IMO 2016 Training Camp-Junior

Trigonometry-ratio hacks

15 January 2016

1 Basics

Just in case you are curious of what is the beauty of this technique, the answer is: you can add some algebraic flavour into pure geometric problems! Synthetic geometry often requires some ingenuity but a slightly less amount of creativity is needed if you know trigs. That having said, some proficiency of geometry is needed, like the ability to draw extra lines or even circles correctly (Mr. Suhaimi calls it *geometric eyes*).

Here are some facts that will be useful:

1. $\sin(A + B) = \sin A \cos B + \cos A \sin B$
2. $\sin(A - B) = \sin A \cos B - \cos A \sin B$
3. $\cos(A + B) = \cos A \cos B - \sin A \sin B$
4. $\cos(A - B) = \cos A \cos B + \sin A \sin B$
5. $\sin A + \sin B = 2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$
6. $\sin A - \sin B = 2 \sin \dfrac{A-B}{2} \cos \dfrac{A+B}{2}$
7. $\cos A + \cos B = 2 \cos \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$
8. $\cos A - \cos B = -2 \sin \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$

   No.5 and No.6 entail that $\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$. No. 7 and No. 8 imply that $\cos^2 A - \cos^2 B = -\sin(A+B)\sin(A-B)$. Also note the relation $\sin(180° - A) = \sin A$, $\cos(180° - A) = -\cos A$, $\sin(90° \pm A) = \cos A$ and $\cos(90° \pm A) = \mp \sin A$. We will use these identities below profusely without proof.

2 Identities

Note: the last identity is not commonly used in contests so make sure you understand the proof!

1. (Sine rule) For any triangle ABC, $\frac{AB}{AC} = \frac{\sin\angle ACB}{\sin\angle ABC}$.
   This is available even in Form 4 textbook, and I shall leave the proof as an exercise.
2. (Cosine rule) For a triangle ABC, we have $BC^2 = AB^2 + CA^2 - 2 \cdot AB \cdot CA \cdot \cos\angle BAC$.
   Corollary. $\cos\angle BAC = \frac{AB^2+CA^2-BC^2}{2\cdot AB\cdot CA}$.

3. Let D be a point on BC of $\triangle ABC$. Then $\frac{BD}{DC} = \frac{AB}{AC} \cdot \frac{\sin\angle BAD}{\sin\angle CAD}$.
Proof: now let $|\triangle ABC|$ be the area of $\triangle ABC$. Also let the perpendicular from points D to lines AB and AC be E and F respectively.Then $\frac{BD}{DC} = \frac{|\triangle ABD|}{|\triangle ADC|} = \dfrac{\frac{1}{2}\cdot AB\cdot DE}{\frac{1}{2}\cdot AC\cdot DF} = \frac{AB\cdot(AD\cdot\sin\angle BAD)}{AC\cdot(AD\cdot\sin\angle CAD)} = \frac{AB}{AC}\cdot\frac{\sin\angle BAD}{\sin\angle CAD}$.

4. Let $ABDC$ be a cyclic quadrilateral. Then $\frac{BD}{DC} = \frac{\sin\angle BAD}{\sin\angle CAD}$.
In fact, chord length=diameter of the circle to which the chord belongs $\times$ sine of the angle subtended by the chord on the arc of the circle (or angle between the chord and the tangent to the circle at either of the chord's endpoint!)

   As long as ratio is concerned, the diameter of the circle is not so relevant in our solution compared to the sines.

5. Let D be in the angle domain of $\angle BAC$ of a triangle BAC and let AD intersect BC at E. Then $\frac{BE}{EC} = \frac{AB}{AC}\cdot\frac{BD}{CD}\cdot\frac{\sin\angle ABD}{\sin\angle ACD}$. (See how it is equivalent to the trigo version of Ceva's theorem).

   Corollary 1. Take D as above. Then $\frac{\sin\angle BAD}{\sin\angle CAD} = \frac{BD}{CD}\cdot\frac{\sin\angle ABD}{\sin\angle ACD}$.

   Corollary 2. Let D and E be in the angle domain of $\angle BAC$ of a triangle BAC. Then $\frac{\sin\angle BAD}{\sin\angle CAD} = \frac{\sin\angle BAE}{\sin\angle CAE} \iff \angle BAD = \angle BAE$ and $\angle CAD = \angle CAE$.

   Corollary 3. Denote D and E the same way as we did in corollary 2. Then A, D, E are collinear iff $\frac{BD\cdot\sin\angle ABD}{CD\cdot\sin\angle ACD} = \frac{BE\cdot\sin\angle ABE}{CE\cdot\sin\angle ACE}$.

