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# Center conditions at infinity for Abel differential equations

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## Abstract

An Abel differential equation  $y' = p(x)y^2 + q(x)y^3$  is said to have a center at a set  $A = \{a_1, \dots, a_r\}$  of complex numbers if  $y(a_1) = y(a_2) = \dots = y(a_r)$  for any solution  $y(x)$  (with the initial value  $y(a_1)$  small enough).

The polynomials  $p, q$  are said to satisfy the “Polynomial Composition Condition” on  $A$  if there exist polynomials  $\tilde{P}, \tilde{Q}$  and  $W$  such that  $P = \int p$  and  $Q = \int q$  are representable as  $P(x) = \tilde{P}(W(x))$ ,  $Q(x) = \tilde{Q}(W(x))$ , and  $W(a_1) = W(a_2) = \dots = W(a_r)$ . We show that for wide ranges of degrees of  $P$  and  $Q$  (restricted only by certain assumptions on the common divisors of these degrees) the composition condition provides a very accurate approximation of the Center one — up to a finite number of configurations not accounted for. To our best knowledge, this is the first “general” (i.e., not restricted to small degrees of  $p$  and  $q$  or to a very special form of these polynomials) result in the Center problem for Abel equations.

As an important intermediate result we show that “at infinity” (according to an appropriate projectivization of the parameter space) the Center conditions are given by a system of the “Moment equations” of the form  $\int_{a_1}^{a_s} P^k q = 0$ ,  $s = 2, \dots, r$ ,  $k = 0, 1, \dots$ .

## 1. Introduction

1.1. *Summary.* In this paper we consider a version of the classical Center-Focus problem, the one for Abel differential equations. This version is closely related to the original problem, and is believed to reflect all its main features and difficulties.

*Definition 1.1.* An Abel differential equation

$$(1.1) \quad y' = p(x)y^2 + q(x)y^3$$

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with polynomial coefficients  $p(x), q(x)$  is said to have a center at a set  $A = \{a_1, \dots, a_r\}$  of complex numbers if  $y(a_1) = y(a_2) = \dots = y(a_r)$  for any its solution  $y(x)$  (with the initial value  $y(a_1)$  small enough).

The Center-Focus problem is to give necessary and sufficient conditions on  $p, q$  for the Abel equation above to have a center.

A more standard setting of the problem is for  $r = 2$ , i.e. for  $A = \{a_1, a_2\}$ . We include this setting as the most important special case (some of our results refer to this case only). However, we believe that study of the more general setting with any number  $r \geq 2$  of the node points  $a_1, \dots, a_r$  may clarify some instances of the problem.

The only (known to us) *sufficient* condition for the Center is the following “Polynomial Composition Condition” (PCC):

*Definition 1.2.* The polynomials  $p = P', q = Q'$  are said to satisfy the Polynomial Composition Condition (PCC) on  $A = \{a_1, \dots, a_r\}$ , if there exist polynomials  $\tilde{P}, \tilde{Q}$  and  $W$  such that

$$P(x) = \tilde{P}(W(x)), \quad Q(x) = \tilde{Q}(W(x)), \quad W(a_1) = W(a_2) = \dots = W(a_r).$$

(PCC) is known also to be necessary for the Center for small degrees of  $p, q$  and in some other very special situations. Notice that (PCC) is described by a finite number of algebraic equations on the coefficients of  $p, q$ . It can be explicitly verified for given  $p, q$ .

In the present paper we show that for wide ranges of degrees of  $P$  and  $Q$  (restricted only by certain assumptions on the common divisors of these degrees) Composition condition (PCC) is “almost necessary”: it provides a very accurate approximation of Center conditions — up to a finite number of configurations, possibly not accounted for.

To get this result we first describe the Center set “at infinity”. It is well known that the necessary and sufficient Center conditions are provided by a complicated infinite set of algebraic equations, known as Center equations. The study of the Center-Focus problem for many years has been concentrated around a detailed investigation of the few initial Center equations. Virtually no information on a general structure of the entire system of the Center equations has been available.

In the present paper we show that “at infinity” (according to an appropriate projectivization of the parameter space) the Center equations are reduced to a much simpler system of the “Moment equations” of the form  $\int_{a_1}^{a_s} P^k q = 0$ .

Thus we reduce the Center-Focus problem at infinity to a difficult, but at least basically tractable Generalized Moment problem, deeply rooted in classical analysis and algebra. This problem is to give necessary and sufficient conditions on  $p, q$  for the Moment equations to hold. Indeed, it is difficult to find a more classically looking question: this is just a question of orthogonality of  $q$  to all the powers of  $P$  on each of the segments  $[a_1, a_s]$ ,  $s = 2, \dots, r$ .

The investigation of the Generalized Moment problem has been well under way for the last ten years ([9], [10], [15]–[19], [28], [29], [49], [53], [54], [61]). Deep relations of this problem to the Cauchy type integrals, with Topological Theory of Polynomials and with Composition Algebra have been found. In particular, Polynomial Composition Condition (PCC) turns out to be sufficient and “almost necessary” also for the Generalized Moment problem. Still, a complete answer has been obtained only very recently ([53]). This answer reduces the Generalized Moment problem to a verification of certain properties of  $P$  and  $q$  in Composition Algebra. Because of abundance of analytic and algebraic tools applicable, we expect now that a detailed understanding of the Generalized Moment problem can be achieved.

In our opinion, the main conclusion from the results of the present paper may be that *any information available in the Generalized Moment problem can be translated in almost a one-to-one way into information about Center conditions at infinity which, in turn, implies information on the affine (original) Center conditions.*

*Remark.* The special form of equation (1.1) is not essential in most of the results below. We can consider a general equation of the form

$$\frac{dy}{dx} = f(x, y) = \sum_{i=0}^{\infty} p_i(x)y^i$$

with exactly the same definition of the center as above. We restrict ourselves to the case of the polynomial Abel equation (1.1) in order to simplify the statements of the results and the proofs, and because it appears in the classical setting of the Center-Focus problem.

