, one may consider the stability property implied by (5) with nonzero μ to be more desirable in practice. With this in mind, all subsequent derivations contain the μV term. The relaxed condition in equation (4) can be recovered by setting $\mu = 0$.

III. Setup and Motivation

We now introduce the uncertainty description used in the rest of the paper and explain its usefulness in ROA analysis based on computing Lyapunov functions using SOS programming. Consider the system governed by

$$\dot{x}(t) = f(x(t)) = f_0(x(t)) + \phi(x(t)) + \psi(x(t)), \tag{6}$$

where $f_0, \phi, \psi : \mathcal{R}^n \to \mathcal{R}^n$ are locally Lipschitz. Assume that f_0 is known, $\phi \in \mathcal{D}_\phi$, and $\psi \in \mathcal{D}_\psi$, where

$$\mathcal{D}_\phi := \{\phi \; : \; \phi_l(x) \preceq \phi(x) \preceq \phi_u(x) \; \forall x \in \mathcal{G}\},$$

$$\mathcal{D}_\psi := \{\psi \; : \; \psi(x) = \Psi(x)\alpha \; \forall x \in \mathcal{G}, \; \alpha_l \preceq \alpha \preceq \alpha_u\}.$$

Here, $\mathcal{G}$ is a given subset of $\mathcal{R}^n$ containing the origin, ϕ_l and ϕ_u are n dimensional vectors of known polynomials satisfying $\phi_l(x) \preceq 0 \preceq \phi_u(x)$ for all $x \in \mathcal{G}$, $\alpha, \alpha_l, \alpha_u \in \mathcal{R}^N$, and Ψ is a matrix of known polynomials. Let ϕ_i, $\phi_{l,i}$, $\phi_{u,i}$, α_i, $\alpha_{l,i}$, and $\alpha_{u,i}$ denote i-th entry of ϕ, ϕ_l, ϕ_u, α, α_l, and α_u, respectively. Define $\mathcal{D} := \mathcal{D}_\phi + \mathcal{D}_\psi$. We assume that $f_0(0) = 0$, $\phi(0) = 0$ for all $\phi \in \mathcal{D}_\phi$ (i.e., $\phi_l(0) = 0$, and $\phi_u(0) = 0$), and $\psi(0) = 0$ for all $\psi \in \mathcal{D}_\psi$ (i.e., $\Psi(0) = 0$), which assures that all systems in (6) have a common equilibrium point at the origin.[1] In order to be able to use SOS programming, we restrict our attention to the case where f_0, ϕ_l, ϕ_u, and Ψ have only polynomial entries and $\mathcal{G}$ is defined as $\mathcal{G} := \{x \in \mathcal{R}^n \; : \; g(x) \succeq 0, \; g_i \in \mathbb{R}[x], \; i = 1, \ldots, m\}$. Note that entries of ϕ do not have to be polynomial but have to satisfy local polynomial bounds.

Motivation for this kind of system description stems from the following sources:

(i) Perturbations as in (6) may be due to modeling errors, aging, disturbances and uncertainties due to environment which may be present in any realistic problem. Prior knowledge about the system may provide local bounds on the entries of ϕ and/or bounds for the parametric uncertainties α. Moreover, uncertainties that do not change system order can always be represented as in (6) (see p.339 in [27]).

(ii) Analysis of dynamical systems using SOS programming is often limited to systems with polynomial or rational vector field. In [28], a procedure for re-casting non-rational vector fields into rational ones at the expense of increasing the state dimension is proposed. Another way to deal with a non-polynomial vector field is to locally approximate the vector field with a polynomial, and bound the error. For practical purposes only finite number of terms can be used. Finite-term approximations are relatively accurate in a restricted region containing the origin. However, they are not exact. On the other hand, it may be possible to represent terms, for which the error between the exact vector field and its finite-term approximation obey local polynomial bounds, using ϕ in (6).

(iii) SOS programming can be used to analyze systems with polynomial vector fields. The number of decision variables N_{decision} and the size N_{SDP} of the matrix in the SDP for checking existence of a SOS decomposition for a degree $2d$ polynomial in n variables grows polynomially with n if d is fixed and vice versa [6]. However, N_{SDP} and N_{decision} get practically intractable for the state-of-the-art SDP solvers even for moderate values of n for fixed d (see Table 2, where solid lines in the table represent a fuzzy boundary between tractable and intractable SDPs). Moreover, using higher degree Lyapunov functions and/or higher degree multipliers (used in the sufficient conditions for certain set containment constraints in section IV) as well as higher degree vector fields increases the problem size, and, in fact, the growth of the problem size with the simultaneous increase in n and d is exponential. Therefore, in order to be able to use SOS programming, one may have to simplify the dynamics by truncating higher degree terms in the vector field. In this case, ϕ_l and ϕ_u provide local bounds on the truncated terms. This is discussed further at the end of section (IV). It is also worth mentioning that bilinearity, a common feature of the optimization problems for local analysis using Lyapunov arguments (see section IV), introduces extra complexity [29] and therefore a further necessity for simplifying the system dynamics.

TABLE I

N_{SDP} (LEFT COLUMNS) AND N_{decision} (RIGHT COLUMNS) FOR DIFFERENT VALUES OF n AND $2d$.

n	2d: 4		6		8		10	
2	6	6	10	27	15	75	21	165
5	21	105	56	1134	126	6714	**252**	**2e4**
9	55	825	**220**	**1e4**	**715**	**2e5**	⋆	⋆
14	120	4200	**680**	⋆	⋆	⋆	⋆	⋆
16	**153**	**6936**	⋆	⋆	⋆	⋆	⋆	⋆

In summary, representation in (6) and definitions of $\mathcal{D}_\phi$ and $\mathcal{D}_\psi$ are motivated by uncertainties introduced due to incapabilities in modeling and/or analysis.

IV. Computation of Robustly Invariant Sets

In this section, we will develop tools for computing invariant subsets of the robust ROA. The robust ROA is the intersection of the ROAs for all possible systems governed by (6) and formally defined, assuming that the origin is an asymptotically equilibrium point of (6) for all $\delta \in \mathcal{D}$, as

[1] The assumption that all possible systems in (6) have a common equilibrium point can be alleviated by generalizing the analysis based on contraction metrics and SOS programming studied in [26] to address local stability (rather than global stability as in [26]). However, this method leads to higher computational cost. Therefore, we do not pursue this direction here.

Definition 4.1: The robust ROA R_0^r of the origin for systems governed by (6) is

$$R_0^r := \bigcap_{\delta\in\mathcal{D}} \{\mathbf{x}_0 \in \mathcal{R}^n \ : \ \lim_{t\to\infty} \varphi(t;\mathbf{x}_0,\delta) = 0\},$$

where $\varphi(t;\mathbf{x}_0,\delta)$ denotes the solution of (6) for $\delta \in \mathcal{D}$ with the initial condition $x(0) = \mathbf{x}_0$. ◁

The robust ROA is an open and connected subset of $\mathcal{R}^n$ containing the origin and is invariant under the flow of all possible systems described by (6) [15]. We focus on computing invariant subsets of the robust ROA characterized by sublevel sets of appropriate Lyapunov functions. Since the uncertainty description for ϕ and ψ holds only for $x \in \mathcal{G}$, we will also require the computed invariant set to be a subset of $\mathcal{G}$. To this end, we modify Lemma 2.2 such that condition (5) holds for (6) for all $\delta \in \mathcal{D}$ (i.e., for all $\phi \in \mathcal{D}_\phi$ and $\psi \in \mathcal{D}_\psi$).

Proposition 4.2: If there exists a continuously differentiable function $V : \mathcal{R}^n \to \mathcal{R}$ and $\mu \geq 0$ such that, for all $\delta \in \mathcal{D}$, conditions (2)-(3),

$$\Omega_V \subseteq \mathcal{G}, \text{ and} \tag{7}$$

$$\Omega_V\setminus\{0\} \subset \{x \in \mathcal{R}^n : \nabla V(x)(f_0(x) + \delta(x)) < -\mu V(x)\} \tag{8}$$

hold, then for all $\mathbf{x}_0 \in \Omega_V$ and for all $\delta \in \mathcal{D}$, the solution of (6) exists, satisfies $\varphi(t;\mathbf{x}_0,\delta) \in \Omega_V$ for all $t \geq 0$, and $\lim_{t\to\infty} \varphi(t;\mathbf{x}_0,\delta) = 0$, i.e., Ω_V is an invariant subset of R_0^r. ◁

Proof: Proposition 4.2 follows from Lemma 2.2. Indeed, for any given system $\dot{x} = f_0(x) + \delta(x)$, (8) assures that (5) is satisfied. Then, for any fixed $\delta \in \mathcal{D}$ and for all $\mathbf{x}_0 \in \Omega_V$, $\varphi(t;\mathbf{x}_0,\delta)$ exists and satisfies $\varphi(t;\mathbf{x}_0,\delta) \in \Omega_V$ for all $t \geq 0$, $\lim_{t\to\infty}\varphi(t;\mathbf{x}_0,\delta) = 0$, and Ω_V is an invariant subset of $\{\mathbf{x}_0 \in \mathcal{R}^n \ : \ \varphi(t;\mathbf{x}_0,\delta) \to 0\}$. Therefore, Ω_V is an invariant subset of R_0^r. ■

Remark 4.3: In fact, Ω_V is invariant for both time-invariant and time-varying perturbations. The conclusion of Proposition 4.2 holds for time-varying $\delta(= \phi + \alpha)$ as long as $\phi_l(x) \preceq \phi(x,t) \preceq \phi_u(x)$ and $\alpha_l \preceq \alpha(t) \preceq \alpha_u$ for all $x \in \mathcal{G}$ and $t \geq 0$. Recall that, in the uncertain linear system literature, e.g. [30], the notion of quadratic stability is similar, where a single quadratic Lyapunov function proves the stability of an entire family of uncertain linear systems. ◁

Note that $\mathcal{D}$ has infinitely many elements; therefore, there are infinitely many constraints in (8). Now, define

$$\mathcal{E}_\phi := \{\phi \ : \ \phi_i \in \mathbb{R}[x] \text{ and } \phi_i \text{ is equal to } \phi_{l,i} \text{ or } \phi_{u,i}\},$$

$$\mathcal{E}_\psi := \{\psi \ : \ \psi(x) = \Psi(x)\alpha, \ \alpha_i \text{ is equal to } \alpha_{l,i} \text{ or } \alpha_{u,i}\},$$

and $\mathcal{E} := \mathcal{E}_\phi + \mathcal{E}_\psi$. $\mathcal{E}$, a finite subset of $\mathcal{D}$, can be used to transform condition (8) to a finite set of constraints that are more suitable for numerical verification:

Proposition 4.4: If

$$\Omega_V\setminus\{0\} \subseteq \{x \in \mathcal{R}^n : \nabla V(x)(f_0(x) + \xi(x)) < -\mu V(x)\} \tag{9}$$

holds for all $\xi \in \mathcal{E}$, then (8) holds for all $\delta \in \mathcal{D}$. ◁

Proof: Let $\tilde{x} \in \Omega_V$ be nonzero and $\delta \in \mathcal{D}$. Then, $\tilde{x} \in \mathcal{G}$ by (7); therefore, there exist $\ell_1, \cdots, \ell_n, k_1, \ldots, k_N$ (depending on $\tilde{x}$) with $0 \leq \ell_i \leq 1$ and $0 \leq k_i \leq 1$ such that $\delta(\tilde{x}) = L\phi_l(\tilde{x}) + (I - L)\phi_u(\tilde{x}) + \Psi(\tilde{x})(K\alpha_l + (I-K)\alpha_u)$, where L and K are diagonal with $L_{ii} = \ell_i$ and $K_{ii} = k_i$. Hence, there exist nonnegative numbers ν_ξ (determined from ℓ's and k's) for $\xi \in \mathcal{E}$ with $\sum_{\xi\in\mathcal{E}} \nu_\xi = 1$ such that $\delta(\tilde{x}) = \sum_{\xi\in\mathcal{E}} \nu_\xi \xi(\tilde{x})$. Consequently, by $\nabla V(\tilde{x})(f_0(\tilde{x}) + \delta(\tilde{x})) = \nabla V(\tilde{x})(f_0(\tilde{x}) + \sum_{\xi\in\mathcal{E}} \nu_\xi\xi(\tilde{x})) = \sum_{\xi\in\mathcal{E}} \nu_\xi \nabla V(\tilde{x})(f_0(\tilde{x}) + \xi(\tilde{x})) < -\sum_{\xi\in\mathcal{E}} \nu_\xi \mu V(\tilde{x}) = -\mu V(\tilde{x})$, (8) follows. ■

In order to enlarge the computed invariant subset of the robust ROA, we define a variable sized region $\mathcal{P}_\beta := \{x \in \mathcal{R}^n \ : \ p(x) \leq \beta\}$, where $p \in \mathbb{R}[x]$ is a fixed, positive definite, convex polynomial, and maximize β while imposing constraints (2)-(3), (7), (9), and $\mathcal{P}_\beta \subseteq \Omega_V$. This can be written as an optimization problem.

$$\beta^*(\mathcal{V}) := \max_{V\in\mathcal{V},\beta>0} \ \beta \ \text{ subject to} \tag{10a}$$

$$\left.\begin{array}{c} V(0) = 0 \text{ and } V(x) > 0 \text{ for all } x \neq 0, \\ \Omega_V = \{x \in \mathcal{R}^n : V(x) \leq 1\} \text{ is bounded,} \end{array}\right\} \tag{10b}$$

$$\left.\begin{array}{c} \mathcal{P}_\beta = \{x \in \mathcal{R}^n : p(x) \leq \beta\} \subseteq \Omega_V, \\ \Omega_V \subseteq \mathcal{G}, \end{array}\right\} \tag{10c}$$

$$\Omega_V\setminus\{0\} \subseteq \bigcap_{\xi\in\mathcal{E}} \{x \in \mathcal{R}^n : \nabla V(f_0(x) + \xi(x)) < -\mu V(x)\}. \tag{10d}$$

Here, $\mathcal{V}$ denotes the set of candidate Lyapunov functions over which the maximum is defined (e.g. $\mathcal{V}$ may be equal to all continuously differentiable functions).

In order to make the problem in (10) amenable to numerical optimization (specifically SOS programming), we restrict V to be a polynomial in x of fixed degree. We use the well-known sufficient condition for polynomial positivity [6]: for any $\pi \in \mathbb{R}[x]$, if $\pi \in \Sigma[x]$, then π is positive semidefinite. Using simple generalizations of the S-procedure (Lemmas 8.1 and 8.2 in the appendix), we obtain sufficient conditions for set containment constraints. Specifically, let l_1 and l_2 be a positive definite polynomials (typically $\epsilon x^T x$ with some (small) real number ϵ). Then, since l_1 is radially unbounded, the constraint

$$V - l_1 \in \Sigma[x] \tag{11}$$

and $V(0) = 0$ are sufficient conditions for the constraints in (10b). By Lemma 8.1, if $s_1 \in \Sigma[x]$ and $s_{4k} \in \Sigma[x]$ for $k = 1, \ldots, m$, then

$$-[(\beta - p)s_1 + (V-1)] \in \Sigma[x] \tag{12}$$

$$g_k - (1 - V)s_{4k} \in \Sigma[x], \ k = 1, \ldots, m, \tag{13}$$

imply the first and second constraints in (10c), respectively. By Lemma 8.2, if $s_{2\xi}, s_{3\xi} \in \Sigma[x]$ for $\xi \in \mathcal{E}$, then

$$-[(1-V)s_{2\xi} + (\nabla V(f_0 + \xi) + \mu V)s_{3\xi} + l_2] \in \Sigma[x] \tag{14}$$

is a sufficient condition for the feasibility of the constraint in (10d). Using these sufficient conditions, a lower bound on $\beta^*(\mathcal{V})$ can be defined as an optimization:

Proposition 4.5: Let β_B^* be defined as

$$\beta_B^*(\mathcal{V}_{poly}, \mathcal{S}) := \max_{V,\beta,s_1,s_{2\xi},s_{3\xi},s_{4k}} \ \beta \text{ subject to } (11) - (14), \tag{15}$$

$V(0) = 0$, $V \in \mathcal{V}_{poly}$, $s_1 \in \mathcal{S}_1$, $s_{2\xi} \in \mathcal{S}_{2\xi}$, $s_{3\xi} \in \mathcal{S}_{3\xi}$, $s_{4k} \in \mathcal{S}_{4k}$, and $\beta > 0$. Here, $\mathcal{V}_{poly} \subset \mathcal{V}$ and $\mathcal{S}$'s are prescribed finite-dimensional subspaces of $\mathbb{R}[x]$ and $\Sigma[x]$, respectively. Then, $\beta_B^*(\mathcal{V}_{poly}, \mathcal{S}) \leq \beta^*(\mathcal{V}_{poly})$. ◁

The optimization problem in (15) provides a recipe to compute subsets of $\mathcal{R}^n$ that are invariant under the flow of all possible systems described by (6). The number of constraints in (15) (consequently the number of decision variables since each new constraint includes new variables) increases exponentially with N and $n - \tilde{n}$ where $\tilde{n}$ is defined as the number of entries of the vectors ϕ_l and ϕ_u satisfying $\phi_l(x) = \phi_u(x) = 0$ for all $x \in \mathcal{G}$. Namely, there are $2^{(n-\tilde{n})+N}$ SOS conditions in (15) due to the constraint in (14). Revisiting the discussion in item (iii) at the end of section III, we note that covering the high degree vector field with low degree uncertainty reduces the dimension of the SOS constraints but increases (exponentially, depending on $n - \tilde{n}$) the number of constraints. Consequently, the utility of this approach will depend on $n - \tilde{n}$ and is problem dependent. Example (3) in section V-A illustrates this technique.

This difficulty can be partially alleviated by accepting suboptimal solutions for (15) in two steps: First compute a Lyapunov function for a finite sample of systems corresponding to the finite set $\mathcal{D}_{sample} \subset \mathcal{D}$ (for example, $\mathcal{D}_{sample}$ can be taken as the singleton corresponding to the "center" of $\mathcal{D}$) solving the problem

$$\begin{array}{rl} \max\limits_{V,\beta,s_1,s_{2\delta},s_{3\delta},s_{4k}} & \beta \text{ subject to} \\ & V - l_1 \in \Sigma[x], \\ & -[(\beta - p)s_1 + (V-1)] \in \Sigma[x] \\ & g_k - (1-V)s_{4k} \in \Sigma[x],\ k = 1,\dots,m, \\ & -[(1-V)s_{2\delta} + \nabla V(f_0+\delta)s_{3\delta} + l_2] \in \Sigma[x], \end{array} \tag{16}$$

for $\delta \in \mathcal{D}_{sample}$, where $s_1 \in \mathcal{S}_1, s_{2\delta} \in \mathcal{S}_2, s_{3\delta} \in \mathcal{S}_3, s_{4k} \in \mathcal{S}_4$ are SOS, $V \in \mathcal{V}_{poly}$, $V(0) = 0$, and let $\tilde{V}$ the Lyapunov function computed by solving (16). In the second step, compute the largest sublevel set $\mathcal{P}_{\beta^{subopt}}$ such that $\tilde{V}$ certifies $\mathcal{P}_{\beta^{subopt}}$ to be in the ROA for every vertex system by solving several smaller decoupled affine SDPs. For $\xi \in \mathcal{E}$, define

$$\begin{array}{rl} \gamma_\xi := \max\limits_{\gamma, s_2\in\mathcal{S}_2, s_3\in\mathcal{S}_3} & \gamma \text{ subject to } s_2, s_3 \in \Sigma[x], \\ & -\left[(\gamma - \tilde{V})s_2 + \nabla\tilde{V}(f_0+\xi)s_3 + l_2\right] \in \Sigma[x], \end{array} \tag{17}$$

and $\gamma^{subopt} := \min\{\gamma_\xi \ :\ \xi \in \mathcal{E}\}$. Then, a lower bound for β^{subopt} can be computed through

$$\begin{array}{rl} \beta^{subopt} := \max\limits_{\beta, s_1\in\mathcal{S}_1} & \beta \text{ subject to } s_1 \in \Sigma[x] \\ & -\left[(\beta - p)s_1 + (\tilde{V} - \gamma^{subopt})\right] \in \Sigma[x]. \end{array} \tag{18}$$

While the two-step procedure sacrifices optimality, it has practical computational advantages. The constraints in (14) decouple in the problem (17). In fact, for each $\xi \in \mathcal{E}_\Delta$, the problem in (17) contains only a single constraint from (14). Therefore, this decoupling enables suboptimal local stability analysis for systems with uncertainty without solving optimization problems larger than those one would have to solve in local stability analysis for systems without uncertainty. Furthermore, problems in (17) can be solved independently for different $\xi \in \mathcal{E}_\Delta$ and therefore computations can be trivially parallelized. Advantages of this decoupling may be better appreciated by noting that one of the main difficulties in solving large-scale SDPs is the memory requirements of the interior-point type algorithms [31]. Consequently, it is possible to perform some ROA analysis on systems with relatively reasonable number of states and/or uncertain parameters using the proposed suboptimal solution technique.

Finally, the following upper bound on the value of μ, for which (14) can be feasible, will be useful in section V.

Proposition 4.6: Let $L_2 \succ 0$ and $l_2(x) := x^T L_2 x$. Then,

$$\begin{array}{rl} \bar{\mu} := \max\limits_{\mu\geq 0, P=P^T\succeq 0} & \mu \quad \text{subject to} \\ & A_\xi^T P + PA_\xi + \mu P \preceq -L_2, \quad \text{for all } \xi \in \mathcal{E}, \end{array} \tag{19}$$

where $A_\xi := \left.\frac{\partial(f_0+\xi)}{\partial x}\right|_{x=0}$, is an upper bound for the values of μ such that (14) can be feasible. ◁

Proof: With l_2 as defined and $s_{2\xi}, s_{3\xi} \in \Sigma[x]$, if $b_\xi(x) := -[(1-V)s_{2\xi} + (\nabla V(f_0+\xi) + \mu V)s_{3\xi} + l_2] \in \Sigma[x]$, then $s_{2\xi}$ and b_ξ cannot contain constant and linear monomials and the quadratic part of b_ξ has to be SOS and equivalently positive semidefinite. Therefore, the result follows from the fact that, for fixed $\mu \geq 0$ and positive definite quadratic l_2, the existence of $P \succeq 0$ satisfying $A_\xi^T P + PA_\xi + \mu P \preceq -L_2$ is necessary for the existence of V, $s_{2\xi}$, and $s_{3\xi}$ feasible for (14). ■

Note that the problem in (19) can be solved as a sequence of affine SDPs by a line search on μ.

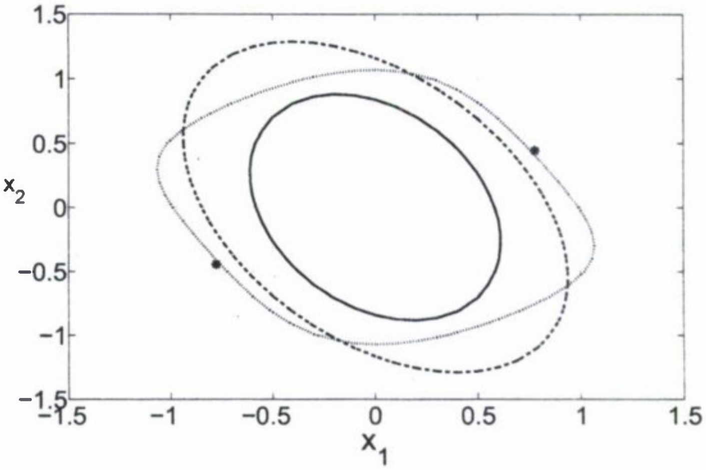

Fig. 1. Invariant subsets of ROA reported in [16] (solid) and those computed solving the problem in (15) with $\partial(V) = 2$ (dash) and $\partial(V) = 4$ (dot) along with initial conditions (stars) for some divergent trajectories of the system corresponding to $\alpha = 1$.

TABLE II

OPTIMAL VALUES OF β IN THE PROBLEM (15) WITH DIFFERENT VALUES OF μ AND $\partial(V) = 2$ AND 4.

μ \ $\partial(V)$	2	4
0	0.623	0.771
0.01	0.603	0.763
0.05	0.494	0.742
0.1	0.404	0.720
0.15	0.277	0.698
0.2	0.137	0.676

V. EXAMPLES

In the following examples, $p(x) = x^T x$ (except for example (2) in section V-A), $l_1(x) = 10^{-6}x^T x$, and $l_2(x) = 10^{-6}x^T x$. All certifying Lyapunov functions and multipliers are available at [32]. All computations use the generalization of the simulation based ROA analysis method from [13], [24]. Representative computation times on 2.0 GHz desktop PC are listed with each example.

A. Examples from the literature

(1) Consider the following system from [16]: $\dot{x}_1 = x_2$ and $\dot{x}_2 = -x_2 + \alpha(-x_1 + x_1^3)$, where $\alpha \in [1,3]$ is a parametric uncertainty. We solved problem (15) with $\partial(V) = 2$ and $\partial(V) = 4$ for $\mu = 0, 0.01, 0.05, 0.1, 0.15$, and 0.2. Note that $\bar{\mu}$ (as defined in Proposition 4.6) is 0.244. Typical computation times are 5 and 8 seconds for $\partial(V) = 2$ and 4, respectively.

Figure 1 shows the invariant subset of the robust ROA reported in [16] (solid) and those computed here with $\partial(V) = 2$ (dash) and $\partial(V) = 4$ (dot) for $\mu = 0$ along with two points (stars) that are initial conditions for divergent trajectories of the system corresponding to $\alpha = 1$. Table II shows the optimal values of β in the problem (15) with $\partial(V) = 2$ and 4 for different values of μ.

(2) Consider the system (from [17]) of $\dot{x}_1 = -x_2 + 0.2\alpha x_2$ and $\dot{x}_2 = x_1 + (x_1^2 - 1)x_2$ where $\alpha \in [-1,1]$. For easy comparison with the results in [17], let $p(x) = 0.378x_1^2 - 0.274x_1x_2 + 0.278x_2^2$ and $\mu = 0$. In [17], it was shown that $\mathcal{P}_{0.545}$ (with a single parameter independent quartic V), $\mathcal{P}_{0.772}$ (with pointwise maximum of two parameter independent quartic V's), $\mathcal{P}_{0.600}$ (with a single parameter

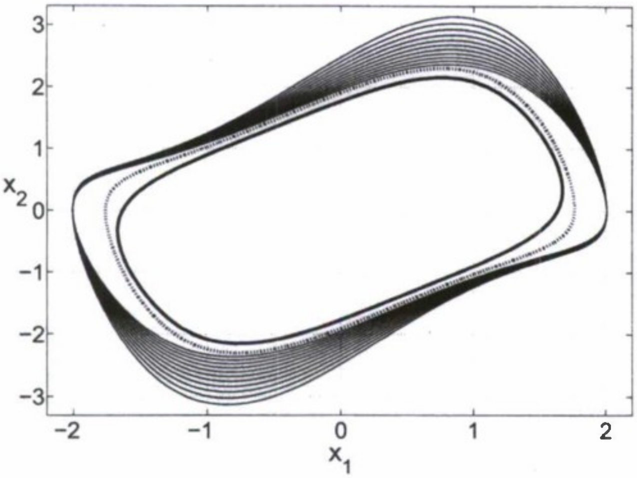

Fig. 2. Invariant subsets of ROA with $\partial(V) = 4$ (inner solid) and $\partial(V) = 6$ (dash) along with the unstable limit cycle (outer solid curves) of the system corresponding to $\alpha = -1.0, -0.8, \ldots, 0.8, 1.0$.