3 Examples and solutions

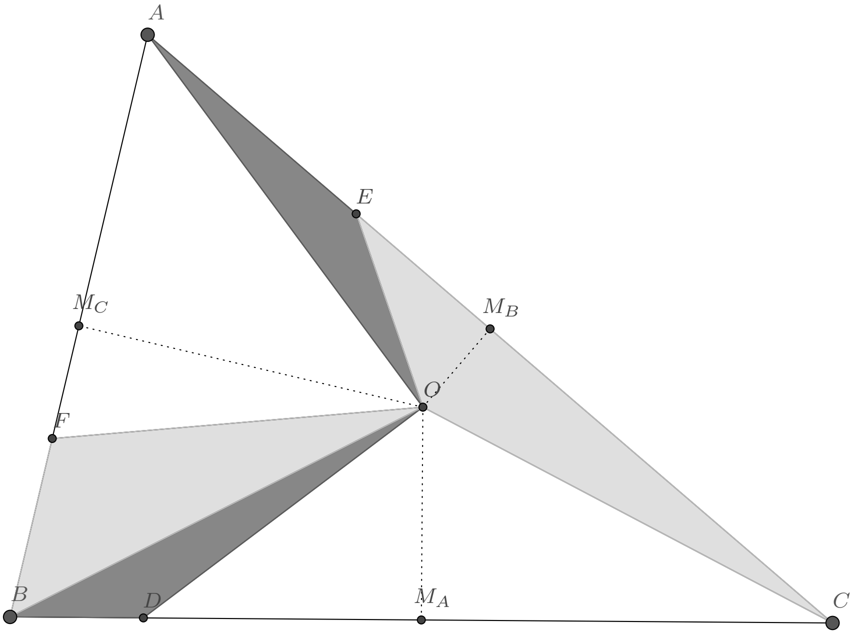
1. *APMO 2013, Problem 1*: Let ABC be an acute triangle with altitudes AD, BE and CF, and let O be the center of its circumcircle. Show that the segments OA, OF, OB, OD, OC, OE dissect the triangle ABC into three pairs of triangles that have equal areas.

   Solution 1: The first obstacale is to "pair" the six triangles, so calculate each of them. Let's pick $\triangle BOD$ for now.

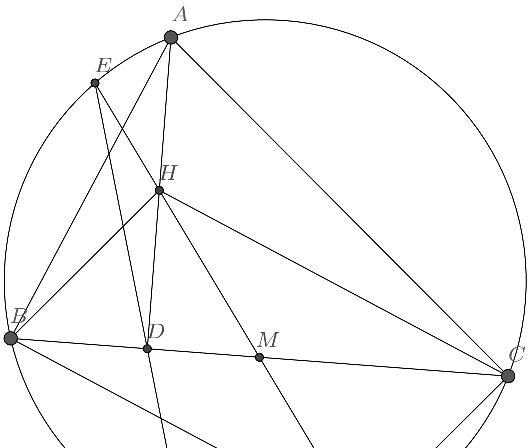
   As we see $|\triangle BOC| = \frac{1}{2}\cdot BO\cdot OC\cdot\sin\angle BOC = \frac{1}{2}\cdot R^2\cdot\sin(2\angle A) = R^2\cdot\sin\angle A\cos\angle A$. Now $|\triangle BOD| = |\triangle BOC|\cdot\frac{BD}{BC} = |\triangle BOC|\cdot\frac{BD}{BD+DC}$. But $\frac{BD}{DC} = \frac{\sin\angle C\cos\angle B}{\sin\angle B\cos\angle C}$ so $\frac{BD}{BC} = \frac{\sin\angle C\cos\angle B}{\sin\angle C\cos\angle B+\sin\angle B\cos\angle C} = \frac{\sin\angle C\cos\angle B}{\sin\angle A}$ (recall that $\angle B + \angle C$ and $\angle A$ are sumpplementary so the sines of these angles must be the same). Combining above we must have $|\triangle BOD| = R^2\cdot\cos\angle B\cos\angle A\sin\angle C$. Now you can verify that this is the same for $|\triangle AOE|$. Similarly, $|\triangle COD| = |\triangle AOF| = R^2\cdot\cos\angle A\cos\angle C\sin\angle B$ and $|\triangle BOF| = |\triangle COE| = R^2\cdot\cos\angle B\cos\angle C\sin\angle A$. ■

   Solution 2: Perhaps we have made our pairing conjecture and want to play with ratios (sides and bases). Let's name our orthocentre H and midpoints of BC, CA, AB as M_A, M_B, M_C respectively. Observe that M_A, M_B, M_C are also the perpendicular of O to sides BC, CA, AB. (Why?)

   Now we shall take $\triangle BOD$ and $\triangle AEO$ again, and $\frac{|\triangle BOD|}{|\triangle AEO|} = \dfrac{\frac{1}{2}\cdot BD\cdot OM_A}{\frac{1}{2}\cdot CE\cdot OM_B}$. Now $\triangle EHA \sim \triangle DHB$, so $\frac{BD}{EA} = \frac{HB}{HA} = \frac{\sin\angle DAB}{\sin\angle EBA} = \frac{\cos\angle B}{\cos\angle A}$ while $OM_A = OB\cos BOM_A = R\cos\frac{\angle BOC}{2} = R\cos\angle A$. In a similar manner $OM_B = R\cos\angle B$. So $\frac{|\triangle BOD|}{|\triangle AEO|} = \frac{BD\cdot OM_A}{CE\cdot OM_B} = \frac{\cos\angle B}{\cos\angle A}\cdot\frac{R\cos\angle A}{R\cos\angle B} = 1$. ■



2. *APMO 2012, Problem 4*: Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC, by M the midpoint of BC, and by H the orthocenter of ABC. Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the half line MH, and F the intersection (other than E) of the line ED and the circle Γ. Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold.