1.2. *Background.* Consider a system of differential equations

$$(1.2) \quad \begin{cases} \dot{x} = -y + F(x, y) \\ \dot{y} = x + G(x, y) \end{cases}$$

with  $F(x, y)$  and  $G(x, y)$  vanishing at the origin with their first derivatives. System (1.2) has a center at the origin if all the solutions around zero are closed. The classical Center-Focus problem, posed by H. Poincaré in the 1880’s ([56]), is to find conditions on  $F$  and  $G$  necessary and sufficient for system (1.2) to have a center at the origin.

This problem together with a closely related second part of Hilbert’s 16-th problem (asking for the maximal possible number of isolated closed trajectories of (1.2) with  $F(x, y)$  and  $G(x, y)$  polynomials of a given degree) have resisted until now all the attacks. Many deep partial results have been obtained (see [7], [8], [28]–[30], [34], [38], [42], [44], [45], [48], [55], [58], [63], [65], [72] and references therein) but general center conditions are not known even for  $F(x, y)$  and  $G(x, y)$  polynomials of degree three.

The classical approach to the Center-Focus problem is to analyze the conditions on the parameters of the system (1.2) provided by the vanishing of the first

several “obstructions” to the existence of the center. (Below we call these “obstructions” the Center equations. They form an infinite system of polynomial equations in the coefficients of  $F(x, y)$  and  $G(x, y)$ , and the degrees of these equations grow linearly with the order number of the equation).

If one can show that the first few of the Center equations imply the existence of the “first integral” of (1.2) then the system has a center and no further analysis of the obstructions is necessary. The problem with this approach is that *there is no known a priori bound on the number of the obstructions to be explicitly analyzed*. Thus as for today, already for  $F(x, y)$  and  $G(x, y)$  polynomials in  $x$  and  $y$  of degree three, all the efforts to reduce the (infinite) system of the Center equations to a certain finite list of integrability conditions failed. Consequently, the Center-Focus problem remains open starting with  $F$  and  $G$  of degree 3.

For some special classes of the system (1.2), like the Liénard equation and some others Center conditions can be given explicitly (see [63], [65], [28]–[33], [3]–[5]). However, in a general situation, and especially for  $F(x, y)$  and  $G(x, y)$  polynomials of high degree, only a part of Center configurations can be described while even a reasonable approximation of the entire Center set is not available. The reason is the one mentioned above: the Center equations are complicated, their degrees grow, and there is no general information on their algebraic-geometric structure.

The Abel equation version of the Center-Focus problem (see §1.2 below) has been studied in [1]–[6], [21]–[26], [28]–[32], [46], [47], [64], [67] (see also [66]) and in many other publications. For  $F$  and  $G$  in (1.2), homogeneous polynomials in  $x, y$ , this system can be reduced by a “Cherkas transformation” ([26]) to the Abel equation (1.1) with  $p, q$ -trigonometric polynomials. It is a general belief that the Abel equation version reflects the main difficulties of the original classical Center-Focus problem.

Around 1995 a new line of investigation in the Center-Focus problem for Abel equations was initiated in [15], [16], concentrating on the analytic and algebraic structure of the Center equations. It was shown in [15] that these equations can be constructed via a simple *linear* recurrence relation. An immediate conclusion was that these coefficients are given by combinations of *iterated integrals in  $p, q$* . (The same conclusion was obtained in [32], [33] by transforming the expressions obtained via the original *nonlinear* recurrence relation. Recently, an iterated integrals representation of the Center equations was revealed in [22], [23] via the Picard iterations). See also [39].

In particular the *first order terms in  $p$  (resp. in  $q$ ) turned out to be “Generalized Moments” of the form  $\int P^k q, \int Q^k p$ . This fact opened important connections with classical analysis and algebra and stressed the role of the “polynomial moment problem” and of composition algebra in the search for Center conditions ([9], [10], [12]–[19]). It allowed for application of strong analytic and algebraic tools, like classical Moment problem techniques, Cauchy type integrals, Ritt’s results*

on polynomial compositions, etc. to the study of the infinitesimal version of the Center-Focus problem ([18], [54], [60], [61], [69]).

New significant progress in the polynomial moment problem has been achieved by F. Pakovich in [49], [53]. In particular, it was shown in [49] that the Composition Condition (PCC) is not always necessary for the vanishing of the moments. New techniques, including topological theory of polynomials, Galois groups and some techniques of number theory have been used, and on this base a rather accurate description of many special cases of the polynomial moment problem has been achieved by F. Pakovich. In particular, many cases have been specified where (PCC) is equivalent to the moment's vanishing. Very recently a complete solution of the polynomial moment problem was shown in [53], which provides the answer in terms of certain properties of  $P$  and  $q$  in composition algebra. In some important cases (as for Chebyshev polynomials) the necessary computations in composition algebra can be pushed forward to give an explicit description of all the cases of the vanishing of  $\int P^k q$  ([49], [50], [53]).

These results opened a way to investigation of a *general structure* of the Center equations. In [11] the following important local interaction pattern was found between the Moment equations, Composition condition and Center equations: *Moment equations form a linear part of the Center ones. Their vanishing (mostly) implies the Composition condition, which, in turn, implies the vanishing of all the high-order terms in the Center equations.* The conclusion, via the algebraic-geometric “Nakayama lemma”, is that locally the zeroes of the Moment equations are the same as for the full Center equations, and they are completely characterized by the Composition condition. On this base, the local coincidence of Center and Composition conditions was shown in [11] under rather general assumptions.

In this paper we continue a general investigation of the algebraic-geometric structure of the Center equations. We show that *also “at infinity” these equations are essentially reduced to the Moment equations.*

Now an application of the presently available results on the polynomial moment problem provides a *coincidence of the Center and Composition conditions at infinity* under rather general assumptions.