TABLE III
OPTIMAL VALUES OF β IN THE PROBLEM (15) WITH DIFFERENT VALUES OF μ AND $\partial(V) = 4$ AND 6.

μ \ $\partial(V)$	4	6
0	0.773	0.826
0.01	0.767	0.820
0.05	0.741	0.803
0.1	0.708	0.787
0.2	0.640	0.750
0.5	0.517	0.651
0.75	0.406	0.573

dependent quartic (in state) V), $\mathcal{P}_{0.806}$ (with pointwise maximum of two parameter dependent quartic (in state) V's) are contained in the robust ROA. On the other hand, the solution of problem (15) with $\partial(V) = 4$ and $\partial(V) = 6$ certifies that $\mathcal{P}_{0.773}$ and $\mathcal{P}_{0.826}$ are subsets of the robust ROA, respectively. Figure 2 shows invariant subsets of the robust ROA computed using $\partial(V) = 4$ (inner solid) and $\partial(V) = 6$ (dash) along with the unstable limit cycle (outer solid curves) of the system corresponding to $\alpha = -1.0, -0.8, \ldots, 0.8, 1.0$. In order to demonstrate the effect of the parameter μ on the size of the invariant subsets of the robust ROA verifiable solving the optimization problem in (15), the analysis is repeated with $\mu = 0.01, 0.05, 0.1., 0.2, 0.5$, and 0.75. Note that $\bar{\mu}$ (as defined in Proposition 4.6) is 0.769. Table III shows the optimal values of β in the problem (15) with $\partial(V) = 4$ and 6 for different values of μ. Typical computation times are 19 and 24 seconds for $\partial(V) = 4$ and 6, respectively.

(3) Consider the system governed by

$$\dot{x} = \begin{bmatrix} -2x_1 + x_2 + x_1^3 + 1.58x_2^3 \\ -x_1 - x_2 + 0.13x_2^3 + 0.66x_1^2 x_2 \end{bmatrix} + \phi(x), \qquad (20)$$

where ϕ satisfies the bounds $-0.76x_2^2 \leq \phi_1(x) \leq 0.76x_2^2$ and $-0.19(x_1^2 + x_2^2) \leq \phi_2(x) \leq 0.19(x_1^2 + x_2^2)$ in the set $\mathcal{G} = \{x \in \mathcal{R}^2 : g(x) = x^T x \leq 2.1\}$. Fig. 3 shows invariant subsets of the robust ROA computed with $\partial(V) = 2$ (solid) and $\partial(V) = 4$ (dash) along with two points that are initial conditions for divergent trajectories ("$*$" for $\phi(x) = (0.76x_2^2, 0.19(x_1^2 + x_2^2))$ and "$\times$" for $\phi(x) = (-0.76x_2^2, -0.19(x_1^2+x_2^2)))$. Typical computation times are 13 and 35 seconds for $\partial(V) = 2$ and 4, respectively.

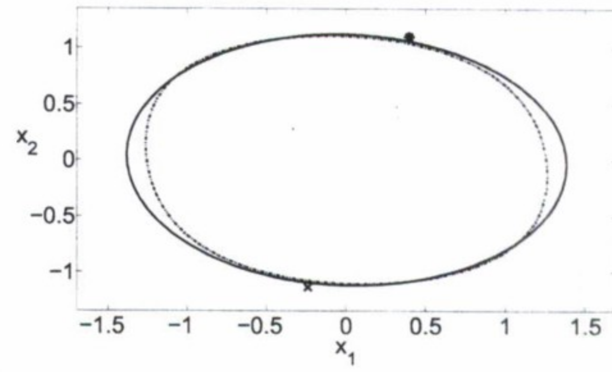

Fig. 3. Invariant subsets of ROA with $\partial(V) = 2$ (solid) and $\partial(V) = 4$ (dash) along with initial conditions for divergent trajectories ("$*$" for $\phi(x) = (0.76x_2^2, 0.19(x_1^2+x_2^2))$ and "$\times$" for $\phi(x) = (-0.76x_2^2, -0.19(x_1^2+x_2^2)))$.

B. Controlled short period aircraft dynamics

Consider the plant dynamics

$$\dot{z} = \begin{bmatrix} -3 & -1.35 & -0.56 \\ 0.91 & -0.64 & -0.02 \\ 1 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 1.35 - 0.04z_2 \\ 0.4 \\ 0 \end{bmatrix} u + \begin{bmatrix} (1+\alpha_1)(0.08z_1z_2 + 0.44z_2^2 + 0.01z_2z_3 + 0.22z_2^3) \\ (1+\alpha_2)(-0.05z_2^2 + 0.11z_2z_3 - 0.05z_3^2) \\ 0 \end{bmatrix} \qquad (21)$$

$y = [z_1 \; z_3]^T$, where z_1, z_2 and, z_3 are the pitch rate, the angle of attack, and the pitch angle, respectively. The input u is the elevator deflection and determined by

$$\dot{\eta} = \begin{bmatrix} -0.60 & 0.09 \\ 0 & 0 \end{bmatrix} \eta + \begin{bmatrix} -0.06 & -0.02 \\ -0.75 & -0.28 \end{bmatrix} y \qquad (22)$$

$u = \eta_1 + 2.2\eta_2$, where η is the controller state. Here, α_1 and α_2 are two uncertain parameters introducing 10% uncertainty for the entries of the plant dynamics that are nonlinear in v, i.e., $\alpha_1 \in [-0.1, 0.1]$ and $\alpha_2 \in [-0.1, 0.1]$. Entries in the vector fields above are shown up to three significant digits. The exact vector field used for this example is available at [32]. The solution of (15) with $\partial(V) = 2$ and $\mu = 0$ verifies that $\mathcal{P}_{7.2} \subset R_0^r$ whereas it can be certified that $\mathcal{P}_{8.6}$ is a subset of the ROA for the nominal system (i.e., for $\alpha_{l,1} = \alpha_{l,2} = \alpha_{u,1} = \alpha_{u,2} = 0$). With $\partial(V) = 4$ the problem in (15) has more than 4000 decision variables. Therefore, we computed a suboptimal solution in two steps for $\mu = 0$: We first computed a Lyapunov function for the nominal system (35 minutes, which certifies that $\mathcal{P}_{15.2}$ is in the ROA for the nominal system) and then verified (3 minutes) that $\mathcal{P}_{9.6}$ is an invariant subset of the ROA for the uncertain system. To assess the suboptimality of the results, we performed extensive simulations for the uncertain system setting α_1 and α_2 to their limit values and found a diverging trajectory with the initial condition satisfying $p(z(0), \eta(0)) \approx 14$. The gap between the value of $\beta \approx 14$ for which $\mathcal{P}_\beta$ cannot be a subset of the robust ROA and the value of $\beta = 9.6$ for which $\mathcal{P}_\beta \subset R_0^r$ is verified may be due to the finite dimensional parametrization for V, the issues mentioned in Remark 4.3, the fact that we only use sufficient conditions and/or suboptimality of the two step procedure used for this example; nevertheless, it demonstrates a necessity of further study to make local system analysis based on Lyapunov functions and SOS relaxations more efficient.

VI. Conclusions

We proposed a method to compute provably invariant subsets of the region-of-attraction for the asymptotically stable equilibrium points of uncertain nonlinear dynamical systems. We considered polynomial dynamics with perturbations that either obey local polynomial bounds or are described by uncertain parameters multiplying polynomial terms in the vector field. This uncertainty description is motivated by both incapabilities in modeling, as well as bilinearity and dimension of the sum-of-squares programming problems whose solutions provide invariant subsets of the region-of-attraction. We demonstrated the method on three examples from the literature and a controlled short period aircraft dynamics example.

VII. Acknowledgements

This work was sponsored by the Air Force Office of Scientific Research, USAF, under grant/contract number FA9550-05-1-0266. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the AFOSR or the U.S. Government. The authors would like to thank Peter Seiler for valuable discussions and Kailash Krishnaswamy for providing the closed-loop aircraft dynamics in section V-B.

References

[1] E. J. Davison and E. M. Kurak, "A computational method for determining quadratic Lyapunov functions for nonlinear systems," *Automatica*, vol. 7, pp. 627–636, 1971.
[2] R. Genesio, M. Tartaglia, and A. Vicino, "On the estimation of asymptotic stability regions: State of the art and new proposals," *IEEE Transaction on Automatic Control*, vol. 30, no. 8, pp. 747–755, 1985.
[3] A. Vannelli and M. Vidyasagar, "Maximal Lyapunov functions and domains of attraction for autonomous nonlinear systems," *Automatica*, vol. 21, no. 1, pp. 69–80, 1985.
[4] J. Hauser and M. C. Lai, "Estimating quadratic stability domains by nonsmooth optimization," in *Proc. American Control Conf.*, Chicago, IL, 1992, pp. 571–576.
[5] H.-D. Chiang and J. S. Thorp, "Stability regions of nonlinear dynamical systems: A constructive methodology," *IEEE Transaction on Automatic Control*, vol. 34, no. 12, pp. 1229–1241, 1989.
[6] P. Parrilo, "Semidefinite programming relaxations for semialgebraic problems," *Mathematical Programming Series B*, vol. 96, no. 2, pp. 293–320, 2003.
[7] J. Lasserre, "Global optimization with polynomials and the problem of moments," *SIAM Journal on Optimization*, vol. 11, no. 3, pp. 796–817, 2001.
[8] T.-C. Wang, S. Lall, and M. West, "Polynomial level-set methods for nonlinear dynamical systems analysis," in *Proc. Allerton Conf. on Communication, Control, and Computing*, Allerton, IL, 2005.
[9] B. Tibken and Y. Fan, "Computing the domain of attraction for polynomial systems via BMI optimization methods," in *Proc. American Control Conf.*, Minneapolis, MN, 2006, pp. 117–122.
[10] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, "LMI-based computation of optimal quadratic Lyapunov functions for odd polynomial systems," *Int. J. Robust Nonlinear Control*, vol. 15, pp. 35–49, 2005.
[11] W. Tan and A. Packard, "Stability region analysis using sum of squares programming," in *Proc. American Control Conf.*, Minneapolis, MN, 2006, pp. 2297–2302.
[12] ——, "Stability region analysis using polynomial and composite polynomial lyapunov functions and sum-of-squares programming," *Automatic Control, IEEE Transactions on*, vol. 53, no. 2, pp. 565–571, 2008.
[13] U. Topcu, A. Packard, P. Seiler, and T. Wheeler, "Stability region analysis using simualtions and sum-of-squares programming," in *Proc. American Control Conf.*, New York, NY, 2007.
[14] F. Camilli, L. Grune, and F. Wirth, "A generalization of Zubov's method to perturbed systems," *SIAM Journal on Control and Optimization*, vol. 40, no. 2, pp. 496–515, 2001.
[15] A. Paice and F. Wirth, "Robustness analysis of domains of attraction of nonlinear systems," in *Proc. of the Mathematical Theory of Networks and Systems*, Padova, Italy, 1998, pp. 353–356.
[16] G. Chesi, "Estimating the domain of attraction of uncertain poynomial systems," *Automatica*, vol. 40, pp. 1981–1986, 2004.
[17] W. Tan, "Nonlinear control analysis and synthesis using sum-of-squares programming," Ph.D. Dissertation, UC, Berkeley, May 2006.
[18] A. Trofino, "Robust stability and domain of attraction of uncertain nonlinear systems," in *Proc. American Control Conf.*, Chicago, IL, 2000, pp. 3707–3711.
[19] W. Tan, A. Packard, and T. Wheeler, "Local gain analysis of nonlinear systems," in *Proc. American Control Conf.*, Minneapolis, MN, 2006, pp. 92–96.
[20] A. Hassibi, J. How, and S. Boyd, "A path-path following method for solving BMI problems in control," in *Proc. American Control Conf.*, San Diego, CA, 1999, pp. 1385–1389.
[21] Z. Jarvis-Wloszek, R. Feeley, W. Tan, K. Sun, and A. Packard, "Control applications of sum of squares programming," in *Positive Polynomials in Control*, D. Henrion and A. Garulli, Eds. Springer-Verlag, 2005, pp. 3–22.
[22] C. Hol, "Structured controller synthesis for mechanical servo systems: Algorithms, relaxations and optimality certificates," Ph.D. Dissertation, Technical University of Delft, 2006.
[23] M. Kočvara and M. Stingl, "*PENBMI User's Guide (Version 2.0)*, available from `http://www.penopt.com`," 2005.
[24] U. Topcu, A. Packard, and P. Seiler, "Local stability analysis using simulations and sum-of-squares programming," 2006, to appear in Automatica.
[25] M. Vidyasagar, *Nonlinear Systems Analysis*, 2^{nd} ed. Prentice Hall, 1993.
[26] E. M. Aylward, P. A. Parrilo, and J.-J. E. Slotine, "Stability and robustness analysis of nonlinear systems via contraction metrics and SOS programming," 2006, available at `http://arxiv.org/abs/math.OC/0603313`.
[27] H. K. Khalil, *Nonlinear Systems*, 3^{rd} ed. Prentice Hall, 2002.
[28] A. Papachristodoulou and S. Prajna, "Analysis of non-polynomial systems using the sum of squares decomposition," in *Positive Polynomials in Control*, D. Henrion and A. Garulli, Eds. Springer-Verlag, 2005, pp. 23–43.
[29] D. Henrion and M. Šebek, "Overcoming nonconvexity in polynomial robust control design," in *Proc. Symp. Math. Theory of Networks and Systems*, Leuven, Belgium, 2006.
[30] B. Barmish, "Necessary and sufficient conditions for quadratic stabilizability of an uncertain system," *Journal of Optimization Theory and Applications*, vol. 46, no. 4, pp. 399–408, 1985.
[31] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*. MPS-SIAM Series on Optimization, 2001.
[32] "`http://jagger.me.berkeley.edu/~pack/certificates`."

VIII. Appendix

Following two lemmas are simple generalizations of the S-procedure. The proof of the first one is trivial. We provide a proof for the second one.

Lemma 8.1: Given $g_0, g_1, \cdots, g_m \in \mathbb{R}[x]$, if there exist $s_1, \cdots, s_m \in \Sigma[x]$ such that $g_0 - \sum_{i=1}^{m} s_i g_i \in \Sigma[x]$, then $\{x \in \mathcal{R}^n : g_1(x), \ldots, g_m(x) \geq 0\} \subseteq \{x \in \mathcal{R}^n : g_0(x) \geq 0\}$. ◁

Lemma 8.2: Let $g \in \mathbb{R}[x]$ be positive definite, $h \in \mathbb{R}[x]$, $\gamma > 0$, $s_1, s_2 \in \Sigma[x]$, $l \in \mathbb{R}[x]$ be positive definite and satisfy $l(0) = 0$. Suppose that $-[(\gamma - g)s_1 + hs_2 + l] \in \Sigma[x]$ holds. Then, $\Omega_{g,\gamma} \backslash \{0\} \subset \{x \in \mathcal{R}^n : h(x) < 0 \text{ and } s_2(x) > 0\}$. ◁

Proof: Let $x \in \Omega_{g,\gamma}$ be nonzero. Then,

$$0 > -l(x) - (\gamma - g(x))s_2(x) \geq h(x)s_2(x),$$

and, consequently, $s_2(x) > 0$ (since $s_2(x) \geq 0$) and $h(x) < 0$. ■

Local Stability Analysis Using Simulations and Sum-of-Squares Programming ⋆

Ufuk Topcu [a], Andrew Packard [a] and Peter Seiler [b]

[a] *Department of Mechanical Engineering, University of California, Berkeley, 94720-1740, USA*

[b] *Honeywell Technology Center, Minneapolis, MN, 55418, USA*

Abstract

The problem of computing bounds on the region-of-attraction for systems with polynomial vector fields is considered. Invariant subsets of the region-of-attraction are characterized as sublevel sets of Lyapunov functions. Finite dimensional polynomial parameterizations for Lyapunov functions are used. A methodology utilizing information from simulations to generate Lyapunov function candidates satisfying necessary conditions for bilinear constraints is proposed. The suitability of Lyapunov function candidates are assessed solving linear sum-of-squares optimization problems. Qualified candidates are used to compute invariant subsets of the region-of-attraction and to initialize various bilinear search strategies for further optimization. We illustrate the method on small examples from the literature and several control oriented systems.

Key words: Local Stability; Region-of-attraction; Nonlinear dynamics; Sum-of-squares programming; Simulations.

1 Introduction

The region-of-attraction (ROA) of a locally asymptotically stable equilibrium point is an invariant set such that all trajectories emanating from points in this set converge to the equilibrium point. Computing the exact ROA for nonlinear dynamics is very hard if not impossible. Therefore, researchers have focused on determining invariant subsets of the ROA. Among all other methods those based on Lyapunov functions are dominant in the literature (Davison and Kurak, 1971; Genesio *et al.*, 1985; Vannelli and Vidyasagar, 1985; Chiang and Thorp, 1989; Chesi *et al.*, 2005; Papachristodoulou, 2005; Tan and Packard, 2006; Hachicho and Tibken, 2002; Tibken and Fan, 2006; Tibken, 2000). These methods compute a Lyapunov function as a local stability certificate and sublevel sets of this Lyapunov function, in which the function decreases along the flow, provide invariant subsets of the ROA.

Using sum-of-squares (SOS) relaxations for polynomial nonnegativity (Parrilo, 2003), it is possible to search for polynomial Lyapunov functions for systems with polynomial and/or rational dynamics using semidefinite programming (Papachristodoulou, 2005; Tan and Packard, 2006; Hachicho and Tibken, 2002). Reliable and efficient solvers for linear semidefinite programs (SDPs) are available (Sturm, 1999). However, the SOS relaxation for the problem of computing invariant subsets of the ROA leads to bilinear matrix inequality (BMI) constraints. BMIs are nonconvex and bilinear SDPs, those with BMI constraints, are known to be NP-hard in general (Toker and Ozbay, 1995). Consequently, the state-of-the-art of the solvers for bilinear SDPs is far behind that for the linear ones. Recently PENBMI, a solver for bilinear SDPs, was introduced (Kočvara and Stingl, 2005) and subsequently used for computing invariant subsets of the ROA (Tan and Packard, 2006; Tibken and Fan, 2006). It is a local optimizer and its behavior (speed of convergence, quality of the local optimal point, etc.) depends on the point from which the optimization starts.

By contrast, simulating a nonlinear system of moderate size, except those governed by stiff differential equations, is computationally efficient. Therefore, extensive simulation is a tool used in real applications. Although the information from simulations is inconclusive, i.e., cannot be used to find provably invariant subsets of the ROA, it provides insight into the system behavior. For example, if, using Lyapunov arguments, a function certifies that a set $\mathcal{P}$ is in the ROA, then that function must be positive and decreasing on any solution trajectory initiating in $\mathcal{P}$. Using a finite number of points on finitely many convergent trajectories and a linear parametrization of the Lya-

⋆ This paper was not presented at any IFAC meeting. Corresponding author A. Packard. Tel. +1 510 642 6152. Fax +1 510 642 6163.

Email addresses: utopcu@jagger.me.berkeley.edu (Ufuk Topcu), pack@jagger.me.berkeley.edu (Andrew Packard), peter.j.seiler@honeywell.com (Peter Seiler).

punov function V, those constraints become affine, and the feasible polytope (in V-coefficient space) is a convex outer bound on the set of coefficients of valid Lyapunov functions. It is intuitive that drawing samples from this set to seed the bilinear SDP solvers may improve the performance of the solvers. In fact, if there are a large number of simulation trajectories, samples from the set often are suitable Lyapunov functions (without further optimization) themselves. Effectively, we are relaxing the bilinear problem (using a very specific system theoretic interpretation of the problem) to a linear problem, and the true feasible set is a subset of the linear problem's feasible set. Information from simulations is also used in (Prokhorov and Feldkamp, 1999) and (Serpen, 2005) for computing *approximate* Lyapunov functions.

Notation: For $x \in \mathcal{R}^n$, $x \succeq 0$ means that $x_k \geq 0$ for $k = 1, \cdots, n$. For $Q = Q^T \in \mathcal{R}^{n\times n}$, $Q \succeq 0$ ($Q \succ 0$) means that $x^TQx \geq 0$ (≤ 0) for all $x \in \mathcal{R}^n$. $\mathbb{R}[x]$ represents the set of polynomials in x with real coefficients. The subset $\Sigma[x] := \{\pi \in \mathbb{R}[x] \ : \ \pi = \pi_1^2 + \pi_2^2 + \cdots + \pi_m^2, \pi_1, \cdots, \pi_m \in \mathbb{R}[x]\}$ of $\mathbb{R}[x]$ is the set of SOS polynomials. For $\pi \in \mathbb{R}[x]$, $\partial(\pi)$ denotes the degree of π. $\mathcal{C}^1$ denotes the space of continuously differentiable functions. We use the term "semidefinite programming" to mean optimization problems with affine objective function and *general* (not necessarily affine) matrix (semi)definiteness constraints. ◁

2 Characterization of invariant subsets of the ROA and bilinear SOS problem

Consider the autonomous nonlinear dynamical system

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the state vector and $f : \mathcal{R}^n \to \mathcal{R}^n$ is such that $f(0) = 0$, i.e., the origin is an equilibrium point of (1), and f is locally Lipschitz. Let $\phi(\xi, t)$ denote the solution to (1) at time t with the initial condition $\phi(\xi, 0) = \xi$. If the origin is asymptotically stable but not globally attractive, one often wants to know which trajectories converge to the origin as time approaches ∞. The region-of-attraction R_0 of the origin for the system (1) is $R_0 := \{\xi \in \mathcal{R}^n : \lim_{t\to\infty} \phi(\xi, t) = 0\}$. A modification of a similar result in (Vidyasagar, 1993) provides a characterization of invariant subsets of the ROA. For $\eta > 0$ and a function $V : \mathcal{R}^n \to \mathcal{R}$, define the η-sublevel set $\Omega_{V,\eta}$ of V as $\Omega_{V,\gamma} := \{x \in \mathcal{R}^n \ : \ V(x) \leq \eta\}$.

Lemma 1 *Let $\gamma \in \mathcal{R}$ be positive. If there exists a $\mathcal{C}^1$ function $V : \mathcal{R}^n \to \mathcal{R}$ such that*

$$\Omega_{V,\gamma} \text{ is bounded, and} \tag{2}$$
$$V(0) = 0 \text{ and } V(x) > 0 \text{ for all } x \in \mathcal{R}^n \tag{3}$$
$$\Omega_{V,\gamma}\backslash\{0\} \subset \{x \in \mathcal{R}^n \ : \ \nabla V(x) f(x) < 0\}, \tag{4}$$

then for all $\xi \in \Omega_{V,\gamma}$, the solution of (1) exists, satisfies $\phi(\xi, t) \in \Omega_{V,\gamma}$ for all $t \geq 0$, and $\lim_{t\to\infty} \phi(\xi, t) = 0$, i.e., $\Omega_{V,\gamma}$ is an invariant subset of R_0. ◁

In order to enlarge the computed invariant subset of the ROA, we define a variable sized region $\mathcal{P}_\beta := \{x \in \mathcal{R}^n \ : \ p(x) \leq \beta\}$, where $p \in \mathbb{R}[x]$ is a fixed positive definite convex polynomial, and maximize β while imposing the constraint $\mathcal{P}_\beta \subseteq \Omega_{V,\gamma}$ along with the constraints (2)-(4). This can be written as

$$\beta^*(\mathcal{V}) := \max_{\beta>0, V\in\mathcal{V}} \beta \text{ subject to } (2)-(4), \ \mathcal{P}_\beta \subseteq \Omega_{V,\gamma}. \tag{5}$$

Here $\mathcal{V}$ denotes the set of candidate Lyapunov functions over which the maximum is defined, for example all $\mathcal{C}^1$ functions. Lemma 1 and the associated optimization problem in (5) provide a characterization of the invariant subsets of the ROA in terms of the sublevel sets of Lyapunov functions.

The problem in (5) is an infinite dimensional problem. In order to make it amenable to numerical optimization (specifically SOS optimization), we restrict $\mathcal{V}$ to be all polynomials of some fixed degree. We use the well-known sufficient condition: for any $\pi \in \mathbb{R}[x]$, if $\pi \in \Sigma[x]$, then π is positive semidefinite (Parrilo, 2003). Using simple generalizations of the S-procedure (Lemmas 2 and 3), we obtain sufficient conditions for set containment constraints. Specifically, let l_1 and l_2 be a positive definite polynomials (typically ϵx^Tx for some small real number ϵ). Then, since l_1 is radially unbounded, the constraint

$$V - l_1 \in \Sigma[x] \tag{6}$$

and $V(0) = 0$ are sufficient conditions for (2) and (3). By Lemma 2, if $s_1 \in \Sigma[x]$, then

$$-[(\beta - p)s_1 + (V - \gamma)] \in \Sigma[x] \tag{7}$$

implies the set containment $\mathcal{P}_\beta \subseteq \Omega_{V,\gamma}$, and by Lemma 3, if $s_2, s_3 \in \Sigma[x]$, then

$$-[(\gamma - V)s_2 + \nabla V f s_3 + l_2] \in \Sigma[x] \tag{8}$$

is a sufficient condition for (4). Using these sufficient conditions, a lower bound on $\beta^*(\mathcal{V})$ can be defined as

$$\begin{aligned} \beta_B^*(\mathcal{V}, \mathcal{S}) := & \max_{V\in\mathcal{V},\beta,s_i\in\mathcal{S}_i} \beta \text{ subject to } (6)-(8), \\ & V(0) = 0, s_i \in \Sigma[x], \beta > 0. \end{aligned} \tag{9}$$

Here, the sets $\mathcal{V}$ and $\mathcal{S}_i$ are prescribed finite-dimensional subspaces of polynomials. Although β_B^* depends on these subspaces, it will not always be explicitly notated. Note that since the conditions (6)-(8) are only sufficient conditions, $\beta_B^*(\mathcal{V}, \mathcal{S}) \leq \beta^*(\mathcal{V}) \leq \beta^*(\mathcal{C}^1)$. The optimization problem in (9) is bilinear because of the product terms βs_1 in (7) and $V s_2$ and $\nabla V f s_3$ in (8). However, the problem has more structure than a general BMI problem. If V is fixed, the problem becomes affine in $S = \{s_1, s_2, s_3\}$ and vice versa. In section 3, we will construct a convex outer bound on the set of feasible V and sample from this outer bound set to obtain candidate V's, and then solve (9) for S, holding V fixed.

