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A MID-AIR COLLISION THREAT ALGORITHM THAT USES BEARING DATA

Irvin W. Kay

November 1976

INSTITUTE FOR DEFENSE ANALYSES
SCIENCE AND TECHNOLOGY DIVISION

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ABSTRACT

This paper derives an algorithm for use by an airborne mid-air collision avoidance system to determine when an alarm should be given in case a mid-air collision is imminent. The algorithm is based on an extension of the standard modified tau alarm criterion used in most collision avoidance system threat logics.

The standard criterion uses only altitude and range data and, as a result, will generate high alarm rates in heavy air traffic. The criterion presented here makes use of bearing data as well as altitude and range data and should, therefore, provide lower alarm rates.

CONTENTS

I. INTRODUCTION

The objective of this paper is to construct a new algorithm for deciding when the danger of collision between two aircraft is great enough to warrant an avoidance maneuver. In making such a decision a collision avoidance system (CAS) in one of the aircraft would measure the other's relative range, altitude, and bearing* to which data it would then apply the algorithm. If the result were consistent with the possibility of an imminent collision the CAS would generate an alarm to alert the pilot.

The scope of the work reported here has been limited deliberately to the mathematical development and justification of the collision threat algorithm. No attempt has been made to assess its potential usefulness in practice, e.g., for reducing the number of false alarms relative to the number generated by other threat criteria which have been implemented or suggested. Hopefully, this can be done in a future study.

ANTC-117 (Ref. 1), the document containing the CAS performance standards recommended by the Air Transport Association, specifies alarm criteria which do not involve bearing data. These criteria are of two types: one is concerned entirely with altitude and altitude rate while the other is concerned entirely with range and range rate. The second is based on the so called modified tau condition which, Ref. 2 has shown,

*The relative range, altitude and bearing rates are also needed, but these quantities can be calculated from measurements of the corresponding static quantities at two different times separated by an accurately determined time interval.

provides the most efficient* possible alarm criterion using range and range rate data alone.

However, this result of Ref. 2 is valid only when the fixed parameters of the modified tau condition are properly selected. Some IDA studies (Ref. 3 and Ref. 4) have applied the analysis of Ref. 2 to show that the fixed parameters selected for the alarm criteria specified in ANTC-117 have values which will not guarantee an alarm for every possible threat. In addition, the studies concluded that in high density traffic such as projected by the FAA for the Los Angeles Basin in 1982 use of the modified tau condition, even with the less than adequate ANTC-117 parameters, would result in an excessive number of alarms, many of which would be unnecessary. Moreover, if the correct modified tau parameter values in accordance with the theory of Ref. 2 were used the expected number of alarms would be even greater.

A CAS can reduce the number of alarms it must give by reducing its uncertainty in estimating the location and movement of intruding aircraft. One way it can do this is to use more information about the position and motion of each intruder, e.g., by adding bearing and bearing rate to the data now required for the ANTC-117 threat determination logic. A CAS, referred to as LCAS, recently proposed by George Litchford, provides bearing data along with the usual range and altitude.

In principle, the new threat determination algorithm derived here makes the most efficient possible use of the range,

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*By "efficient," as the term is used here, it is meant that an alarm is given if and only if, in view of the data available to the CAS, a collision is possible unless an avoidance maneuver begins before the advent of the next measurement cycle, which is sometimes referred to as an "epoch."

altitude and bearing data assumed to be available.* Hopefully, this will result in significantly fewer unnecessary alarms when implemented by a CAS than would be the case if the modified tau condition were used instead.

The data provided by LCAS are sufficient to locate and track an intruding aircraft with a precision that is limited only by measurement errors. However, the intruder, whose intentions are assumed to be unknown, may deviate at any time from linear flight, accelerating in the process, usually by turning, as much as 1/2 g. The threat determination algorithm must recognize a hazardous situation, allowing for measurement errors and the possibility of such an unknown acceleration, in sufficient time to permit the successful completion of an avoidance maneuver.

Thus, even when bearing data are available and the most efficient possible threat analysis is used there will still exist enough uncertainty concerning the intruder's flight path to cause an alarm on some occasions when a collision would not have occurred had the alarm been ignored. No attempt has been made here to estimate how often this will happen nor whether, in fact, the lower alarm rate generated by the new algorithm will not still be too high for satisfactory CAS performance in heavy traffic.

Section II discusses a mathematical condition for the occurrence of a mid-air collision at a given instant of time when the relative acceleration between the encountering aircraft is bounded by a known constant. Section III establishes an efficient alarm criterion based on that condition and an assumed

*This is true if no measurement errors are assumed to exist. When measurement errors are taken into account in Section V it is expedient to sacrifice some efficiency in order to avoid mathematical complications; i.e., some alarms will be given which theoretically might be avoided by more effective use of the available data.

minimum escape time, i.e., the time required for a CAS to generate the alarm and the aircraft to maneuver to safety.

A threat algorithm, which does not take into account measurement errors, will be derived (with the aid of the appendices) and presented in Section IV. In Section V, a revised threat algorithm will be given that does include the effect of measurement errors.

II. CONDITION FOR A MID-AIR COLLISION

In CAS threat analyses (e.g., Ref. 2) it is customary to base the concept of a hazard upon the possibility of a collision between two aircraft assuming that both obey regulations which limit an aircraft's speed and acceleration. The hazard may exist if the collision can occur for an allowable relative acceleration although, since it is assumed that at least one pilot's intentions are unknown, the collision is not actually predicted.

A CAS officially recognizes a hazardous situation requiring an alarm when it determines that after its next observation such a collision could occur before the maximum estimated time (at some confidence level) needed for maneuvering to safety has elapsed. In this context "maneuvering to safety" means adopting a flight path (usually in a vertical plane) which guarantees that the encountering aircraft remain separated by at least 150 ft.*

Following Ref. 2, the condition for a collision can be stated mathematically in terms of the initial position vector $\underset{\sim}{R}$ of the intruder aircraft relative to the CAS, the initial relative velocity vector $\dot{\underset{\sim}{R}}$, and the relative acceleration vector $\underset{\sim}{A}(t)$. The acceleration $\underset{\sim}{A}(t)$ can vary as a function of time t but, by assumption, remains bounded in magnitude. The bound is a predetermined constant U; i.e.,

$$\left|\underset{\sim}{A}(t)\right| \leq U$$

*Cf. Ref. 1.

for all t. The condition for collision before escape is possible is then

$$\underset{\sim}{R} + t\,\dot{\underset{\sim}{R}} + \int_0^t \int_0^s \underset{\sim}{A}\,(\tau)\; d\tau ds = 0 \tag{1}$$

for some t in the interval

$$0 \le t < t_e,$$

where t_e is the estimated time required for escape.*

In Ref. 2 it was shown that whenever there exists an acceleration $\underset{\sim}{A}(t)$ bounded in magnitude by a constant U, i.e.,

$$\left|\underset{\sim}{A}(t)\right| \le U,$$

for which the collision condition (1) holds, then there exists a constant vector $\underset{\sim}{A}$ also bounded in magnitude by U, i.e.,

$$\left|\underset{\sim}{A}\right| \le U,$$

such that (1) holds for some time t' in the interval

$$0 \le t' \le t_e$$

with $\underset{\sim}{A}(t)$ replaced by $\underset{\sim}{A}$; i.e., (1) takes the form

$$\underset{\sim}{R} + t'\,\dot{\underset{\sim}{R}} + 1/2t'^2\,\underset{\sim}{A} = 0. \tag{2}$$

Therefore, without loss of generality (2), which is more explicit than (1), can be used as the collision condition for CAS threat analyses.

*Not only the time required for any necessary avoidance maneuvers but also all delays such as those due to pilot and aircraft reactions, as well as an allowance of one full epoch for updating, are included in t_e.

III. THE CAS ALARM CRITERION

In order to specify a CAS alarm criterion suitable for translation into an algorithm, following Ref. 2 an attempt can be made to replace the vector collision condition (2) with a scalar condition which does not contain the unknown equivalent acceleration vector $\underset{\sim}{A}$. The new condition is*

$$|\underset{\sim}{R} + t_e \dot{\underset{\sim}{R}}|^2 < 1/4\, t_e^4\, U^2. \tag{3}$$

Indeed, it can be shown that if (3) holds there exists a $\bar{t}$ in the interval

$$0 \leq \bar{t} < t_e$$

and a constant vector $\underset{\sim}{A}$, such that

$$\left|\underset{\sim}{A}\right| \leq U,$$

for which the collision condition (2) is satisfied, i.e., for which

$$\underset{\sim}{R} + \bar{t}\,\dot{\underset{\sim}{R}} + \frac{\bar{t}^2}{2}\,\underset{\sim}{A} = o. \tag{4}$$

The proof is not difficult. Let b(t) be defined by

$$b(t) = \frac{t^4 U^2}{4} - \left|\underset{\sim}{R} + t\,\dot{\underset{\sim}{R}}\right|^2. \tag{5}$$

*As Ref. 4 has observed, a condition such as (3) is technically not the most efficient one possible, since it does not include the effect of limiting each aircraft's speed, although it may be presumed that a maximum speed will be as well enforced as the maximum acceleration which is taken into account. As Ref. 4 has indicated, however, the inclusion of a speed limit should not greatly reduce the number of expected alarms generated by an otherwise efficient threat algorithm.