Next, an application of the methods and results of [11] allows us to extend this coincidence *to the entire neighborhood of the infinite hyperplane.*

Finally we use the following elementary remark from algebraic geometry: if a complex algebraic set  $\mathcal{A}$  does not touch the infinite hyperplane, then  $\dim \mathcal{A} = 0$ . In our case, if the Center set has a component  $\mathcal{A}$  not inside the Composition set, then  $\mathcal{A}$  does not come to infinity. Indeed, otherwise  $\mathcal{A}$  would cross the neighborhood of the infinity — in contradiction to the previous step. Therefore,  $\dim \mathcal{A} = 0$ .

We believe that Theorems 1.1–1.6 below are the first “general” results in the Center-Focus problem for the Abel equation.

1.3. *Main results.* A simple, but basic fact relating Composition and Center conditions formulated above is given by the following proposition:

PROPOSITION 1.1. *If  $p, q$  satisfy the Polynomial Composition Condition (PCC) on  $A = \{a_1, \dots, a_r\}$  then the Abel equation (1.1) has a center on  $A$ .*

*Proof.* Indeed, after a change of variables  $w = W(x)$  we get a new polynomial Abel equation

$$(1.3) \quad \frac{d\tilde{y}}{dw} = \tilde{p}(w)\tilde{y}^2 + \tilde{q}(w)\tilde{y}^3,$$

with  $\tilde{p} = \tilde{P}'$ ,  $\tilde{q} = \tilde{Q}'$ . All the solutions of (1.1) are obtained from the solutions of (1.3) by the same substitution  $y(x) = \tilde{y}(W(x))$ . Since  $W(a_i) = W(a_j)$ ,  $i, j = 1, \dots, r$ , we conclude that  $y(a_i) = y(a_j)$ ,  $i, j = 1, \dots, r$ .  $\square$

A composition condition similar to (PCC) was introduced for a trigonometric Abel equation in [1], [6] (see also [5]). The condition (PCC) itself has been introduced and intensively studied in [15]–[18], [9], [10], [29], [67].

Now we can state the main results of this paper in the original “affine” setting, before passing to the projectivization of the parameter space and to the infinite hyperplane. Consider first of all the “classical” case  $A = \{a_1, a_2\}$ . The following theorem gives an accurate description of the Center set of (1.1), up to a finite number of points, possibly not accounted for, in the arbitrarily large ranges of the degrees of  $p$  and  $q$  (if we put certain restrictions on the possible common divisors of these degrees).

For two natural numbers  $m$  and  $n$ ,  $m \leq n$ , denote by  $\Omega_{m,n}$  the set of polynomials  $p$  of the form  $p(x) = \sum_{i=m}^n \alpha_i x^i$ .

THEOREM 1.1. *Let  $A = \{a_1, a_2\}$  and let a polynomial  $q$  of an arbitrary degree  $d$  be fixed, together with the natural numbers  $m$  and  $n$ ,  $m \leq n$ . Assume that the interval  $[m + 1, n + 1]$  does not contain any nontrivial multiples of the prime divisors of  $d + 1$  (but possibly contains some of these prime divisors themselves). Then the set of  $p \in \Omega_{m,n}$  for which equation (1.1) has a center on  $\{a_1, a_2\}$  consists of all such  $p \in \Omega_{m,n}$  that  $p, q$  satisfy (PCC) on  $\{a_1, a_2\}$ , and possibly of a finite number of additional polynomials  $p_1, \dots, p_s$ .*

In Section 6 below we give an explicit description of the Composition set arising in Theorem 1.1. Basically, it is a union of the linear subspaces in  $\Omega_{m,n}$  corresponding to the “prime composition divisors” of  $Q = \int q$ . Let us give now a few corollaries in order to illustrate some special cases of Theorem 1.1.

COROLLARY 1.1. *Let  $A = \{a_1, a_2\}$  and let a polynomial  $q$  of an arbitrary degree  $d$  be fixed, together with the degree  $d_1$  of  $p$ . Assume that each prime divisor of  $d + 1$  is greater than  $\lfloor \frac{d_1 + 1}{2} \rfloor$ . Then the set of  $p$  of degree at most  $d_1$  for which equation (1.1) has a center on  $\{a_1, a_2\}$  consists of all  $p$  such that  $p$  and  $q$  satisfy the Polynomial Composition Condition (PCC) on  $\{a_1, a_2\}$ , and possibly of a finite number of additional polynomials  $p_1, \dots, p_s$ .*

COROLLARY 1.2. *Let  $A = \{a_1, a_2\}$  and let a polynomial  $q$  of an arbitrary degree  $d$  be fixed, together with the natural numbers  $m$  and  $n$ ,  $m \leq n$ . Assume that*

all the numbers in the interval  $[m + 1, n + 1]$  are relatively prime with  $d + 1$ . Then the set of  $p \in \Omega_{m,n}$  for which equation (1.1) has a center on  $\{a_1, a_2\}$  consists of a finite number of isolated polynomials  $p_1, \dots, p_s$ .

**COROLLARY 1.3.** *Let  $A = \{a_1, a_2\}$  and let a polynomial  $q$  of degree  $d$  be fixed, with  $d + 1$  prime. Then the set of  $p$  of degree at most  $2d$  for which equation (1.1) has a center on  $\{a_1, a_2\}$  consists of all the  $p$  of the form  $p = \alpha q$ ,  $\alpha \in \mathbb{C}$ , and possibly of a finite number of additional polynomials  $p_1, \dots, p_s$ .*

Let us now state our main result in the setting of the Center-Focus problem on a general set  $A = \{a_1, \dots, a_r\}$ :

**THEOREM 1.2.** *Let  $A = \{a_1, \dots, a_r\}$ ,  $r \geq 3$ , and let a polynomial  $q$  of an arbitrary degree  $d$  be fixed. Then the set of  $p$  of degree at most  $2r - 2$  for which equation (1.1) has a center on  $A$  consists of all  $p$  such that  $p$  and  $q$  satisfy the Polynomial Composition Condition (PCC) on  $A$ , and possibly of a finite number of additional polynomials  $p_1, \dots, p_s$ .*

In a special case when  $r = 2$ , the degree bound  $2r - 2$  of Theorem 1.2 is two. However, in this case the following stronger result is valid:

**THEOREM 1.3.** *Let  $A = \{a_1, a_2\}$  and let a polynomial  $q$  of arbitrary degree  $d$  be fixed. Then the set of  $p$  of degree at most four for which the equation (1.1) has a center on  $\{a_1, a_2\}$  consists of all  $p$  such that  $p, q$  satisfy (PCC) on  $\{a_1, a_2\}$  and possibly of a finite number of additional polynomials  $p_1, \dots, p_s$ .*

In each of the cases covered by Theorems 1.1–1.3 there is

$$N = N(\deg p, \deg q, r)$$

such that the zero set of the first  $N$  Center equations differs from the Composition set by at most a finite number of points. This follows immediately from the Hilbert finiteness theorem. However, we do not know *a priori* any explicit bound on  $N$ . We recall that bounding the necessary number of the Center equations (or of the “obstructions to the existence of the first integral”) is one of the major difficulties in the Center-Focus problem.

We can show (see §8 below) that in all the cases covered in the present paper the problem of explicitly bounding  $N = N(\deg p, \deg q, r)$  can be reduced to a similar problem for the moments. In this last problem we have some partial results (see [11], [20]), and we expect that an explicit bound for the necessary number of the Moment equations can be given also in the general case. However, in the present paper we restrict ourselves only to the following special situation:

**THEOREM 1.4.** *In the case covered by Theorem 1.2 let  $N = (r - 1)(d + r - 1)$ . Then the zero set of the first  $N$  Center equations differs from the Composition set by at most a finite number  $s$  of points, where*

$$s = \frac{1}{2}4^{(r-1)}(d + r - 1)^{2(r-1)}.$$



We prove Theorems 1.1–1.4 in Section 8.

Notice that equation (1.1) is not symmetric with respect to  $p$  and  $q$ . However, it turns out that if we fix  $p$  in (1.1) and consider  $q$  as a free parameter, the results of Theorems 1.1–1.4 remain valid, with  $p$  and  $q$  interchanged. This fact is in no way clear *a priori*, and we prove it below, using some specific algebraic properties of the Center equations.

*Remark.* One can ask whether the “finite number of additional points” in the results above may indeed appear, or are they artifacts of our approach. The only indication in this regard we have at this moment is that in all the specific situations covered in [15]–[19], [9], [10], [29], [67], [69] no additional center configurations appear, besides the Composition ones.

All the information we have today on the Center conditions for the polynomial Abel equation, including Theorems 1.1–1.3 above, and their counterparts for  $p$  fixed and  $q$  free, supports the following conjecture:

COMPOSITION CONJECTURE. *The Abel equation (1.1) with polynomial coefficients  $p, q$  has a center on the set of points  $A = \{a_1, \dots, a_r\}$  if and only if the Composition Condition (PCC) holds for  $p$  and  $q$  on  $A$ .*

This conjecture has been verified for small degrees of  $p$  and  $q$  and in many special cases in [11], [15]–[19], [9], [10], [29], [67], [69], [71].

It is important to stress once more that *equation (1.1) is not symmetric with respect to  $p$  and  $q$ , while the Composition Condition is.*

However, one of the results of this paper (Theorem 1.6 below) shows that *the Composition Conjecture is not true “at infinity”*. Still, there are indications that all the non-Composition Center components at infinity may disappear in the affine part of the parameter space.

Now let us pass to the description of the Center equations and the Center set at infinity. Let the set  $A = \{a_1, \dots, a_r\}$ ,  $r \geq 2$ , be fixed.

From now on we shall always assume that the polynomials  $p = P'$  and  $q = Q'$  satisfy the following condition:

$$(1.4) \quad P(a_1) = \dots = P(a_r) = Q(a_1) = \dots = Q(a_r) = 0.$$

Condition (1.4) follows from the Center equations. Indeed, the second and the third Center equations on  $A = \{a_1, a_2\}$  are shown in Section 2 below to be  $P(a_1) - P(a_2) = 0$  and  $Q(a_1) - Q(a_2) = 0$ . Therefore, in general, a few initial Center equations imply  $P(a_1) = \dots = P(a_r)$  and  $Q(a_1) = \dots = Q(a_r)$ . Given  $p$  and  $q$  we can always choose their primitives  $P$  and  $Q$  to satisfy  $P(a_1) = 0$ ,  $Q(a_1) = 0$ , and the condition (1.4) follows.

However, the relation between the Center equations and the Moment equations becomes much more complicated, than presented below, if we do not assume (1.4). Consequently, without this assumption the statement of most of our results would be much more cumbersome. Notice that in general it is an open question, whether

the Moment equations by themselves imply (1.4). See [54] for a discussion and some partial results in this direction.

As far as an accurate definition of the “Center set at infinity” is concerned, let us notice that there are several possible ways of defining a compactification of the parameter space of (1.1). In a detailed presentation below we have to consider and to compare some of them (in particular, in order to prove Theorems 1.1–1.4 and their “symmetric” counterparts). However, here in the introduction we present our results only for one specific compactification where the coefficients of  $p$  and  $q$  are scaled proportionally.

Let  $V_d$  denote the space of complex polynomials of degree at most  $d$ . We assume that the coefficients  $p, q$  of the Abel equation (1.1) are polynomials of the degrees  $d_1$  and  $d_2$ , respectively:  $p \in V_{d_1}, q \in V_{d_2}$ . We denote by  $V_{d_1, d_2}(A)$  the linear subspace in  $V_{d_1} \times V_{d_2} \cong C^{d_1+d_2+2}$  of all the couples  $(p, q)$ , for which their primitives  $P = \int_{a_1} p$  and  $Q = \int_{a_1} q$  satisfy condition (1.4).

*Definition 1.3.* The Center set of the Abel equation (1.1) on  $A = \{a_1, \dots, a_r\}$  is the set  $\mathcal{C} \subset V_{d_1, d_2}(A)$  of all  $(p, q) \in V_{d_1, d_2}(A)$  for which equation (1.1) has a center on  $A$ .