Solution: Let G be a point such that $BHCG$ is a parallelogram. One crucial identity is that: $\angle BAC + \angle BGC = \angle BAC + \angle BHC = 180°$ so G is on Γ. Now HG passes through M (recall one property of a parallelogram), we know that G is also meeting point of line

MH and Γ besides E.

Now combining third and fourth identity in section 2 we have $1 = \dfrac{BM}{MC} = \dfrac{BE \cdot \sin \angle BEM}{CE \cdot \sin \angle CEM} = \dfrac{BE \cdot BG}{CE \cdot CG} = \dfrac{BE \cdot HC}{CE \cdot BH}$ (recall another property of a parallelogram!) so

$$\frac{BE}{CE} = \frac{BH}{CH} = \frac{\sin \angle HCB}{\sin \angle HBC} = \frac{\sin(90^\circ - \angle B)}{\sin(90^\circ - \angle C)} = \frac{\cos \angle B}{\cos \angle C} ...(1)$$

. Also notice that

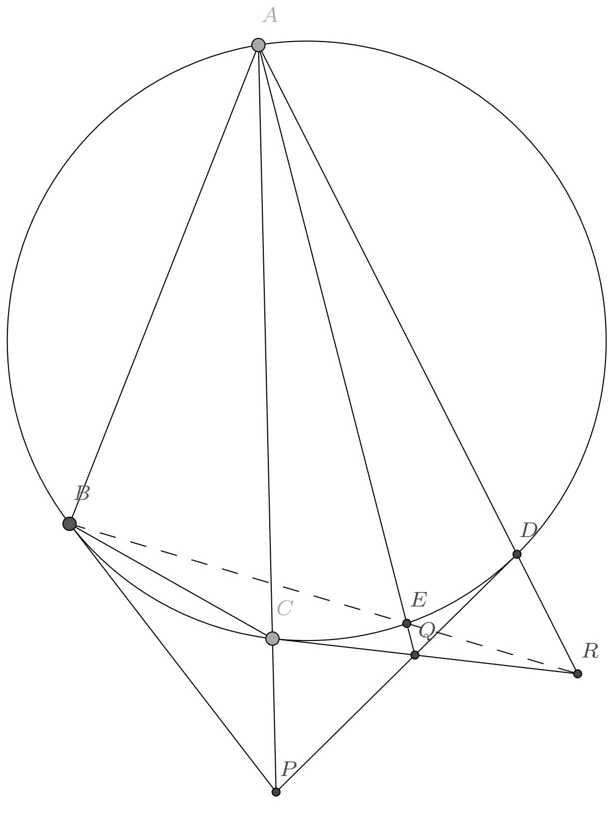
$$\frac{BD}{DC} = \frac{AB \cdot \sin \angle BAD}{CA \cdot \sin \angle CAD} = \frac{\sin \angle C \sin \angle(90^\circ - \angle B)}{\sin \angle B \sin \angle(90^\circ - \angle C)} = \frac{\sin \angle C \cos \angle B}{\sin \angle B \cos \angle C} ...(2)$$

and $\dfrac{BD}{BC} = \dfrac{BE}{CE} \cdot \dfrac{BF}{CF}$. This, combined with (1) and (2), entails that $\dfrac{BF}{CF} = \dfrac{\sin \angle C}{\sin \angle B} = \dfrac{AB}{AC}$. ■

Below, a daunting problem 5 on the APMO can be vulnerable under the hands of a calculation-hacker. This, however, requires the flexibility of converting ratios in terms of sines to those in terms of lengths (and vice versa).

3. *APMO 2013, Problem 5.* Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.

Solution.



It suffices to prove that $\frac{\sin \angle CBE}{\sin \angle ABE} = \frac{\sin \angle CBR}{\sin \angle ABR}$. Taking $\triangle ABC$ as a reference, the left hand side is simply $\frac{CE}{AE}$ (identity 4) while the right hand side is