Then (3) is equivalent to

$$b(t_e) > o, \tag{6}$$

which, because b(t) is continuous, implies that there is a $\bar{t}$, with

$$0 \leq \bar{t} < t_e,$$

such that

$$b(\bar{t}) > 0.$$

Then if $\underset{\sim}{A}$ is given by

$$\underset{\sim}{A} = -\frac{2}{\bar{t}^2}(\underset{\sim}{R} + \bar{t}\,\dot{\underset{\sim}{R}}),$$

it follows from (5) that

$$\left|\underset{\sim}{A}\right| < U$$

and, by substitution, that (4) is satisfied as stated. Thus, if, for a given U and t_e, for some pair of vectors $\underset{\sim}{R}$ and $\dot{\underset{\sim}{R}}$ (3) is satisfied, then a hazard as defined by (2) exists. However, the converse is not necessarily true. That is, if condition (2) holds, so that a hazard exists, it does not necessarily follow that (3) holds. Therefore, if (3) were used alone as an algorithm to determine the existence of a threat, in some cases it would fail to do so.

In order to simplify the condition and remedy this defect Ref. 2 replaced (3) with the modified tau criterion

$$R + \dot{R}\,t_e < 1/2\;U\,t_e^2\;. \tag{7}$$

This condition involves the scalar range R, which is the magnitude of the relative position vector $\underset{\sim}{R}$, and the scalar range rate $\dot{R}$, which is the component of the relative velocity vector $\dot{\underset{\sim}{R}}$ along $\underset{\sim}{R}$.

The modified tau condition (7) holds whenever (2) holds and therefore always predicts a hazard when one exists. In addition, Ref. 2 showed that when (7) is violated a hazard as defined by (2) does not exist. A proof of this fact is also sketched here in Appendix B.

However, a hazard does not necessarily exist when (7) is satisfied. For this reason that condition is expected to predict too many false threats in heavy traffic (cf. Ref. 3), although Ref. 2 showed that (7) is the most efficient threat determination criterion possible, given only a knowledge of R and $\dot{R}$.

If the complete vectors $\underset{\sim}{R}$ and $\dot{\underset{\sim}{R}}$ are known it should be possible to replace (7) with a more efficient algorithm. A candidate for this role is (3); however, it has already been observed that (3) by itself is not adequate because it fails to detect a threat in some cases. This defect can be remedied by introducing some conditions to be used in addition to (3). It is then possible to formulate a satisfactory algorithm which detects all true and no false threats as they have been defined here.

To this end it is useful to consider, first, a class of cases for which a violation of (3) does imply that no hazard exists. Ref. 2 proved that this is always true for the modified τ criterion (7) (cf. also Appendix B here), and that proof depends on the fact that the equation

$$1/2Ut^2 - \dot{R}t - R = 0$$

always has exactly one positive root. If the parameters which determine the function b(t) defined by (5) are such that the equation

$$b(t) = 0 \tag{8}$$

has exactly one positive root, the same conclusion would follow for the criterion (3). That is, if a collision could occur at a time $\bar{t}$ in the interval

$$0 \leq \bar{t} < t_e,$$

i.e.,

$$b(\bar{t}) = 0,$$

then, because b(t) becomes arbitrarily large as t becomes large, if b(t) cannot become zero again for t positive it follows that

$$b(t_e) > 0$$

as required.

In summary, if (3) is satisfied, then an alarm should be given, and if (7) is violated, i.e., if

$$R + \dot{R}t_e \geq 1/2\ Ut_e^2\ , \tag{9}$$

no alarm should be given. It remains to investigate what happens when (3) is violated and (7) is satisfied, and the answer depends upon the positive roots of equation (8).

In Appendix B, a condition which determines whether (8) has exactly one positive root is derived. In the alternative case, in which it has more than one positive root, special conditions to determine whether one of those roots does, in fact, lie between 0 and t_e are also derived. Together with (3) and (9) these additional conditions comprise an algorithm which satisfies the requirements of a CAS alarm criterion.

Given a knowledge of the vectors $\underset{\sim}{R}$ and $\underset{\sim}{\dot{R}}$ and the fact that the relative acceleration between the encountering aircraft is bounded by a specified constant U, but ignoring any speed limit and measurement errors, this algorithm provides the most efficient alarm criterion possible. In the next section it is given explicitly in terms of the slant range R from the CAS to the intruder, the range rate $\dot{R}$, the bearing rate $\dot{\phi}$, the relative altitude z, and the altitude rate $\dot{z}$, all of which are assumed to be quantities known through measurement.

IV. THE THREAT ALGORITHM

The following parameters are assumed to be known through measurement:

R, slant range;
$\dot{R}$, range rate;
ϕ, bearing;
$\dot{\phi}$, bearing rate;
z, relative altitude (altitude difference);
$\dot{z}$, altitude range

These quantities are also defined in Appendix A in terms of the coordinates of an intruder aircraft relative to a CAS located at the origin of the appropriate coordinate system. In addition, it is assumed that there is a known bound U on the magnitude of the relative acceleration between encountering aircraft and a known time t_e, the escape time, required for an aircraft to maneuver to safety after it receives an alarm.

A polynomial f(t) in the time variable t with coefficients determined by the measured parameters and the quantities U and t_e is defined by setting

$$b(t) = \frac{U^2}{4} f(t),$$

so that, in accordance with (5), f(t) has the form

$$f(t) = t^4 - a_2 t^2 + a_1 t - a_o, \quad a_2 \geq o, \quad a_o > o.$$

According to Appendix A,

$$a_o = \frac{4R^2}{U^2}, \qquad (10)$$

$$a_1 = -\frac{8R\dot{R}}{U^2},$$

$$a_2 = \frac{4}{U^2}\left[\dot{R}^2 + \dot{\phi}^2\,(R^2 - z^2) + \frac{(\dot{R}z - \dot{z}R)^2}{R^2 - z^2}\right]^*$$

In the absence of measurement errors the algorithm which determines whether a hazard exists and therefore an alarm should be given is derived in Appendix B. The algorithm is defined by the following sequence of tests:

(1) $R + \dot{R}\,t_e < 1/2\ Ut_e^2$?

If <u>no</u> then <u>no alarm</u>;

if <u>yes</u> then (2).

(2) $t_e^4 - a_2\,t_e^2 + a_1\,t_e - a_o > o$?

If <u>yes</u> then <u>alarm</u>;

if <u>no</u> then (3).

(3) $a_1 > o$?

If <u>no</u> then <u>no alarm</u>;

if <u>yes</u> then (4).

(4) $8a_2^{\,3} > 27a_1^{\,2}$?

If <u>no</u> then <u>no alarm</u>;

if <u>yes</u> then (5).

(5) $a_o \leq \frac{1}{12}\,a_2^{\,2}$?

If <u>no</u> then <u>no alarm</u>;

*It will be observed that when the intruder is directly overhead or directly below the CAS, i.e., when $R = |z|$, the parameter a_2 is either ambiguous or infinite. It may also be expected that a_2 will be very sensitive to error when $R \sim |z|$. Therefore, as a practical matter unless R differs from $|z|$ by more than the maximum combined error of the measured parameters R and z the usual modified tau criterion should be used instead of the threat algorithm. This is, in fact, equivalent to letting a_2 become infinite in the threat algorithm.

if yes then (6).

$$\text{(6) for } s_o = \sqrt{\frac{1}{6}\left[1 - \left(1 - \frac{12a_o}{a_2^{\,2}}\right)^{\frac{1}{2}}\right]},$$

$$\frac{a_1}{a_2^{\,3/2}} \geq 2s_o - 4s_o^{\,3}?$$

If no then no alarm;

if yes then (7).

$$\text{(7) } 6t_e^{\,2} > a_2?$$

If yes then alarm;

if no* then (8).

$$\text{(8) } 4t_e^{\,3} - 2a_2\, t_e + a_1 < 0?^{**}$$

If yes then alarm;

if no then no alarm.

Example 1.

As an example of how this algorithm would be applied, suppose that the following CAS parameters are measured for an intruder:

$$\text{range } R = 4000 \text{ ft},$$
$$\text{range rate } \dot{R} = -590 \text{ ft/sec},$$
$$\text{relative altitude } z = 200 \text{ ft}$$
$$\text{relative altitude rate } \dot{z} = -1600 \text{ ft/min} = -26.67 \text{ ft/sec}$$
$$\text{bearing rate } \dot{\phi} = 1^{\circ}/\text{sec} = 0.01745 \text{ rad/sec}$$

If the maximum relative acceleration is assumed to be 1g, then

$$U = 32.2 \text{ ft/sec}^2.$$

---

*$6t_e^{\,2} \neq a_2$ because of steps (4) and (5).