In a similar way we define the Composition set:

*Definition 1.4.* The *Composition set* on  $A = \{a_1, \dots, a_r\}$  is the set  $\mathcal{L} \subset V_{d_1, d_2}(A)$  of all  $(p, q) \in V_{d_1, d_2}(A)$ , satisfying the Polynomial Composition Condition (PCC) on  $A$ .

Before stating the results, we have to introduce the last main ingredient in our approach, which is provided by the Moment vanishing conditions.

*Definition 1.5.* We say that the *first Moment vanishing condition* (or the first system of the Moment equations) is satisfied for the polynomials  $p = P', P(a_1) = 0$ , and  $q = Q'$  on the set  $A = \{a_1, \dots, a_r\}$  if the following equalities are true:

$$(1.5) \quad m_k(p, q, a_1, a_s) = \int_{a_1}^{a_s} P^k(x)q(x)dx = 0, \quad s = 2, \dots, r, \quad k = 0, 1, \dots$$

Exactly as in Proposition 1.1 we can show that the Polynomial Composition Condition (PCC) implies the Moment vanishing condition (1.5).

*Definition 1.6.* The *Moment set* on  $A = \{a_1, \dots, a_r\}$  is the set  $\mathcal{M} \subset V_{d_1, d_2}(A)$  of all  $(p, q) \in V_{d_1, d_2}(A)$  satisfying the first Moment vanishing condition (1.5) on  $A$ .

Notice once more that the Composition condition (and the set) is symmetric with respect to  $p$  and  $q$ , while the Center condition and the Moment vanishing conditions (and the corresponding sets) apparently are not.

Now we consider the standard projectivization  $PV_{d_1, d_2}(A)$  of the parameter space  $V_{d_1, d_2}(A)$  (see §3 below). The Center, Composition, and Moment sets at infinity  $\mathcal{C}_\infty, \mathcal{L}_\infty$  and  $\mathcal{M}_\infty$  are defined as the intersections  $\mathcal{C}_\infty = \mathcal{C} \cap H, \mathcal{L}_\infty = \mathcal{L} \cap H, \mathcal{M}_\infty = \mathcal{M} \cap H$ , where  $H$  is the infinite hyperplane in the projective space  $PV_{d_1, d_2}(A)$ .

The following is the main result of this paper, as the Center problem at infinity is concerned:

**THEOREM 1.5.** *The Center set at infinity  $\mathcal{C}_\infty$  coincides with the Moment set  $\mathcal{M}$  defined by the first Moment vanishing condition (1.5).*

We prove Theorem 1.5 in Section 4. It completely reduces the Center problem “at infinity” to the Polynomial moment problem.

An important property of the system of the Moment equations (1.5) is that it is *linear with respect to  $q$  and highly nonlinear with respect to  $p$* . This property underlines many of the techniques and results below.

Notice also that the Moment equations are homogeneous with respect to each of  $p$  and  $q$  separately, in contrast to the Center equations (which are quasi-homogeneous with respect to  $p$  and  $q$  jointly, but not separately).

The next step in our approach is to compare the Moment sets  $\mathcal{M}$  with the Composition sets  $\mathcal{C}$  at infinity. The first very important conclusion from Theorem 1.5 and from the Pakovich results is that the Center set at infinity  $\mathcal{C}_\infty$  may be strictly larger than the Composition set  $\mathcal{L}_\infty$ :

**THEOREM 1.6.** *For  $A = \{a_1, a_2\}$  and for any fixed degrees  $d_1 \geq 5$  and  $d_2 \geq 2$ , the Center set at infinity  $\mathcal{C}_\infty \subset H \subset PV_{d_1, d_2}(A)$  is strictly larger than the Composition set  $\mathcal{L}_\infty \subset H \subset PV_{d_1, d_2}(A)$ .*

We prove Theorem 1.6 in Section 5 below.

So the Composition Conjecture is not true anymore for the Center set at infinity! Still, the non-Composition components may appear only on the infinite hyperplane  $H$ , and the affine Composition conjecture may still be true.

On the other hand, we show that under the appropriate restrictions on the degrees  $d_1$  and  $d_2$  the Center set and the Composition set at infinity coincide:

**THEOREM 1.7.** 1. *Let  $A = \{a_1, \dots, a_r\}$ . For  $d_1 \leq 2r - 2$  and any  $d_2$  the Center set and the Composition set at infinity coincide.*

2. *Let  $A = \{a_1, a_2\}$ . For  $d_1 \leq 4$  and any  $d_2$  the Center set and the Composition set at infinity coincide.*

3. *Let  $A = \{a_1, a_2\}$ . Assume that  $q$  of degree  $d_2$  is fixed. If each of the prime divisors of  $d_2 + 1$  is larger than  $\lfloor \frac{1}{2}d_1 \rfloor$  then the Center set and the Composition set at infinity coincide.*

There are some additional situations where we can show the coincidence of the Center set and the Composition set at infinity. We prove Theorem 1.7 and its corresponding extensions in Section 5.

In Section 7 we extend the coincidence of the Center set and the Composition set to the entire neighborhood of infinity. However, these results form just a technical step towards Theorems 1.1–1.3 (and follow from these theorems); so we do not state them here.

1.4. *Organization of the paper.* The rest of the paper is devoted to the detailed proof of the results stated in the introduction and certain of their extensions.

In Section 2 we derive and study the Center equations. To overcome combinatorial difficulties arising in the determination of the explicit form of these equations, we study analytically certain generating functions (derivatives of the Poincaré first return mapping with respect to small parameters).

In Section 3 a detailed description is given of different settings of the Center-Focus problem at infinity, and of the corresponding projectivizations of the parameter space. In particular, we derive in each case the Center equations at infinity.

In Section 4 we prove Theorem 1.5 and some of its extensions.

In Section 5 we use the results on Moment and Composition Conditions in order to prove Theorems 1.6 and 1.7 and some additional results providing the coincidence, under proper assumptions, of the Center and Composition sets at infinity.