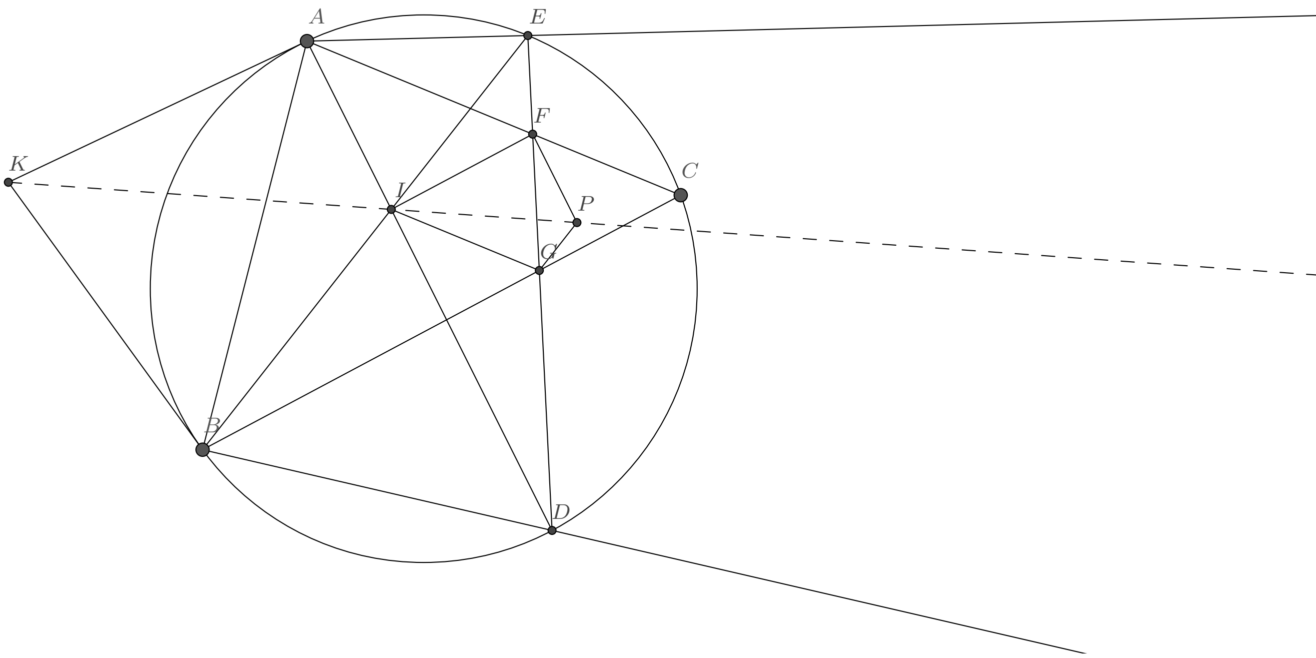
$$\frac{CR}{AR} \cdot \frac{\sin \angle BCR}{\sin \angle BAR} = \frac{\sin \angle CAD}{\sin \angle ACR} \cdot \frac{BC}{BD} = \frac{CD}{AC} \cdot \frac{BC}{BD}$$

due to identity 4 and corollary 1 of identity 5. Now to prove that the ratios are the same, we need $\frac{CD \cdot BC \cdot AE}{CE \cdot CA \cdot BD} = 1$. From the statment of the problem, both $ABCD$ and $ACED$ are harmonic quadrilaterals, and by Ptolemy's theorem, $AC \cdot BD = AB \cdot CD + AD \cdot BC = AD \cdot BC + AD \cdot BC = 2AD \cdot BC$. Likewise, $AE \cdot CD = 2AD \cdot CE$. Therefore, $\frac{CD \cdot BC \cdot AE}{CE \cdot CA \cdot BD} = \frac{2AD \cdot CE \cdot BC}{2AD \cdot BC \cdot CE} = 1$. ■

Now brace yourselves. We are going to use the first corollary of identity 5 twice.

4. *IMO 2011, G5.* Let ABC be a triangle with incenter I and circumcircle ω. Let D and E be the second intersection points of ω with the lines AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and at B meet at a point K. Prove that the three lines AE, BD, and KP are either parallel or concurrent.

Solution: Let's split the problems into two parts:
Part 1: prove that K, I, P are collinear.
Proof: First notice that $\angle DEB = \angle DAB = \frac{\angle A}{2} = \angle CAD = \angle CED$ and similarly $\angle CDE = \frac{\angle B}{2} = \angle EDA$. Therefore $\triangle CED \cong \triangle IED$ and ED is the perpendicular bisector of segment CI. This entails that $CF = CG$ and $\angle CFG = \angle CGF = \angle IFG = \angle IGF = 90° - \frac{\angle C}{2}$.Knowing that $FP \parallel AI$ and $GP \parallel BI$, we want to prove that $\angle FPI = \angle AIK$, or equivalently $\angle GPI = \angle BIK$. It suffices to prove that $\frac{\sin \angle FPI}{\sin \angle GPI} = \frac{\sin \angle AIK}{\sin \angle BIK}$.



Now from the corollary we have:

$$\frac{\sin\angle FPI}{\sin\angle GPI} = \frac{FI}{GI}\cdot\frac{\sin\angle PFI}{\sin\angle PGI} = \frac{\sin(\angle CFI - \angle CFP)}{\sin(\angle CGI - \angle CGP)} = \frac{\sin(180^\circ - \angle C - \frac{\angle A}{2})}{\sin(180^\circ - \angle C - \frac{\angle B}{2})} = \frac{\sin(\angle C + \frac{\angle A}{2})}{\sin(\angle C + \frac{\angle B}{2})}$$

(Why? Recall that $FP \parallel AI \Rightarrow \angle CFP = \angle CAI$, $GP \parallel BI \Rightarrow \angle CPG = \angle CBI$, and that $CFIG$ is a rhombus so $FI = GI$). Then

$$\frac{\sin\angle AIK}{\sin\angle BIK} = \frac{AK}{BK}\cdot\frac{\sin\angle IAK}{\sin\angle IBK} = \frac{\sin(\angle C + \frac{\angle A}{2})}{\sin(\angle C + \frac{\angle B}{2})} = \frac{\sin\angle FPI}{\sin\angle GPI},$$

as desired. (Don't ask me why $AK = BK$. You should know it.)