**$4t_e^{\,3} - 2a_2t_e + a_1 \neq 0$ because of steps (6) and (7).

Then from (10) it is found that

$$a_o = 6.1726 \times 10^4 \text{ sec}^4,$$
$$a_1 = 1.8209 \times 10^4 \text{ sec}^3,$$
$$a_2 = 1361.7 \text{ sec}^2.$$

If it is assumed that the escape time t_e is 25 sec, then at step (1) of the algorithm it is found that

$$R + \dot{R}t_e = -10750 \text{ ft} < 10,062 = \tfrac{1}{2}Ut_e^2.$$

Thus, an alarm would be given by the modified tau criterion, but for the threat algorithm of this section it would be necessary at this point to consider step (2). Then it would be found that

$$t_e^4 - a_2 t_e^2 + a_1 t_e - a_o = -6.6938 \times 10^4 \text{ sec}^4 < 0.$$

Thus, it would be necessary to consider step (3):

$$a_1 = 1.8209 \times 10^4 \text{ sec}^3 > o,$$

which leads to step (4). It is found that

$$8a_2^3 = 2.0199 \times 10^{10} > 8.9523 \times 10^9 = 27a_1^2,$$

which leads to step (5). For that test

$$a_o = 6.1726 \times 10^4 < 1.5452 \times 10^5 = \frac{1}{12} a_2^2.$$

Thus, it is necessary to go to step (6). For this purpose s_o must be calculated:

$$s_o = \sqrt{\frac{1}{6}\left[1 - \left(1 - \frac{12a_o}{a_2^2}\right)^{\frac{1}{2}}\right]} = 0.19367$$

Then step (6) gives

$$\frac{a_1}{a_2^{3/2}} = 0.36238 \geq 0.35828 = 2s_o - 4s_o^3.$$

Thus, according to the test at step (6) step (7) must be considered. At step (7) it is found that

$$6t_e^2 = 3750 > 1361.7 = a_2;$$

hence an alarm must be given.

This result can be verified by inspection of Fig. 1 which shows the curve

$$f(t) = t^4 - a_2t^2 + a_1t - a_o.$$

It may be observed that f(t) vanishes at t = 6 sec and again at a little more than 8 sec, both of which represent possible collision times before t_e = 25 sec.

Example 2.

The purpose of this example is to see what happens if the bearing rate is changed but all of the other parameters of example 1 are retained. The modified tau criterion is unaffected by such a change. However, if the bearing rate $\dot{\phi}$ is taken to be 2^o/sec (0.034907 rad/sec), then a_o and a_1 remain the same but now

$$a_2 = 1418 \text{ sec}^2.$$

In this case it will be found that the threat algorithm indicates "no alarm" at step (6).

Indeed, Fig. 1, which contains the curve for f(t) shows that f(t) has a maximum at about 7 sec, is negative between 0 and 25 sec, and has positive curvature at 25 sec. Appendix B demonstrates that these facts should account for the indicated behavior of the threat algorithm. Since, as the curve shows, f(t) does not vanish between 0 and 25 sec there is no threat of collision before the escape time has elapsed.

Example 3.

If instead of the bearing rate $\dot{\phi}$ changing, it is the range which is changed to 3 nmi (18,240 ft) while the other parameters

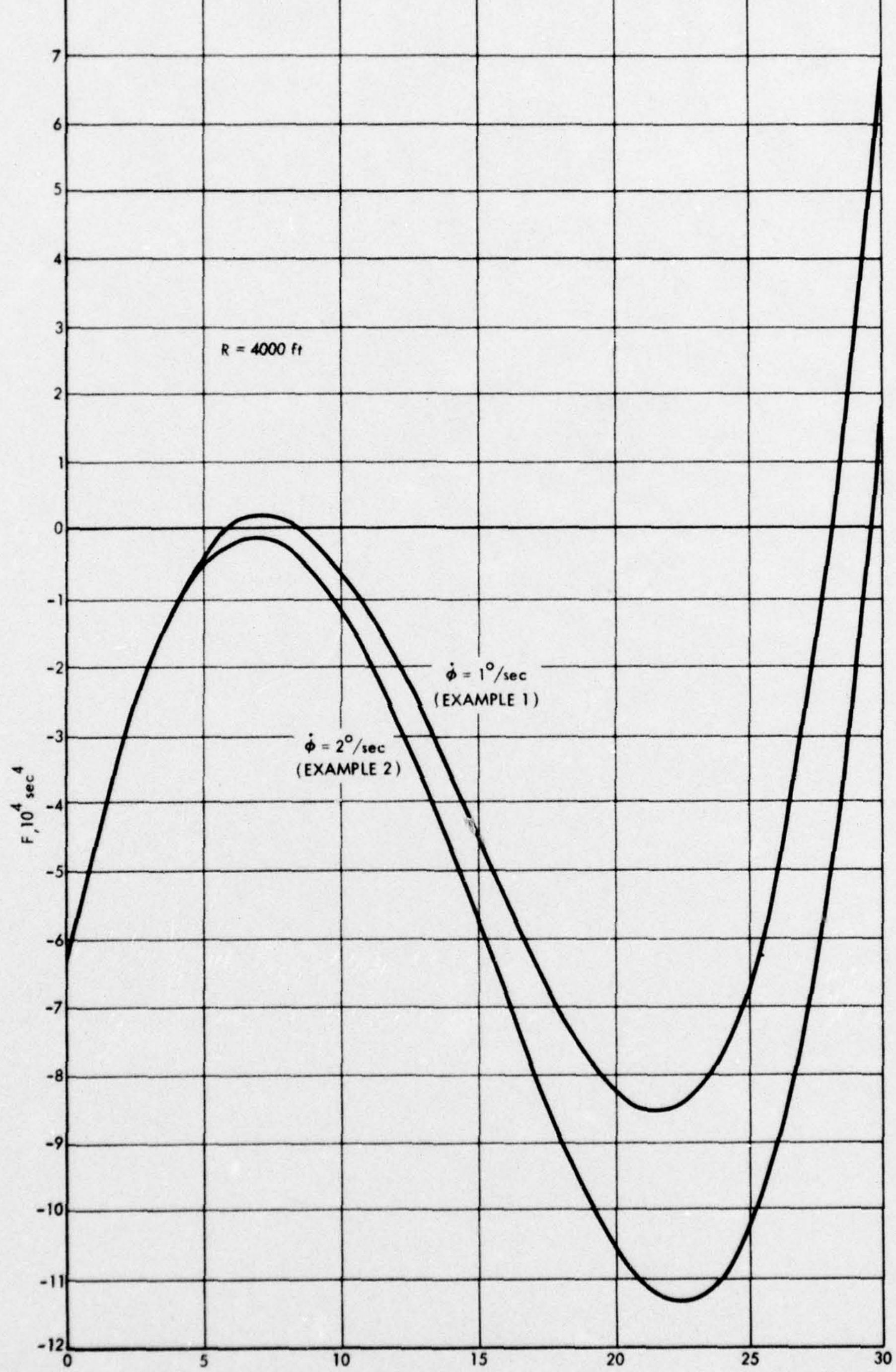

FIGURE 1. Collision Threat Criterion Polynomial F = f(t)

remain the same (e.g., $\dot{\phi}$ stays 1°/sec), the threat algorithm indicates an alarm at step (2). That is, the coefficients obtained from (10) are

$$a_o = 1.2835 \times 10^8 \text{ sec}^4,$$
$$a_1 = 8.3034 \times 10^4 \text{ sec}^3,$$
$$a_2 = 1735.4 \text{ sec}^2,$$

so that

$$f(t_e) = t_e^{\,4} - a_2 t_e^{\,2} + a_1 t_e - a_o = 9.8320 \times 10^4 > 0.$$

Example 4.

On the other hand, if the range R is increased to 4 nmi (24,320 ft) with the other parameters left unchanged, then

$$a_o = 2.2818 \times 10^6 \text{ sec}^4,$$
$$a_1 = 1.1071 \times 10^5 \text{ sec}^3,$$
$$a_2 = 2039.8 \text{ sec}^2,$$

and at step (4) it is found that

$$8 a_2^{\,3} = 6.7897 \times 10^{10} < 3.3093 \times 10^{11} = 27 a_1^{\,2}.$$

Thus, no alarm would be generated for the larger range. Fig. 2 shows the curves for f(t) corresponding to the 3 and 4 nmi ranges. Neither has a maximum (as expected, since both fail the test at step (4)), the 3 nmi curve is zero at a time slightly beyond 23 sec, indicating a collision threat before the escape time of 25 sec, and the 4 nmi curve remains negative between 0 and 25 sec, indicating no threat.