3 Relaxation of the bilinear SOS problem using simulation data

The usefulness of simulation in understanding the ROA for a given system is undeniable. Faced with the task of performing a stability analysis (eg., "for a given p, is $\mathcal{P}_\beta$ contained in the ROA?"), a pragmatic, fruitful and wise approach begins with a linearized analysis and at least a modest amount of simulation runs. Certainly, just one divergent trajectory starting in $\mathcal{P}_\beta$ certifies that $\mathcal{P}_\beta \not\subset R_0$. Conversely, a large collection of only convergent trajectories hints to the likelihood that indeed $\mathcal{P}_\beta \subset R_0$. Suppose this latter condition is true, let $\mathbf{C}$ be the set of N_{conv} trajectories $\mathbf{c}$ converging to the origin with initial conditions in $\mathcal{P}_\beta$. In the course of simulation runs, divergent trajectories $\mathbf{d}$ whose initial conditions are not in $\mathcal{P}_\beta$ may also get discovered, so let the set of $\mathbf{d}$'s be denoted by $\mathbf{D}$ and N_{div} be the number of elements of $\mathbf{D}$. Although $\mathbf{C}$ and $\mathbf{D}$ depend on β and the manner in which $\mathcal{P}_\beta$ is sampled, this is not explicitly notated.

With β and γ fixed, the set of Lyapunov functions which certify that $\mathcal{P}_\beta \subset R_0$, using conditions (6)-(8), is simply $\{V \in \mathbb{R}[x] : (6) - (8) \text{ hold for some } s_i \in \Sigma[x]\}$. Of course, this set could be empty, but it must be contained in the convex set $\{V \in \mathbb{R}[x] \ : \ (10) \text{ holds}\}$, where

$$\begin{aligned} &\nabla V(\mathbf{c}(t)) f(\mathbf{c}(t)) < 0, \\ &l_1(\mathbf{c}(t)) \leq V(\mathbf{c}(t)), \text{ and } V(\mathbf{c}(0)) \leq \gamma, \\ &\gamma + \delta \leq V(\mathbf{d}(t)), \end{aligned} \tag{10}$$

for all $\mathbf{c} \in \mathbf{C}$, $\mathbf{d} \in \mathbf{D}$, and $t \geq 0$, where δ is a fixed (small) positive constant. Informally, these conditions simply say that any V which verifies that $\mathcal{P}_\beta \subset R_0$ using the conditions (6)-(8) must, on the trajectories starting in $\mathcal{P}_\beta$, be decreasing and take on values between 0 and γ. Moreover, V must be greater than γ on divergent trajectories. In fact, with the exception of the strengthened lower bound on V (beyond mere positivity), the conditions in (10) are even necessary conditions for any $V \in \mathcal{C}^1$ which verify $\mathcal{P}_\beta \subset R_0$ using conditions (2)-(4).

3.1 Affine relaxation using simulation data

Let $\mathcal{V}$ be linearly parameterized as $\mathcal{V} := \{V \in \mathbb{R}[x] \ : \ V(x) = \varphi(x)^T\alpha\}$, where $\alpha \in \mathcal{R}^{n_b}$ and φ is n_b-dimensional vector of polynomials in x. Given $\varphi(x)$, constraints in (10) can be viewed as constraints on $\alpha \in \mathcal{R}^{n_b}$ yielding the convex set $\{\alpha \in \mathcal{R}^{n_b} \ : \ (10) \text{ holds for } V = \varphi(x)^T\alpha\}$. For each $\mathbf{c} \in \mathbf{C}, \mathbf{d} \in \mathbf{D}$, let $\mathcal{T}_\mathbf{c}$ and $\mathcal{T}_\mathbf{d}$ be finite subsets of the interval $[0, \infty)$ including the origin. A polytopic outer bound for this set described by finitely many constraints is $\mathcal{Y}_{sim} := \{\alpha \in \mathcal{R}^{n_b} \ : \ (11) \text{ holds}\}$, where

$$\begin{aligned} &[\nabla\varphi(\mathbf{c}(\tau_c)) f(\mathbf{c}(\tau_c))]^T \alpha < 0, \\ &l_1(\mathbf{c}(\tau_c)) \leq \varphi(\mathbf{c}(\tau_c))^T\alpha, \text{ and } \varphi(\mathbf{c}(0))^T\alpha \leq \gamma, \\ &\varphi(\mathbf{d}(\tau_d))^T\alpha \geq \gamma + \delta \end{aligned} \tag{11}$$

for all $\mathbf{c} \in \mathbf{C}$, $\tau_c \in \mathcal{T}_\mathbf{c}$, $\mathbf{d} \in \mathbf{D}$, and $\tau_d \in \mathcal{T}_\mathbf{d}$. Note that $\varphi(\mathbf{c}(0))^T\alpha \leq \gamma$ in (11) provides necessary conditions for $\mathcal{P}_\beta \subseteq \Omega_{V,\gamma}$ since $\mathbf{c}(0) \in \mathcal{P}_\beta$ for all $\mathbf{c} \in \mathbf{C}$. In practice, we replace the strict inequality in (11) by $[\nabla\varphi(\mathbf{c}(\tau_c)) f(\mathbf{c}(\tau_c))]^T \alpha \leq -l_3(\mathbf{c}(\tau_c))$, where l_3 is a fixed, positive definite polynomial imposing a bound on the rate of decay of V along the trajectories.

The constraint that $\nabla V f$ be negative on a sublevel set of V implies that $\nabla V f$ is negative on a neighborhood of the origin. While a large number of sample points from the trajectories will approximately enforce this, in some cases (eg. exponentially stable linearization) it is easy to analytically express as a constraint on the low order terms of the polynomial Lyapunov function. For instance, assume V has a positive-definite quadratic part, and that separate eigenvalue analysis has established that the linearization of (1) at the origin, i.e., $\dot{x} = \nabla f(0)x$, is asymptotically stable. Define $\mathcal{L}(P) := (\nabla f(0))^T P + P(\nabla f(0))$, where $P^T = P \succ 0$ is such that $x^T Px$ is the quadratic part of V. Then, if (8) holds, it must be that

$$\mathcal{L}(P) \prec 0. \tag{12}$$

Let $\mathcal{Y}_{lin} := \{\alpha \in \mathcal{R}^{n_b} \ : \ P = P^T \succ 0 \text{ and (12) holds}\}$. It is well-known that $\mathcal{Y}_{lin}$ is convex (Boyd and Vandenberghe, 2004). Again, in practice, (12) is replaced by the condition $\mathcal{L}(P) \preceq -\epsilon I$, for some small real number ϵ. Furthermore, define $\mathcal{Y}_{SOS} := \{\alpha \in \mathcal{R}^{n_b} \ : \ (6) \text{ holds}\}$. By (Parrilo, 2003), $\mathcal{Y}_{SOS}$ is convex. Since $\mathcal{Y}_{sim}$, $\mathcal{Y}_{lin}$ and $\mathcal{Y}_{SOS}$ are convex, $\mathcal{Y} := \mathcal{Y}_{sim} \cap \mathcal{Y}_{lin} \cap \mathcal{Y}_{SOS}$ is a convex set in $\mathcal{R}^{n_b}$. Equations (11) and (12) constitute a set of necessary conditions for (6)-(8); thus, we have $\mathcal{Y} \supseteq \mathcal{B} := \{\alpha \in \mathcal{R}^{n_b} \ : \exists s_2, s_3 \in \Sigma[x] \text{ such that } (6) - (8) \text{ hold}\}$. Since (8) is not jointly convex in V and the multipliers, $\mathcal{B}$ may not be a convex set and even may not be connected.

A point in $\mathcal{Y}$ can be computed solving an affine (feasibility) SDP with the constraints (6), (11) and (12). An arbitrary point in $\mathcal{Y}$ may or may not be in $\mathcal{B}$. However, if we generate a collection $\mathcal{A} := \{\alpha^{(k)}\}_{k=0}^{N_V - 1}$ of N_V points distributed approximately uniformly in $\mathcal{Y}$, it may be that some of the points are in $\mathcal{B}$. To this end, we use the so-called "Hit-and-Run" ($H\&R$) random point generation algorithm as described in (Tempo *et al.*, 2005). When applied to generate a sample of $\mathcal{Y}$, each step of $H\&R$ algorithm requires solving four small affine SDPs.

3.2 Algorithms

Since a feasible value of β is not known a priori, an iterative strategy to simulate and collect convergent and divergent trajectories is necessary. This process when coupled with the $H\&R$ algorithm constitutes the Lyapunov function candidate generation.

Simulation and Lyapunov function generation (SimLFG) algorithm: Given positive definite convex $p \in \mathbb{R}[x]$, a vector of polynomials $\varphi(x)$ and constants β_{SIM}, N_{conv}, N_V, $\beta_{shrink} \in (0, 1)$, and empty sets $\mathbf{C}$ and $\mathbf{D}$, set $\gamma = 1$, $N_{more} = N_{conv}$, $N_{div} = 0$.

i. Integrate (1) from N_{more} initial conditions in the set $\{x \in \mathcal{R}^n : p(x) = \beta_{SIM}\}$.

ii. If there is no diverging trajectory, add the trajectories to $\mathbf{C}$ and go to (iii). Otherwise, add the divergent trajectories to $\mathbf{D}$ and the convergent trajectories to $\mathbf{C}$, let N_d denote the number of diverging trajectories found in the last run of (i) and set N_{div} to $N_{div} + N_d$. Set β_{SIM} to the minimum of $\beta_{shrink}\beta_{SIM}$ and the minimum value of p along the diverging trajectories. Set N_{more} to $N_{more} - N_d$, and go to (i).

iii. At this point $\mathbf{C}$ has N_{conv} elements. For each $i = 1, \ldots, N_{conv}$, let $\overline{\tau}_i$ satisfy $\mathbf{c}_i(\tau) \in \mathcal{P}_{\beta_{SIM}}$ for all $\tau \geq \overline{\tau}_i$. Eliminate times in $\mathcal{T}_i$ that are less than $\overline{\tau}_i$.

iv. Find a feasible point for (6), (11), and (12). If (6), (11), and (12) are infeasible, set $\beta_{SIM} = \beta_{shrink}\beta_{SIM}$, and go to (iii). Otherwise, go to (v).

v. Generate N_V Lyapunov function candidates using H&R algorithm, and return β_{SIM} and Lyapunov function candidates. ◁

The suitability of a Lyapunov function candidate is assessed by solving two optimization problems. Both problems require bisection and each bisection step involves a linear SOS problem. Alternative linear formulations appear in the appendix. These do not require bisection, but generally involve higher degree polynomial expressions.

Problem 1: Given $V \in \mathbb{R}[x]$ (from *SimLFG* algorithm) and positive definite $l_2 \in \mathbb{R}[x]$, define

$$
\begin{aligned}
\gamma_L^* := \max_{\gamma, s_2, s_3} \ \gamma \text{ subject to } s_2, s_3 \in \Sigma[x],\ \gamma > 0, \\
-[(\gamma - V)s_2 + \nabla V f s_3 + l_2] \in \Sigma[x].
\end{aligned} \tag{13}
$$

If *Problem 1* is feasible, then $\gamma_L^* > 0$ and define

Problem 2: Given $V \in \mathbb{R}[x]$, $p \in \mathbb{R}[x]$, and γ_L^*, solve

$$
\begin{aligned}
\beta_L^* := \max_{\beta, s_1} \ \beta \text{ subject to } s_1 \in \Sigma[x],\ \beta > 0, \\
-[(\beta - p)s_1 - (V - \gamma_L^*)] \in \Sigma[x].
\end{aligned} \tag{14}
$$

Although γ_L^* and β_L^* depend on the allowable degree of s_1, s_2, and s_3, this is not explicitly notated.

Assuming *Problem 1* is feasible, it is true that $\mathcal{P}_{\beta_L^*}\backslash\{0\} \subseteq \Omega_{V,\gamma_L^*}\backslash\{0\} \subset \{x \in \mathcal{R}^n : \nabla V(x) f(x) < 0\}$, so V certifies that $\mathcal{P}_{\beta_L^*} \subset R_0$. Solutions to *Problems 1* and *2* provide a feasible point for the problem in (9). This feasible point can be further improved by solving the problem in (9) using PENBMI and/or iterative coordinate-wise linear optimization schemes, one of which is given next.

Coordinate-wise optimization (CWOpt) algorithm:

Given $V \in \mathbb{R}[x]$, positive definite $l_1, l_2 \in \mathbb{R}[x]$, a constant ε_{iter}, and maximum number of iterations N_{iter}, set $k = 0$

i. Solve *Problems 1* and *2*.

ii. Given s_1, s_2, s_3, and γ_L^* from step (i), set γ in (7)-(8) to γ_L^*, solve (9) for V and β, and set $\beta_L^* = \beta_B^*$.

iii. If $k = N_{iter}$ or the increase in β_L^* between successive applications of (ii) is less than ε_{iter}, return V, γ_L^*, and β_L^*. Otherwise, set k to $k+1$ and go to (i). ◁

The algorithms (*SimLFG*, *Problems 1* and *2*, and *CWOpt*) yield lower bounds on $\beta^*(\mathcal{C}^1)$, as they produce a Lyapunov function which certifies that a particular value of β satisfies $\mathcal{P}_\beta \in R_0$. Upper bounds (i.e., values of β that are not certifiable) may also be obtained. More specifically, diverging trajectories found in the course of simulation runs provide upper bounds on $\beta^*(\mathcal{C}^1)$ while inconsistency of the constraints (6), (11), and (12) provide upper bounds on β_B^*. A diverging trajectory with the initial condition x_0 satisfying $p(x_0) = \beta$ proves that $\mathcal{P}_\beta$ cannot be a subset of the ROA, i.e., $\beta^*(\mathcal{C}^1) < \beta$. Furthermore, restricting Lyapunov function candidates to $\mathcal{V}_\varphi := \{\varphi(x)^T\alpha : \alpha \in \mathcal{R}^{n_b}\}$ has additional implications. Infeasibility of any of the constraints (6), (11), and (12) for some value of β (recall (11) implicitly depends on β) verifies $\beta_B^*(\mathcal{V}_\varphi, \mathcal{S}) \leq \beta^*(\mathcal{V}_\varphi) < \beta$, regardless of the subspaces constituting $\mathcal{S}$. Moreover, the gap between the value of β proven unachievable and what we actually certify, namely a lower bound to $\beta_B^*(\mathcal{V}_\varphi, \mathcal{S})$, can be used as a measure of suboptimality introduced due to the finiteness of the degree of the multipliers and the fact that the bilinear search and the coordinate-wise linear search are only local optimization schemes. Finally, *H&R*, *SimLFG* and *CWOpt* algorithms become more efficient using parallel computing.

4 Examples

Certifying Lyapunov functions, multipliers and missing parameters for all examples in this paper are available at `http://jagger.me.berkeley.edu/~pack/certify`. In the examples, $l_i(x) = 10^{-6}x^Tx$ for $i = 1, 2, 3$.

4.1 Van der Pol dynamics

The Van der Pol dynamics $\dot{x}_1 = -x_2$, $\dot{x}_2 = x_1 + (x_1^2 - 1)x_2$ have a stable equilibrium point at the origin and an unstable limit cycle. The limit cycle is the boundary of the ROA. We applied *SimLFG* algorithm with $p(x) = x^Tx$ and the parameters $N_{conv} = 200$, $\beta_{SIM} = 3.0$ (initial value), $\beta_{shrink} = 0.9$, and $N_V = 50$ for $\partial(V) = 2, 4$, and 6. We found $N_{div} = 21$ diverging trajectories during the simulation runs and feasible solutions for (6), (11), and (12) in step (iv) with $\beta_{SIM} = 1.44$, 1.97, and 2.19 for $\partial(V) = 2, 4$, and 6, respectively. We assessed (computed corresponding values of β_L^* for) the Lyapunov function candidates generated in step (v) solving *Problems 1* and *2* and further optimized initializing PENBMI with the solutions of these problems. Fig. 1 shows β_L^* and corresponding β_B^* values for $\partial(V) = 4$ and 6. Practically, every seeded PENBMI run terminated with the same β_B^* value which is the largest known (at least by us) value of β for which (9) is feasible with the prescribed families of Lyapunov functions and multipliers. In addition, we performed 10 unseeded PENBMI runs for $\partial(V) = 4$ and 6. Of these runs 90% and 50%, respectively, terminated successfully (with an optimal value of β equal to that from the seeded PENBMI runs). Moreover, unseeded PENBMI runs took longer compu-

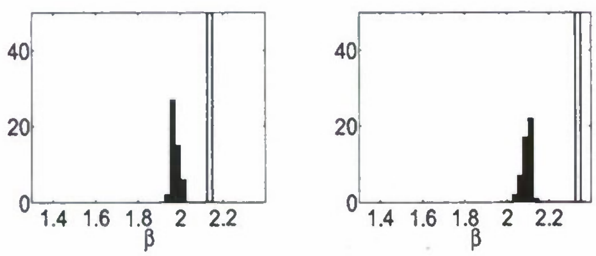

Fig. 1. Histograms of β_L^* (black bars) and β_B^* (white bars) from seeded PENBMI runs for $\partial(V) = 4$ (left), 6 (right).

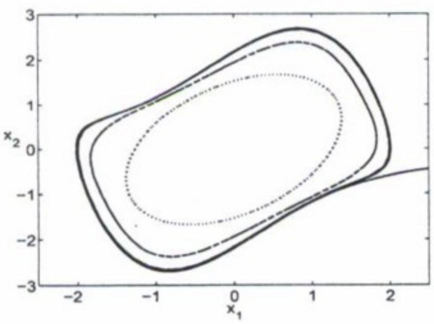

Fig. 2. The invariant subsets of the ROA (dot: $\partial(V) = 2$, dash: $\partial(V) = 4$, and solid: $\partial(V) = 6$ (indistinguishable from the outermost curve for the limit cycle)).

tation times than seeded PENBMI runs. For comparison, seeded PENBMI runs took $3-8$ and $11-24$ seconds for $\partial(V) = 4$ and 6, respectively, on a desktop PC, whereas they took $50-250$ and $1000-2500$ seconds, respectively, for unseeded PENBMI runs. Fig. 2 shows the level sets of the Lyapunov functions corresponding to the value of β_B^*.

4.2 *Examples from the literature*

We present results obtained using the method from the previous section for the systems in (15). (E_1)-(E_3) are from (Chesi *et al.*, 2005), (E_4) and (E_7) are from (Vannelli and Vidyasagar, 1985), and (E_5) and (E_6) are from (Hauser and Lai, 1992) and (Hachicho and Tibken, 2002), respectively. Since the dynamics in (E_1)-(E_7) have no physical meaning and there is no p given, we applied *SimLFG* algorithm sequentially: Apply *SimLFG* algorithm with $p(x) = x^T x$ and $N_V = 1$ for $\partial(V) = 2$. Call the quadratic Lyapunov function obtained $\hat{V}$. Set p to $\hat{V}$ and apply *SimLFG* algorithm with this p and $N_V = 1$ for $\partial(V) = 4$. For (E_5)-(E_7), we further applied *CWOpt* algorithm with $N_{iter} = 10$. Table 1 shows the ratio of the volume of the invariant subset of the ROA obtained using this procedure to that reported in the corresponding references. Empirical volumes of sublevel sets of V are computed by randomly sampling a hypercube containing the sublevel set. Values in Table 1 are volumes normalized by π and $4\pi/3$ for 2 and 3 dimensional problems, respectively. For (E_4), (E_6), and (E_7), we also empirically verified that the invariant subsets of the ROA reported in the corresponding references are contained in those computed by this sequential procedure.

Table 1
Volume ratios for (E_1)-(E_7).

example	volume ratio	example	volume ratio
(E_1)	16.7/10.2	(E_2)	0.99/0.85
(E_3)	37.2/23.5	(E_4)	1.00/0.28
(E_5)	62.3 /7.3	(E_6)	35.0/15.3
(E_7)	1.44/0.70		

$$
\begin{aligned}
(E_1) &: \{\ \dot{x}_1 = x_2,\ \dot{x}_2 = -2x_1 - 3x_2 + x_1^2 x_2. \\
(E_2) &: \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -2x_1 - x_2 + x_1 x_2^2 - x_1^5 + x_1 x_2^4 + x_2^5. \end{cases} \\
(E_3) &: \begin{cases} \dot{x}_1 = x_2,\ \dot{x}_2 = x_3, \\ \dot{x}_3 = -4x_1 - 3x_2 - 3x_3 + x_1^2 x_2 + x_1^2 x_3. \end{cases} \\
(E_4) &: \begin{cases} \dot{x}_1 = -x_2,\ \dot{x}_2 = -x_3, \\ \dot{x}_3 = -0.915x_1 + (1 - 0.915x_1^2)x_2 - x_3. \end{cases} \\
(E_5) &: \begin{cases} \dot{x}_1 = x_2 + 2x_2x_3,\ \dot{x}_2 = x_3, \\ \dot{x}_3 = -0.5x_1 - 2x_2 - x_3. \end{cases} \\
(E_6) &: \{\ \dot{x}_1 = -x_1 + x_2 x_3^2,\ \dot{x}_2 = -x_2 + x_1 x_2, \dot{x}_3 = -x_3. \\
(E_7) &: \begin{cases} \dot{x}_1 = -0.42x_1 - 1.05x_2 - 2.3x_1^2 - 0.5x_1x_2 - x_1^3, \\ \dot{x}_2 = 1.98x_1 + x_1 x_2. \end{cases}
\end{aligned}
\tag{15}
$$

4.3 *Controlled short period aircraft dynamics*

The closed-loop dynamics in (16) have an asymptotically stable equilibrium point at the origin.

$$
\dot{x} = \begin{bmatrix} \sum_{i=1}^5 a_{1i}x_i + \sum_{i=1}^5 r_{1i}x_i x_2 + r_{16}x_2^3 \\ \sum_{i=1}^5 a_{2i}x_i + \sum_{i,j=2,5} r_{ij}x_i x_j \\ A_{345}x \end{bmatrix} \tag{16}
$$

Here, $x = [x_1, x_2, x_3, x_4, x_5]^T$ is the state vector, x_1, x_2, x_5 are pitch rate, angle of attack, and pitch angle, respectively, x_3 and x_4 are the controller states, and $A_{345} \in \mathcal{R}^{3\times 5}$. Before applying our method, we performed excessive simulations and found a diverging trajectory whose initial condition x_0 satisfies $x_0^T x_0 = 16.1$; therefore, initialized β_{SIM} with 16.0. We applied algorithm *SimLFG* with $p(x) = x^T x$, $\beta_{shrink} = 0.85$, $N_{conv} = 4000$, $N_V = 1$ for $\partial(V) = 2$ and 4. We assessed the Lyapunov function candidates solving *Problems A.1* and *A.2* and further optimized using *CWOpt* algorithm with $N_{iter} = 6$. Certified values of β before and after applying iterations and from unseeded PENBMI runs are shown in Table 2. Unseeded PENBMI runs led to slightly higher values of β. However, this benefit was at the expense of high computational effort. For example, the unseeded PENBMI run took 38 hours for $\partial(V) = 4$

Table 2
Certified values of β before and after applying *CWOpt* algorithm and from unseeded PENBMI run.

	$\partial(V)=2$	$\partial(V)=4$
before iterations	6.56	8.99
after iterations	8.56	14.4
PENBMI (unseeded)	8.60	15.2

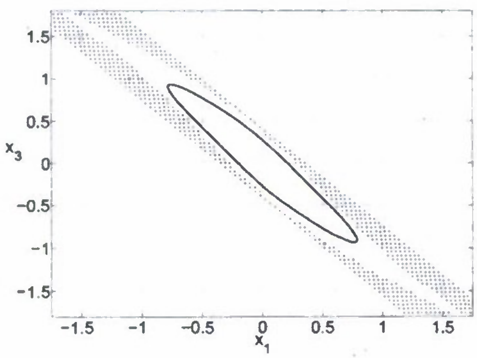

Fig. 3. A slice of the invariant subset of the ROA (solid line) and initial conditions (with $x_2 = 0$ and $x_4 = 0$) for diverging trajectories (dots).

whereas our method took 36 minutes (15 minutes for the *SimLFG* algorithm and 21 minutes for the *CWOpt* algorithm). Finally, the dependence that the starting point of *CWOpt* algorithm has on its performance is significant. For example, simply initializing *CWOpt* algorithm with $V(x) = x^T Px + 0.001\sum_{i=1}^{5} x_i^4$, where $P^T = P \succ 0$ satisfies $\mathcal{L}(P) = -I$ yields poor results. After 30 iterations, the *CWOpt* iteration converges, but the resultant Lyapunov function only certifies $\mathcal{P}_{8.5} \subset R_0$.

4.4 Pendubot dynamics

The pendubot is an underactuated two-link pendulum with torque action only on the first link. We designed an LQR controller to balance the two-link pendulum about its upright position. Third order polynomial approximation of the closed-loop dynamics is $\dot{x}_1 = x_2$, $\dot{x}_2 = 782x_1 + 135x_2 + 689x_3 + 90x_4$, $\dot{x}_3 = x_4$ and $\dot{x}_4 = 279x_1x_3^2 - 1425x_1 - 257x_2 + 273x_3^3 - 1249x_3 - 171x_4$. Here, x_1 and x_3 are angular positions of the first link and the second link (relative to the first link). We applied *SimLFG* algorithm sequentially exactly as described in section 4.2 and *CWOpt* algorithm with 10 iterations and obtained $\beta_L^* = 1.69$. Conversely, we found a diverging trajectory with the initial condition $\bar{x}$ with $p(\bar{x}) = 1.95$ proving that $1.69 \leq \beta^*(\mathcal{C}^1) < 1.95$. Fig. 3 shows the $x_2 = 0$ and $x_4 = 0$ slice of the invariant subset of the ROA along with initial conditions (with $x_2 = 0$ and $x_4 = 0$) for some diverging trajectories.

4.5 Closed-loop dynamics with nonlinear observer based controller

For the dynamics $\dot{x}_1 = u$, $\dot{x}_2 = -x_1 + x_1^3/6 - u$ and $y = x_2$, where x_1 and x_2 are the states, u is the control input and y is the output, an observer L with polynomial vector field $\dot{z} = L(y,z)$ with $\partial(L) = 3$ and a control law in the form $u = -145.9z_1 + 12.3z_2$, where z_1 and z_2 are the observer states, were computed in (Tan, 2006). The application of *SimLFG* algorithm with $\partial(V) = 2$ and p from (Tan, 2006) and *CWOpt* algorithm with $N_{iter} = 4$ lead to $\beta_L^* = 0.32$. We also applied *CWOpt* algorithm (initialized with the quadratic V found in the first application) with $\partial(V) = 4$ and $N_{iter} = 6$ and obtained $\beta_L^* = 0.52$. Conversely, we found a diverging trajectory with the initial condition $(\bar{x}, \bar{z})$ satisfying $p(\bar{x},\bar{z}) = 0.54$ proving that $0.52 \leq \beta^*(\mathcal{C}^1) < 0.54$.