Part 2: prove that AE, BD and KI are concurrent (or parallel).

Proof: For convenience we assume that AE and BD are not parallel; the limit case (i.e. parallel) happens when $\angle C = 60^\circ$ and quadrilateral $AIBK$ is cyclic, which is left as an exercise (no trigo needed because angle chasing method becomes straightforward). Now let AE and BD intersect at Q. We need $\frac{\sin\angle AKQ}{\sin\angle BKQ} = \frac{\sin\angle AKI}{\sin\angle BKI}$, or $\frac{AQ}{BQ}\cdot\frac{\sin\angle QAK}{\sin\angle QBK} = \frac{AI}{BI}\cdot\frac{\sin\angle IAK}{\sin\angle IBK}$. Now $\frac{AQ}{BQ} = \frac{\sin\angle QBA}{\sin\angle QAB} = \frac{\sin\angle DBA}{\sin\angle EAB} = \frac{DA}{EB}$ and $\angle QAK = \angle EAK = 180^\circ - \angle EBA$ so $\sin\angle EAK = \sin\angle EBA$. Similarly $\sin\angle QBK = \sin\angle DAB$. Thus,

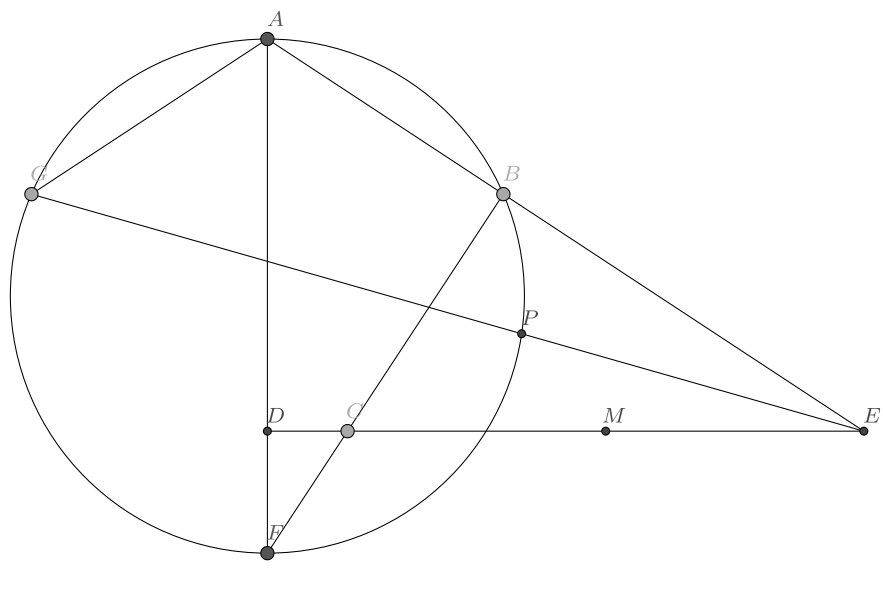
$$\frac{\sin\angle QAK}{\sin\angle QBK} = \frac{\sin\angle EBA}{\sin\angle DAB} = \frac{AI}{BI} = \frac{\sin\frac{\angle B}{2}}{\sin\frac{\angle A}{2}}$$

and $\frac{\sin\angle IAK}{\sin\angle IBK} = \frac{\sin\angle DAK}{\sin\angle EBK} = \frac{AD}{EB}$. Therefore $\frac{AQ}{BQ}\cdot\frac{\sin\angle QAK}{\sin\angle QBK} = \frac{\sin\frac{\angle B}{2}}{\sin\frac{\angle A}{2}}\cdot\frac{AD}{EB} = \frac{AI}{BI}\cdot\frac{\sin\angle IAK}{\sin\angle IBK}$. ■

If you enjoy some trigonometric brute force, here you are:

5. *RIMO 2014, Day 1, Problem 2.* In a quadrilateral $ABCD$ with $\angle B = \angle D = 90^\circ$, the extensions of AB and DC meet at E; and the extensions of AD and BC meet at F. A line through B parallel to CD intersects the circumcircle ω of the triangle ABF at G distinct from B; the line EG intersects ω at P distinct from G; and the line AP intersects CE at M. Prove that M is the midpoint of CE.

Solution.