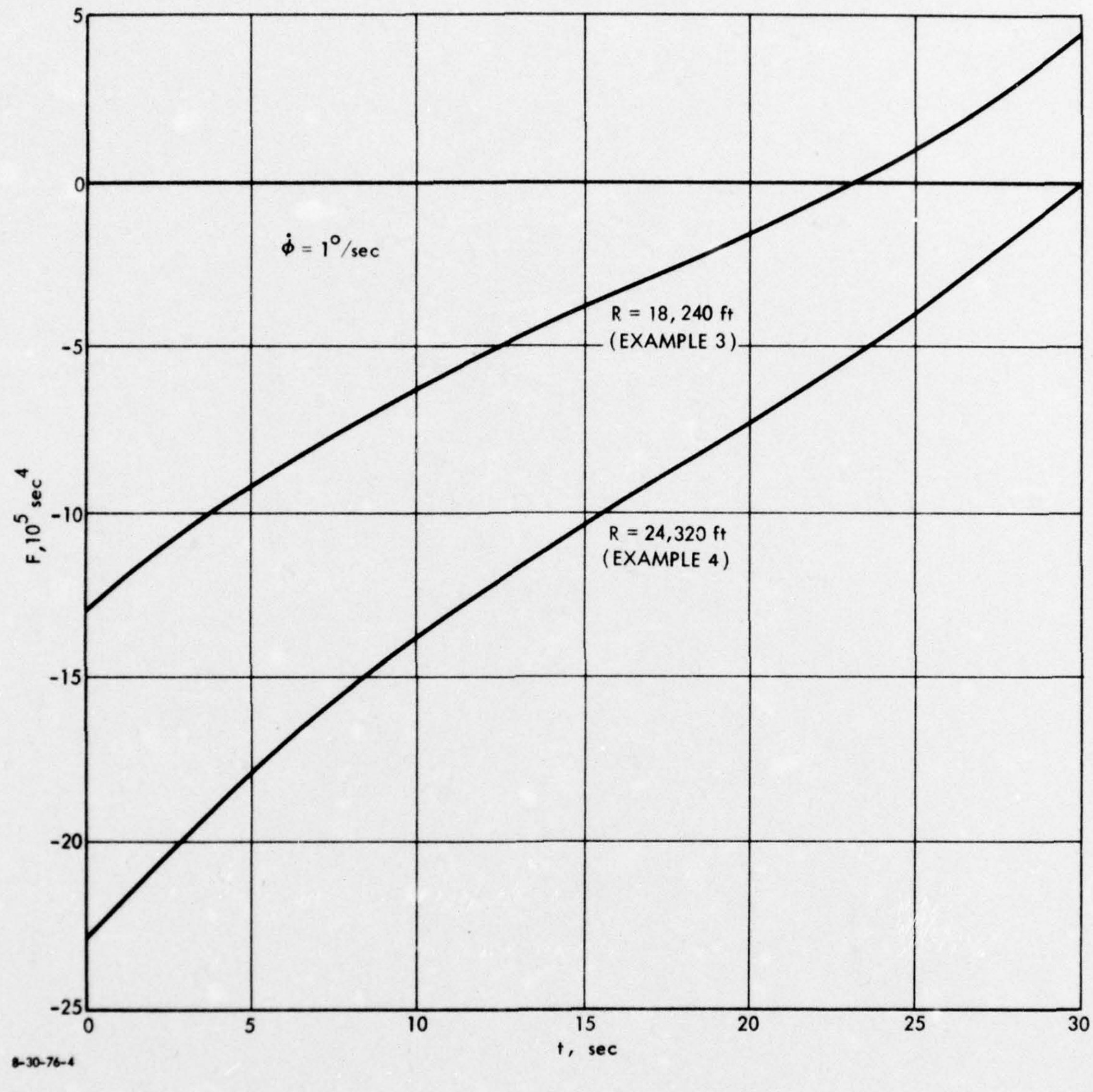

FIGURE 2. Collision Threat Criterion Polynomial
F = f(t)

V. THREAT ALGORITHM WITH MEASUREMENT ERRORS

A. ERROR TOLERANCES AND PARAMETER UPPER AND LOWER BOUNDS

Each of the parameters measured by a CAS will have some specified error tolerance, so that a single measurement will actually define a range of possible values for a parameter rather than a single value. Therefore, it is necessary to replace the algorithm given in Section IV by one which will generate an alarm whenever a measurement and the error tolerances are consistent with any set of independent parameter values that satisfy an alarm condition of the original algorithm.

If the revised algorithm were, indeed, restricted to indicating an alarm only in such a circumstance its efficiency would remain unimpaired, although the number of alarms would necessarily increase because of the contribution of measurement error to the uncertainty in estimating an intruder's position and movement. However, it will be found mathematically convenient to widen the effective tolerance ranges of certain derived quantities, replacing their possible maximum and minimum values with grosser estimates which are no more than upper and lower bounds. This will reduce the algorithm's efficiency somewhat, thereby increasing the number of alarms it will generate. The result is equivalent to an increase in the error sensitivity of the affected quantities.

In revising the threat algorithm of Section IV it will be convenient to use the following notation. When the subscript m is added to the symbol for a measured quantity the new symbol will designate the quantity's minimum value as determined by the specified error tolerance. Similarly, adding the subscript M will provide a symbol for the quantity's maximum. Thus, for example, if a range measurement is R_o and the error tolerance is $\pm\,\varepsilon$, so that the range R would be properly indicated by

$$R = R_o \pm \varepsilon,$$

then

$$R_m = R_o - \varepsilon$$

and

$$R_M = R_o + \varepsilon\ ,$$

where the usual convention that ε is non negative has been assumed.*

On the other hand, for symbols representing quantities which are derived rather than measured directly, the addition of the subscripts m and M will indicate, less precisely, just lower and upper bounds. These may or may not be the true minima and maxima implied by the given measurement error tolerances.

For example, the quantity a_o defined by (10) will lead naturally to the definitions

$$a_{om} = \frac{4R_m^{\ 2}}{U^2}$$

and

$$a_{oM} = \frac{4R_M^{\ 2}}{U^2},$$

since R is non negative by virtue of its geometrical meaning. However, for a_1, defined by (10), it would be permissible, for

*The error ε is a fixed bound on the deviation of R from R_o. A statistical error fluctuation would be taken into account by assigning some kind of confidence limit along with ε, e.g., by stating that ε bounds the error with 99% probability or that ε is a 3σ bound on the error.

example, (although not recommended in this case) to define

$$a_{1m} = \frac{-8\ R_M \left| \dot{R}_M \right|}{U^2},$$

which is a lower bound smaller than the minimum of a_1 whenever

$\dot{R}_M < 0$.*

A known physical or geometrical relationship may play a role in determining the minimum or maximum value of a derived quantity's tolerance interval. For example, the quantity ρ^2 defined in terms of the relative range R and relative altitude z by

$$\rho^2 = R^2 - z^2$$

must always be positive, whatever the tolerance limits may be on R and z separately. Thus, since ρ^2 cannot be less than zero it follows that

$$\rho_m{}^2 = R_m{}^2 - |z|_M{}^2 \text{ **} \qquad (11)$$

if

$$R_m{}^2 - |z|_M{}^2 \geq 0$$

but

$$\rho_m{}^2 = 0 \qquad (12)$$

if

$$R_m{}^2 - |z|_M{}^2 \leq 0.$$

*In this case the true minimum value of a_1 would be $\frac{-8R_m\dot{R}_M}{U^2}$, which is positive.

**Since z can be either positive or negative $\left|z\right|_M{}^2$ is only equal to $z_M{}^2$ if z_M is positive; if z_M is negative, $\left|z\right|_M{}^2 = z_m{}^2$.

On the other hand, it is always understood that

$$\rho_M^2 = R_M{}^2 - |z|_m{}^2. \tag{13}$$

B. AUXILIARY SUBALGORITHMS

Before attempting to revise the threat algorithm of Section IV it will be found useful to formulate some auxiliary algorithms which will help to simplify the description of the necessary logic. These cover the arithmetical steps which are implied when the subscripts m and M are added to some simple algebraic expressions.

The addition of a subscript to an expression of the form $(u\ v)$ or $|u + v|$ has the effect of operating on the expression to produce an algebraic function of the quantities u_m, u_M, v_m, and v_M. Four such operations are possible: $(u\ v)_m$, $(u\ v)_M$, $|u + v|_m$, and $|u + v|_M$. The results, which may be regarded as algorithms for the calculation of the subscripted quantities, are given in Tables 1, 2 and 3.

It should be emphasized here that these results can be relied upon to provide actual maxima and minima, in general, only when u and v are independent. If u and v are dependent inefficient bounds may be given instead, e.g., in the case $(u^2)_m$ which can be calculated from Table 2 by setting $u = v$. When $u_m < 0 < u_M$, Table 2 gives a lower bound for $(u^2)_m$ below the true minimum, which is min $(u_m{}^2, u_M{}^2)$.