5 Critique and Conclusions

5.1 Sampling vs. Simulating

A common question we get is "why simulate to get the sample points? – just sample some region, and impose $\nabla V(x)f(x) < 0$ there." There are a few answers to this. Intuitively, even running a few simulations gives insight into the system behavior. Engineers commonly use simulation to assess rough measures of stability robustness and ROA. Moreover, as converse Lyapunov theorems (Vidyasagar, 1993) implicitly define a certifying Lyapunov function in terms of the flow, it makes sense to sample the flow when looking for a Lyapunov function of a specific form. Furthermore, we have the following observation demonstrating that merely sampling some region and imposing $\nabla V(x)f(x) < 0$ there carries misleading information. Consider the Van der Pol dynamics with $p(x) = x^Tx$ and let $\mathbf{S}_\beta$ denote a finite sample of $\mathcal{P}_\beta$. It can be shown that the set of quadratic positive definite functions V that satisfy

$$\mathbf{S}_{1.8}\backslash\{0\} \subset \{x \in \mathcal{R}^n \ : \ \nabla V(x)f(x) < 0\} \tag{17}$$

is nonempty. In fact, for $V(x) = 0.32x_1^2 - 0.25x_1x_2 + 0.31x_2^2$, (17) is satisfied (actually for all $x \in \mathcal{P}_{1.8}$, $\nabla V(x)f(x) \leq -l_3(x)$). This naively suggests to draw samples from the set of quadratic positive definite functions satisfying (17) in order to try to prove that $\mathcal{P}_{1.8} \subset R_0$. However, simulations reveal a contradicting fact: Using trajectories with initial conditions in $\mathbf{S}_{1.8}$ for $\partial(V) = 2$, i.e., with $\varphi(x) = [x_1^2,\ x_1x_2,\ x_2^2]^T$, constraints (6), (11) (with $\gamma = 1$), and (12) turn out to be infeasible. This verifies that *no quadratic Lyapunov function can prove* $\mathcal{P}_{1.8} \subset R_0$ using conditions (6)-(8), with the additional constraint that $\dot{V}(x) \leq -10^{-6}x^Tx$ on all trajectories starting in $\mathcal{P}_{1.8}$. Recall though, that using quartic Lyapunov functions we know $\beta^*(\mathcal{V}_\varphi, \mathcal{S}) \geq 2.14$. By these observations, we have the following series of inclusions for the subsets of the positive definite quadratic polynomials

$$\begin{aligned}
&\{V \ : \ V \text{ certifies } \mathcal{P}_\beta \subset R_0 \text{ using } (6)-(8)\} \\
&\quad \subset \{V \ : \ \nabla V(\mathbf{c_s}(\tau))f(\mathbf{c_s}(\tau)) < 0 \ \forall\tau,\ \forall \mathrm{s} \in \mathbf{S}_\beta\} \\
&\qquad \subset \{V \ : \ \nabla V(\mathbf{s})f(\mathbf{s}) < 0 \ \forall\ \mathbf{s} \in \mathbf{S}_\beta\},
\end{aligned}$$

where $\mathbf{c_s}$ denotes the trajectory with the initial condition $s \in \mathbf{S}_\beta$. Therefore, merely sampling instead of using simulations leads to a larger outer set from which the samples for V are taken in step (v) of *SimLFG* algorithm and it is less likely to find a function that certifies that $\mathcal{P}_\beta \subset R_0$.

5.2 Conclusions

We proposed a method for computing invariant subsets of the region-of-attraction for asymptotically stable equilibrium points of dynamical systems with polynomial vector fields. We used polynomial Lyapunov functions as local stability certificates whose certain sublevel sets are invariant subsets of the region-of-attraction. Similar to many local analysis problems, this is a nonconvex problem. Furthermore, its sum-of-squares relaxation leads to a bilinear optimization problem. We developed a method utilizing information from simulations for easily generating Lyapunov function candidates. For a given Lyapunov function candidate, checking its feasibility and assessing the size of the associated invariant subset are affine sum-of-squares optimization problems. Solutions to these problems provide invariant subsets of the region-of-attraction directly and/or they can further be used as seeds for local bilinear search schemes or iterative coordinate-wise linear search schemes for improved performance of these schemes. We reported promising results in all these directions.

Acknowledgements

This work was sponsored by the Air Force Office of Scientific Research, USAF, under grant/contract number FA9550-05-1-0266. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the AFOSR or the U.S. Government.

References

Boyd, S. and L. Vandenberghe (2004). *Convex Optimization*. Cambridge Univ. Press.

Chesi, G., A. Garulli, A. Tesi and A. Vicino (2005). LMI-based computation of optimal quadratic Lyapunov functions for odd polynomial systems. *Int. J. Robust Nonlinear Control* **15**, 35–49.

Chiang, H.-D. and J. S. Thorp (1989). Stability regions of nonlinear dynamical systems: A constructive methodology. *IEEE Transaction on Automatic Control* **34**(12), 1229–1241.

Davison, E. J. and E. M. Kurak (1971). A computational method for determining quadratic Lyap. functions for nonlinear systems. *Automatica* **7**, 627–636.

Genesio, R., M. Tartaglia and A. Vicino (1985). On the estimation of asymptotic stability regions: State of the art and new proposals. *IEEE Transaction on Automatic Control* **30**(8), 747–755.

Hachicho, O. and B. Tibken (2002). Estimating domains of attraction of a class of nonlinear dynamical systems with LMI methods based on the theory of moments. In: *Proc. CDC*. Las Vegas. pp. 3150–5.

Hauser, J. and M.C. Lai (1992). Estimating quadratic stability domains by nonsmooth optimization. In: *Proc. ACC*. Chicago, IL. pp. 571–576.

Kočvara, M. and M. Stingl (2005). *PENBMI User's Guide*.

Papachristodoulou, A. (2005). Scalable analysis of nonlinear systems using convex optimization. Ph.D. dissertation. Caltech.

Parrilo, P. (2003). Semidefinite programming relaxations for semialgebraic problems. *Mathematical Programming Series B* **96**(2), 293–320.

Prokhorov, D.V. and L.A. Feldkamp (1999). Application of SVM to Lyapunov function approximation. In: *Proc. Int. Joint Conf. on Neural Networks*. Washington, DC.

Serpen, G. (2005). Search for a Lyapunov function through empirical approximation by artificial neural nets: Theoretical framework. In: *Proc. Int. Joint Conf. on Artificial Neural Networks*. Montreal, Canada. pp. 735–740.

Sturm, J. (1999). Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software* **11**, 625–653.

Tan, W. (2006). Nonlinear control analysis and synthesis using sum-of-squares programming. Ph.D. dissertation. UC, Berkeley.

Tan, W. and A. Packard (2006). Stability region analysis using sum of squares programming. In: *Proc. ACC*. Minneapolis, MN. pp. 2297–2302.

Tempo, R., G. Calafiore and F. Dabbene (2005). *Randomized Algorithms for Analysis and Control of Uncertain Systems*. Springer.

Tibken, B. (2000). Estimation of the domain of attraction for polynomial systems via LMIs. In: *Proc. CDC*. Sydney, Australia. pp. 3860–3864.

Tibken, B. and Y. Fan (2006). Computing the domain of attraction for polynomial systems via BMI optimization methods. In: *Proc. ACC*. Minneapolis, MN. pp. 117–122.

Toker, O. and H. Ozbay (1995). On the NP-hardness of solving bilinear matrix inequalities and simultaneous stabilization with static output feedback. In: *Proc. ACC*. pp. 2525–2526.

Vannelli, A. and M. Vidyasagar (1985). Maximal Lyapunov functions and domains of attraction for autonomous nonlinear systems. *Automatica* **21**(1), 69–80.

Vidyasagar, M. (1993). *Nonlinear Systems Analysis*. 2^{nd} ed.. Prentice Hall.

A Appendix

Lemma 2 *Given $g_0, g_1, \cdots, g_m \in \mathbb{R}[x]$, if there exist $s_1, \cdots, s_m \in \Sigma[x]$ such that $g_0 - \sum_{i=1}^m s_i g_i \in \Sigma[x]$, then* $\{x \in \mathcal{R}^n : g_1(x), \ldots, g_m(x) \geq 0\} \subseteq \{x \in \mathcal{R}^n : g_0(x) \geq 0\}$.

Lemma 3 *Given $g_0, g_1, g_2 \in \mathbb{R}[x]$ such that g_0 is positive definite and $g_0(0) = 0$, if there exist $s_1, s_2 \in \Sigma[x]$ such that $g_1 s_1 + g_2 s_2 - g_0 \in \Sigma[x]$, then $\{x \in \mathcal{R}^n : g_1(x) \leq 0\}\backslash\{0\} \subset \{x \in \mathcal{R}^n : g_2(x) > 0\}$.* ◁

Problems 1 and *2* in section 3 compute lower bounds on the largest value of γ and β such that, for given V and p, $\Omega_{V,\gamma}\backslash\{0\} \subset \{x \in \mathcal{R}^n : \nabla V(x) f(x) < 0\}$ and $\mathcal{P}_\beta \subset \Omega_{V,\gamma}$. We propose alternative formulations, that do not require line search, to compute similar lower bounds. Labeled γ_a^* and β_a^*, these are generally different than γ_L^* and β_L^*. For $h, g \in \mathbb{R}[x]$ and a positive integer d, define

$\mu^o(h,g) := \inf_{x\neq 0} h(x)$ such that $g(x)=0$, and

$$\mu^*(h,g,d) := \sup_{\mu>0, r\in\mathbb{R}[x]} \mu \text{ subject to}$$
$$(h-\mu)\left(x_1^{2d}+\cdots+x_n^{2d}\right) - gr \in \Sigma[x].$$

Note that $\mu^*(h,g,d) \leq \mu^o(h,g)$.

Lemma 4 *Let g, $h : \mathcal{R}^n \to \mathcal{R}$ be continuous, h be positive definite, $g(0)=0$, and $g(x)<0$ for all nonzero $x \in \mathcal{O}$, a neighborhood of the origin. Define $\gamma^o := \mu^o(h,g)$. Then, the connected component of $\{x \in \mathcal{R}^n : h(x) < \gamma^o\}$ containing the origin is a subset of $\{x \in \mathcal{R}^n : g(x) < 0\} \cup \{0\}$.* ◁

Proof: Suppose not and let $\overline{x} \neq 0$ be in the connected component of $\{x \in \mathcal{R}^n : h(x) < \gamma^o\}$ containing the origin but $g(\overline{x}) \geq 0$. Then, there exists a continuous function $\vartheta : [0,1] \to \mathcal{R}^n$ such that $\vartheta(0)=0$, $\vartheta(1)=\overline{x}$, and $h(\vartheta(t)) < \gamma^o$ for all $t \in [0,1]$. Since $g(0)=0$ and $g(x)<0$ for all nonzero $x \in \mathcal{O}$, there exists $0<\epsilon<1$ such that $g(\vartheta(\epsilon))<0$. Since $\overline{x}$ is not in $\{x \in \mathcal{R}^n : g(x)<0\}$, $g(\vartheta(1)) \geq 0$. Since g and ϑ are continuous, there exists $t^* \in (0,1]$ such that $g(\vartheta(t^*))=0$, which implies $h(\vartheta(t^*)) \geq \gamma^o$. This contradiction leads to $\overline{x} \in \{x \in \mathcal{R}^n : g(x)<0\}$. □

Corollary 5 *Let $V \in \mathbb{R}[x]$ be a positive definite C^1 function and satisfy (12) and $V(0)=0$. Then, for all γ such that $0<\gamma<\mu^o(V,\nabla V f)$, the connected component of $\Omega_{V,\gamma}$ containing the origin is an invariant subset of the ROA.* ◁

Proof: Since the quadratic part of V is a Lyapunov function for the linearized system, there exists a neighborhood $\mathcal{O}$ of the origin such that $\nabla V(x) f(x) < 0$ for all nonzero $x \in \mathcal{O}$. By Lemma (4), the connected component of $\Omega_{V,\gamma}$ containing the origin, a subset of the connected component of $\{x \in \mathcal{R}^n : V(x) < \mu^o(V,\nabla V f)\}$ containing the origin, is contained in $\{x \in \mathcal{R}^n : \nabla V(x) f(x) < 0\} \cup \{0\}$. Corollary (5) follows from regular Lyapunov arguments (Vidyasagar, 1993). □

Corollary 6 *For some positive integer d_1, define $\gamma_a^* := \mu^*(V,\nabla V f, d_1)$. Then, if $\gamma < \gamma_a^*$ for some positive integer d_1, then the connected component of $\Omega_{V,\gamma}$ containing the origin is an invariant subset of the ROA.* ◁

Corollary 7 *Let $0<\gamma<\gamma_a^*$, d_2 be a positive integer, $V, p \in \mathbb{R}[x]$ be positive definite and p be convex. Define $\beta_a^* := \mu^*(p, V-\gamma, d_2)$. Then for any $\beta < \beta_a^*$, $\mathcal{P}_\beta \subset \Omega_{V,\gamma}$ and $\mathcal{P}_\beta \subset R_0$.* ◁

Stability Region Analysis using polynomial and composite polynomial Lyapunov functions and Sum of Squares Programming

Weehong Tan and Andrew Packard

Abstract

We propose using (bilinear) sum-of-squares programming for obtaining inner bounds of regions-of-attraction for dynamical systems with polynomial vector fields. We search for polynomial as well as composite Lyapunov functions, comprised of pointwise maximums of polynomial functions. Results for several examples from the literature are presented using the proposed methods and the PENBMI solver.

I. Introduction

Finding the stability region or region-of-attraction (ROA) of a nonlinear system is a topic of significant importance and has been studied extensively, for example in [1-12]. It also has practical applications, such as determining the operating envelope of aircraft and power systems.

Most computational methods aim to compute an inner bound on the region-of-attraction, namely a set that contains the equilibium point, and is contained in the region-of-attraction. The methods above can roughly be split into Lyapunov and non-Lyapunov methods. Lyapunov methods (the focus of this paper) are based on local stability theorems and search for functions satisfying conditions which quantitatively prove local stability. Nonlinear programming is used in [1] to optimize (by choice of positive definite matrix) the volume of an ellipsoid contained in the region-of-attraction. Rational Lyapunov functions that approach ∞ on the boundary of the region-of-attraction are constructed iteratively in [2], motivated from Zubov's work. Computational considerations limit the degree of the rational function, and inner estimates to the ROA are obtained.

Department of Mechanical Engineering, University of California, Berkeley. Email: {weehong, pack}@jagger.me.berkeley.edu

Easy to compute estimates are considered in [3], which restricts the Lyapunov function search to low-dimensional manifold of quadratic Lyapunov functions, obtaining analytical simplifications. Following [1], but employing semidefinite programming techniques, [4] aims to maximize the volume of an ellipsoid whose containment in the region-of-attraction can be ascertained with sum-of-squares (SOS) decompositions. Attention is restricted to odd, polynomial vector fields, and SOS optimization is combined with general nonlinear programming. A sequence of functions, called *nested* Lyapunov functions, are introduced in [5] to derive stability region and rate-of-convergence estimates. Both [6] and [7] solve a sequence of linear semidefinite programs, iteratively searching over Lyapunov function candidates and sum-of-squares multipliers. The "coordinatewise" ascent method is generally effective, though no convergence result holds. By contrast, the formulation here is more direct, but yields a single bilinear (nonconvex) SOS program. Closely related to Lyapunov methods are viability methods, which effectively integrate an invariant set backwards in time, obtaining increasing estimates for the region-of-attraction. Both [8] and [9] use discretization (in time) to flow invariant sets backwards along the flow of the vector field, obtaining larger and larger estimates for the region-of-attraction. In [8], the invariant sets are restricted to be sublevel sets of polynomials, and the discretized backwards flow is approximated with semidefinite programming. The approach of [9] also requires discretization in space and suffers from exponential growth in state dimension. Generally, the method is exact, but computation may require exponential growth in state dimension. Depending on the user's point of view, problems of modest (between 4 and 8) state dimension are intractable. Non-Lyapunov methods, [10] and [11] focus on topological properties of regions of attraction. A survey of results, as well as an extensive set of examples and new ideas, is presented in [12].

In this paper, we present a method of using sum-of-squares (SOS) programming to search for polynomial Lyapunov functions that enlarge an inner estimate of the region-of-attraction of nonlinear systems with polynomial vector fields. SOS programming, coupled with polynomial Lyapunov functions has roots that can be traced back at least to Bose and Li [13] and Brockett, [14] and the power transform of Barkin et.al [15], which was used in [16] to find non-quadratic Lyapunov functions for uncertain linear systems. Recent theoretical work, [17], [7] and [18], continues to further the role of this approach. An impediment to using high degree Lyapunov functions is the extremely rapid increase in the number of optimization decision variables as the state dimension and the degree of the Lyapunov function (and the vector field) increase. Here, we propose using pointwise maximums of polynomial functions to obtain rich functional forms while keeping the number of optimization decision variables relatively low. Pointwise maximum and other composite Lyapunov functions have been used in many instances, [19], [20], [21], including stability and performance analysis of constrained systems and robustness analysis of uncertain systems, where affine and polynomial parameter-dependent Lyapunov functions are also used, [22], [23]. The notation is generally standard, with $\mathcal{R}_n$ denoting the set of polynomials with real coefficients in n variables and $\Sigma_n \subset \mathcal{R}_n$ denoting the subset of SOS polynomials.

II. Estimating a Region of Attraction

Consider an autonomous dynamical system of the form

$$\dot{x}(t) = f(x(t)) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and f is an n-vector of elements of $\mathcal{R}_n$ with $f(0) = 0$. The following lemma on invariant subsets of the region-of-attraction is a modification of ideas from [24, pg. 167] and [25, pg. 122]:

Lemma 1: If there exist continuously differentiable functions $\{V_i\}_{i=1}^{q} : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(x) := \max_{1 \le i \le q} V_i(x) \text{ is positive definite,} \tag{2}$$

$$\Omega := \{x \in \mathbb{R}^n \,|\, V(x) \le 1\} \text{ is bounded,} \tag{3}$$

$$L_i := \{x \in \mathbb{R}^n \,|\, \max_{1 \le j \le q} V_j(x) \le V_i(x) \le 1\}, \qquad i = 1, \ldots, q \tag{4}$$

$$L_i \setminus \{0\} \subset \{x \in \mathbb{R}^n \,|\, \tfrac{\partial V_i}{\partial x} f(x) < 0\}, \qquad i = 1, \ldots, q, \tag{5}$$

then for all $x(0) \in \Omega$, the solution of (1) exists, satisfies $x(t) \in \Omega$, and $\lim_{t \to \infty} x(t) = 0$. As such, Ω is invariant, and a subset of the region-of-attraction for (1).

Proof: The proof is written for $q = 2$. The extension to $q = 1$ or $q > 2$ is straightforward. Since $L_1 \cup L_2 = \Omega$, condition (5) insures that if $x(0) \in \Omega$, $V(x(t)) \le V(x(0)) \le 1$ while the solution exists. Solutions starting in Ω remain in Ω while the solution exists. Since Ω is compact, the system (1) has an unique solution defined for all $t \ge 0$ whenever $x(0) \in \Omega$.

Take $\epsilon > 0$. Define $S_\epsilon := \{x \in \mathbb{R}^n \,|\, \frac{\epsilon}{2} \le V(x) \le 1\}$, so $S_\epsilon \subseteq (L_1 \cup L_2) \setminus \{0\}$. Note that for each i, $(S_\epsilon \cap L_i) \subseteq L_i \setminus \{0\} \subset \{x \in \mathbb{R}^n \,|\, \frac{\partial V_i}{\partial x} f(x) < 0\}$, so on the compact set $S_\epsilon \cap L_i$, $\exists r_{i,\epsilon}$, such that $\frac{\partial V_i}{\partial x} f(x) \le -r_{i,\epsilon} < 0$. Consequently, if $x(t) \in S_\epsilon \cap L_1$ on $[t_A, t_B]$, then $V(x(t_B)) \leq -r_{1,\epsilon}(t_B - t_A) + V(x(t_A))$. Similarly, if $x(t) \in S_\epsilon \cap L_2$ on $[t_A, t_B]$, then $V(x(t_B)) \leq -r_{2,\epsilon}(t_B - t_A) + V(x(t_A))$. Therefore, if $x(t) \in S_\epsilon \cap (L_1 \cup L_2)$ on $[t_A, t_B]$, then $V(x(t_B)) \leq -r_\epsilon(t_B - t_A) + V(x(t_A))$, where $r_\epsilon = \min(r_{1,\epsilon}, r_{2,\epsilon})$. Since $r_\epsilon > 0$, this implies that $\exists\, t^*$ such that $V(x(t)) < \epsilon$ for all $t > t^*$, i.e. $x(t) \in T_\epsilon := \{x \in \mathbb{R}^n \,|\, V(x) < \epsilon\}$ for all $t > t^*$. This shows that if $x(0) \in \Omega$, $V(x(t)) \to 0$ as $t \to \infty$.

Finally, let $\epsilon > 0$. Define $\Omega_\epsilon := \{x \in \mathbb{R}^n \,|\, \|x\| \geq \epsilon, V(x) \leq 1\}$. Ω_ϵ is compact, with $0 \notin \Omega_\epsilon$. Since V is continuous and positive definite, $\exists\, \gamma$ such that $V(x) \geq \gamma > 0$ on Ω_ϵ. We have already established that $V(x(t)) \to 0$ as $t \to \infty$, so $\exists\, \hat{t}$ such that for all $t > \hat{t}$, $V(x(t)) < \gamma$ and hence $x(t) \notin \Omega_\epsilon$, which means $\|x(t)\| < \epsilon$. So $x(t) \to 0$ as $t \to \infty$. ■

Remarks: Standard modifications to the hypothesis of Lemma 1 can yield global stability conditions as well. However, neither formulation can yield exact results for systems whose region-of-attraction is unbounded, but not all of $\mathbf{R}^n$ (since in Lemma 1, Ω must be is bounded). See section III-C for further details. The constraints in equations (2)-(5) are not convex constraints on V, as illustrated by a 1-dimensional example, [26]. Take $f(x) = -x$, $q = 1$ and $V_1^a(x) = 16x^2 - 19.95x^3 + 6.4x^4$ and $V_1^b(x) = 0.1x^2$. Then V_1^a and V_1^b satisfy (2)-(5), but $0.58V_1^a + 0.42V_1^b$ does not.

In order to enlarge Ω (by choice of V), we define a variable sized region $P_\beta := \{x \in \mathbb{R}^n \,|\, p(x) \leq \beta\}$, and maximize β while imposing the constraint $P_\beta \subseteq \Omega$. Here, $p(x)$ is a fixed, positive definite polynomial, chosen to reflect the relative importance of the states. Applying Lemma 1, the problem is posed as an optimization:

$$\max_{\beta \in \mathbb{R}, V_i \in \mathcal{R}_n} \beta \qquad \text{s.t.} \qquad V_i(0) = 0$$

$$V(x) := \max_{1 \leq i \leq q} V_i(x) \text{ is positive definite,} \tag{6}$$

$$\Omega := \{x \in \mathbb{R}^n \,|\, V(x) \leq 1\} \text{ is bounded,} \tag{7}$$

$$P_\beta \subseteq \Omega \tag{8}$$

$$\{x \in \mathbb{R}^n \,|\, \max_{1 \leq j \leq q} V_j(x) \leq V_i(x) \leq 1\} \setminus \{0\} \subset \{x \in \mathbb{R}^n \,|\, \tfrac{\partial V_i}{\partial x} f(x) < 0\} \tag{9}$$

where (9) holds for $i = 1, \ldots, q$. Let $l_1(x)$ be a fixed, positive definite polynomial. For each V_i, if we require $V_i - l_1 \in \Sigma_n$ for $i = 1, \ldots, q$, then both (6) and (7) are satisfied. Clearly, (8) holds if and only if

$$\{x \in \mathbb{R}^n \,|\, p(x) \leq \beta\} \subseteq \bigcap_{i=1}^{q} \{x \in \mathbb{R}^n \,|\, V_i(x) \leq 1\}, \tag{10}$$

Introducing another fixed, positive definite polynomial, $l_2(x)$, we can apply Lemmas 2 and 3 (see appendix) to obtain sufficient conditions which ensure constraints (9) and (10) hold. Written as an optimization, the problem is

$$\max \beta \quad \text{over } \beta \in \mathbb{R}, V_i \in \mathcal{R}_n,\ V_i(0) = 0,\ s_{1i}, s_{2i}, s_{3i}, s_{0ij} \in \Sigma_n, i = 1, \ldots q$$

such that

$$V_i - l_1 \in \Sigma_n, \tag{11}$$

$$-\Big((\beta - p)s_{1i} + (V_i - 1)\Big) \in \Sigma_n, \tag{12}$$

$$-\left[(1 - V_i)s_{2i} + \tfrac{\partial V_i}{\partial x} f s_{3i} + l_2\right] - \sum_{\substack{j=1 \\ j \neq i}}^{q} s_{0ij}(V_i - V_j) \in \Sigma_n. \tag{13}$$

All constraints are sum-of-square constraints, however (even for $q = 1$) products of decision variables are present. Therefore, the optimization cannot be translated into a linear semidefinite program, but is converted to a bilinear semidefinite program. Two of the conditions require positivity (beyond nonnegativity), and the fixed positive-definite polynomials, l_1 and l_2 are introduced as offsets to enforce this. Next we present results from several small problems. We have chosen to rely on the PENBMI solver [27], a local bilinear matrix inequality solver from PENOPT to attack our problems. This uses a penalty method. Alternate approaches to BMIs, such as linearization and homotopy, [28] and interior point methods, [29, Chap 7], may yield improved results and/or superior computational efficiency. Resolving these questions is left for further research.

III. Examples

All of the systems considered are locally exponentially stable. The notation n_V denotes the degree of V, specifically each V_i includes all monomials of degree 2 up through n_V. In all examples, p is quadratic, and the degree of s_{1i} is chosen so that the degree of the polynomial in equation (12) is equal to n_V. The integer n_A denotes degree of the polynomial in (13). Once n_V is chosen, and the vector field f is fixed, n_A limits the degrees of the multipliers s_{2i}, s_{3i} and s_{0ij} through simple degree counting. In each case, the positive definite polynomials l_1 and

l_2 are of the form $l_k(x) = \sum_{i=1}^{n} e_{k,i}x_i^2$. For the purposes of computation, the $e_{k,i}$ are treated as additional decision variables, and constrained to satisfy $e_{k,i} \geq 10^{-7}$.

A. Example 1 - Van der Pol equations

The system is $\dot{x}_1 = -x_2, \dot{x}_2 = x_1 + (x_1^2 - 1)x_2$. It has an unstable limit cycle and a stable equilibrium point at the origin. Finding its region-of-attraction has been studied extensively, for example, in [1], [12], [11]. The region-of-attraction for this system is the region enclosed by its limit cycle, which is easily visualized from the numerical solution of the ODE. However, our goal is to use the bilinear SOS formulation. For this example, p is chosen to be $x^T R x$, for two different $R_i \in \mathbb{R}^{2\times 2}$,

$$R_1 := \begin{bmatrix} 0.38 & -0.14 \\ -0.14 & 0.28 \end{bmatrix}, \quad \text{and} \quad R_2 := \begin{bmatrix} 0.28 & 0.14 \\ 0.14 & 0.38 \end{bmatrix},$$

The results using shape factor defined by R_1 using the pointwise maximum of two fixed degree polynomials are listed in Table I. Fig. 1 shows the limit cycle and the level sets of the certifying Lyapunov functions[1]. The level set of pointwise maximum of two 6th degree polynomial functions includes nearly the entire actual region-of-attraction. The dashed line is the level set of p (for $n_V = 6$), which clearly shows that our p has been preselected to "align" closely with the actual region-of-attraction. Of course, this would be impossible to do in general, and we discuss the implications of this later in this section. Our results compare favorably with [11] as well as the degree 6 solution from [7], and the final (40^{th}) iterate from degree 6 solutions of [8], all of which are shown in Figure 2. Clearly, the solution of [8] is a very high quality estimate of the true ROA. Parametrizing the boundaries using polar coordinates reveals that as a function of angle, the radius of [8] exceeds our $n_V = 6$ radius on 52.6% of $[0\ 2\pi]$; is 0.22% larger, on average, than our $n_V = 6$ radius; exceeds our $n_V = 6$ radius by as much as 1.4% in some directions; is smaller than our $n_V = 6$ radius by as much as 0.8% in other directions. We conclude that the result in [8] is very similar, though slightly superior to our result.