For brevity denote $\angle EAF, \angle FEA, \angle AFE$ as $\angle A, \angle E, \angle F$ respectively. Now

$$\frac{DM}{ME} = \frac{AD}{AE} \cdot \frac{\sin \angle PAF}{\sin \angle BAP} = \cos \angle EAD \cdot \frac{\sin \angle PGF}{\sin \angle BGP} = \cos \angle A \cdot \frac{EF}{EB} \cdot \frac{\sin \angle EFG}{\sin \angle EBG}$$

$$= \cos \angle A \cdot \frac{1}{\cos \angle E} \cdot \frac{\sin(\angle F + \angle AFG)}{\cos \angle A} = \frac{\sin(\angle F - \angle A + 90^\circ)}{\cos(180^\circ - \angle F - \angle A)}$$
$$= \frac{\cos \angle F \cos \angle A + \sin \angle F \sin \angle A}{\sin \angle F \sin \angle A - \cos \angle F \cos \angle A}.$$

(Well just in case you are lost on how I obtain the equivalence of the angles, $\angle EBG = 180^\circ - \angle ABG = 90^\circ + \angle A$ and $\angle AFG = \angle BFA = 90^\circ - \angle A$).

Now

$$\frac{CD}{DE} = \frac{\tan \angle FAC}{\tan \angle FAE} = \frac{\tan(90^\circ - \angle F)}{\tan \angle A} = \frac{1}{\tan \angle A \tan \angle F} = \frac{\cos \angle A \cos \angle F}{\sin \angle A \sin \angle F}.$$

Therefore

$$\frac{CM}{ME} = \frac{DM}{ME} - \frac{DC}{ME} = \frac{\cos \angle F \cos \angle A + \sin \angle F \sin \angle A}{\sin \angle F \sin \angle A - \cos \angle F \cos \angle A} - \frac{\cos \angle A \cos \angle F}{\sin \angle A \sin \angle F} \cdot \frac{DE}{ME}$$

. Now

$$\frac{DE}{ME} = \frac{DM + ME}{ME} = 1 + \frac{DM}{ME} = 1 + \frac{\cos \angle F \cos \angle A + \sin \angle F \sin \angle A}{\sin \angle F \sin \angle A - \cos \angle F \cos \angle A}$$

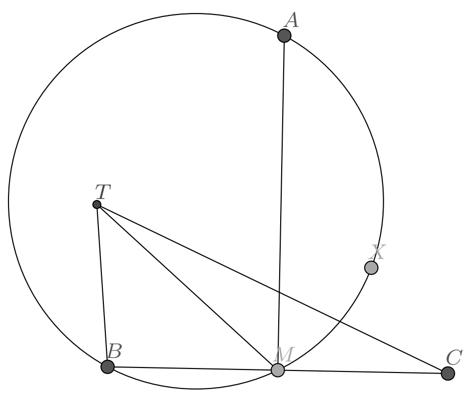
so our original ratio $\frac{CM}{ME}$ becomes $\frac{\cos \angle F \cos \angle A + \sin \angle F \sin \angle A}{\sin \angle F \sin \angle A - \cos \angle F \cos \angle A} - \frac{\cos \angle A \cos \angle F}{\sin \angle A \sin \angle F}$
$-\frac{\cos \angle A \cos \angle F}{\sin \angle A \sin \angle F} \cdot \frac{\cos \angle F \cos \angle A + \sin \angle F \sin \angle A}{\sin \angle F \sin \angle A - \cos \angle F \cos \angle A} = 1.$ ■

Not convinced? If $\cos \angle F \cos \angle A = z$ and $\sin \angle F \sin \angle A = y$ then what we have is $\frac{y+z}{y-z} - \frac{z}{y} - \frac{z}{y} \cdot \frac{y+z}{y-z} = 1 + \frac{2z}{y-z} - \frac{z}{y}(1 + \frac{y+z}{y-z}) = 1 + \frac{2z}{y-z} - \frac{z}{y}(\frac{2y}{y-z}) = 1 + \frac{2z}{y-z} - \frac{2z}{y-z} = 1.$

For the final blow we are going to be crazy: virtually turning a geometry problem into an algebra one. This is, of course, not the most desirable solution but what else can we do when we are desperate?

6. *IMO 2007, G2.* Given an isosceles triangle ABC with $AB = AC$. The midpoint of side BC is denoted by M. Let X be a variable point on the shorter arc MA of the circumcircle of triangle ABM. Let T be the point in the angle domain BMA, for which $\angle TMX = 90^\circ$ and $TX = BX$. Prove that $\angle MTB - \angle CTM$ does not depend on X.

Solution.



Well we can prove that $\cos(\angle MTB - \angle CTM)$ is constant, but we need to have both sines and coses of the two angles. Name $\angle BAM = b$ and $\angle AMX = \angle BMT = x$, assuming $AB = 1$, all chords on the circumcircle of BAM is equal to the sine of angle subtended by that chord. Therefore $CM = BM = \sin b$, $TX = BX = \sin(90^\circ + x) = \cos x$ and $MX = \sin(180^\circ - b - (90^\circ + x)) = \cos(b + x)$. Now $TM = \sqrt{TX^2 - MX^2} = \sqrt{\cos^2 x - \cos^2(b + x)} = \sqrt{-\sin(-b)\sin(b + 2x)} = \sqrt{\sin b \sin(b + 2x)}$ by the final identity in section 1 above. First of all let's prove that $BT \cdot CT = \sin b \sin(2x)$. Observe that:

$$BT^2 = BM^2 + MT^2 - 2BM \cdot MT \cdot \cos \angle BMT$$

$$= \sin^2 b + \sin b \sin(b + 2x) - 2 \sin b \sqrt{\sin b \sin(b + 2x)} \cos x$$

while

$$CT^2 = CM^2 + MT^2 - 2CM \cdot MT \cdot \cos \angle CMT$$

$$= \sin^2 b + \sin b \sin(b + 2x) + 2 \sin b \sqrt{\sin b \sin(b + 2x)} \cos x.$$