TABLE 1. Subalgorithm for $|u+v|_m$ and $|u+v|_M$

	if $u_M+v_M < 0$	if $u_m+v_m > 0$	if $u_m+v_m \le 0 \le u_M+v_M$
$\|u + v\|_m$	$-(u_M + v_M)$	$u_m + v_m$	0
$\|u + v\|_M$	$-(u_m + v_m)$	$u_M + v_M$	max $\left[-(u_m+v_m), (u_M+v_M)\right]$

TABLE 2. Subalgorithm for $(uv)_m$

	if $v_M < 0$	if $v_m > 0$	if $v_m \leq 0 \leq v_M$
if $u_M < 0$	$u_M v_M$	$u_m v_M$	$u_m v_M$
if $u_m > 0$	$u_M v_m$	$u_m v_m$	$u_M v_m$
if $u_m \leq 0 \leq u_M$	$u_M v_m$	$u_m v_M$	$\min (u_m v_M, u_M v_m)$

TABLE 3. Subalgorithm for $(uv)_M$

	if $v_M < 0$	if $v_m > 0$	if $v_m \leq 0 \leq v_M$
if $u_M < 0$	$u_m v_m$	$u_M v_m$	$u_m v_m$
if $u_m > 0$	$u_m v_M$	$u_M v_M$	$u_M v_M$
if $u_m \leq 0 \leq u_M$	$u_m v_m$	$u_M v_M$	$\max (u_m v_m, u_M v_M)$

In addition to the algorithms defined by the tables, formulas for the expressions $(\frac{u}{v})_m$ and $(\frac{u}{v})_M$, where u and v are both non negative, are needed. These are

$$(\frac{u}{v})_m = \frac{u_m}{v_M} \tag{14}$$

and

$$(\frac{u}{v})_M = \frac{u_M}{v_m} . \tag{15}$$

Most of the subscripting required for the revised threat algorithm can be accomplished by repeated applications of the subalgorithms defined by Tables 1, 2, and 3, the formulas (14) and (15), and the trivial relations

$$(u + v)_m = u_m + v_m,$$

$$(u + v)_M = u_M + v_M.$$

It should be observed, however, that this procedure is not uniquely defined for a given expression and, therefore, may lead to different results, depending upon the grouping of terms and the order in which the subscripting is done.

If at each application of a subalgorithm in this procedure u and v are independent, the true minimum or maximum of the original expression will be obtained. If, however, at some step u and v are dependent the final result may only be a bound, i.e., a quantity which is smaller than the true minimum or larger than the true maximum. As an example, consider the calculation of $(x^2 + x)_M$, where

$$x_m = -2, \; x_M = 0 .$$

If the calculation is done as follows:

$$(x^2 + x)_M = (x^2)_M + x_M = 4 + 0 = 4,$$

the result is not a true maximum but an upper bound. In fact, it is easy to see that the true maximum occurs when $x = x_m = -2$ and that its value is 2.

The calculation can also be done by first setting

$$(x^2 + x)_M = (x(x + 1))_M,$$

next identifying

$$u = x, \; v = x + 1,$$

and finally using Table 3. It will be found, first, that in this case

$$u_m = -2, \; u_M = 0$$

and

$$v_m = -1, \; v_M = 1.$$

Table 3 applied to $(uv)_M$ will then give the true maximum 2 as the result.

The calculation of the extrema a_{om}, a_{oM}, a_{1m}, a_{1M}, a_{2m}, a_{2M} corresponding to the coefficients given by (10) could be accomplished by repeated applications of the subalgorithms but with some unnecessary loss of efficiency. Instead, it is preferable to use

$$a_{om} = \frac{4R_m^{\,2}}{U^2}, \quad a_{oM} = \frac{4R_M^{\,2}}{U^2} \tag{16}$$

for a_{om} and a_{oM}. Tables 2 and 3 can be used to calculate a_{1m} and a_{1M} without loss of efficiency, however.

For a_{2m} and a_{2M}, accepting some loss of efficiency is probably worthwhile for the sake of mathematical simplicity in the calculations. A reasonable form of a_2 for the calculation is

$$a_2 = \frac{4}{U^2}\left[|\dot{R}|^2 + |\dot{\phi}|^2\,\rho^2 + \frac{|\dot{R}z - \dot{z}R|^2}{\rho^2}\right],$$

so that

$$a_{2m} = \frac{4}{U^2}\left[|\dot{R}|_m^{\,2} + |\dot{\phi}|_m^{\,2}\,\rho_m^{\,2} + \frac{|\dot{R}z - \dot{z}R|_m^{\,2}}{\rho_M^{\,2}}\right] \tag{17}$$

and

$$a_{2M} = \frac{4}{U^2}\left[|\dot{R}|_M^{\,2} + |\dot{\phi}|_M^{\,2}\,\rho_M^{\,2} + \frac{|\dot{R}z - \dot{z}R|_M^{\,2}}{\rho_m^{\,2}}\right]. \tag{18}$$

In (17) and (18), the second and third terms can be calculated from Tables 1, 2, and 3 with the aid of (11), (12), and (13) for calculating $\rho_m^{\,2}$ and $\rho_M^{\,2}$.

Expressions for the extrema s_{om} and s_{oM} of the quantity s_o, which was defined in step (6) of the original threat algorithm of Section IV, can be obtained by inspection. For this purpose it should be recalled that a_o and a_2 are both non negative. Although some loss of efficiency occurs because a_o and

a_2 are dependent, reasonable expressions are

$$s_{oM} = \sqrt{\frac{1}{6}\left[1 - \left(1 - \frac{12a_{oM}}{a_{2m}^{\ 2}}\right)^{\frac{1}{2}}\right]} \qquad (19)$$

and

$$s_{om} = \sqrt{\frac{1}{6}\left[1 - \left(1 - \frac{12a_{om}}{a_{2M}^{\ 2}}\right)^{\frac{1}{2}}\right]} \qquad (20)$$

It will be seen later that the values of s_{om} and s_{oM} given by (19) and (20) are guaranteed to be real because the revised threat algorithm would otherwise have terminated at step (5) before a need for these quantities could have arisen.

Algorithms are also needed for the extrema $f(t_e)_m$, $f(t_e)_M$, $f'(t_e)_m$, and $f'(t_e)_M$ of the polynomials $f(t_e)$ and $f'(t_e)$ which appear in steps (2) and (8) of the threat algorithm of Section IV. The original polynomials were defined by

$$f(t_e) = t_e^{\ 4} - a_2 t_e^{\ 2} + a_1 t_e - a_o \qquad (21)$$

and

$$f'(t_e) = 4t_e^{\ 3} - 2a_2 t_e + a_1. \qquad (22)$$

For $f'(t_e)$ it is probably advisable to accept

$$f'(t_e)_m = 4t_e^{\ 3} - 2a_{2M} t_e + a_{1m} \qquad (23)$$

and

$$f'(t_e)_M = 4t_e^{\ 3} - 2a_{2m} t_e + a_{1M}. \qquad (24)$$

However, somewhat more efficient bounds can be obtained for $f(t_e)$ by first expressing it in the form, using (10),

$$f(t_e) = \alpha - \frac{4}{U^2}\,|R + t_e\dot{R}|^2, \qquad (25)$$

where

$$\alpha = t_e^{\ 4} - \frac{4}{U^2}\left[|\dot{\phi}|^2\,\rho^2 + \frac{|\dot{R}z - \dot{z}R|^2}{\rho^2}\right]t_e^{\ 2}\ . \qquad (26)$$

Then

$$\alpha_m = t_e^4 - \frac{4}{U^2}\left[|\dot{\phi}|_M^2 \rho_M^2 + \frac{|\dot{R}z-\dot{z}R|_M^2}{\rho_m^2}\right] t_e^2 \qquad (27)$$

and

$$\alpha_M = t_e^4 - \frac{4}{U^2}\left[|\dot{\phi}|_m^2 \rho_m^2 + \frac{|\dot{R}z-\dot{z}R|_m^2}{\rho_M^2}\right] t_e^2 , \qquad (28)$$

wherein further subscripting can be carried out with the aid of the previously derived subalgorithms. The subalgorithms can be applied along with (27) and (28) to (25) in order to calculate $f(t_e)_m$ and $f(t_e)_M$.

C. THE REVISED THREAT ALGORITHM

In order to obtain the revised threat algorithm it is only necessary to strengthen the inequality condition for no alarm or weaken the condition for an alarm at each step of the threat algorithm in Section III by applying the appropriate extremum operations. The subalgorithms derived in Section VB now make this a completely mechanical procedure.

The result is:

(1) $R_m + \dot{R}_m t_e < \frac{1}{2} U t_e^2$?

If no then no alarm;

if yes then (2).

(2) $f(t_e)_M > 0$?

If yes then alarm;

if no than (3).

(3) $a_{1M} > 0$?

If no then no alarm;

if yes then (4).

(4) $8a_{2M}^3 > 27a_{1m}^2$?