It is interesting to observe how the V_i functions interact in, for example, the 6th degree case. Figure 3 shows the level sets $\{x \mid V_i(x) \leq 1\}$. For V_1, there are 3 connected components, one

[1]The certifying Lyapunov functions and SOS multipliers for all examples in this paper are available at `http://jagger.me.berkeley.edu/~pack/certificates`

"large" component centered at the origin (whose boundary is essentially the limit cycle), and 2 "islands" in the 2nd and 4th quadrants. For V_2, the level set is one connected component centered at the origin, visually the same as the large component of V_1. Label the two islands as I_1 and I_2, and the intersection of the two (nearly identical) centered components as Ω.

Inside I_1 and I_2, $\dot{V}_1 \not< 0$, but $V_2 > 1 > V_1$, so I_1 and I_2 are excluded in the set Ω. Moreover, on Ω, $\dot{V}_1 < 0$ where $V_1 \geq V_2$, and $\dot{V}_2 < 0$ where $V_2 \geq V_1$, proving that Ω is a region-of-attraction. Since $\{x \,|\, V_2(x) \leq 1\} \approx \Omega$, it is tempting to assume that V_2 alone can prove the stability claim. However, many points have $\dot{V}_2 \geq 0$ (the shaded region in Ω).

In this example, using pointwise maximum of three polynomials does not offer additional benefits (row 1 and 4 of Table I). Better results are obtained (row 5) by increasing the degree of the $\{s_i\}$, but this increases the number of decision variables, so the computational benefit is effectively erased.

Finally, optimizing with the shape factor defined by R_2 yields almost identical results (in terms of Ω). Fig. 4 illustrates the analogous level sets of V, and also shows a level set for this p. Clearly, the level sets for this shape factor are not aligned with the actual region-of-attraction, nevertheless, the optimization performs quite well.

B. *6 examples from [4]*

Reference [4] aims to *maximize the volume* of an inner *ellipsoidal* estimate of the region of attraction, presenting results from 6 examples. The volume reported in [4] is normalized: in $\mathbf{R}^2$ it is 2-dimensional area divided by π, while in $\mathbf{R}^3$ it is 3-dimensional volume divided by $\frac{4\pi}{3}$. As an exercise, we solve the same problems here. The results are summarized in Table II. Maximizing volume is not directly compatible with our scalar objective involving the function p (whose level sets may or may not be ellipsoidal). We began with a simple approach: using a spherical shape factor, $p(x) := x^T x$, solve the optimization problem and then compute the volume of the level set $\{x : V(x) \leq 1\}$ (easily computed for a quadratic V, and estimated with Monte Carlo integration for high degree and pointwise-max V's). Problems S1, S2, S3 and S4 are successfully addressed using this approach. Note the improvement for S2 when the degree of the multipliers is increased (via n_A) even though n_V is held constant. Problem S5 required an alteration, referred to as *bootstrap*, to obtain large volumes. In this calculation, the initial optimization was as above, with a spherical p, using quadratic Lyapunov function candidates.

Subsequent optimizations, with richer Lyapunov function candidates used, for p, the obtained quadratic Lyapunov function (as opposed to $x^T x$). Problem S6 is more challenging and the methods we present here do not obtain volumes as large as those reported in [4]. The S6 table entry involving quartic functions is empty, as PENBMI exhibited unreliable behavior on this problem, exposing some genuine deficiences in our overall approach.

C. Unbounded Region-of-Attraction

Consider $\dot{x}_1 = x_2, \dot{x}_2 = -(1 - x_1^2)x_1 - x_2$ from [30]. The region of attraction to the stable equilibrium at $x = 0$ is unbounded, but not all of $\mathbb{R}^2$. Exact methods, such as those in [9], may obtain the correct answer in this problem. By contrast, the formulation in equations (11)-(13) cannot, since Ω is necessarily compact. Using a simple $p(x) := x_1^2 + x_2^2$ shape factor, we obtain nearly identical results for quadratic and pointwise-max quadratic Lyapunov functions, yielding β such that P_β nearly touches the stability boundary, and the bounded level set $\{x : V(x) \leq 1\}$ is ellipsoidal, roughly aligned with the true region-of-attraction. Using the bootstrap, with $n_V = 6$ yields significant improvement. The two level sets are shown in the left panel of figure 5, along with some trajectories of the system.

D. An example from reference [2]

Another 2-state example with polynomial vector field comes from example 4 in [2]. The dynamics are $\dot{x}_1 = -0.42x_1 - 1.05x_2 - 2.3x_1^2 - 0.5x_1x_2 - x_1^3; \dot{x}_2 = 1.98x_1 + x_1x_2$. The inner estimate from [2] along with our estimate using quadratic, quartic, pointwise-max quartic, and degree 6 functions are shown in the right panel of Figure 5. Pointwise-max ($q = 2$) degree 6 solutions yielded no appreciable improvement over the $q = 1$ case, and are not shown.

IV. Benchmark Study

There are several drawbacks to our approach, most notably searching over the non-convex decision variable space. Given this deficiency, it is useful to investigate how equations (11)-(13), coupled with the PENBMI solver perform on an "easy" nonlinear problem, with respect to "arbitrary" data and increasing problem size. Let $\dot{x} = -Ix + (x^T Bx)x$ where $x(t) \in \mathbb{R}^n$, and $B \in \mathbb{R}^{n \times n}$, $B \succ 0$. For this system, inspired by Example 5 of [1], the set $\{x \in \mathbb{R}^n \mid x^T Bx < 1\}$ is the exact region-of-attraction for the $x = 0$ equilibrium point (use $V(x) := x^T Bx$ to prove this). Let $P_\beta := \{x \in \mathbb{R}^n \,|\, x^T R x \leq \beta\}$, $R \in \mathbb{R}^{n \times n}$, $R \succ 0$. The supremum value for β such that $P_\beta \subseteq \{x \in \mathbb{R}^n \,|\, x^T B x < 1\}$ is $\beta = [\lambda_{\max}(R^{-\frac{1}{2}} B R^{-\frac{1}{2}})]^{-1}$. Equations (11)-(13) *can* yield this answer, specifically, take $q = 1$ and for any $\gamma > 1$, choose τ such that $1 < \tau < \gamma$. Then for large enough α (depending on fixed choice of quadratic l_2) the choices $V(x) := \gamma x^T B x$, $s_2 := 2\alpha\tau x^T B x$ and $s_3 := \alpha$ satisfy (13), prove that $\{x \,|\, x^T B x < 1\}$ is in the region-of-attraction. Hence, this class of problems provides a test for any specified BMI solver to actually discover the known-to-exist solution. For each n, 100 trials are performed. Each trial consists of a random choice of positive definite B and R, each with eigenvalues $\exp(2r_i)$ where each r_i is picked from a normal distribution with zero mean and unit variance, and random, orthonormal eigenvectors. For each trial, we run the PENBMI optimizer 3 times (initial point randomly chosen each run). Table III shows the results of the test.

A run is classified *successful* if the solver returns the message "No problems detected", and classified *failure* otherwise. Except for the case of $n = 6$, there are no trials that fail for all 3 runs (for $n = 6$, one trial did fail in all 3 runs, and note that this single instance, 3-trial failure is not taken into account in the table entries described below). Among the successful runs, the quality of the answer is assessed by the nearness of $\beta \times \lambda_{\max}$ to 1. The worst case (smallest) value among the (296-300) successful runs is given. The next column shows the worst case $\beta \times \lambda_{\max}$ over 100 trials, exploiting the 3 repeated attempts and the randomized initial starting point chosen by PENBMI. The entries are ≈ 1, which indicates that repeated runs of the same problem eventually lead to the optimal solution for this example. For this limited benchmark example, although our problem formulation is bilinear in the decision polynomials and the bilinear solver, PENBMI, is a local solver, the results obtained are encouraging.

V. Conclusions

In this paper, we presented techniques using sum-of-squares programming for finding provable regions-of-attraction for nonlinear systems with polynomial vector fields. Several small examples are presented. For systems with cubic vector fields, analyzing local stability using Lyapunov functions which are the pointwise-max of quadratic and quartic functions appears to be a useful, and modestly tractable extension to simply using polynomial Lyapunov functions.

VI. ACKNOWLEDGEMENTS

Weehong Tan's scholarship is sponsored by DSO National Laboratories, Singapore. This work was sponsored (in part) by the Air Force Office of Scientific Research, USAF, under grant/contract number FA9550-05-1-0266. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the AFOSR or the U.S. Government.

The authors would like to thank Johan Löfberg for incorporating bilinear SOS parameterization into YALMIP [31] and PENOPT for the use of an academic license for PENBMI [27]. The authors would also like to thank Ufuk Topcu for help in preparing the examples, and Pete Seiler, Ufuk Topcu and Gary Balas for valuable discussions, and the reviewers for several helpful suggestions and comments.

VII. APPENDIX

A *monomial* m_α in n variables is a function defined as $m_\alpha(x) = x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $\alpha \in \mathbb{Z}_+^n$. The degree of a monomial is defined, $\deg m_\alpha := \sum_{i=1}^n \alpha_i$. A *polynomial* f in n variables is a finite linear combination of monomials, with $c_\alpha \in \mathbb{R}$:

$$f := \sum_\alpha c_\alpha m_\alpha = \sum_\alpha c_\alpha x^\alpha.$$

Define $\mathcal{R}_n$ to be the set of all polynomials in n variables. The degree of f is defined as $\deg f := \max_\alpha \deg m_\alpha$ (provided the associated c_α is non-zero). Additionally define Σ_n to be the set of sum-of-squares (SOS) polynomials in n variables.

$$\Sigma_n := \left\{ p \in \mathcal{R}_n \,\middle|\, p = \sum_{i=1}^t f_i^2 \quad , \, f_i \in \mathcal{R}_n, \, i = 1, \ldots, t \right\}.$$

Obviously if $p \in \Sigma_n$, then $p(x) \geq 0 \; \forall x \in \mathbb{R}^n$. A polynomial, $p \in \Sigma_n$ iff $\exists \, 0 \preceq Q \in \mathbb{R}^{r \times r}$ such that $p(x) = z^T(x) Q z(x)$, with $z(x)$ a vector of suitable monomials. The set of Q that satisfies $p(x) = z^T(x) Q z(x)$ is an affine subspace, so that semidefinite programming plays the key role in deciding if a given polynomial is SOS. The lemmas below are elementary extensions of the $\mathcal{S}$-procedure, [32], and very limited special cases of the Positivstellensatz, [33, Theorem 4.2.2]. In both cases, the SOS polynomials $\{s_k\}_{i=1}^m$ are often called the "SOS multipliers."

Lemma 2: Given $p_1, p_2 \in \mathcal{R}_n$, and positive definite $h \in \mathcal{R}_n$, with $h(0) = 0$. If $s_1, s_2 \in \Sigma_n$ satisfy $p_1 s_1 + p_2 s_2 - h \in \Sigma$ then $\{x : p_1(x) \leq 0\} \setminus \{0\} \subset \{x : p_2(x) > 0\}$.

Lemma 3: Given $\{p_i\}_{i=0}^m \in \mathcal{R}_n$. If there exist $\{s_k\}_{i=1}^m \in \Sigma_n$ such that $p_0 - \sum_{i=1}^m s_i p_i \in \Sigma_n$, then $\bigcap_{i=1}^m \{x \in \mathbb{R}^n \,|\, p_i(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n \,|\, p_0(x) \geq 0\}$.

SOSTOOLS, [34], [35], GloptiPoly, [36], and YALMIP, [31] automate the translation from SOS programs to semidefinite programs, converting to solver-specific, e.g., SeDuMi [37] or SDPT3 [38], syntax. YALMIP also handles bilinear decision polynomials, using PENBMI [27].

Despite these software tools, and even ignoring the nonconvexity of our formulation, there are significant dimensionality problems as well: [39, Table 6.1] illustrates the unpleasant growth in the number of decision variables with n and the polynomial degree.

REFERENCES

[1] E. J. Davison and E. M. Kurak, "A computational method for determining quadratic Lyapunov functions for non-linear systems," *Automatica*, vol. 7, pp. 627–636, 1971.

[2] A. Vannelli and M. Vidyasagar, "Maximal Lyapunov functions and domains of attraction for autonomous nonlinear systems," *Automatica*, vol. 21, no. 1, pp. 69–80, 1985.

[3] A. Levin, "An analytical method of estimating the domain of attraction for polynomial differential equations," *IEEE Transactions on Automatic Control*, vol. 39, no. 12, pp. 2471–2475, 1994.

[4] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, "LMI-based computation of optimal quadratic Lyapunov functions for odd polynomial systems," *Int. J. Robust and Nonlinear Control*, vol. 15, pp. 35–49, 2005.

[5] N. Peterfreund and Y. Baram, "Convergence analysis of nonlinear dynamical systems by nested Lyapunov functions," *IEEE Transactions on Automatic Control*, vol. 43, no. 8, pp. 1179–1184, 1998.

[6] Z. Jarvis-Wloszek, "Lyapunov based analysis and controller synthesis for polynomial systems using sum-of-squares optimization," Ph.D. Dissertation, University of California, Berkeley, Dec 2003.

[7] A. Papachristodoulou, "Scalable analysis of nonlinear systems using convex optimization," Ph.D. Thesis, California Institute of Technology, Jan 2005.

[8] T. Wang, S. Lall, and M. West, "Polynomial level-set methods for nonlinear dynamical systems analysis," in *Proc. of Allerton Conference on Communication. Control and Computing*, 2005.

[9] E. Cruck, R. Moitie, and N. Suebe, "Estimation of basins of attraction for uncertain systems with affine and Lipschitz dynamics," *Dynamics and Control*, vol. 11, pp. 211–227, 2001.

[10] H.-D. Chiang, M. Hirsch, and F. Wu, "Stability regions of nonlinear autonomous dynamical systems," *IEEE Transactions on Automatic Control*, vol. 33, no. 1, pp. 16–27, 1988.

[11] H.-D. Chiang and J. S. Thorp, "Stability regions of nonlinear dynamical systems: A constructive methodology," *IEEE Transactions on Automatic Control*, vol. 34, no. 12, pp. 1229–1241, 1989.

[12] R. Genesio, M. Tartaglia, and A. Vicino, "On the estimation of asymptotic stability regions: State of the art and new proposals," *IEEE Transactions on Automatic Control*, vol. 30, no. 8, pp. 747–755, 1985.

[13] N. Bose and C. Li, "A quadratic form representation of polynomials of several variables and its applications," *IEEE Transactions on Automatic Control*, vol. 13, no. 8, pp. 447–448, 1968.

[14] R. Brockett, *Lie Algebras and Lie Groups in Control Theory.* Dordrecht: Reidel, in Geometric Methods in System Theory, 1973, pp. 213–225.

[15] A. Barkin and A. Zelentsovsky, "Method of power transformations for analysis and stability of nonlinear control systems," *Systems & Control Letters*, vol. 3, pp. 303–310, 1983.

[16] A. Zelentsovsky, "Nonquadratic Lyapunov functions for robust stability analysis of linear uncertain systems," *IEEE Transactions on Automatic Control*, vol. 39, no. 1, pp. 135–138, 1994.

[17] P. Parrilo, "Structured semidefinite programs and semialgebraic methods in robustness and optimization," Ph.D. Dissertation, California Institute of Technology, 2000.

[18] S. Prajna, "Optimization-based methods for nonlinear and hybrid systems verification," Ph.D. Dissertation, California Institute of Technology, 2005.

[19] T. Hu and Z. Lin, "Composite quadratic Lyapunov functions for constrained control systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 3, pp. 440–450, 2003.

[20] L. Xie, S. Shishkin, and M. Fu, "Piecewise Lyapunov functions for robust stability of linear time-varying systems," *Systems & Control Letters*, vol. 31, pp. 165–171, 1997.

[21] A. Hassibi and S. Boyd, "Quadratic stabilization and control of piecewise linear systems," in *Proceedings of the American Control Conference*, 1998, pp. 3659–3664.

[22] D. Henrion, D. Arzelier, D. Peaucelle, and J. Lasserre, "On parameter-dependent Lyapunov functions for robust stability of linear systems," in *Proceedings of the Conference on Decision and Control*, 2004, pp. 887–892.

[23] R. Oliveira and P. Peres, "Stability of polytopes of matrices via affine parameter-dependent Lyapunov functions: Asymptotically exact LMI conditions," *Linear Algebra and its Applications*, vol. 405, pp. 209–228, August 2005.

[24] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed. Prentice Hall, 1993.

[25] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.

[26] P. Seiler, June 2005, personal communication.

[27] M. Kočvara and M. Stingl, *PENBMI User's Guide (Version 2.0)*, Available from `http://www.penopt.com`, 2005.

[28] A. Hassibi, J. How, and S. Boyd, "A path-following method for solving BMI problems in control," in *Proceedings of the American Control Conference*, 1999, pp. 1385–1389.

[29] C. Hol, "Structured controller synthesis for mechanical servo systems: Algorithms, relaxations and optimality certificates," Ph.D. Dissertation, Technical University of Delft, Aug 2006.

[30] T. Vincent and W. Grantham, *Nonlinear and Optimal Control Systems*. New York: Wiley-Interscience, 1997.

[31] J. Löfberg, "YALMIP : A toolbox for modeling and optimization in MATLAB," in *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004, available from `http://control.ee.ethz.ch/~joloef/yalmip.php`.

[32] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia: SIAM, 1994.

[33] J. Bochnak, M. Coste, and M.-F. Roy, *Géométrie algébrique réelle*. Berlin: Springer, 1986.

[34] S. Prajna, A. Papachristodoulou, and P. Parrilo, "Introducing SOSTOOLS: A general purpose sum of squares programming solver," in *Proc of the Conf on Decision and Control*, 2002, pp. 741–746.

[35] S. Prajna, A. Papachristodoulou, P. Seiler, and P. A. Parrilo, *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*, Available from `http://www.cds.caltech.edu/sostools` and `http://www.mit.edu/~parrilo/sostools`, 2004.

[36] D. Henrion and J. Lasserre, "Solving nonconvex optimization problems," *IEEE Control Systems Magazine*, vol. 24, pp. 72–83, June 2004.

[37] J. Sturm, "Using SeDuMi 1.02, a Matlab toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11–12, pp. 625–653, 1999, available at `http://sedumi.mcmaster.ca/`.

[38] K. Toh, M. Todd, and R. Tütüncü, "SDPT3–a Matlab software package for semidefinite programming," *Optimization Methods and Software*, vol. 11, pp. 545–581, 1999.

[39] P. Parrilo, "Semidefinite programming relaxations for semialgebraic problems," *Mathematical Programming Series B*, vol. 96, no. 2, pp. 293–320, 2003.

TABLES

TABLE I

POINTWISE MAX FOR VAN DER POL

	degree of						total no. of
q	V	s_{1i}	s_{2i}	s_{3i}	s_{0ij}	β	decision variables
2	2	0	2	0	2	0.75	38
2	4	2	2	0	2	0.93	120
2	6	4	2	0	2	1.03	338
3	2	0	2	0	2	0.75	73
3	2	0	4	2	4	0.82	265

TABLE II

CERTIFIED NORMALIZED VOLUME ON EXAMPLES S1-S6 FROM [4]. THE VECTOR FIELDS FOR EXAMPLES S3 AND S4 HAVE DEGREE EQUAL TO 5, WHILE ALL OTHERS HAVE DEGREE EQUAL TO 3.

from [4]	(n_V, n_A)	q	Vol	from [4]	(n_V, n_A)	q	Vol	from [4]	(n_V, n_A)	q	Vol
S1(10.2)	(2,4)	1	7.5	S2(27.1)	(2,4)	1	24.9	S3(9.51)	(2,6)	1	1.68
	(2,4)	2	13.7		(2,4)	2	24.9		(2,6)	2	global
					(2,6)	2	43.0				
S4(0.85)	(2,6)	1	0.83	S5(23.5)	(2,4)	1	21.3	S6(10.9)	(2,4)	1	8.5
	(2,6)	2	0.92		(2,4)	2	21.3		(2,4)	2	9.4
	(4,8)	1	1.12		(4,6)	1	32.9		(4,6)	1	-
	(4,8)	2	1.16						(4,6)	2	-

TABLE III

COMPUTATION STATISTICS FOR THE BENCHMARK EXAMPLE

n	variables	successes	worst (in 300)	over 100	time (sec)
2	13	298	0.99995	1.00000	0.70
3	25	296	0.90955	0.99984	1.12
4	48	297	0.07687	0.99999	2.14
6	157	297	0.99997	0.99998	11.2
8	420	300	0.99989	0.99992	99.7

Figure Captions

Fig. 1. Provable ROA using pointwise maximum of two polynomial functions, with shape factor $x^T R_1 x$

Fig. 2. Provable ROA, from [11], [7] and [8].

Fig. 3. VDP: Level sets of two 6th degree polynomials at $V_1, V_2 = 1$

Fig. 4. Provable ROA using pointwise max of two polynomial functions, with shape factor $x^T R_2 x$

Fig. 5. Trajectories and level sets $V \leq 1$. LEFT panel: example from Section III-C; RIGHT panel: example from Section III-D

FIGURES

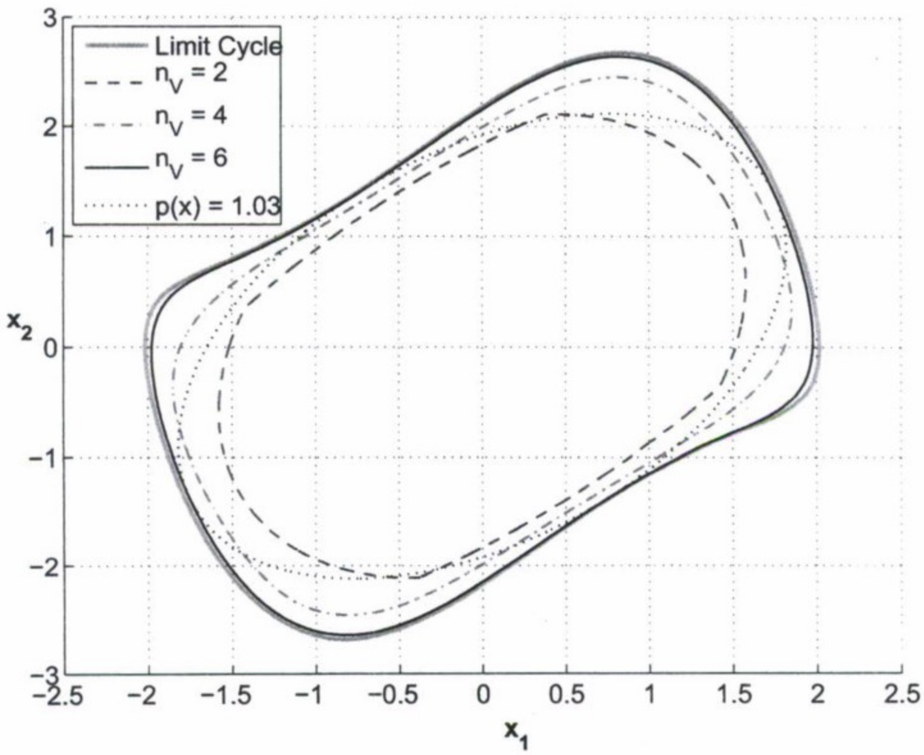

Fig. 1. Provable ROA using pointwise maximum of two polynomial functions, with shape factor $x^T R_1 x$

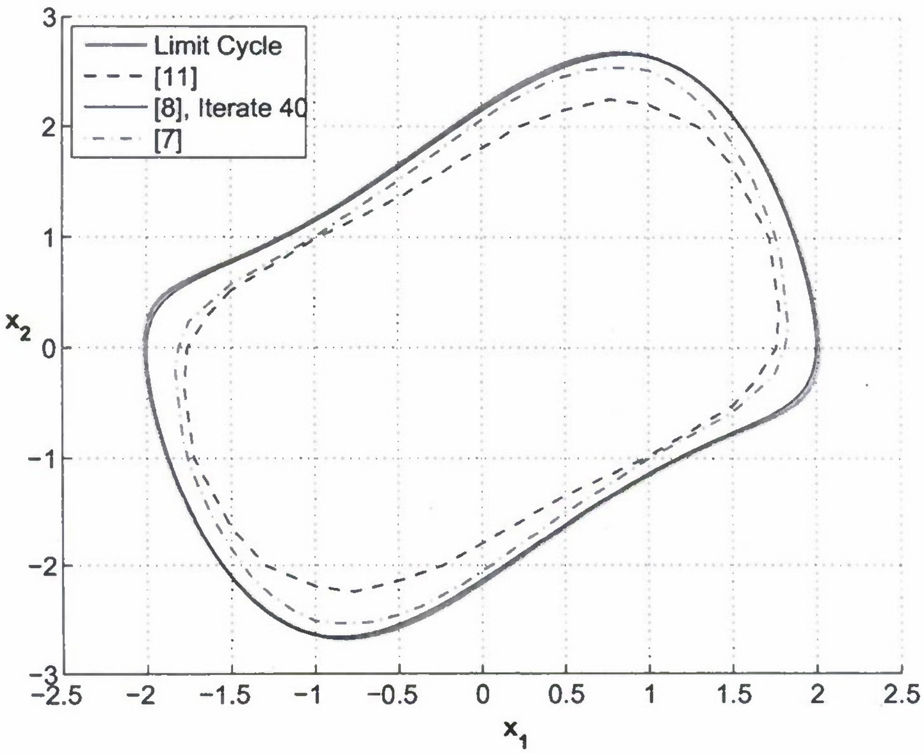

Fig. 2. Provable ROA, from [11], [7] and [8].