Multiplying the two yields

$(\sin^2 b + \sin b \sin(b + 2x))^2 - (2 \sin b \sqrt{\sin b \sin(b + 2x)} \cos x)^2$
$= \sin^4 b + \sin^2 b \sin^2(b + 2x) + 2 \sin^3 b \sin(b + 2x) - 4 \sin^3 b \sin(b + 2x) \cos^2 x$
$= \sin^2 b(\sin^2 b + \sin^2(b + 2x) + 2 \sin b \sin(b + 2x)(1 - 2 \cos^2 x))$
$= \sin^2 b(\sin^2 b + \sin^2(b + 2x) - 2 \sin b \sin(b + 2x) \cos(2x))$
because $\cos 2x = \cos^2 x - 1$. Now expand $\sin(b + 2x)$, we get $\sin b \cos(2x) + \cos b \sin(2x)$ and $\sin^2(b+2x) = \sin^2 b \cos^2(2x)+\cos^2 b \sin^2(2x)+2 \cos b \sin b \cos(2x) \sin(2x)$. The original expression then becomes

$\sin^2 b(\sin^2 b - \sin^2 b \cos^2(2x) + \cos^2 b \sin^2(2x))$
$= \sin^2 b(\sin^2 b(1 - \cos^2(2x)) + \cos^2 b \sin^2(2x))$
$= \sin^2 b(\sin^2 b \sin^2(2x) + \cos^2 b \sin^2(2x))$
$= \sin^2 b \sin^2(2x)$

because $\sin^2 a + \cos^2 a \equiv 1$ for any a.

Now $\cos \angle BTM = \frac{BT^2+TM^2-BM^2}{2 \cdot BT \cdot TM}$, but since $BT^2 = BM^2 + MT^2 - 2BM \cdot MT \cdot \cos \angle BMT$ we can write the cosine as $\frac{2TM^2-2BM \cdot MT \cos \angle BMT}{2 \cdot BT \cdot TM} = \frac{TM-BM \cdot \cos \angle BMT}{BT} = \frac{TM-\sin b \cos x}{BT}$. Similarly $\cos \angle CTM = \frac{TM+\sin b \cos x}{CT}$. Therefore $\cos \angle BTM \cdot \cos \angle CTM = \frac{TM^2-\sin^2 b \cos^2 x}{BT \cdot CT} = \frac{\sin b \sin(b+2x)-\sin^2 b \cos^2 x}{\sin b \sin(2x)}$. Meanwhile, $\sin \angle BTM = \sin \angle TMB \cdot \frac{BM}{TB} = \frac{\sin b \sin x}{BT}$ and $\sin \angle CTM = \sin \angle TMC \cdot \frac{CM}{TC} = \frac{\sin b \sin x}{CT}$ so $\sin \angle BTM \cdot \sin \angle CTM = \frac{\sin^2 b \sin^2 x}{BT \cdot CT} = \frac{\sin^2 b \sin^2 x}{\sin b \sin(2x)}$. Therefore

$\cos(\angle MTB - \angle CTM) = \cos \angle BTM \cdot \cos \angle CTM + \sin \angle BTM \cdot \sin \angle CTM$
$= \frac{\sin b \sin(b+2x)-\sin^2 b \cos^2 x}{\sin b \sin(2x)} + \frac{\sin^2 b \sin^2 x}{\sin b \sin(2x)}$
$= \frac{\sin(b+2x)-\sin b \cos^2 x+\sin b \sin^2 x}{\sin(2x)}$
$= \frac{\sin b \cos(2x)+\cos b \sin(2x)-\sin b \cos(2x)}{\sin(2x)}$
$= \cos b$. Independent of x isn't it? In fact it is now evident that the angle difference is indeed b. (phew!) ■

4 Practice problems.

It's finally your turn to demonstrate your ability in problem solving with one extra tool! While you may enjoy solving these problems using trigonometry, please look up the official solutions to understand the synthetic solutions after you have finished them.

1. Let $ABCD$ be a quadrilateral with $\angle B = \angle D = 90^\circ$. Let E and F be the feet of altitude from A and C to line BD, respectively. Prove that $BE = DF$.

2. (Proof of harmonic quadrilateral identity) Let $ABCD$ be a cyclic quadrilateral, inscribed in 'circumcircle' Γ. (Assume no angle of the cyclic quadrilateral is right). Then the tangent to Γ at A, the tangent to Γ at C and line BD are concurrent if and only if $AB \cdot CD = AD \cdot BC$. Notice that this also entails that the tangent at B, the tangent at D and line AC are concurrent.