In Section 6 a detailed “geometric” description of the arising Composition sets is given. Indeed, from the point of view of Section 5 the Composition set arises as the set of zeroes of a complicated nonlinear system of the Moment equations. This system by itself does not say much (at once) about the structure of its solutions.

To avoid this difficulty, in Section 6, instead of analyzing the Moment equations we study the Composition set *ad hoc* using the original results of Ritt ([57], [62]).

In Section 7 the coincidence of the Center and the Composition sets is extended from the infinite hyperplane  $H$  to its entire neighborhood via the results of Section 6 and the approach of [11]. This requires, in particular, a computation of the first (and higher) derivatives of the Center equations at infinity.

Finally, in Section 8 we prove Theorems 1.1–1.4.

## 2. Center equations

In this section we derive the Center equations and obtain the Center set as the algebraic set of zeroes of the Center equations. We investigate the structure of the Center equations in enough detail to allow us later to obtain the precise form of the Center equations at infinity.

2.1. *Poincaré first return mapping.* Let  $V_d$ , as above, denote the space of complex polynomials of degree at most  $d$ . We assume that the coefficients  $p, q$  of the Abel equation (1.1) are polynomials of the degrees  $d_1$  and  $d_2$ , respectively:  $p \in V_{d_1}$ ,  $q \in V_{d_2}$ .

The *Poincaré first return mapping*  $G(y, a, b)$  of the Abel equation (1.1) at  $a, b \in \mathbb{C}$  associates to each  $y = y_a$  the value  $G(y) = y(b)$  at the point  $b$  of the solution  $y(x)$  of (1.1) satisfying  $y(a) = y_a$  at the point  $a$ . For  $y = y_a$  sufficiently small the singularities of the solution  $y(x)$  are “far away” from the origin. Hence  $G(y) = y(b)$  does not depend on the continuation path from  $a$  to  $b$ , and so  $G(y)$

is a regular function for  $y$  near zero and is given by a convergent power series

$$(2.1) \quad G(y, a, b) = y + \sum_{k=2}^{\infty} v_k(\lambda, \mu, a, b)y^k,$$

where  $(\lambda, \mu) = (\lambda_0, \dots, \lambda_{d_1}, \mu_0, \dots, \mu_{d_2})$  is the set of coefficients of the polynomials  $(p, q) \in V_{d_1} \times V_{d_2}$ .

Clearly, the solution  $y(x)$  of (1.1) is “periodic” at  $(a, b)$  (i.e.  $y(a) = y(b)$ ) if and only if  $G(y(a), a, b) = y(a)$ . In the same way, the solutions  $y(x)$  of (1.1) satisfy  $y(a) \equiv y(b)$  if and only if  $G(y) \equiv y$ . In turn, this last condition is equivalent to the vanishing of all the coefficients  $v_k(\lambda, \mu, a, b)$  of (2.1). Therefore we get the following fact:

**PROPOSITION 2.1.** *Abel equation (1.1) has a center on  $A = \{a_1, \dots, a_r\}$  if and only if an infinite sequence of equations*

$$(2.2) \quad v_k(\lambda, \mu, a_1, a_j) = 0, \quad k = 2, \dots, \quad j = 2, \dots, r$$

*is satisfied.*

We call equations (2.2) the Center equations. We remind the reader that the Center set of the Abel equation (1.1) on  $A = \{a_1, \dots, a_r\}$  is the set  $\mathcal{C} \subset V_{d_1, d_2}(A)$  of  $(p, q) \in V_{d_1, d_2}(A)$  for which (1.1) has a center on  $A$ .

By Proposition 2.1,  $\mathcal{C}$  is defined in  $V_{d_1, d_2}(A)$  by an infinite system of equations (2.2). It is shown in Proposition 2.2 below that the equations in (2.2), i.e., the Taylor coefficients  $v_k(\lambda, \mu, a_1, a_j)$  of the Poincaré mapping (2.1), are polynomials in all their parameters, in particular, in the parameters  $\lambda, \mu$  of  $p$  and  $q$ . The degree of these polynomials grows linearly with the index  $k$ . We get the following:

**COROLLARY 2.1.** *The Center set  $\mathcal{C}$  is an algebraic subset in  $V_{d_1, d_2}(A)$ .*

*Proof.* By Hilbert’s finiteness theorem,  $\mathcal{C}$  is defined by a finite subsystem of the polynomial equations (2.2). □

Notice that Hilbert’s theorem provides no bound on the number of equations in this finite subsystem. As mentioned in the introduction, this fact presents one of the major problems in the treatment of the Center equations (or of the obstructions to the existence of the first integral). In this paper we provide some information in this direction (see Theorem 1.4).

As we shall see below, Center equations (2.2) have a pretty complicated form (especially if we write them explicitly in the parameters  $\lambda, \mu$ ). So their straightforward analysis is difficult. In the next section we investigate the analytic structure of these equations, while we insist on writing them through the iterated integrals in  $p, q$  and *not translating explicitly* the obtained expressions into the coefficients  $\lambda, \mu$  of  $p$  and  $q$ .

Let us stress that below we present only one of many possible approaches to the study of the coefficients of the Poincaré first return mapping. Let us mention

just a small sample of related results [21]–[25], [39], [40], [41], [20], [37], [59], [61], [73]. One can hope that a combination of different approaches will bring a better understanding of this subject.

2.2. *The structure of the Center equations.* From now until the end of Section 2, we restrict ourselves to the case  $A = \{a, b\}$ . By Proposition 2.1 the Center equations for any  $A = \{a_1, \dots, a_r\}$  are given by the vanishing of the Taylor coefficients of the Poincaré mapping on each of the intervals  $[a_1, a_i]$ , so that the results below are translated automatically to the general case.