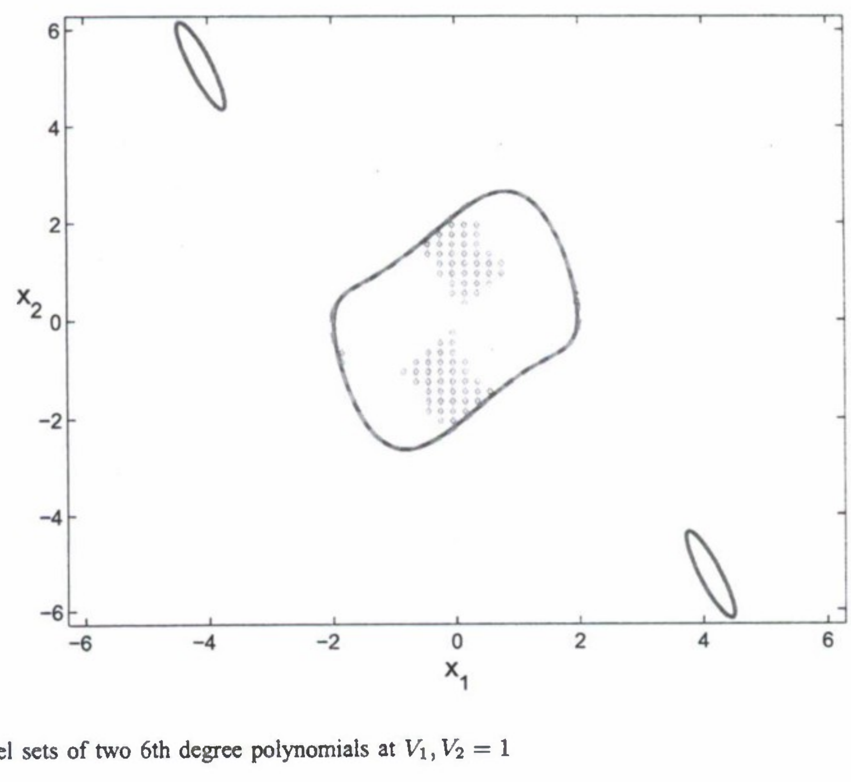

Fig. 3. VDP: Level sets of two 6th degree polynomials at $V_1, V_2 = 1$

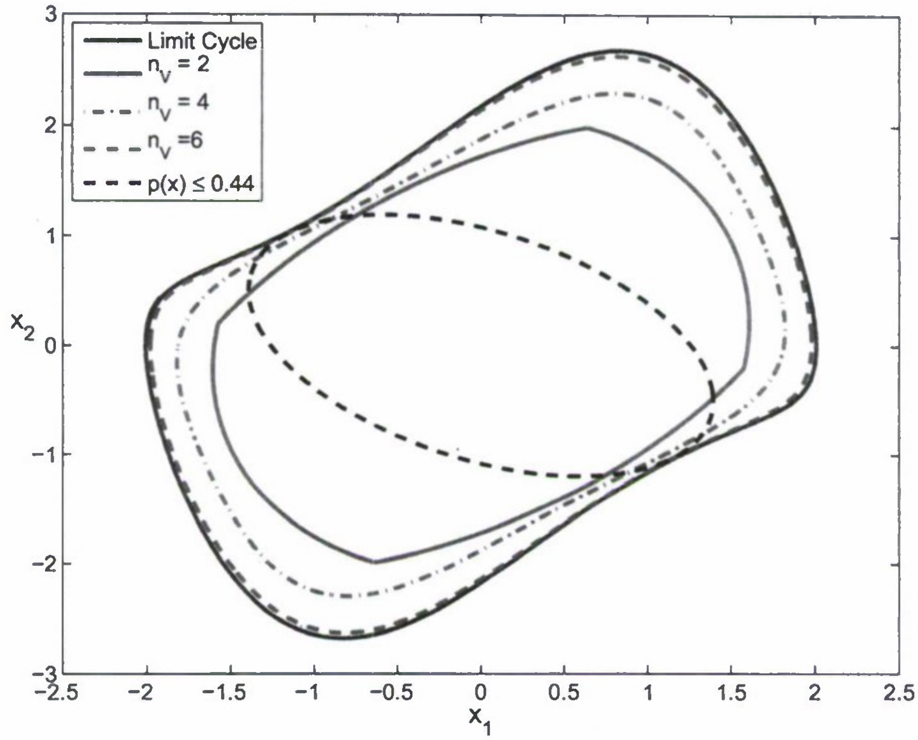

Fig. 4. Provable ROA using pointwise max of two polynomial functions, with shape factor $x^T R_2 x$

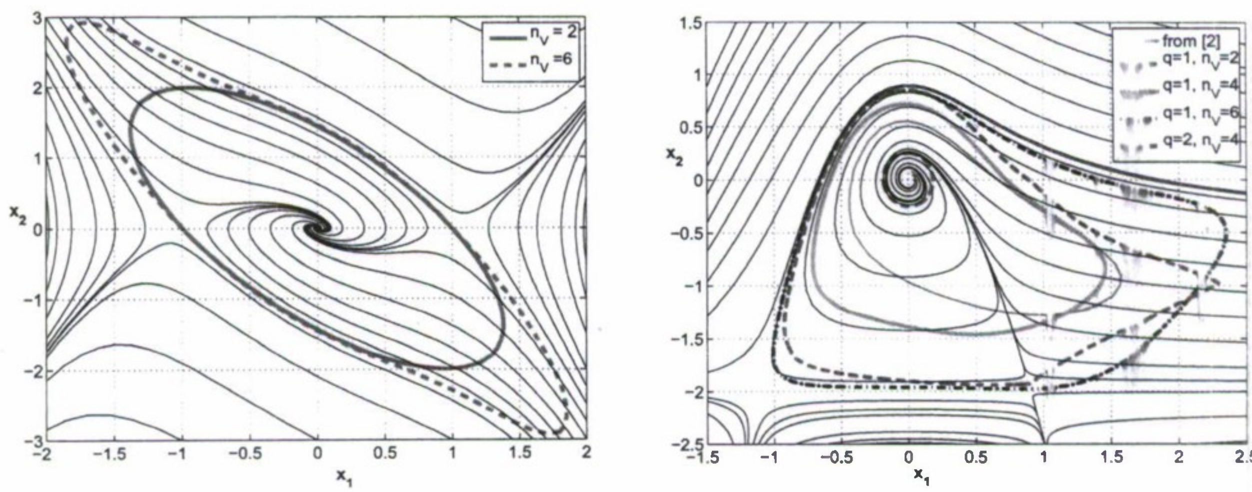

Fig. 5. Trajectories and level sets $V \leq 1$. LEFT panel: example from Section III-C; RIGHT panel: example from Section III-D

Local Robust Performance Analysis for Nonlinear Dynamical Systems

Ufuk Topcu and Andrew Packard

Abstract— **We propose a computational method for local robust performance analysis of nonlinear systems with polynomial dynamics. Specifically, we characterize upper bounds for local $\mathcal{L}_2 \to \mathcal{L}_2$ input-output gains using polynomial Lyapunov/storage functions satisfying certain dissipation inequalities and compute safe approximations for these upper bounds via sum-of-squares programming problems. We consider both bounded parametric uncertainties and bounded uncertainties due to unmodeled dynamics.**

I. Introduction

We consider the problem of quantifying robust performance properties of uncertain nonlinear dynamical systems with polynomial vector fields around asymptotically stable equilibrium points. The amount of amplification of bounded $\mathcal{L}_2$ input norms at the output channels is used as a measure of performance. Two types of uncertainties are considered: (1) bounded uncertainties due to unmodeled dynamics and (2) bounded parametric uncertainties. Following [1], [2], [3], we characterize upper bounds on local input-output gains due to bounded $\mathcal{L}_2$ disturbances by Lyapunov/storage functions which satisfy certain "local" dissipation inequalities [4]. Similar problems were studied in [1], [2], [5], [6], [7] mainly for systems with no uncertainty. Input-output properties of uncertain nonlinear systems were examined in [8] (for discrete time nonlinear systems with a finite-time horizon performance metric) and [9] (input-output gains for sufficiently small input signals).

In this paper, we use polynomial Lyapunov/storage function candidates, simple generalizations of the S-procedure [10], and sum-of-squares (SOS) relaxations for polynomial nonnegativity [11] and compute upper bounds on the input-output gains via (bilinear) SOS programming problems. Uncertainties due to unmodeled dynamics are accounted for in the setting [12] shown in Figure 1 where M models the nominal part and Φ is an unknown operator satisfying certain relations between the input z and the output w_2. The objective is to compute upper bounds on the $\mathcal{L}_2$ norm of the exogenous output e in terms of the $\mathcal{L}_2$ norm of the exogenous input w_1. The approach is composed of two steps: first bound the $\mathcal{L}_2$ norm of the internal input w_2 to M in terms of the $\mathcal{L}_2$ norm of w_1 and then perform an input-output gain analysis for M from the inputs (w_1, w_2) to the output e.

The approach for the bounded parametric uncertainties is similar to that developed in [13], [14] in the context of robust region-of-attraction analysis. Namely, a parameter-independent Lyapunov/storage function is used to characterize input-output properties of uncertain systems over the entire set of admissible values of uncertain parameters. The input-output relations characterized by a single parameter-independent certificate may be more conservative compared to those by parameter-dependent certificates. This potential conservatism is simply reduced by partitioning the set of uncertain parameters into subregions and computing parameter-independent certificates for each subregion. The partition of the uncertainty set can be refined following ideas parallel to branch-and-bound algorithm [15] to further reduce the conservatism. Although it is simplistic (compared to techniques based on parameter-dependent Lyapunov functions), this approach offers certain computational advantages as discussed in [14] for robust region-of-attraction analysis. In fact, in robustness analysis involving time-invariant unknown parameters, it is common, [16], [17], to combine easily-computable sufficient conditions with branch-and-bound strategies, often yielding improved analysis results.

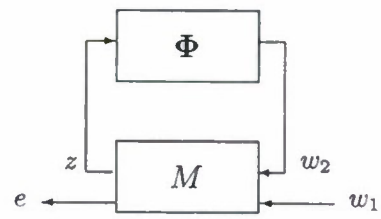

Fig. 1. Input-output system with the feedback interconnection of Φ and M.

The rest of the paper is organized as follows: A characterization of upper bounds for $\mathcal{L}_2 \to \mathcal{L}_2$ input-output gains by Lyapunov/storage functions is discussed in section II. Section III is devoted to the development of the results for the case with unmodeled dynamics and this is followed by the method to account for parametric uncertainties in section IV. Implementation details are given in section V. Demonstration of the methodology with examples in section VI precedes the concluding remarks.

Notation: For $\xi \in \mathcal{R}^n$, $\xi \succeq 0$ means that $\xi_k \geq 0$ for $k = 1, \cdots, n$. For $Q = Q^T \in \mathcal{R}^{n\times n}$, $Q \succeq 0$ ($Q \succ 0$) means that $\xi^T Q \xi \geq 0$ (> 0) for all $\xi \in \mathcal{R}^N$. $\mathbb{R}[\xi]$ represents the set of polynomials in ξ with real coefficients. The subset $\Sigma[\xi] := \{\pi = \pi_1^2 + \pi_2^2 + \cdots + \pi_M^2 \ : \ \pi_1, \cdots, \pi_M \in \mathbb{R}[\xi]\}$ of $\mathbb{R}[\xi]$ is the set of SOS polynomials. For $\pi \in \mathbb{R}[\xi]$, $\partial(\pi)$ denotes the degree of π. For $\eta > 0$ and a function $g : \mathcal{R}^n \to \mathcal{R}$, define the η-sublevel set $\Omega_{g,\eta}$ of g as

$$\Omega_{g,\eta} := \{x \in \mathcal{R}^n \ : \ g(x) \leq \eta\}.$$

U. Topcu is with Control and Dynamical Systems at California Institute of Technology, Pasadena, CA, 91125 (utopcu@cds.caltech.edu) and A. Packard is with the Department of Mechanical Engineering, The University of California, Berkeley, CA, 94720-1740, USA (pack@jagger.me.berkeley.edu).

In several places, a relationship between an algebraic condition on some real variables and state properties of a dynamical system is claimed, and same symbol for a particular real variable in the algebraic statement as well as the state of the dynamical system is used. This could be a source of confusion, so care on the reader's part is required. $\triangleleft$

II. Upper bounds on the $\mathcal{L}_2 \to \mathcal{L}_2$ input-output gain

Consider the dynamical system governed by

$$\begin{aligned} \dot{x}(t) &= f(x(t), w(t)) \\ z(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $x(t) \in \mathcal{R}^n$, $w(t) \in \mathcal{R}^{n_w}$, and f is a n-vector with elements in $\mathbb{R}[(x, w)]$ such that $f(0,0) = 0$ and h is an n_z-vector with elements in $\mathbb{R}[x]$ such that $h(0) = 0$. Let $\phi(t; \mathbf{x}_0, w)$ denote the solution to (1) at time t with the initial condition $x(0) = \mathbf{x}_0$ driven by the input/disturbance w. For a piecewise continuous map $u : [0, \infty) \to \mathcal{R}^m$, define the (truncated) $\mathcal{L}_2$ norm as

$$\|u\|_{2,T} := \sqrt{\int_0^T u(t)^T u(t) dt}.$$

For notational simplicity, denote $\|u\|_{2,\infty}$ by $\|u\|_2$.

A. Upper bounds on local $\mathcal{L}_2 \to \mathcal{L}_2$ gain

Lemma II.1. *[2] If there exist a real scalar $\gamma > 0$ and a continuously differentiable function V such that, for $R > 0$, Ω_{V,R^2} is bounded,*

$$V(0) = 0 \text{ and } V(x) > 0 \quad \text{for all nonzero } x \in \mathcal{R}^n, \tag{2}$$

$$\nabla V f(x, w) \leq w^T w - \gamma^{-2} z^T z \quad \forall x \in \Omega_{V,R^2} \text{ and } \forall w \in \mathcal{R}^{n_w}, \tag{3}$$

then it holds that for the system in (1) and for all $T \geq 0$

$$\|w\|_{2,T} \leq R \text{ and } x(0) = 0 \Rightarrow \|z\|_{2,T} \leq \gamma \|w\|_{2,T}. \tag{4}$$

$\triangleleft$

In other words, γ is a *local* upper bound for the input-output gain for the system in (1). We call γ to be a local upper bound because the upper bound on the norm of the output z is only supposed to hold whenever the norm of the input is bounded by R. This is unlike the input-output gains for linear systems which hold for all values of input norms.

Let $\gamma > 0$ be fixed and $\mathcal{V}$ be the space of continuously differentiable functions. Define $R_{\mathcal{L}_2,opt}(\mathcal{V}, \gamma)$ be the maximum value of R such that the conditions in Lemma II.1 hold for some $V \in \mathcal{V}$. Let $\mathcal{V}_{poly}$ be a subset of $\mathcal{V}$ that is composed of all polynomials in x of some fixed finite degree (omitted in notation). By restricting the search for V satisfying the conditions in Lemma II.1 to $\mathcal{V}_{poly}$, utilizing a generalization of the S-procedure (see Lemma VIII.1 in the Appendix) to obtain sufficient conditions for the set containment constraints in Lemma II.1 and SOS relaxations for polynomial nonnegativity, the following proposition provides an upper bound on $R_{\mathcal{L}_2,opt}(\mathcal{V}, \gamma)$.

Proposition II.1. *For given $\gamma > 0$ and positive definite polynomial l, let $R_{\mathcal{L}_2}$ be defined through*

$$R^2_{\mathcal{L}_2}(\mathcal{V}_{poly}, \mathcal{S}, \gamma, l) := \max_{V \in \mathcal{V}_{poly}, R \geq 0, s \in \mathcal{S}} R^2 \quad s.\ t.$$

$$V(0) = 0, \quad s \in \Sigma[(x, w)], \tag{5}$$

$$V - l \in \Sigma[x], \tag{6}$$

$$-\left[(R^2 - V)s + \nabla V f - w^T w + \gamma^{-2} z^T z\right] \in \Sigma[(x, w)], \tag{7}$$

where $\mathcal{V}_{poly} \subseteq \mathcal{V}$ is as defined above and $\mathcal{S}$ is a prescribed finite-dimensional subset of $\mathbb{R}[(x, w)]$. Then, $R_{\mathcal{L}_2}(\mathcal{V}_{poly}, \mathcal{S}, \gamma, l) \leq R_{\mathcal{L}_2,opt}(\mathcal{V}, \gamma)$. $\triangleleft$

Note that $R_{\mathcal{L}_2,opt}$ depends on γ and $\mathcal{V}$ and $R_{\mathcal{L}_2}$ depends on $\mathcal{V}_{poly}$, $\mathcal{S}$, ϵ, l, and γ. Hereafter, this dependence will not be notated explicitly unless it causes confusion.

The optimization problem in Proposition II.1 can be cast in a bilinear SDP (i.e., nonconvex in general). Bilinear SDPs are known to be harder than linear ones [18]. Consequently, the state-of-the-art of the solvers for bilinear SDPs is far behind that for the linear ones and methods for bilinear SDPs are generally based on heuristics such as coordinate-wise affine search or specialized solvers e.g. PENBMI[19]. Although these techniques are local search schemes and convergence to a global optimum is not guaranteed, coupled with efficient initializations, they have been effectively used for several system analysis questions [20], [21]. For the examples in this paper, we use a coordinatewise affine search scheme as detailed in section V.

For given $\gamma > 0$, the optimization problem in Proposition II.1 maximizes R (that can be verified through the families of admissible Lyapunov function candidates (V) and S-procedure multipliers (s)) such that $\|z\|_2 \leq \gamma \|w\|_2$ whenever $\|w\|_2 \leq R$. One can also choose to minimize γ for a given value of R and this can be formulated as an optimization problem similar to that in Proposition II.1 with minor changes.

III. Robust performance in the presence of unmodeled dynamics

Consider the input-output system in Figure 1. Let

$$\begin{aligned} \dot{x}(t) &= f(x(t), w_1(t), w_2(t)) \\ z(t) &= h_1(x(t)) \\ e(t) &= h_2(x(t)) \end{aligned} \tag{14}$$

be a realization of M, where $w_1(t) \in \mathcal{R}^{n_{w_1}}$, $w_2(t) \in \mathcal{R}^{n_{w_2}}$, f is an n-vector of polynomials in $\mathbb{R}[(x, w_1, w_2)]$ with $f(0,0,0) = 0$, h_1 and h_2 are n_z and n_e dimensional vectors with entries in $\mathbb{R}[x]$ satisfying $h_1(0) = 0$ and $h_2(0) = 0$. Furthermore, assume that $\mathbf{\Phi}$ is causal and, starting from rest, satisfies

$$\|\mathbf{\Phi}(z)\|_{2,T} = \|w_2\|_{2,T} \leq \|z\|_{2,T} \tag{15}$$

for all $T \geq 0$. Lemma III.1 provides a bound on $\|w_2\|_{2,T}$ in terms of $\|w_1\|_{2,T}$ for $T \geq 0$. In the following proposition, this result will be used to establish a local upper bound on the norm of the exogenous output e in terms of the norm of the exogenous input w_1.

$$R_1^2 := \max_{V\in\mathcal{V}_{poly}, R\geq 0, s\in\mathcal{S}} \quad R^2 \quad \text{subject to} \tag{8}$$

$$V(0)=0, \quad s\in\Sigma[(x,w_1,w_2)], \quad V-l\in\Sigma[x], \tag{9}$$

$$-\left[(R^2-V)s+\nabla V f(x,w_1,w_2)-(\beta^2 w_1^T w_1+w_2^T w_2)+\alpha^{-2}z^T z\right]\in\Sigma[(x,w_1,w_2)], \tag{10}$$

$$\gamma_1 := \min_{Q\in\mathcal{V}_{poly},\gamma>0,s\in\mathcal{S}} \quad \gamma \quad \text{subject to} \tag{11}$$

$$Q(0)=0, \quad s\in\Sigma[(x,w_1,w_2)], \quad Q-l\in\Sigma[x], \tag{12}$$

$$-\left[\left(\tfrac{R_1^2}{1-\alpha^2}-Q\right)s+\nabla Q f(x,w_1,w_2)-(\beta^2 w_1^T w_1+w_2^T w_2)+\gamma^{-2}e^T e\right]\in\Sigma[(x,w_1,w_2)]. \tag{13}$$

Lemma III.1. *For $R>0$, $0<\alpha<1$ and $\beta>0$, if there exists a continuously differentiable, positive definite function V such that $V(0)=0$, Ω_{V,R^2} is bounded, and*

$$\nabla V f(x,w_1,w_2)\leq\beta^2 w_1^T w_1+w_2^T w_2-\frac{1}{\alpha^2}z^T z$$

for all $x\in\Omega_{V,R^2}$, $w_1\in\mathcal{R}^{n_{w_1}}$, and $w_2\in\mathcal{R}^{n_{w_2}}$, then for Φ starting from rest and for all $T\geq 0$

$$x(0)=0 \text{ and } \|w_1\|_{2,T}\leq\frac{R}{\beta}\Rightarrow\|w_2\|_{2,T}\leq\frac{\alpha R}{\sqrt{1-\alpha^2}}.$$

◁

Proof: While solutions to (14) exist, for $T\geq 0$

$$\beta^2\|w_1\|_{2,T}^2+\|w_2\|_{2,T}^2-\tfrac{1}{\alpha^2}\|z\|_{2,T}^2 \leq\beta^2\|w_1\|_{2,T}^2-\tfrac{1-\alpha^2}{\alpha^2}\|z\|_{2,T}^2\leq\beta^2\|w_1\|_{2,T}^2. \tag{16}$$

Since Ω_{V,R^2} is bounded, as long as

$$\|w_1\|_{2,T}\leq\frac{\sqrt{R^2-V(x(0))}}{\beta}$$

solutions to (14) exist for all $T\geq 0$ and satisfy

$$\|w_2\|_{2,T}^2\leq\|z\|_{2,T}^2\leq\frac{\alpha^2\beta^2}{1-\alpha^2}\|w_1\|_{2,T}^2+\frac{\alpha^2}{1-\alpha^2}V(x(0)).$$

In particular, for $x(0)=0$, $V(x(0))=$ and $\|w_2\|_{w,T}\leq\frac{\alpha R}{\sqrt{1-\alpha^2}}$. □

Proposition III.1. *In addition to the conditions in Lemma III.1, if there exists a continuously differentiable, positive definite function Q such that $Q(0)=0$ and*

$$\nabla Q f(x,w_1,w_2)\leq\beta^2 w_1^T w_1+w_2^T w_2-\frac{1}{\gamma^2}e^T e$$

for all $x\in\Omega_{Q,R^2/(1-\alpha^2)}$, $w_1\in\mathcal{R}^{n_{w_1}}$, $w_2\in\mathcal{R}^{n_{w_2}}$, then for Φ starting from rest and for all $T\geq 0$

$$\|w_1\|_{2,T}\leq\frac{R}{\beta} \text{ and } x(0)=0\Rightarrow\|e\|_{2,T}\leq\gamma R/\sqrt{1-\alpha^2}.$$

◁

Proof: By Lemma III.1,

$$\|w_1\|_{2,T}\leq R/\beta\Rightarrow\beta^2\|w_1\|_{2,T}^2+\|w_2\|_{2,T}^2\leq\frac{R^2}{1-\alpha^2}.$$

Consequently, the result follows from Lemma II.1. □

Lemma III.1 and Proposition III.1 can be used to construct relations between $\|w_1\|_{2,T}$ and $\|e\|_{2,T}$. Similar to Proposition II.1, one can obtain sufficient conditions for those in Lemma III.1 and Proposition III.1 using Lemma VIII.1 and SOS relaxations for polynomial nonnegativity. For given $\beta>0$, $\alpha>0$, and $l(x)=\epsilon x^T x$ (with $\epsilon>0$ fixed), solve the problems in (8)-(10) and (11)-(13). Then, for Φ starting from rest and for all $T\geq 0$

$$x(0)=0 \text{ and } \|w_1\|_{2,T}\leq\frac{R_1}{\beta}\Rightarrow\|e\|_{2,T}\leq\frac{\gamma R_1}{\sqrt{1-\alpha^2}}.$$

When Φ is unknown but a global gain relation between its inputs and outputs is known, then the results of this section provide a framework for robust performance analysis for the feedback interconnection between M and Φ. On the other hand, even when the operator Φ is known, the procedure outlined in this section can be used as a framework for compositional performance analysis. Note that conditions in Lemma III.1 and Proposition III.1 do not involve the states of (the realization of) Φ. When Φ has the state space realization $\dot{x}_2(t)=f_2(x_2(t),z(t))$, then the gain relation $\|\Phi(z)\|_{2,T}\leq\|z\|_{2,T}$ can be established by determining a positive definite, continuously differentiable function V_2 satisfying

$$\nabla V_2 f_2(x_2,z)\leq z^T z-w_2^T w_2 \qquad \forall x_2\in\mathcal{R}^{n_2},\ \forall z\in\mathcal{R}^{n_z}$$

which does not depend on the states x_1 of M. Consequently, Lemma III.1 and Proposition III.1 enable performance analysis for the feedback interconnection between M and Φ based on the input-output properties of individual blocks. Of course, this analysis may be conservative. Nevertheless, compositional analysis may be a fruitful direction which extends the applicability of SOS programming based nonlinear analysis tools for reasonably larger dimensional systems whenever it is possible to establish an interconnection structure as in Figure 1. Furthermore, it may be possible to refine the input-output relations between w_1 and e by using transformations at the interconnections similar to the D scales in linear robustness analysis [12].

IV. Robust performance analysis in the presence of parametric uncertainties

We now generalize the development in section II to the case where the vector field contains unknown but fixed and bounded parameters. Following the methodology proposed in [14] in the context of robust stability analysis, we first restrict our attention to

$$\begin{aligned} \dot{x}(t) &= f(x,w,\delta) \\ &:= f_0(x(t),w(t)) + \textstyle\sum_{i=1}^{m} \delta_i f_i(x(t),w(t)) \\ z(t) &= h(x(t)), \end{aligned} \tag{17}$$

where $f_0, f_1, \ldots, f_m$ are n-vectors with elements in $\mathbb{R}[(x,w)]$ such that $f_0(0,0,\delta) = f_1(0,0,\delta) = \ldots = f_m(0,0,\delta) = 0$, for all $\delta \in \mathbf{\Delta} \subset \mathcal{R}^m$, and $\mathbf{\Delta}$ is a known bounded polytope. Let $\phi(t; \mathrm{x}_0, w, \delta)$ denote the solution of (17) for δ at time t with the initial condition $x(0) = \mathrm{x}_0$ driven by the input/disturbance w and $\mathcal{E}_{\mathbf{\Delta}}$ denote the set of vertices of $\mathbf{\Delta}$.

Proposition IV.1. *If there exist a real scalar $\gamma > 0$ and a continuously differentiable function V such that $V(0) = 0$, $V(x) > 0$ for all nonzero $x \in \mathcal{R}^n$, Ω_{V,R^2} is bounded, and*

$$\nabla V f(x,w,\delta) \leq w^T w - \gamma^{-2} z^T z \tag{18}$$

for all $x \in \Omega_{V,R^2}$, $w \in \mathcal{R}^{n_w}$, and $\delta \in \mathcal{E}_{\mathbf{\Delta}}$, then the system in (17) with $x(0) = 0$ satisfies $\|z\|_2 \leq \gamma \|w\|_2$ whenever $\|w\|_2 \leq R$ and $\delta \in \mathbf{\Delta}$. ◁

Proof: Since the vector field is affine in δ and $\mathbf{\Delta}$ is a bounded polytope, it follows that, for $\delta \in \mathbf{\Delta}$, $\nabla V f(x,w,\delta) \leq w^T w$ for all $x \in \Omega_{V,R^2}$, $w \in \mathcal{R}^{n_w}$. By Lemma II.1, for each $\delta \in \mathbf{\Delta}$, $\|z\|_2 = \|h(\phi(\cdot; 0, w, \delta))\|_2 \leq \gamma \|w\|_2$ whenever $\|w\|_2 \leq R$. □

Note that restricting the attention to affine uncertainty δ dependence and polytopic $\mathbf{\Delta}$, Proposition IV.1 enables to compute upper bounds on $\mathcal{L}_2 \to \mathcal{L}_2$ gain for the system in (17) by imposing the conditions in (18) at finitely many $\delta \in \mathcal{E}_{\mathbf{\Delta}}$ instead of at infinitely many $\delta \in \mathbf{\Delta}$. Furthermore, sufficient conditions for those in Proposition IV.1 can be obtained using Lemma VIII.1 and SOS relaxations.