3. *TOT, Fall 2010, Junior A-Level Problem 6.* In acute triangle ABC, an arbitrary point P is chosen on altitude AH. Points E and F are the midpoints of sides CA and AB respectively. The perpendiculars from E to CP and from F to BP meet at point K. Prove that $KB = KC$.

4. *JOM 2013, G6.* Consider a triangle ABC. Points P, Q lie on AB, AC respectively such that the four points B, C, P, Q are concyclic. The reflection of P across AC is P_0, and the reflection of Q across AB is Q_0. The circumcircles γ_1 and γ_2 of APP' and AQQ', respectively, intersect again at S distinct from A. Furthermore, BS intersects γ_1 again at X, and CS intersects γ_2 again at Y.

   Prove that the four points P, Q, X, Y lie on a circle.

5. *IMO 2012, G2.* Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet t F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under the reflection in AD. Prove that D, H, F, G are concyclic.

6. *IMO 2012, G4.* Let ABC be a triangle with $AB \neq AC$ and circumcenter O. The bisector of $\angle BAC$ intersects BC at D. Let E be the reflection of D with respect to the midpoint of BC. The lines through D and E perpendicular to BC intersect the lines AO and AD at X and Y respectively. Prove that the quadrilateral $BXCY$ is cyclic.

7. *IMO 2012, G3.* In an acute triangle ABC the points D, E, and F are the feet of altitudes through A, B, and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 an O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

8. *IMO 2008, G4.* In an acute triangle ABC segments BE and CF are altitudes. Two circles passing through the points A and F are tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that the lines PE and QF intersect on the circumcircle of triangle AEF.

9. *IMO 2011, G7.* Let$ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with center O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

5 Hints (or outlines) to practice problems.

1. No hint (straightforward application of third identity in section 2).

2. Let tangents at A and C intersect at P. Try to prove that $\frac{\sin\angle APD}{\sin\angle CPD} = (\frac{AD}{DC})^2$ using the first corollary of fact 5 at section 2.

3. Relate $\frac{\tan\angle ABC}{\tan\angle ACB}$ to $\frac{BH}{HC}$. What can you say about $\frac{\tan\angle PBC}{\tan\angle PCB}$? As we want to show that K is on the perpendicular bisector of AB, it will be extremely useful to draw the midpoint of BC and the altitude from it.

4. By radical axis identity we need to have PY, QX, AS concurrent. Now we know that PY divides AS in the ratio $\frac{AP}{AY} \cdot \frac{SP}{SY}$. (Notice that $\sin\angle PAY = \sin\angle PSY$ for they are supplementary. Similarly QX divides AS in the ratio $\frac{AQ}{AX} \cdot \frac{SQ}{SX}$. We want the two ratios to be the same.

5. One important lemma is $\triangle FDG \sim \triangle FBE$. To prove that $\angle DFG = \angle BFE$, bear in mind the second corollary following from identity 5 in section 2 (twist, and use it in 'reflection') and finish the lemma using the first corollary.

6. Locate the second intersection of AD and the circumcircle of triangle ABC. Reflect Y in the perpendicular bisector of BC. Then do power of point. (Remember to write each relevant side lengths in terms of the product of other known length and sine/cos of some angle, if necessary!)

7. The main part of it is to prove that AI_1I_2B is cyclic, and finish things off using radical axis identity. Remember, triangles AEF and ABC are similar with similitude $\cos\angle$???, so this is the ratio $\frac{AI_1}{AI}$ (I is the incenter of ABC). The identity $1 - \cos(2A) = 2\sin^2 A$ will also be useful here.

8. Our aim is to prove that $\angle QEB = \angle CEP$. Now use fact 3 of section 2 on triangle BFC and point Q, then on triangle BEC and point P. How are we going to write BQ and BP in terms of AB, BC, CA, $\sin\angle A$, $\cos\angle A$, $\sin\angle B$, $\cos\angle B$, $\sin\angle C$, $\cos\angle C$?

9. A problem that befits G7 should be hard, but there is a striaghtforward trigonometric hack. Now find three pairs of sides of equal length, and locate three angles that are the same. Aren't our job reduced to proving $CJ = EL$? Now with $DF \perp BK$ we have $BF^2 - BD^2 = KF^2 - KD^2$. Try to express the squares of four lengths interms of AF, AB, BC, CD, $EF, EK, DE, \angle A$ and $\frac{A}{2}$ so that we obtain EK (and hence EL) in relation to BC. Be sure to cover cases of $\angle C$ being acute, obtuse and right.

6 Final advice

Not all geometric problems are trigo-friendly as shown in the examples here; otherwise, I would have solved most hard geometry problems on the IMO. Therefore this hack is not a panacea for geometry and students should still improve on other geometric skills (particularly synthetic geometry). Technically speaking, some other problems may still be solvable using trigs, but may require skilful manipulations like expanding $\sin(3a + 2b)$. Other times, useful geometric obsevartions like similar triangles and cyclic quadrilaterals may help in making our trigo work clean.

"So what should I do?"

Practice! Read! Understand the motivation behind every single solution!