It is convenient to “free” the endpoint  $b$  in the definition (2.1) of the Poincaré first return mapping  $G$ : if we denote by  $G(y, x)$  the Poincaré mapping  $G(y)$  from  $a$  to  $x$  we obtain the following convergent Taylor representation which can be used to express both the Poincaré mapping (as we fix  $x$ ) and the solutions of (1.1) (as we fix  $y$ ):

$$(2.3) \quad G(y, x) = y + \sum_{k=2}^{\infty} v_k(\lambda, \mu, a, x)y^k.$$

One can easily show (by substituting expansion (2.3) into equation (1.1)) that  $v_k(x) = v_k(\lambda, \mu, a, x)$  satisfy the recurrence relation

$$(2.4) \quad \begin{cases} v_0(x) \equiv 0, \\ v_1(x) \equiv 1, \\ v_n(0) = 0, \quad \text{and} \\ v'_n(x) = p(x) \sum_{i+j=n} v_i(x)v_j(x) + q(x) \sum_{i+j+k=n} v_i(x)v_j(x)v_k(x), \quad n \geq 2. \end{cases}$$

An immediate consequence is the following:

PROPOSITION 2.2. *The Taylor coefficients  $v_k(\lambda, \mu, a, x)$  are polynomials in  $a, x$  and in  $\lambda, \mu$ . In particular, for  $x = b$  the coefficients  $v_k(\lambda, \mu) = v_k(\lambda, \mu, a, b)$  are polynomials in all the parameters  $a, b, \lambda, \mu$ .*

*Proof.* This follows from the recurrence relation (2.4) via induction by  $k$ .  $\square$

It was shown in [15] that recurrence relation (2.4) can be linearized in the following sense: consider the inverse Poincaré mapping  $G^{-1}$  associating to the end value  $y(x) = y_x$  of each solution  $y$  of (1.1) its initial value  $y(a) = y_a$ . We have a Taylor expansion

$$(2.5) \quad y_a = G^{-1}(y_x) = y_x + \sum_{k=2}^{\infty} \psi_k(\lambda, \mu, a, x)y_x^k.$$

In particular, for  $x = b$  we get the inverse to the Poincaré mapping  $G$  at  $a, b$ . Hence the Center condition  $y(a) \equiv y(b)$  is equivalent to another infinite system

of polynomial in  $\lambda, \mu$  equations

$$(2.6) \quad \psi_k(\lambda, \mu, a, b) = \psi_k(\lambda, \mu) = 0, \quad k = 2, \dots$$

In fact, one can show (see [15]) that for each  $k = 2, \dots$  the ideals  $I_k = \{v_2(\lambda, \mu), \dots, v_k(\lambda, \mu)\}$  and  $I'_k = \{\psi_2(\lambda, \mu), \dots, \psi_k(\lambda, \mu)\}$  in the ring of polynomials in  $\lambda, \mu$  coincide. It was shown in [15] that for fixed  $\lambda, \mu$  the Taylor coefficients  $\psi_k(x) = \psi_k(\lambda, \mu, a, x)$  satisfy a *linear* recurrence relation

$$(2.7) \quad \begin{cases} \psi_0(x) \equiv 0, \\ \psi_1(x) \equiv 1, \\ \psi_n(0) = 0, \quad \text{and} \\ \psi'_n(x) = -(n-1)\psi_{n-1}(x)p(x) - (n-2)\psi_{n-2}(x)q(x), \quad n \geq 2. \end{cases}$$

Now one can see that each  $\psi_k(\lambda, \mu, a, b) = \psi_k(\lambda, \mu)$  can be written as a sum of iterated integrals: each summand has the form  $\text{Const} \cdot \int q \int p \dots \int p \int q$  (the order and the number of the integrands  $p$  and  $q$  vary). More accurately, the iterated integrals entering the polynomials  $\psi_k(\lambda, \mu)$  are given by

$$(2.8) \quad I_\alpha = \int_a^b h_{\alpha_1}(x_1)dx_1 \left( \int_a^{x_1} h_{\alpha_2}(x_2)dx_2 \dots \left( \int_a^{x_{s-1}} h_{\alpha_s}(x_s)dx_s \right) \dots \right).$$

Here  $\alpha = (\alpha_1, \dots, \alpha_s)$  with  $\alpha_j = 1$  or  $2$ , and  $h_1 = p, h_2 = q$ .

Formally integrating recurrence relation (2.7) we can obtain in a combinatorial way the “symbolic” expressions for  $\psi_k$  through the sums of the iterated integrals (2.8). The first few of these expressions for  $\psi_k$  are as follows:

$$\begin{aligned} \psi_0 &\equiv 0, \\ \psi_1 &\equiv 1, \\ \psi_2 &= - \int p = I_1, \\ \psi_3 &= 2 \int p \int p - \int q = 2I_{11} - I_2, \\ \psi_4 &= -6 \int p \int p \int p + 3 \int p \int q + 2 \int q \int p = -6I_{111} + 3I_{12} + 2I_{21}, \\ \psi_5 &= 24I_{1111} - 12I_{112} - 8I_{121} - 6I_{211} + 3I_{22}, \\ \psi_6 &= -120I_{11111} + 60I_{1112} + 40I_{1121} + 30I_{1211} - 15I_{122}, \\ &\quad + 24I_{2111} - 12I_{212} - 8I_{221}. \end{aligned}$$

*Remark.* Recurrence relation (2.4) produces more complicated expressions for  $v_k$ : they contain iterated integrals of the *products* of iterated integrals of lower orders. On the other hand, interchanging the endpoints  $a, b$  of the integration, we express  $v_k$  via recurrence relation (2.7) as the *sums* of iterated integrals. Thus we get for each  $k$  an identity between certain expressions in iterated integrals of  $p, q$  on  $(a, b)$ . Presumably, these identities correspond to a known fact in the theory of

iterated integrals: their products can be expressed as linear combinations of longer iterated integrals. See [25], [27].