The approach proposed here is restrictive: (1) only affine dependence on δ and polytopic $\mathbf{\Delta}$ are allowed (2) SOS relaxations for the conditions in Proposition IV.1 may include a large number of SDP constraints (3) single (δ-independent) Lyapunov/storage function is to certify properties for an entire family of systems. These limitations can be partially alleviated using techniques proposed in [14] in the context of robust region-of-attraction analysis. For example, polynomial dependence on δ in the vector field and the output map can be handled by covering the graph of non-affine functions δ (in the conditions in Proposition IV.1) by bounded polytopes. Furthermore, the fact that constraints in the SOS relaxations for the conditions in IV.1 are only coupled through the Lyapunov/storage functions (which include relatively small portion of all decision variables in associated SDPs) can be exploited through a suboptimal two-step procedure: pick a point in $\mathbf{\Delta}$, compute a Lyapunov/storage function for the system corresponding to that point, and then in the second step determine an input-output relation certified by the Lyapunov/storage function determined (fixed) in the first step which holds for the entire family of admissible systems. This procedure effectively decouples the large number of constraints in the second step enabling use of trivial parallelization. Finally, conservatism (due to using a single parameter-independent Lyapunov function and due to the suboptimal two-step procedure) can be reduced by an informal branch-and-bound type refinement procedure where $\mathbf{\Delta}$ is partitioned into smaller subregions and a different Lyapunov/storage function is computed for each subregion. See [14] which develops a similar methodology in the context of robust region-of-attraction analysis.

V. Implementation Issues

The SOS relaxations in (5)-(7) lead to bilinear SDPs due to the multiplication between the decision variables in V and the multipliers. Therefore, solution techniques for these problems are usually limited to local search schemes such as PENBMI [19] or coordinate-wise affine search based on the observation that, for given V and R, constraints in these problems are affine in the decision variables in the multipliers. For example, one can obtain a suboptimal solution for the problem in (5)-(7) by alternatingly solving the following two problems until a maximum number of iterations is reacaed or the increase in the value of certified R becomes smaller than a pre-specified tolerance . For given V

$$\max_{R>0, s\in\mathcal{S}} R^2 \text{ subject to } s \in \Sigma[(x,w)] \text{ and } (7), \tag{19}$$

which can be solved using an off-the-shelf affine SDP solver through a line search on R, and for given (feasible) multiplier s

$$\max_{R>0, V\in\mathcal{V}_{poly}} R^2 \text{ subject to } V - l \in \Sigma[x],\ V(0) = 0, \text{ and } (7). \tag{20}$$

Furthermore, by a change of variables, it is possible to iterate without a line search in the first step. Indeed, for $\beta > 0$, if the problem in (5)-(7) has the solution R_l^2, V_l and s_l, then

$$\begin{aligned} \tfrac{1}{R_l^2} = & \min_{K\in\mathcal{V}_{poly}, 1/R^2>0, \tilde{s}\in\mathcal{S}} \frac{1}{R^2} \text{ subject to} \\ & \tilde{s} \in \Sigma[(x,w)],\ K \in \mathbb{R}[x],\ K - l_1/R^2 \in \Sigma[x], \\ & -\left[(1-K)\tilde{s} + \nabla K f - \tfrac{1}{R^2}(w^T w - \gamma^{-2} z^T z)\right] \\ & \qquad \in \Sigma[(x,w)]. \end{aligned} \tag{21}$$

Note that for given K constraints in (21) are affine in $1/R^2$ and $\tilde{s}$. In fact, optimal values of $\tilde{s}$ and K are $\tilde{s} = R_l^2 s_l$, and $K = V_l / R_l^2$.

VI. Examples

Consider the controlled short period aircraft dynamics in Figure 2 where $x_p := [x_1\ x_2\ x_3]^T$, x_1, x_2, and x_3 denote the pitch rate, the angle of attack, and the pitch angle,

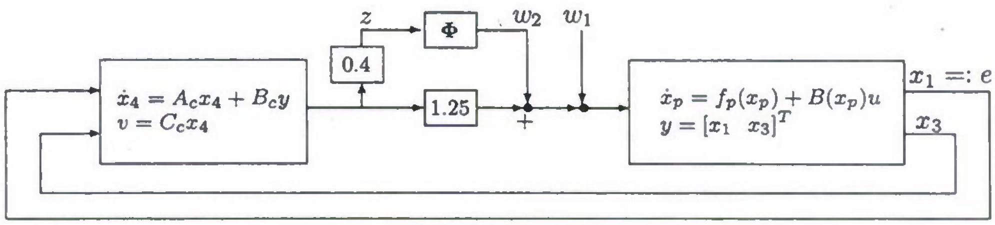

Fig. 2. Controlled short period aircraft dynamics with unmodeled dynamics.

respectively, and

$$\dot{x}_p = \begin{bmatrix} c(x_p) \\ q(x_p) \\ x_1 \end{bmatrix} + \begin{bmatrix} \ell_b^T x_p + b_1 \\ b_2 \\ 0 \end{bmatrix} u, \qquad (22)$$

where, c and q are cubic and quadratic polynomials, respectively, $\ell_b \in \mathcal{R}^3$, b_1 and b_2 are real scalars (see [22] for the values of the missing parameters). The plant output is $[x_1 \; x_3]^T$. The input u to the plant is

$$u = 1.25v + w_1 + w_2$$

where v, the elevator deflection, is the controller output determined by

$$\begin{aligned} \dot{x}_4 &= -0.864y_1 - 0.321y_2 \\ v &= 2x_4, \end{aligned}$$

where x_4 is the controller state. Assume that $\Phi : \mathcal{R} \to \mathcal{R}$ satisfies, starting from rest,

$$\|\Phi(z)\|_{2,T} = \|w_2\|_{2,T} \leq \|z\|_{2,T}$$

for all $T \geq 0$. We performed the following analysis:

(i) For several values of $\alpha \in [0.55, 0.9]$, solve the problems in (8)-(10) and (11)-(13).

(ii) Apply linearized robust performance analysis for the feedback interconnection [23] and fit a first order stable minimum phase transfer function, say $H(s)$, to the optimal D-scales. For several values of $\alpha \in [0.55, 0.9]$, solve the problems in (8)-(10) and (11)-(13) for the system HMH^{-1} with a minimal realization for H.

(iii) Solve the problem in Proposition II.1 for the system with no uncertainty for several values of γ.

Figure 3 shows the $\mathcal{L}_2$ norms of the exogenous outputs e versus the $\mathcal{L}_2$ norms of the exogenous inputs w_1 in each of these cases: (i) with marker "+", (ii) with marker "•", and (iii) with marker "×".

Figure 3 illustrates the trade off between the robustness and performance: As α gets larger, the gap between the nominal performance level and the "robust" performance level increases deduced from the divergence between the curve with "+" and other two curves.

VII. Conclusion

We proposed a computational method for local robust performance analysis of nonlinear systems with polynomial dynamics. Specifically, we characterized upper bounds for local $\mathcal{L}_2 \to \mathcal{L}_2$ input-output gains using polynomial Lyapunov/storage functions satisfying certain dissipation inequalities and computed safe approximations for these upper bounds via sum-of-squares programming problems. We considered both bounded parametric uncertainties and bounded uncertainties due to unmodeled dynamics.

Acknowledgements

This work was sponsored by the Air Force Office of Scientific Research, USAF, under grant/contract number FA9550-05-1-0266. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the AFOSR or the U.S. Government.

U. Topcu acknowledges partial support from the Boeing Corporation. A. Packard acknowledges partial support from NASA under the contract NNX08AC80A.

VIII. Appendix

The following lemma is a simple generalization of the S-procedure [10] and is used to obtain sufficient conditions for certain set containment constraints throughout the paper.

Lemma VIII.1. *For* $g_0, g_1, \cdots, g_m \in \mathbb{R}[x]$, *if there exist* $s_1, \cdots, s_m \in \Sigma[x]$ *such that*

$$g_0 - \sum_{i=1}^{m} s_i g_i \in \Sigma[x],$$

then

$$\{x \in \mathcal{R}^n : g_1(x), \ldots, g_m(x) \geq 0\} \subseteq \{x \in \mathcal{R}^n : g_0(x) \geq 0\}.$$

◁

References

[1] Z. Jarvis-Wloszek, R. Feeley, W. Tan, K. Sun, and A. Packard, "Control applications of sum of squares programming," in *Positive Polynomials in Control*, D. Henrion and A. Garulli, Eds. Springer-Verlag, 2005, pp. 3–22.

[2] W. Tan, A. Packard, and T. Wheeler, "Local gain analysis of nonlinear systems," in *Proc. American Control Conf.*, Minneapolis, MN, 2006, pp. 92–96.

[3] W. Tan, U. Topcu, P. Seiler, G. Balas, and A. Packard, "Simulation-aided reachability and local gain analysis for nonlinear dynamical systems," in *Proc. Conf. on Decision and Control*, 2008.

[4] J. C. Willems, "Dissipative dynamical systems I: Genaral theory," *Archive for Rational Mechanics and Analysis*, vol. 45, pp. 321–343, 1972.

[5] S. Prajna, "Barrier certificates for nonlinear model validation," vol. 3, 2003, pp. 2884–2889.

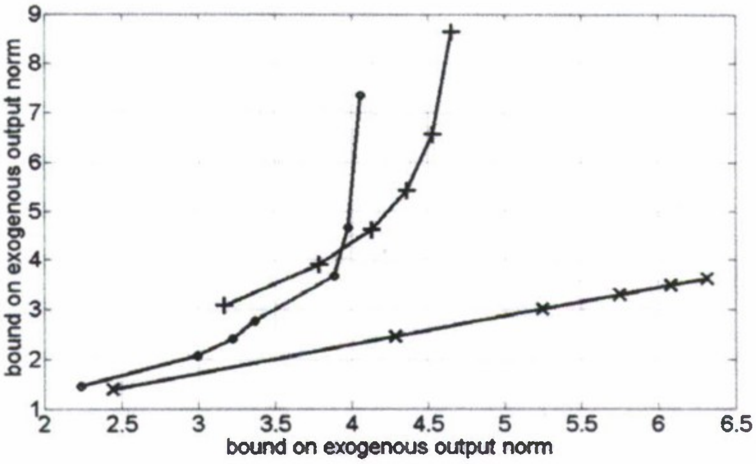

Fig. 3. The $\mathcal{L}_2$ norms of the exogenous outputs e versus the $\mathcal{L}_2$ norms of the exogenous inputs w_1 in each of these cases: (i) with marker "+", (ii) with marker "•", and (iii) with marker "×".

[6] D. F. Coutinho, M. Fu, A. Trofino, and P. Danes, "$\mathcal{L}_2$-gain analysis and control of uncertain nonlinear systems with bounded disturbance inputs," *International Journal of Robust and Nonlinear Control*, vol. 18, pp. 88–110, 2007.

[7] R. Vinter, "A characterization of the reachable set for nonlinear control systems," *SIAM Journal on Control and Optimization*, vol. 18, no. 6, pp. 599–610, 1980.

[8] J. Tierno and R. M. Murray, "Robust performance analysis for a class of uncertain nonlinear systems," in *Proc. Conf. on Decision and Control*, New Orleans, LA, 1995, pp. 1684–1689.

[9] H. Ito, "Local stability and performance robustness of nonlinear systems with structured uncertainty," *IEEE Transactions on Automatic Control*, vol. 44, no. 2, pp. 1250–1254, 1999.

[10] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*. Philadelphia: SIAM, 1994.

[11] P. Parrilo, "Semidefinite programming relaxations for semialgebraic problems," *Mathematical Programming Series B*, vol. 96, no. 2, pp. 293–320, 2003.

[12] K. Zhou and J. C. Doyle, *Essentials of Robust Control*. Prentice Hall, 1998.

[13] U. Topcu and A. Packard, "Local stability analysis for uncertain nonlinear systems," 2007, to appear in IEEE Transaction on Automatic Control.

[14] U. Topcu, A. Packard, P. Seiler, and G. Balas, "Robust region-of-attraction estimation," 2008, submitted to IEEE Transaction on Automatica Control.

[15] E. Lawler and D.E.Wood, "Branch-and-bound methods: a survey," *Operations Research*, vol. 14, no. 4, pp. 679–719, 1966.

[16] V. Balakrishnan, S. Boyd, and S. Balemi., "Branch and bound algorithm for computing the minimum stability degree of parameter-dependent linear systems," *Int. J. of Robust and Nonlinear Control*, vol. 1, no. 4, pp. 295–317, 1991.

[17] P. M. Young, M. P. Newlin, and J. C. Doyle, "μ analysis with real parametric uncertainty," in *Proc. Conf. on Decision and Control*, 1991, pp. 1251–1256.

[18] O. Toker and H. Ozbay, "On the NP-hardness of solving bilinear matrix inequalities and simultaneous stabilization with static output feedback," in *Proc. American Control Conf.*, 1995, pp. 2525–2526.

[19] M. Kočvara and M. Stingl, "*PENBMI User's Guide (Version 2.0)*, available from `http://www.penopt.com`," 2005.

[20] W. Tan and A. Packard, "Stability region analysis using sum of squares programming," in *Proc. American Control Conf.*, Minneapolis, MN, 2006, pp. 2297–2302.

[21] U. Topcu, A. Packard, and P. Seiler, "Local stability analysis using simulations and sum-of-squares programming," *Automatica*, vol. 44, pp. 2669 – 2675, 2008.

[22] U. Topcu, "Quantitative local analysis of nonlinear systems," Ph.D. Dissertation, UC, Berkeley, 2008, available at `http://jagger.me.berkeley.edu/~utopcu/dissertation`.

[23] G. Balas, J. Doyle, K. Glover, A. Packard, and R. Smith, "μ-Analysis and Synthesis Toolbox - User's Guide," 1998.

Linearized Analysis versus Optimization-based Nonlinear Analysis for Nonlinear Systems

Ufuk Topcu and Andrew Packard

Abstract— **For autonomous nonlinear systems stability and input-output properties in small enough (infinitesimally small) neighborhoods of (linearly) asymptotically stable equilibrium points can be inferred from the properties of the linearized dynamics. On the other hand, generalizations of the S-procedure and sum-of-squares programming promise a framework potentially capable of generating certificates valid over quantifiable, finite size neighborhoods of the equilibrium points. However, this procedure involves multiple relaxations (unidirectional implications). Therefore, it is not obvious if the sum-of-squares programming based nonlinear analysis can return a feasible answer whenever linearization based analysis does. Here, we prove that, for a restricted but practically useful class of systems, conditions in sum-of-squares programming based region-of-attraction, reachability, and input-output gain analyses are feasible whenever linearization based analysis is conclusive. Besides the theoretical interest, such results may lead to computationally less demanding, potentially more conservative nonlinear (compared to direct use of sum-of-squares formulations) analysis tools.**

I. Introduction

Internal stability, input-to-state, and input-to-output properties of dynamical systems are commonly analyzed by constructing Lyapunov/storage functions satisfying certain conditions (such as dissipation inequalities) [1], [2], [3], [4]. Generalizations of the S-procedure [5], [4] and sum-of-squares (SOS) relaxations for polynomial nonnegativity [6] provide a framework for the search of such Lyapunov/storage functions for systems with polynomial vector fields based on (linear or bilinear) semidefinite programming (SDP) problems [7], [8], [9], [10], [11], [12], [13], [14], [16], [17], [18].

On the other hand, it is well known that if there exist Lyapunov/storage functions for the linearized dynamics (around an asymptotically stable equilibrium point) then, by certain continuity assumptions, these functions (always) serve as Lyapunov/storage functions for the nonlinear system possibly only locally, i.e., corresponding Lyapunov or dissipation inequalities only hold in a "sufficiently small" neighborhood of the equilibrium point. The promise of SOS programming based nonlinear analysis is that it may be possible to construct Lyapunov/storage functions that satisfy the Lyapunov or dissipation inequalities not only in a "sufficiently small" neighborhood of the equilibrium point but also over quantifiable, non-trivial subsets of the state space. However, the transformation from system analysis questions to corresponding SDP problems (in nonlinear analysis) involves a series of sufficient (but not necessarily necessary) conditions. For example, except certain special or hypothetical cases, S-procedure is not lossless and not all nonnegative polynomials are SOS [9], [6], [19]. Therefore, it is not obvious if (SOS programming based) nonlinear analysis yields a certificate for the nonlinear system whenever the linear analysis does.

In this paper, we propose conditions for the feasibility of SDP problems (equivalently SOS programming problems), proposed in [5], [20], [7] in the context of stability robustness, reachability, and input-output gain analyses of nonlinear systems around asymptotically stable equilibrium points, based on the properties of the corresponding linearized dynamics. We focus on systems with cubic polynomial vectors fields mainly due to practical reasons. Although SOS programming based analysis can theoretically be used for systems with polynomial vector fields of any finite degree, there are practical bounds on the degree imposed by the capabilities of current SDP solvers and computational resources (see [7], [14] for a more detailed discussion). Therefore, nonlinear analysis with cubic vectors fields is a pragmatic extension for linearization based analysis with tighter approximations for the actual dynamics and richer families of Lyapunov/storage functions.

The motivation is primarily theoretical, showing that the optimization-based (S-procedure/SOS) methods for nonlinear analysis (as proposed in [5], [20], [7]) always involve feasible bilinear SDP problems whenever the linearization is asymptotically stable. Furthermore, these results may also have some limited practical value in actually constructing (possibly conservative) quantitative results for the nonlinear system as outlined in section VI.

Notation: For $\xi \in \mathcal{R}^n$, $\xi \succeq 0$ means that $\xi_k \geq 0$ for $k = 1, \cdots, n$. For $Q = Q^T \in \mathcal{R}^{n\times n}$, $Q \succeq 0$ ($Q \succ 0$) means that $\xi^T Q \xi \geq 0$ (> 0) for all $\xi \in \mathcal{R}^n$. For $x_1 \in \mathcal{R}^{n_1}$ and $x_2 \in \mathcal{R}^{n_2}$, $[x_1; x_2] \in \mathcal{R}^{n_1+n_2}$ denotes the concatenation of x_1 and x_2. $\mathbb{R}[\xi]$ represents the set of polynomials in ξ with real coefficients. The subset $\Sigma[\xi] := \{\pi = \pi_1^2 + \pi_2^2 + \cdots + \pi_M^2 \ : \ \pi_1, \cdots, \pi_M \in \mathbb{R}[\xi]\}$ of $\mathbb{R}[\xi]$ is the set of SOS polynomials. For $\eta > 0$ and a function $g : \mathcal{R}^n \to \mathcal{R}$, define the η-sublevel set $\Omega_{g,\eta}$ of g as

$$\Omega_{g,\gamma} := \{x \in \mathcal{R}^n \ : \ g(x) \leq \eta\}.$$

For a piecewise continuous map $u : [0, \infty) \to \mathcal{R}^m$, define the $\mathcal{L}_2$ norm as

$$\|u\|_2 := \sqrt{\int_0^\infty u(t)^T u(t) dt}.$$

U. Topcu is with Control and Dynamical Systems at California Institute of Technology, Pasadena, CA, 91125 (utopcu@cds.caltech.edu) and A. Packard is with the Department of Mechanical Engineering, The University of California, Berkeley, CA, 94720-1740, USA (pack@jagger.me.berkeley.edu).

In several places, a relationship between an algebraic condition on some real variables and state properties of a dynamical system is claimed, and same symbol for a particular real variable in the algebraic statement as well as the state of the dynamical system is used. This could be a source of confusion, so care on the reader's part is required. $\triangleleft$

II. Preliminaries

Following two lemmas are straightforward generalizations of the S-procedure [4]. See [21], [7] for the proofs.

Lemma II.1. *Given* $g_0, g_1, \cdots, g_m \in \mathbb{R}[x]$, *if there exist* $s_1, \cdots, s_m \in \Sigma[x]$ *such that* $g_0 - \sum_{i=1}^m s_i g_i \in \Sigma[x]$, *then* $\{x \in \mathcal{R}^n : g_1(x), \ldots, g_m(x) \geq 0\} \subseteq \{x \in \mathcal{R}^n : g_0(x) \geq 0\}$.

Lemma II.2. *Given* $g_0, g_1, g_2 \in \mathbb{R}[x]$ *such that* g_0 *is positive definite and* $g_0(0) = 0$, *if there exist* $s_1, s_2 \in \Sigma[x]$ *such that* $g_1 s_1 + g_2 s_2 - g_0 \in \Sigma[x]$, *then* $\{x \in \mathcal{R}^n : g_1(x) \leq 0\} \backslash \{0\} \subset \{x \in \mathcal{R}^n : g_2(x) > 0\}$. $\triangleleft$

The following fact will be used in the subsequent sections.

Lemma II.3. *Let* $Q = Q^T \in \mathcal{R}^{n \times n}$ *be positive definite,* $f : \mathcal{R}^n \to \mathcal{R}$ *be defined as* $f(x) = x^T Q x$, $c_1, \ldots, c_m$ *be positive real numbers, and* $g : \mathcal{R}^m \to \mathcal{R}$ *be defined as* $g(y) = c_1 y_1^2 + c_2 y_2^2 + \ldots + c_m y_m^2$. *Then,* $f(x)g(y)$ *can be written in the form*

$$f(x)g(y) = \mathbf{z}(x,y)^T H \mathbf{z}(x,y),$$

where $\mathbf{z}(x,y) = y \otimes x$ *and* $H \succ 0$. $\triangleleft$

Proof:

$$\begin{aligned} f(x)g(y) &= x^T Q x (c_1 y_1^2 + \ldots + c_m y_m^2) \\ &= \textstyle\sum_{i=1}^m c_i (y_i x)^T Q (y_i x) \\ &= \mathbf{z}(x,y)^T H \mathbf{z}(x,y), \end{aligned}$$

where $H = H^T \in \mathcal{R}^{nm \times nm}$ is

$$H := \begin{bmatrix} c_1 Q & & \\ & \ddots & \\ & & c_n Q \end{bmatrix}.$$

Clearly, H is positive definite since Q is positive definite. $\square$

Lemma II.4. *Let* Q *and* f *be as in Lemma II.3,* $c_1, \ldots, c_n$ *be positive real numbers, and* $g : \mathcal{R}^n \to \mathcal{R}$ *be defined as* $g(x) = c_1 x_1^2 + \ldots + c_n x_n^2$. *Then,* $f(x)g(x)$ *can be written in the form*

$$f(x)g(x) = \mathbf{z}(x)^T H \mathbf{z}(x),$$

where $\mathbf{z}(x)$ *is a vector of all monomials of the form* $x_i y_j$ *for* $i = 1, \ldots, n$ *and* $j \geq i$ *with no repetition.* $\triangleleft$

Lemma II.5. *Let* Q *and* f *be as in Lemma II.3,* $c_1, \ldots, c_{n+m}$ *be positive real numbers, and* $g : \mathcal{R}^{m+n} \to \mathcal{R}$ *be defined as* $g(x,y) = c_1 y_1^2 + c_2 y_2^2 + \ldots + c_m y_m^2 + c_{m+1} x_1^2 + \ldots + c_{m+n} x_n^2$. *Then,* $f(x)g(x,y)$ *can be written in the form*

$$f(x)g(x,y) = \mathbf{z}(x,[x;y])^T H \mathbf{z}(x,[x;y]),$$

where $\mathbf{z}(x,[x;y])$ *is a vector of all monomials of the form* x_i^2 *for* $i = 1, \ldots, n$ *and* $x_i y_j$ *for* $i = 1, \ldots n$ *and* $j = 1, \ldots m$ *with no repetition.* $\triangleleft$

Although $\mathbf{z}$ (as defined above)depends on x and/or y, this dependence will not be explicitly notated whenever it is convenient and does not cause confusion.

III. $\mathcal{L}_2 \to \mathcal{L}_2$ Input-Output Gain Analysis

Consider the dynamical system governed by

$$\begin{aligned} \dot{x}(t) &= f(x(t), w(t)) \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $x(t) \in \mathcal{R}^n$, $w(t) \in \mathcal{R}^{n_w}$, and f is a n-vector with elements in $\mathbb{R}[(x,w)]$ such that $f(0,0) = 0$ and h is an n_y-vector with elements in $\mathbb{R}[x]$ such that $h(0) = 0$. Let $\phi(t; \mathbf{x}_0, w)$ denote the solution to (1) at time t with the initial condition $x(0) = \mathbf{x}_0$ driven by the input/disturbance w.

Lemma III.1. *[22] If there exist real scalars* $\gamma > 0$ *and* $R \geq 0$ *and a continuously differentiable function* V *such that*

$$V(0) = 0 \text{ and } V(x) > 0 \quad \text{for all nonzero } x \in \mathcal{R}^n, \tag{2}$$

$$\Omega_{V,R^2} \text{ is bounded}, \tag{3}$$

$$\nabla V f(x,w) \leq w^T w - \gamma^{-2} y^T y \quad \forall x \in \Omega_{V,R^2},\ w \in \mathcal{R}^{n_w}, \tag{4}$$

then for the system in (1)

$$x(0) = 0 \text{ and } \|w\|_2 \leq R \Rightarrow \|y\|_2 \leq \gamma \|w\|_2. \tag{5}$$

$\triangleleft$

In other words, γ is an upper bound for the "local" input-output gain for the system in (1). For given $\gamma > 0$, we restrict V to be a polynomial of some fixed degree and use Proposition III.1 to compute lower bounds on the maximum value of R such that (2)-(4) hold.

Proposition III.1. *[21] For given* $\gamma > 0$ *and positive definite polynomial* l_1 *satisfying* $l_1(0) = 0$, *let* $R_{\mathcal{L}_2}$ *be defined through*

$$R_{\mathcal{L}_2}^2 := \max_{V \in \mathcal{V}_{poly}, R \geq 0, s \in \mathcal{S}} R^2 \quad \text{subject to} \tag{6}$$

$$V(0) = 0, \quad s_1 \in \Sigma[(x,w)], \tag{7}$$

$$V - l_1 \in \Sigma[x], \tag{8}$$

$$-\left[(R^2 - V)s_1 + \nabla V f(x,w) - w^T w + \gamma^{-2} y^T y\right] \in \Sigma[(x,w)], \tag{9}$$

where $\mathcal{V}_{poly} \subseteq \mathcal{V}$ *and* $\mathcal{S}$ *are prescribed finite-dimensional subsets of* $\mathbb{R}[x]$. *Then,*

$$x(0) = 0 \text{ and } \|w\|_2 \leq R_{\mathcal{L}_2} \Rightarrow \|y\|_2 \leq \gamma \|w\|_2.$$

$\triangleleft$

Now, consider the case where f and h are of the form

$$\begin{aligned} f(x,w) &= Ax + Bw + f_2(x) + f_3(x) \\ &\quad + (g_1(x) + g_2(x))w, \\ h(x) &= Cx + h_2(x), \end{aligned} \tag{10}$$

where f_2, f_3, g_1, g_2, and h_2 are matrices (of appropriate dimension) of (purely) quadratic, cubic, linear, quadratic, and quadratic polynomials in their arguments respectively and A, B, and C are matrices (of reals) of appropriate dimension. Then, the following proposition gives conditions on the feasibility of the constraints in (6)-(9) based on the analysis of the corresponding linearized dynamics.