If no then no alarm;

if yes then (5).

$$(5)\ a_{om} \le \frac{1}{12} a_{2M}^{2}\ ?$$

If no then no alarm;

if yes then (6).

$$(6)\ \frac{a_{1M}}{a_m^{3/2}} \ge 2s_{om} - 4s_{oM}^{3}\ ?$$

If no then no alarm;

if yes then (7).

$$(7)\ 6t_e^{2} > a_{2m}?$$

If yes then alarm;

if no then (8).

$$(8)\ f'(t_e)_m < 0?$$

If yes then alarm;

if no then no alarm.

For an example to illustrate the use of the revised threat algorithm, the following error tolerances will be assumed:

range (R) ± 200 ft,
range rate ($\dot{R}$) ± 50 ft/sec (about ± 30kn),
altitude (z) ± 200 ft,
altitude rate ($\dot{z}$) ± 56.6 ft/sec
bearing rate ($\dot{\phi}$) ± 5°/sec.

Then, for measured parameters having the values

R = 12160 ft (2 nmi),
$\dot{R}$ = -590 ft/sec,
z = 200 ft,
$\dot{z}$ = -26.67 ft/sec,
$\dot{\phi}$ = 16°/sec,

the extrema which must be used in the threat algorithm will be given by

$$\begin{aligned}
R_m &= 11{,}960 \text{ ft}, & R_M &= 12{,}360 \text{ ft}, \\
\dot{R}_m &= -640 \text{ ft/sec}, & \dot{R}_M &= -540 \text{ ft/sec}, \\
z_m &= 0 \text{ ft}, & z_M &= 400 \text{ ft}, \\
\dot{z}_m &= -83.3 \text{ ft/sec}, & \dot{z}_M &= 29.9 \text{ ft/sec}, \\
\dot{\phi}_m &= 11^\circ/\text{sec}, & \dot{\phi}_M &= 21^\circ/\text{sec}.
\end{aligned}$$

Using the methods described in this section for calculating the derived parameter extrema, it will be found that

$$\begin{aligned}
\rho_m^2 &= 1.4288\text{x}10^8 \text{ ft,}^2 & \rho_M^2 &= 1.5277\text{x}10^8 \text{ ft,}^2 \\
a_{om} &= 5.5184\text{x}10^5 \text{ sec}^4, & a_{OM} &= 5.8937\text{x}10^5 \text{ sec}^4, \\
a_{1m} &= 4.9831\text{x}10^4 \text{ sec}^3, & a_{1M} &= 6.1035\text{x}10^4 \text{ sec}^3, \\
a_{2m} &= 2.1442\text{x}10^4 \text{ sec}^2, & a_{2M} &= 8.0782\text{x}10^4 \text{ sec}^2, \\
s_{om} &= 9.1970\text{x}10^{-3}, & s_{OM} &= 3.5873\text{x}10^{-2} \\
f(t_e)_M &= 1.2313\text{x}10^7 \text{sec}^4, & f'(t_e)_m &= -3.9268\text{x}10^6 \text{sec}^3.
\end{aligned}$$

The threat algorithm applied to these quantities then leads to the conclusion at step (8), that an alarm should be generated.

If the measured bearing rate $\dot{\phi}$ is increased to 17°/sec, so that

$$\dot{\phi}_m = 12^\circ/\text{sec and } \dot{\phi}_M = 22^\circ/\text{sec},$$

while the other measured parameters are left unchanged, then the derived parameters ρ_m^2, ρ_M^2, a_{om}, a_{oM}, a_{1m}, and a_{1M} remain unchanged but new values for a_{2m}, a_{2M}, s_{om}, and s_{oM} are obtained. These are given by

$$\begin{aligned}
a_{2m} &= 2.5304\text{x}10^4 \text{sec}^2, & a_{2M} &= 8.8502\text{x}10^4 \text{sec}^2, \\
s_{om} &= 8.3946\text{x}10^{-3}, & s_{oM} &= 3.0381\text{x}10^{-2}
\end{aligned}$$

On applying the threat algorithm to the new values of the derived parameter extrema it will be found that "no alarm" is indicated at step (6).

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APPENDIX A

COEFFICIENTS IN THE THREAT ALGORITHM

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COEFFICIENTS IN THE THREAT ALGORITHM

The condition (3) of Section II is the basis for the hazard alarm algorithm given in Section IV. That algorithm operates on the coefficients a_o, a_1, and a_2 of a polynomial f(t) derived from (3), which, in turn, depends upon the components of the range vector $\underset{\sim}{R}$ between the CAS and an intruder aircraft and upon the time derivative $\dot{\underset{\sim}{R}}$ of $\underset{\sim}{R}$. The components of these vectors depend upon a set of quantities measured by the CAS:

R, the range between the CAS and the intruder,

$\dot{R}$, the range rate,

ϕ, the bearing of the intruder

$\dot{\phi}$, the bearing rate,

z, the difference in altitude between the intruder and the CAS,

$\dot{z}$, the relative altitude rate.

Expressions for the coefficients a_o, a_1, and a_2 in terms of the measured quantities will be derived in this appendix.

It will be convenient to use a right-handed spherical coordinate system centered at the CAS. The polar axis is perpendicular to the ground. The polar angle θ is zero along the axis in the upward direction and increases downward to its extreme value of 180^o along the axis toward the ground. The azimuth angle ϕ measures the absolute bearing of the intruder and is therefore zero at some predetermined geographical direction, e.g., magnetic north. Looking down, a horizontal vector which emanates from the CAS and rotates counterclockwise will have an azimuth ϕ that increases as it rotates. The radial coordinate R is the intruder range. A corresponding vector coordinate system is based upon a set of three orthogonal unit vectors $\underset{\sim}{a}_\theta$, $\underset{\sim}{a}_\phi$, and $\underset{\sim}{a}_R$ pointing in the direction of increasing θ, ϕ, and R.

Condition (3) of Section II depends upon the vectors $\underset{\sim}{R}$ and $\dot{\underset{\sim}{R}}$. The vector $\underset{\sim}{R}$ is given by

$$\underset{\sim}{R} = R \, \underset{\sim}{a}_R \tag{A-1}$$

in terms of the spherical coordinate system. The vector $\dot{\underset{\sim}{R}}$ is given by

$$\dot{\underset{\sim}{R}} = \dot{R} \, \underset{\sim}{a}_R + R \, \dot{\underset{\sim}{a}}_R . \tag{A-2}$$

The unit vector $\underset{\sim}{a}_R$ changes direction from point to point and therefore, because of aircraft motion, it does not remain constant in time. For this reason, as indicated in (A-2), its time derivative $\dot{\underset{\sim}{a}}_R$ is not zero in general.

For the purpose of calculating $\dot{\underset{\sim}{a}}_R$ it is helpful to use a Cartesian coordinate system, for which the unit orthogonal basis vectors $\underset{\sim}{a}_x$, $\underset{\sim}{a}_y$, and $\underset{\sim}{a}_z$ are constant throughout space and, thus, do not change in time. A right-handed Cartesian coordinate system for this purpose may be defined by letting the unit vector $\underset{\sim}{a}_z$ point in the direction $\theta = o$, the unit vector $\underset{\sim}{a}_x$ point in the direction $\phi = o$ in the plane $\theta = 90^o$, and the unit vector $\underset{\sim}{a}_y$ point in the direction $\phi = 90^o$ in the plane $\theta = 90^o$. The coordinate systems are illustrated in Fig. A-1.

The relationship between the basis vectors of the spherical coordinate system and those of the Cartesian coordinate system is given by the following equations:

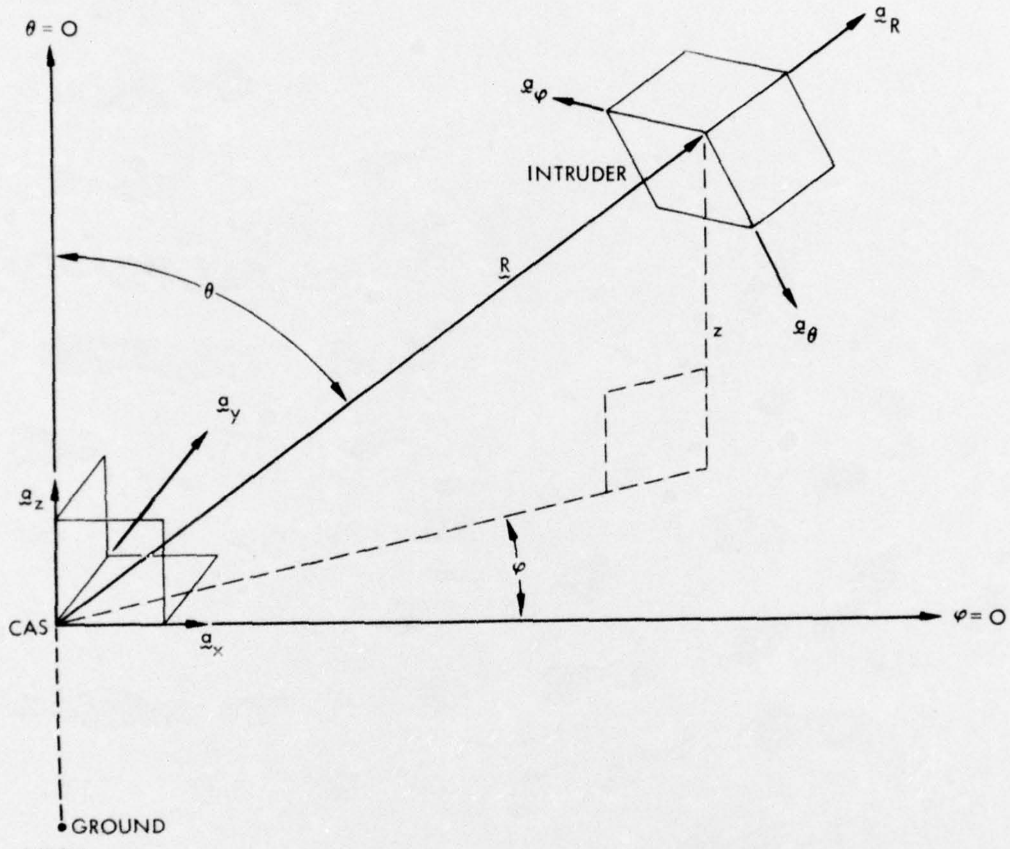

FIGURE A-1. CAS - Intruder Coordinate Systems For Range Vector

$$a_R = (\cos\phi \sin\theta)\, a_x + (\sin\phi \sin\theta)\, a_y + (\cos\theta)\, a_z ,$$

$$a_\phi = (-\sin\phi) a_x + (\cos\phi) a_y , \tag{A-3}$$

$$a_\theta = (\cos\phi \cos\theta) a_x + (\sin\phi \cos\theta) a_y - (\sin\theta) a_z ,$$

and

$$a_x = (-\sin\phi) a_\phi + (\cos\phi \cos\theta) a_\theta + (\cos\phi \sin\theta) a_R , \tag{A-4}$$

$$a_y = (\cos\phi) a_\phi + (\sin\phi \cos\theta) a_\theta + (\sin\phi \sin\theta) a_R ,$$

$$a_z = (-\sin\theta) a_\theta + (\cos\theta)\, a_R .$$

It follows from (A-3) and (A-4) that

$$\dot{a}_R = (-\dot{\phi} \sin\phi \sin\theta + \dot{\theta} \cos\phi \cos\theta) a_x$$

$$+ (\dot{\phi} \cos\phi \sin\theta + \dot{\theta} \sin\phi \cos\theta) a_y$$

$$- (\dot{\theta} \sin\theta) a_z = (\dot{\phi} \sin\theta) a_\phi + \dot{\theta} a_\theta .$$

Then, with the aid of this result, (A-2) can be written

$$\dot{R} = \dot{R}\, a_r + (R\, \dot{\phi} \sin\theta) a_\phi + R\dot{\theta}\, a_\theta \tag{A-5}$$

Actually, while ϕ, $\dot{\phi}$, R, and $\dot{R}$ are assumed to be measured quantities, θ and $\dot{\theta}$ are not measured directly. However, z and $\dot{z}$ are, and θ and $\dot{\theta}$ can be expressed in terms of z, $\dot{z}$, R, and $\dot{R}$. The pertinent relationship is given by

$$z = R \cos\theta , \tag{A-6}$$

from which it follows that

$$\cos\theta = \frac{z}{R}$$

and

$$\sin\theta = \frac{\sqrt{R^2 - z^2}}{R} \tag{A-7}$$

Then, from (A6),

$$\dot{z} = \dot{R}\cos\theta - R\dot{\theta}\sin\theta,$$

and, from (A7)

$$\dot{\theta} = \frac{\dot{R}z - R\dot{z}}{R\sqrt{R^2 - z^2}} .$$

If these relations are substituted into (A-5) the result will be an expression for $\dot{\underset{\sim}{R}}$,

$$\dot{\underset{\sim}{R}} = \dot{R}\, \underset{\sim}{a}_R + (\dot{\phi}\sqrt{R^2 - z^2})\underset{\sim}{a}_\phi + \left(\frac{\dot{R}z - \dot{z}R}{\sqrt{R^2 - z^2}}\right)\underset{\sim}{a}_\theta, \tag{A-8}$$

in terms of measured quantities alone.

The condition (3) of Section II expanded in terms of vector dot products is

$$R^2 + 2(\underset{\sim}{R} \cdot \dot{\underset{\sim}{R}})t_e + |\dot{\underset{\sim}{R}}|^2 t_e^2 < \frac{U^2 t_e^4}{4} . \tag{A-9}$$

This inequality can be written in the form

$$f(t_e) > o ,$$

where

$$f(t) = t^4 - a_2t^2 + a_1t - a_o$$

with

$$a_o = \frac{4R^2}{U^2} ,$$

$$a_1 = - \frac{8(\underset{\sim}{R}\cdot\dot{\underset{\sim}{R}})}{U^2} ,$$

$$a_2 = \frac{4|\dot{\underset{\sim}{R}}|^2}{U^2} .$$

With the aid of (A-1) and (A-8), remembering that the basis vectors have unit magnitude and are mutually orthogonal, the coefficients of f(t) can then be written in terms of the measured quantities alone:

$$a_o = \frac{4R^2}{U^2} ,$$

$$a_1 = - \frac{8R\dot{R}}{U^2} , \qquad \text{(A-10)}$$

$$a_2 = \frac{4}{U^2}\left[\dot{R}^2+\dot{\phi}^2(R^2-z^2) + \frac{(\dot{R}z-R\dot{z})^2}{R^2-z^2}\right] .$$

APPENDIX B

NECESSARY CONDITION FOR A COLLISION THREAT

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NECESSARY CONDITION FOR A COLLISION THREAT

A. SUFFICIENT CONDITIONS FOR A THREAT AND FOR NO THREAT

Relation (3) of Section III is a sufficient condition, depending on range, altitude, and bearing data, for a collision threat. The condition is also equivalent to

$$f\ (t_e) > 0,$$

where f(t) is the biquadratic polynomial

$$f(t) = t^4 - a_2\ t^2 + a_1\ t - a_o \qquad \text{(B-1)}$$

defined in Section IV. The coefficients a_o, a_1, and a_2 are given in terms of the measured data by (10) of Section IV or (A-10) of Appendix A. From their definitions it is clear that

$$a_2 \geq o \text{ and } a_o > o,$$

while a_1 can be positive, negative, or zero.

On the other hand, according to Ref. 2, if the modified tau condition (7) is violated, i.e., if

$$R + \dot{R}\ t_e \geq \frac{1}{2}\ U\ t_e{}^2\ , \qquad \text{(B-2)}$$

no threat exists. This can be seen from the argument of Section III applied to a function b(t) which is redefined for this purpose by

$$b(t) = \frac{1}{2}\ Ut^2 - \dot{R}t - R.$$

The result then follows from the fact, remarked in Section III, that the equation

$$b(t) = \frac{1}{2}\ Ut^2 - \dot{R}t - R = o$$

always has exactly one positive root. This is because the quadratic polynomial $b(t)$ is negative at $t = o$ and increases without limit as either t or $-t$ becomes large. Since, according to (B-2), $b(t_e)$ is negative the positive root must occur at a time later than t_e.

The question now left to be answered in this appendix is what happens when neither (3) nor (B-2) are satisfied. That is, what can be concluded about the existence of a threat if

$$f(t_e) \leq 0? \qquad \text{(B-3)}$$

B. CRITICAL POINTS AND ROOTS

It was shown in Section III that when the equation

$$f(t) = o \qquad \text{(B-4)}$$

has only one positive root and the condition (B-3) is satisfied, no collision can occur at a time earlier than t_e. Thus, in order to determine whether an alarm should be given when (B-3) is satisfied, a first step might be to find out if (B-4) has more than one positive root. If not, the question will be answered in the negative; otherwise, it will be necessary to determine whether one or more of the roots occur for values of t between 0 and t_e.

Since $f(t)$ is negative at $t = 0$ and becomes arbitrarily large as either t or $-t$ becomes large, (B-4) has at least one positive and one negative root. It may, of course, have as many as four distinct real roots.

It also follows that $f(t)$ must have at least one minimum. This corresponds to the fact that the derivative $f'(t)$, being a cubic polynomial, must vanish for at least one real value of t.

If $f'(t)$ vanishes for only one real value t_o of t then the minimum of $f(t)$ must occur at t_o and there can be no other; nor, for that matter, can $f(t)$ have a maximum in such a case. Furthermore, (B-4) will then have just the one positive and the one negative root. This situation is illustrated in Fig. 3 in which the minima of both curves occur at negative values of t.

On the other hand, if $f'(t)$ vanishes at three distinct values of t, $f(t)$ will have two minima separated by a maximum. This relationship is topologically clear; it can also be verified by considering the sign changes of the second derivative $f''(t)$.