Proposition III.2. *For given $\gamma > 0$, $l_1(x) = x^T L_1 x$ with $L_1 \succ 0$, f and h in the form in (10), if there exist a symmetric matrix Q and $\epsilon > 0$ such that $Q \succeq L_1$ and*

$$D_0 := \begin{bmatrix} A^TQ + QA + \gamma^{-2}C^TC & QB \\ B^TQ & -I \end{bmatrix} \preceq -\epsilon I,$$

then the constraints in (6)-(9) are feasible. ◁

Proof: Define $V(x) := x^TQx$. Let $\mathbf{z} = \mathbf{z}(x, [x; w])$ be as defined in section II. Then, there exist $H \succ 0$, M_1, and M_2 such that

$$\begin{aligned} \mathbf{z}^T H \mathbf{z} &= (w^Tw + x^Tx)(x^TQx) \\ x^T M_1 \mathbf{z} &= x^TQ(f_2(x) + g_1(x)w) + x^TC^Th_2(x) \\ \mathbf{z}^T M_2 \mathbf{z} &= 2x^TQ(f_3(x) + g_2(x)w) + h_2(x)^Th_2(x). \end{aligned}$$

Let $\alpha > 0$ be such that

$$D_1 := \begin{bmatrix} -D_0 & -M_1 \\ -M_1^T & \alpha H - M_2 \end{bmatrix} \succeq \epsilon I$$

and $R := \sqrt{\epsilon/(2\alpha)}$. Define

$$s_1(x, w) := \alpha(x^Tx + w^Tw).$$

Then, $V - l_1$ and s_1 are SOS. Consider

$$\begin{aligned} b(x, w) := &- \left[\nabla V f(x, w) - w^Tw + h^T(x)h(x)\right] \\ &- \alpha(x^Tx + w^Tw)\left(R^2 - V\right), \end{aligned}$$

which can be decomposed as

$$b(x, w) = [x; w; \mathbf{z}]^T D_2 [x; w; \mathbf{z}],$$

where

$$D_2 := D_1 - \begin{bmatrix} \alpha R^2 I & 0 \\ 0 & 0 \end{bmatrix} \succeq \epsilon I - \begin{bmatrix} \alpha R^2 I & 0 \\ 0 & 0 \end{bmatrix} \succeq \frac{\epsilon}{2} I.$$

Hence, b is SOS. □

IV. Reachability Analysis

For $R \geq 0$ and $\|w\|_2 \leq R$, the set $\mathcal{G}_{R^2}$ of points reachable from the origin under the flow of (1) is defined as

$$\mathcal{G}_{R^2} := \{\phi(T; 0, w) \in \mathcal{R}^n \ : \ T \geq 0, \ \|w\|_2 \leq R\}.$$

Lemma IV.1 adapted from a Lyapunov-like argument in [4, §6.1.1] provides a characterization of sets containing $\mathcal{G}_{R^2}$ [5], [22].

Lemma IV.1. *If there exists a continuously differentiable function V such that*

$$V(x) > 0 \text{ for all } x \in \mathcal{R}^n \backslash \{0\} \text{ with } V(0) = 0, \qquad (11)$$

$$\Omega_{V,R^2} \text{ is bounded}, \qquad (12)$$

$$\nabla V f(x, w) \leq w^Tw \quad \forall x \in \Omega_{V,R^2}, \ w \in \mathcal{R}^{n_w}, \qquad (13)$$

then $\mathcal{G}_{R^2} \subseteq \Omega_{V,R^2}$. ◁

For given $\beta > 0$ and positive definite, convex polynomial p, the following proposition provides a lower bound for the maximum value of R such that $\mathcal{G}_{R^2} \subseteq \Omega_{p,\beta}$.

Proposition IV.1. *[22] Let $\beta > 0$, l_1 be a positive definite polynomial satisfying $l_1(0) = 0$, R_{reach} be defined through*

$$R^2_{reach} := \max_{V \in \mathcal{V}_{poly}, R \geq 0, s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2} R^2 \text{ subject to} \qquad (14)$$

$$V(0) = 0, \ s_1 \in \Sigma[x], \text{ and } s_2 \in \Sigma[(x, w)], \qquad (15)$$

$$V - l_1 \in \Sigma[x], \qquad (16)$$

$$(\beta - p) - (R^2 - V)s_1 \in \Sigma[x], \qquad (17)$$

$$-\left[(R^2 - V)s_2 + \nabla V f(x, w) - w^Tw\right] \in \Sigma[(x, w)], \qquad (18)$$

where $\mathcal{V}_{poly} \subset \mathcal{V}$ and $\mathcal{S}_i$ are prescribed finite-dimensional subsets of $\mathbb{R}[x]$. Then,

$$\mathcal{G}_{R^2_{reach}} \subseteq \Omega_{V,R^2_{reach}} \subseteq \Omega_{p,\beta}.$$

◁

Proposition IV.2. *For $p(x) = x^TPx$ with $P \succ 0$, $l_1(x) = x^TL_1x$ with $L_1 \succ 0$, and f of the form in (10), if there exist $\epsilon > 0$ and $Q \succeq L_1$ such that*

$$D_0 := \begin{bmatrix} A^TQ + QA & QB \\ B^TQ & -I \end{bmatrix} \preceq -\epsilon I,$$

then the constraints in (14)-(18) are feasible. ◁

Proof: Define $V(x) := x^TQx$. Let $\mathbf{z} = \mathbf{z}(x, [x; w])$ be as defined in section II. Then, there exist $H \succ 0$ (by Lemma II.3), M_1, and M_2 such that

$$\begin{aligned} \mathbf{z}^T H \mathbf{z} &= (w^Tw + x^Tx)(x^TQx) \\ x^T M_1 \mathbf{z} &= x^TQ(f_2(x) + g_1(x)w) \\ \mathbf{z}^T M_2 \mathbf{z} &= 2x^TQ(f_3(x) + g_2(x)w). \end{aligned}$$

Let $\alpha > 0$ be such that

$$D_1 := \begin{bmatrix} -D_0 & -M_1 \\ -M_1^T & \alpha H - M_2 \end{bmatrix} \succeq \epsilon I,$$

and $R := \sqrt{\epsilon/(2\alpha)}$. Define

$$\begin{aligned} s_1(x) &:= \lambda_{max}(P)/\lambda_{min}(Q) \\ s_2(x, w) &:= \alpha(x^Tx + w^Tw). \end{aligned}$$

Then, $V - l_1$, s_1, s_2, and $(\beta - p) - (R^2 - V)s_1$ are SOS. Consider

$$b(x, w) := -\nabla V f(x, w) + w^Tw - \alpha(x^Tx + w^Tw)\left(R^2 - V\right),$$

which can be decomposed as

$$b(x, w) = [x; w; \mathbf{z}]^T D_2 [x; w; \mathbf{z}],$$

where

$$D_2 := D_1 - \begin{bmatrix} \alpha R^2 I & 0 \\ 0 & 0 \end{bmatrix} \succeq \epsilon I - \begin{bmatrix} \alpha R^2 I & 0 \\ 0 & 0 \end{bmatrix} \succeq \frac{\epsilon}{2} I.$$

Hence, b is SOS. □

A. Extensions of the reachability analysis for systems with degenerate linearization

Consider the system

$$\begin{aligned}\dot{x}(t) &= A_m x(t) + B\Lambda \hat{K}_x^T(t)x(t) + Ew(t) \\ \dot{\hat{K}}_x &= -\Gamma_x x(t)x^T(t)PB,\end{aligned} \quad (19)$$

where $x(t) \in \mathcal{R}^n$, $B \in \mathcal{R}^{n\times m}$, $w(t) \in \mathcal{R}^{n\times n_w}$, and P, E, Λ, A_m, and Γ_x are matrices of appropriate dimension with Hurwitz A_m. The dynamics in (19) can be considered as the closed loop dynamics for the system $\dot{x}(t) = A_m x(t) + B\Lambda u(t)$ regulated to the origin by a model reference adaptive controller of the form[23]

$$u(t) = \hat{K}_x(t)x(t)$$

in the presence of the disturbance w. Note that the results in Proposition IV.2 is not applicable to the system in (19) because its linearization at the origin is not asymptotically stable. Nevertheless, the nonlinear reachability analysis as outlined in Proposition IV.1 is still applicable.

Proposition IV.3. *Let $x_1 \in \mathcal{R}^{n_1}$, $x_2 \in \mathcal{R}^{n_2}$, and $w \in \mathcal{R}^{n_w}$ and consider*

$$\begin{aligned}\dot{x}_1(t) &= Ax_1(t) + b(x_1(t), x_2(t)) + Ew(t) \\ \dot{x}_2(t) &= q(x_1(t))\end{aligned} \quad (20)$$

where $b : \mathcal{R}^{n_1+n_2} \rightarrow \mathcal{R}^{n_1}$ whose entries are bilinear polynomials in x_1 and x_2, $q : \mathcal{R}^{n_1} \rightarrow \mathcal{R}^{n_2}$ whose entries are quadratic polynomials in x_1, and E and A are real matrices of appropriate dimension such that there exist $Q_1 = Q_1^T \succ 0$ and $\epsilon > 0$ with

$$\begin{bmatrix} A^TQ_1 + Q_1A & Q_1E \\ E^TQ_1 & -I \end{bmatrix} \preceq -\epsilon I.$$

. Then, there exist positive definite $V \in \mathbb{R}[(x_1, x_2)]$, $s \in \Sigma[x_1]$, and $R > 0$ such that $b_m \in \Sigma[(x_1, x_2, w)]$ where

$$b_m(x_1, x_2, w) := -\left[\nabla V f(x_1, x_2, w) - w^Tw + (R^2 - V)s\right].$$

◁

Proof: Let $V(x) := x_1^TQ_1x_1 + x_2^TQ_2x_2$, where $Q_2 = Q_2^T \succ 0$. Then, there exist B_1, B_2, $H_1 \succ 0$, and $H_2 \succ 0$ such that

$$\begin{aligned} x_1^TQ_1b(x_1, x_2) &= x_1^TB_1\mathbf{z}(x_1, x_2) \\ x_2^TQ_2q(x_1) &= x_1^TB_2\mathbf{z}(x_1, x_2) \\ x_1^TQ_1x_1x_1^Tx_1 &= \mathbf{z}(x_1, x_1)^TM_1\mathbf{z}(x_1, x_1) \\ x_2^TQ_2x_2x_1^Tx_1 &= \mathbf{z}(x_1, x_2)^TM_2\mathbf{z}(x_1, x_2)\end{aligned}$$

and $-b_m$ can be decomposed as

$$b_m = \begin{bmatrix} x_1 \\ w \\ \mathbf{z}(x_1, x_1) \\ \mathbf{z}(x_1, x_2)\end{bmatrix}^T D \begin{bmatrix} x_1 \\ w \\ \mathbf{z}(x_1, x_1) \\ \mathbf{z}(x_1, x_2)\end{bmatrix},$$

where D is

$$\begin{bmatrix} A^TQ_1 + Q_1A + \alpha R^2 I & Q_1E & 0 & B_1 + B_2 \\ E^TQ_1 & -I & 0 & 0 \\ 0 & 0 & -\alpha M_1 & 0 \\ B_1^T + B_2^T & 0 & 0 & -\alpha M_2 \end{bmatrix}$$

and D negative semidefinite by proper choice of α (sufficiently large) and R (sufficiently small). Consequently, $b_m \in \Sigma[(x_1, x_2, w)]$. □

V. Region-of-attraction analysis

The material of this section is adapted from [24] where similar results were proven in the context of robust region-of-attraction analysis for systems with parametric uncertainty. For simplicity, we focus on the case without uncertainty. Consider the autonomous nonlinear dynamical system

$$\dot{x}(t) = f(x(t)), \quad (21)$$

where $x(t) \in \mathcal{R}^n$ is the state vector and f is an n-vector with entries in $\mathbb{R}[x]$ satisfying $f(0) = 0$, i.e., the origin is an equilibrium point of (21). Let $\phi(t; \mathrm{x}_0)$ denote the solution to (21) at time t with the initial condition $x(0) = \mathrm{x}_0$. The region-of-attraction of the origin for the system (21) is

$$\left\{\mathrm{x}_0 \in \mathcal{R}^n : \lim_{t\to\infty} \phi(t; \mathrm{x}_0) = 0\right\}.$$

A modification of a similar result in [2] provides a characterization of invariant subsets of the ROA in terms of sublevel sets of appropriately chosen Lyapunov functions.

Lemma V.1. *Let $\gamma \in \mathcal{R}$ be positive. If there exists a continuously differentiable function $V : \mathcal{R}^n \rightarrow \mathcal{R}$ such that*

$$\Omega_{V,\gamma} \text{ is bounded, and} \quad (22)$$

$$V(0) = 0 \text{ and } V(x) > 0 \text{ for all } x \in \mathcal{R}^n \quad (23)$$

$$\Omega_{V,\gamma}\backslash\{0\} \subset \{x \in \mathcal{R}^n : \nabla V(x)f(x) < 0\}, \quad (24)$$

then $\Omega_{V,\gamma}$ is an invariant subset of the ROA. ◁

In order to enlarge the computed invariant subset of the ROA, we define a variable sized region $\Omega_{p,\beta} = \{x \in \mathcal{R}^n : p(x) \leq \beta\}$, where $p \in \mathbb{R}[x]$ is a fixed positive definite convex polynomial, and maximize β while imposing the constraint $\Omega_{p,\beta} \subseteq \Omega_{V,\gamma}$ along with the constraints (22)-(24).

SOS programming and simple generalizations of the S-procedure (namely Lemmas II.1 and II.2) provide algebraic sufficient conditions for the constraints in Lemma V.1. Specifically, let l_1 and l_2 be a positive definite polynomials. Then, since l_1 is radially unbounded, the constraint

$$V - l_1 \in \Sigma[x] \quad (25)$$

and $V(0) = 0$ are sufficient conditions for (22) and (23). By Lemma II.1, if $s_1 \in \Sigma[x]$, then

$$-\left[(\beta - p)s_1 + (V - \gamma)\right] \in \Sigma[x] \quad (26)$$

implies the set containment $\Omega_{p,\beta} \subseteq \Omega_{V,\gamma}$, and by Lemma

II.2, if $s_2, s_3 \in \Sigma[x]$, then

$$-[(\gamma - V)s_2 + \nabla V f s_3 + l_2] \in \Sigma[x] \qquad (27)$$

is a sufficient condition for (24). Consequently, $\Omega_{p,\beta^*_{ROA}}$ is a subset of the ROA and Ω_{V^*,γ^*} is an invariant subset of the ROA, where

$$\beta^*_{ROA} := \begin{array}{l} \max\limits_{V\in\mathcal{V},\beta,s_i\in\mathcal{S}_i} \beta \text{ subject to (25)} - (27), \\ V(0)=0, s_i \in \Sigma[x], \beta > 0 \end{array} \qquad (28)$$

and V^* and γ^* are optimal values of V and γ in (28). Here, the sets $\mathcal{V}$ and $\mathcal{S}_i$ are prescribed finite-dimensional subspaces of polynomials.

We now focus on systems governed by ordinary differential equations of the form

$$\dot{x} = f(x) = Ax + f_2(x) + f_3(x), \qquad (29)$$

where f_2 and f_3 are vectors of (purely) quadratic and cubic polynomials, respectively, and $A \in \mathcal{R}^{n\times n}$, and prove that asymptotic stability of the linearized dynamics is also sufficient for the feasibility of the constraints in (28) (for sufficiently small $\gamma > 0$).

Proposition V.1. *Let f be an n-vector of cubic polynomials in x satisfying $f(0)=0$, and let $P \succ 0$, $R_1 \succ 0$, $R_2 \succ 0$,*

$$p(x) := x^TPx, \quad l_1(x) := x^TR_1x, \quad l_2(x) := x^TR_2x.$$

If there exists $Q \succ 0$ such that

$$A^TQ + QA \prec 0,$$

then the constraints in (28) are feasible for some $R > 0$. ◁

Proof: The proof is constructive. Let $\mathbf{z} = \mathbf{z}(x)$ be as defined in Lemma II.4, $\tilde{Q} \succ 0$ satisfy $A^T\tilde{Q} + \tilde{Q}A \preceq -2R_2$ and $\tilde{Q} \succeq R_1$ (such $\tilde{Q}$ can be obtained by properly scaling Q). Let

$$\epsilon := \lambda_{min}(R_2), \quad V(x) := x^T\tilde{Q}x,$$

and $H \succ 0$ be such that $(x^Tx)V(x) = \mathbf{z}^TH\mathbf{z}$ (which exists by Lemma II.4). Let $M_2 \in \mathcal{R}^{n\times n_z}$ and symmetric $M_3 \in \mathcal{R}^{n_z\times n_z}$ satisfy

$$\begin{aligned} \nabla V f_2(x) &= x^TM_2\mathbf{z} \\ \nabla V f_3(x) &= \mathbf{z}^TM_3\mathbf{z}. \end{aligned}$$

Define

$$\begin{aligned} s_1(x) &:= \tfrac{\lambda_{max}(\tilde{Q})}{\lambda_{min}(P)} \\ c_2 &:= \tfrac{\lambda_{max}(M_3^+ + \frac{1}{2\epsilon}M_2^TM_2)}{\lambda_{min}(H)} \\ s_2(x) &:= c_2x^Tx \\ \gamma &:= \tfrac{\epsilon}{2c_2} \\ \beta &:= \tfrac{\gamma}{2s_1} \\ s_3(x) &:= 1, \end{aligned}$$

where for a symmetric matrix M, M^+ denotes its projection into the positive semidefinite cone. Clearly, $s_1 \in \Sigma[x]$, $s_2 \in \Sigma[x]$, and $s_3 \in \Sigma[x]$. Note that

$$V(x) - l_1(x) = x^T(\tilde{Q} - R_1)x \in \Sigma[x],$$

since $\tilde{Q} - R_1 \succeq 0$.

$$\begin{aligned} b_1(x) &:= -[(\gamma - V)s_2 + \nabla V f s_3 + l_2] \\ &= \begin{bmatrix} x \\ \mathbf{z} \end{bmatrix}^T B_1 \begin{bmatrix} x \\ \mathbf{z} \end{bmatrix}, \end{aligned}$$

where

$$B_1 := \begin{bmatrix} -\gamma c_2I - R_2 - (A^T\tilde{Q} + \tilde{Q}A) & -M_2/2 \\ -M_2^T/2 & c_2H - M_3 \end{bmatrix}$$

and

$$B_1 \succeq \begin{bmatrix} \frac{\epsilon}{2}I & -M_2/2 \\ -M_2^T/2 & c_2H - M_3 \end{bmatrix} \succeq 0$$

by the Schur's complement formula. Consequently, $b_1(x) \in \Sigma[x]$. Finally,

$$\begin{aligned} &-[(\beta - p)s_1 + (V - \gamma)] \\ &\qquad = \begin{bmatrix} 1 \\ x \end{bmatrix}^T \underbrace{\begin{bmatrix} -\beta s_1 + \gamma & 0 \\ 0 & s_1P - \tilde{Q} \end{bmatrix}}_{B_2} \begin{bmatrix} 1 \\ x \end{bmatrix}, \end{aligned} \qquad (30)$$

where $B_2 \succeq 0$ and consequently $b_2 \in \Sigma[x]$. □

VI. Interpretation and Demonstration of Results

It is worth re-stating that the motivation here is theoretical rather than practical. The conclusions can be summarized as that the nonlinear local analysis (based on S-procedure and SOS programming relaxations as proposed in [5], [20], [7]) is always capable of returning a feasible result (i.e., corresponding optimization problems are feasible) whenever corresponding conditions for the linearized dynamics are feasible. Alternatively, these results may also have some limited practical value in constructing (possibly conservative) quantitative results for the nonlinear system. For example, Propositions V.1, III.2, and IV.2 can be directly used to construct feasible solutions for the problems in Eq. (28) and Propositions III.1 and IV.1, respectively. Proofs of Propositions V.1, III.2, and IV.2 also suggest a recipe for constructing less conservative feasible solutions for these problems by searching for an "optimal" quadratic Lyapunov function (instead of fixing V to a Lyapunov function for the linearization). A construction in the case of region-of-attraction analysis can be summarized as follows: Choose the multipliers s_1, s_2, and s_3 in the form given in the proof of Proposition V.1 with the free parameter c_2. Affinely parameterize H, M_2, and M_3 in terms of Q (note that there may be multiple possible parameterizations for M_2 and M_3 and the choice may change the quantitative results - here we arbitrarily choose one parametrization). Then, Ω_{p,β^*} is a subset of the ROA where

$$\begin{gathered} \beta^* := \max_{\gamma,c_2,\beta,Q=Q^T\succeq R_1} \beta \quad \text{subject to} \\ \begin{bmatrix} -\gamma c_2I - R_2 - A^TQ - QA & -M_2(Q)/2 \\ -M_2(Q)^T/2 & c_2H(Q) - M_3(Q) \end{bmatrix} \succeq 0 \\ \begin{bmatrix} -\beta + \gamma & 0 \\ 0 & P - Q \end{bmatrix} \succeq 0. \end{gathered} \qquad (31)$$

Note that the above problem can be solved through a series of convex SDP problems by a line search on c_2. Construction of feasible solutions for the problems in Propositions III.1 and IV.1 can be developed in a similar manner.

The value of such "suboptimal" construction of feasible solutions for the problems in the context of nonlinear system analysis may be better appreciated by recalling the fact that one of the main difficulties in SOS programming based nonlinear system analysis is the computational complexity of the SOS programming. The procedure outlined above provides an ad hoc way of generating (possibly high quality) solutions for the corresponding optimization problems or initial seeds for further optimization. The following example demonstrates this construction for ROA analysis and compares the results with "optimal"solutions from (28).

Example VI.1. *Consider the Van der Pol dynamics*

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2, \end{aligned}$$

which have a stable equilibrium point at the origin and an unstable limit cycle around the origin which is the boundary of the ROA of the equilibrium point. In this example, we will construct invariant subsets of the ROA using the problem in Eq. (28) and Proposition V.1. Let $x = [x_1\ x_2]^T$, $p(x) = x^T x$, $l_1(x) = l_2(x) = 10^{-6} x^T x$. *The solution of the problem in Eq. (28) with a quadratic Lyapunov function candidate, (purely) quadratic* s_2, *and scalar* s_1 *and* s_3 *certifies* $\Omega_{p,1.57}$ *to be a subset of the ROA. The feasible solution provided in Proposition V.1 certifies* $\Omega_{p,0.20}$ *to be a subset of the ROA. Alternatively, by the procedure outlined above certifies that* $\Omega_{p,0.65}$ *is in the ROA.* ◁

VII. CONCLUSIONS

Sum-of-squares programming based analysis of nonlinear systems with polynomial vector fields may be regarded superior to analysis based on linearized dynamics in the sense that the former is capable of generating quantitative certificates as opposed to conclusions from the latter valid only over infinitesimally small neighborhoods of the equilibrium points. However, sum-of-squares based approach involves multiple relaxations. Therefore, it is not obvious if the sum-of-squares programming based nonlinear analysis can return feasible answers whenever linearization based analysis does. In this paper, we proved that, for a restricted but practically useful class of systems, conditions in sum-of-squares programming based region-of-attraction, reachability, and input-output gain analyses are feasible whenever linearization based analysis is conclusive. Besides the theoretical interest, such results may lead to computationally less demanding, potentially more conservative nonlinear (compared to direct use of sum-of-squares programming formulations) analysis tools.

Acknowledgements

This work was sponsored by the Air Force Office of Scientific Research, USAF, under grant/contract number FA9550-05-1-0266. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the AFOSR or the U.S. Government.

U. Topcu acknowledges partial support from the Boeing Corporation. A. Packard acknowledges partial support from NASA under the contract NNX08AC80A.

REFERENCES

[1] J. C. Willems, "Dissipative dynamical systems 1: Genaral theory," *Archive for Rational Mechanics and Analysis*, vol. 45, pp. 321–343, 1972.

[2] M. Vidyasagar, *Nonlinear Systems Analysis*, 2^{nd} ed. Prentice Hall, 1993.

[3] H. K. Khalil, *Nonlinear Systems*, 3^{rd} ed. Prentice Hall, 2002.

[4] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*. Philadelphia: SIAM, 1994.

[5] Z. Jarvis-Wloszek, R. Feeley, W. Tan, K. Sun, and A. Packard, "Control applications of sum of squares programming," in *Positive Polynomials in Control*, D. Henrion and A. Garulli, Eds. Springer-Verlag, 2005, pp. 3–22.

[6] P. Parrilo, "Semidefinite programming relaxations for semialgebraic problems," *Mathematical Programming Series B*, vol. 96, no. 2, pp. 293–320, 2003.

[7] U. Topcu and A. Packard, "Local stability analysis for uncertain nonlinear systems," 2007, to appear in IEEE Transaction on Automatic Control.

[8] T.-C. Wang, S. Lall, and M. West, "Polynomial level-set methods for nonlinear dynamical systems analysis," in *Proc. Allerton Conf. on Communication, Control, and Computing*, Allerton, IL, 2005.

[9] J. Lasserre, "Global optimization with polynomials and the problem of moments," *SIAM Journal on Optimization*, vol. 11, no. 3, pp. 796–817, 2000.

[10] E. M. Aylward, P. A. Parrilo, and J. E. Slotine, "Stability and robustness analysis of nonlinear systems via contraction metrics and SOS programming," 2008, doi:10.1016/j.automatica.2007.12.012.

[11] A. Papachristodoulou and S. Prajna, "On the construction of Lyapunov functions using the sum of squares decomposition," in *Proceedings of the IEEE Conference on Decision and Control*, 2002, pp. 3482–3487.

[12] P. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization," Ph.D. Dissertation, California Institute of Technology, May 2000.

[13] S. Prajna, "Barrier certificates for nonlinear model validation," vol. 3, 2003, pp. 2884–2889.

[14] U. Topcu, A. Packard, and P. Seiler, "Local stability analysis using simulations and sum-of-squares programming," DOI: 10.1016/j.automatica.2008.03.010.

[15] W. Tan, "Nonlinear control analysis and synthesis using sum-of-squares programming," Ph.D. Dissertation, UC, Berkeley, May 2006.

[16] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, "LMI-based computation of optimal quadratic Lyapunov functions for odd polynomial systems," *Int. J. Robust Nonlinear Control*, vol. 15, pp. 35–49, 2005.

[17] G. Chesi, "On the estimation of the domain of attraction for uncertain polynomial systems via lmis," in *Proc. Conf. on Decision and Control*, Bahamas, 2004, pp. 881–886.

[18] C. Ebenbauer and F. Allgower, "A dissipation inequality for the minimum phase property," *IEEE Transactions on Automatic Control*, vol. 53, pp. 821–826, 2008.

[19] V. A. Yakubovich, "The S-procedure in non-linear control theory," *Vestnik Leningrad Univ. Math.*, vol. 4, pp. 73–93, 1977.

[20] W. Tan and A. Packard, "Stability region analysis using sum of squares programming," in *Proc. American Control Conf.*, Minneapolis, MN, 2006, pp. 2297–2302.

[21] W. Tan, U. Topcu, P. Seiler, G. Balas, and A. Packard, "Simulation-aided reachability and local gain analysis for nonlinear dynamical systems," in *Proc. Conf. on Decision and Control*, Cancun, Mexico, 2008.

[22] W. Tan, A. Packard, and T. Wheeler, "Local gain analysis of nonlinear systems," in *Proc. American Control Conf.*, Minneapolis, MN, 2006, pp. 92–96.

[23] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Prentice Hall, Inc, 1996.

[24] U. Topcu, A. Packard, P. Seiler, and G. Balas, "Robust region-of-attraction estimation," 2008, submitted to IEEE Transaction on Automatica Control.