From (B-1),

$$f''(t) = 12t^2 - 2a_2. \tag{B-5}$$

Thus $f''(t)$ vanishes at two values $t_\pm$ of t given by

$$t_\pm = \pm\sqrt{\frac{a_2}{6}}\ . \tag{B-6}$$

These are the inflection points of $f(t)$ and are evidently located symmetrically about zero.

It can be seen by inspection of (B-5) that the slope $f''(t)$ of $f'(t)$, where from (B-1)

$$f'(t) = 4t^3 - 2a_2\, t + a_1, \tag{B-7}$$

is negative for t between t_- and t_+ and is positive for all other values. Moreover, $f'(t)$, itself, has a maximum at t_- and a minimum at t_+. Thus, if $f'(t)$ vanishes at three points, one lies to the left of t_- and must correspond to a minimum of $f(t)$, one lies between t_- and t_+ and must correspond to a maximum of $f(t)$, and the third lies to the right of t_+ and must correspond to another minimum of $f(t)$.

Since

$$f'(0) = a_1$$

it is also clear that if a_1 is negative, the maximum of $f(t)$ lies between t_- and 0, because $f'(t)$ will change sign between these two values of t. Similarly, if a_1 is positive, the maximum of $f(t)$ lies between 0 and t_+. This situation is illustrated in Figs. 1 and 2.

Since $f(0)$ is negative, it then follows that for a_1 negative, (B-4) can only have one positive root. For a_1 equal to zero the maximum of $f(t)$ will be at zero and will therefore be negative. Thus, $f(t)$ can have more than one positive root only if a_1 is positive. It follows that when (B-3) is satisfied a hazard condition can only exist when

$$a_1 > 0.$$

This condition is equivalent to requiring a negative value for $\dot{R}$, which would imply that the intruder is approaching.

C. THE CUBIC DISCRIMINANT

Since $f(t)$ cannot vanish at more than two points if it has just one minimum and lacks a maximum, a necessary condition that (B-4) have more than one positive root is that $f'(t)$ vanish at more than one real value of t. In fact, the condition may be stated more restrictively: the equation

$$f'(t) = 0 \tag{B-8}$$

must have three distinct real roots. This follows from the observation that a multiple root of (B-8) must occur at an inflection point of $f(t)$, and at such a point $f(t)$ will be stationary but will have neither a maximum nor a minimum.

The condition that $f'(t)$ have three real roots could be obtained from the well known cubic discriminant (cf. Ref. B-1, p. 432). However, it may be instructive to derive it here directly in terms of the coefficients in $f(t)$.

It has been observed that $f'(t)$ has two critical points $t_\pm$, given by (B-6). It is easily verified that t_- corresponds to a maximum of $f(t)$ and t_+ to a minimum. Thus, $f'(t)$ always has one maximum and one minimum. Clearly, then, $f'(t)$ will have three distinct real roots if and only if it is positive at t_- and negative at t_+.

By substituting from (B-6) into (B-7) it will be found that

$$f'(t_-) = \frac{4a_2^{3/2}}{3\sqrt{6}} + a_1 \text{ (a maximum)}$$

and

$$f'(t_+) = \frac{-4a_2^{3/2}}{3\sqrt{6}} + a_1 \text{ (a minimum)}.$$

From these expressions it can easily be verified that when

$$a_1 < 0$$

the minimum of $f'(t)$ is positive if and only if

$$27a_1^2 > 8a_2^3. \tag{B-9}$$

It can also be verified that when

$$a_1 > 0$$

the maximum of $f'(t)$ is negative if and only if (B-9) holds. Conversely, $f'(t)$ will have a negative minimum and a positive maximum if and only if

$$8a_2^3 > 27a_1^2. \tag{B-10}$$

Since $f'(t)$ can vanish at three distinct values of t only when its minimum is negative and its maximum is positive (B-10) is a necessary and sufficient condition that (B-8) have three distinct roots. As remarked earlier, this is also a necessary condition that (B-4) have more than one positive root.

D. NECESSARY CONDITION FOR A HAZARD

While (B-10) is a necessary condition that (B-4) have more than one positive root, it is not sufficient. Moreover, assuming that (B-10) is satisfied, not only must additional conditions be met before it can be concluded that more than one positive root of (B-4) exists, but even when that fact is established the possibility of a collision before the time t_e will still remain open. In order to assert that such a collision is possible, so that a CAS alarm should occur, at least one of the positive roots of (B-4) must be known to lie between 0 and t_e.

Before proceeding with a more detailed analysis of the biquadratic polynomial f(t) it will be convenient to change variables in order to reduce the number of its coefficients from three to two. This purpose is achieved by substituting

$$t = s\sqrt{a_2}$$

into f(t) and dividing the result by a_2^2. It may be observed that when (B-10) is satisfied as assumed a_2 cannot be zero; therefore, this division is possible. The resulting polynomial g(s) will have the form

$$g(s) = s^4 - s^2 + a_1 s - a_o = 0, \qquad \text{(B-11)}$$

where

$$A_1 = \frac{a_1}{a_2^{3/2}}, \quad A_o = \frac{a_o}{a_2^{2}} .$$

In terms of the new coefficients (B-10) can be expressed in the form

$$A_1^2 < \frac{8}{27} ,$$

or

$$A_1 < \frac{2\sqrt{6}}{9} \sim 0.5443. \qquad \text{(B-12)}$$

Since g(s) is negative at s=o and becomes arbitrarily large for large s, if the equation

$$g(s) = o$$

has more than one positive root, g(s) must have a maximum at some positive value s_M of s. The maximum occurs when the second derivative g"(s) is negative, i.e., when

$$12s_M^2 - 2 < 0,$$

or

$$0 < s_M < \frac{1}{\sqrt{6}} . \qquad \text{(B-13)}$$

Also, of course,

$$g'(s_M) = 4s_M^3 - 2s_M + A_1 = 0; \text{ i.e.,}$$

$$A_1 = 2s_M - 4s_M^3. \qquad \text{(B-14)}$$

From (B-11) and (B-12) it follows that a positive maximum can occur only if A_1 satisfies

$$0 < A_1 < \frac{2\sqrt{6}}{9} , \qquad \text{(B-15)}$$

a condition already partly known from (B-12).

Even when g(s) has a maximum and s_M is positive, (B-11) will have more than one positive root if and only if $g(s_M)$ is non negative. After substituting for A_1 from (B-14) into $g(s_M)$ this condition becomes

$$g(s_M) = -3s_M^4 + s_M^2 - A_o \geq 0,$$

or

$$A_o \leq s_M^2 - 3s_M^4 = h(s_M). \qquad \text{(B-16)}$$

For the whole possible range of values for s_M given by (B-13) the function $h(s_M)$ has a positive derivative and therefore is always increasing. Thus if (B-16) is satisfied for any s_M it must also be satisfied when $\sqrt{\frac{1}{6}}$ is substituted for s_M; i.e.,

$$A_o < \frac{1}{6} - \frac{3}{36} = \frac{1}{12} . \qquad \text{(B-17)}$$

Let s_o be the smallest positive root of the equation

$$A_o = h(s_o) = s_o^2 - 3s_o^4; \qquad \text{(B-18)}$$

i.e.,

$$s_o = \sqrt{\frac{1 - \sqrt{1-12A_o}}{6}}\,, \tag{B-19}$$

which must be real because of (B-17). Since h(s) is increasing over the range of s_M values (B-16) will be satisfied if and only if

$$s_o \le s_M,$$

i.e., if and only if

$$2s_o - 4s_o^3 \le 2s_M - 4s_M^3 = A_1.$$

Therefore, g(s) has more than one positive root if and only if

$$A_1 \ge 2s_o - 4s_o^3, \tag{B-20}$$

where s_o is given by (B-19).

When (B-20) is satisfied the maximum of g(s) is either positive or zero. In the first case (B-11) has three positive roots and in the second case (B-11) has two positive roots, the smaller of which occurs at s_M where the maximum also occurs.

If t_e is less than the first positive root of (B-4) no alarm should occur. In this case, since f(0) is negative and the maximum of f(t) occurs after the first root of (B-4), the slope of f(t) at t_e is positive; i.e.,

$$f'(t_e) = 4t_e^3 - 2a_2t_e + a_1 > 0. \tag{B-21}$$

In addition, since the maximum of f(t) occurs before the positive inflection point the second derivative of f(t) does not change sign between zero and the maximum and therefore would have to be negative at t_e; i.e.,

$$f''(t_e) = 12t_e^2 - 2a_2 < 0. \tag{B-22}$$

When (B-4) has more than one positive root, i.e., (B-11) has more than one positive root and either of the conditions (B-21) or (B-22) are violated, (B-4) will have a root between zero and t_e and therefore an alarm should occur.

REFERENCES

B-1. Birkhoff, G. and McLane, S., "A Survey of Modern Algebra," Macmillan, New York, 1948.