In this paper we use the following fact concerning the structure of the Center equations:

**PROPOSITION 2.3.** *If the polynomials  $P$  and  $Q$  satisfy Polynomial Composition Condition (PCC) on  $A = \{a_1, a_2, \dots, a_r\}$  then for  $p = P'$  and  $q = Q'$  all the iterated integrals  $I_\alpha$  on  $a_i, a_j$  vanish. In particular, (PCC) on  $A$  implies vanishing of each of the terms in the Center equations (2.2) on  $A$ .*

*Proof.* It is enough to consider  $A = \{a, b\}$ . Under the factorization  $P(x) = \tilde{P}(W(x))$ ,  $Q(x) = \tilde{Q}(W(x))$  provided by the Polynomial Composition Condition (PCC) we can make a change of the independent variable  $x \rightarrow w = W(x)$  in the iterated integrals. We get

$$(2.9) \quad I_\alpha = \int_{W(a)}^{W(b)} \tilde{h}_{\alpha_1}(w_1)dw_1 \int_{W(a)}^{w_1} \tilde{h}_{\alpha_2}(w_2)dw_2 \dots \int_{W(a)}^{w_{s-1}} \tilde{h}_{\alpha_s}(w_s)dw_s.$$

Here  $\tilde{h}_{\alpha_j}(w) = \tilde{p}(w) = \tilde{P}'(w)$  for  $\alpha_j = 1$  and  $\tilde{h}_{\alpha_j}(w) = \tilde{q}(w) = \tilde{Q}'(w)$  for  $\alpha_j = 2$ . Now since  $\tilde{P}'(w)$  and  $\tilde{Q}'(w)$  are polynomials, all the subsequent integrands in (2.9) are polynomials. But by the conditions, we have  $W(a) = W(b)$  and the most exterior integral must be zero, being the integral of a certain polynomial over a closed contour. □

The basic combinatorial structure of the “symbolic” expressions for  $\psi_k$  produced via the recurrence relation (2.7) is given by the following proposition:

**PROPOSITION 2.4.** *For each  $k \geq 2$  the Poincaré coefficient  $\psi_k$  is given as the integer linear combination of the iterated integrals of  $p$  and  $q$ :*

$$(2.10) \quad \psi_k = \sum n_\alpha I_\alpha,$$

with the sum running over all the multi-indices  $\alpha = (\alpha_1, \dots, \alpha_s)$  for which  $\sum_1^s \alpha_j = k - 1$ . The number of the terms in the expression for  $\psi_k$  is the  $(k - 1)$ -th Fibonacci number. The integer coefficients  $n_\alpha$  are given as the products

$$(2.11) \quad n_\alpha = (-1)^s \prod_{r=1}^s (k - \sum_{j=1}^r \alpha_j).$$

The proof of this proposition is given in [11]. Another derivation of the iterated integrals form of the Center equations was obtained in [22], [23] by a completely different method.

Explicit analysis of the symbolic expressions for  $\psi_k$  is not easy. Integration by parts can be used to simplify them but ultimately this leads to a “word problem” which has been analyzed only partly (and only for the recurrence relation (2.4)) in [32], [33], [2].

In particular, some iterated integrals above containing more than one appearance of both  $p$  and  $q$  cannot be reduced to the one-sided or double moments by



“symbolic” operations (including integration by parts). This follows, in particular, from the example (given in [10]) of the Abel equation (1.1) with the coefficients  $p$ - and  $q$ -elliptic functions, for which all the double moments vanish while the Center equations are not satisfied.

Assuming that  $P(a) = Q(a) = 0$  and simplifying the subsequent equations via the preceding ones we obtain the following explicit form for the first seven Center equations in (2.6):

$$\begin{aligned}
 0 &= \psi_2(b) = -P(b), \\
 0 &= \psi_3(b) = -m_0 = -Q(b), \\
 0 &= \psi_4(b) = -m_1, \\
 0 &= \psi_5(b) = -m_2, \\
 0 &= \psi_6(b) = -m_3 + \frac{1}{2} \int_a^b p Q^2, \\
 0 &= \psi_7(b) = -m_4 - 2 \int_a^b P p Q^2, \\
 0 &= \psi_8(b) = -m_5 + \frac{1}{2} \int_a^b Q^3 p + \int_a^b P^3 Q q \\
 &\quad - 320 \int_a^b P^2(t) q(t) dt \int_a^t P q + 185 \int_a^b P(t) q(t) dt \int_a^t P^2 q.
 \end{aligned}$$

Here  $m_k = \int_a^b P^k(x) q(x) dx$ .

The form of these initial Center equations suggests some important general patterns which can be proved by a combination of integration by parts and of some combinatorial analysis. In particular, the iterated integrals where  $p$  (resp.  $q$ ) appear only once can be transformed via integration by parts to the moments form:

**THEOREM 2.1.** *For  $\alpha = (1, \dots, 1, 2, 1, \dots, 1)$  with  $k - 1$  indices 1 and exactly one index 2 appearing at the  $j^{\text{th}}$  place,*

$$(2.12) \quad I_\alpha = r(j) m_k(p, q),$$

with the coefficient  $r(j) = \frac{(-1)^{j-1}}{(j-1)!(k-j)!}$ .

*Proof.* The result is obtained via integration by parts  $(k - j)$  times from the right and  $(j - 1)$  times from the left, taking into account that  $P(a) = Q(a) = P(b) = Q(b) = 0$ . □

More generally, let the multi-index  $\alpha$  have exactly  $r$  indices 2 appearing with the consequent intervals  $m_i$  between them,  $m_i \geq 0$ . In other words, the index 2 appears in  $\alpha$  exactly at the places  $\sum_{j=0}^i (m_j + 1)$ ,  $i = 0, \dots, r - 1$ . Then  $\alpha$  has exactly  $\sum_{i=0}^r m_i$  indices 1. In particular, 1 appears on the  $m_r$  last places. Let us introduce the multi-indices  $\beta = (\beta_0, \beta_1, \dots, \beta_r)$  with  $\beta_0 = 0, \beta_r = m_r, 0 \leq \beta_i \leq m_i$

for  $1 \leq i \leq r - 1$ . We have the following result (which we do not use in the present paper):

**THEOREM 2.2.** *Any iterated integral  $I_\alpha$  can be transformed via integration by parts to the sum of the iterated integrals of the form*

$$I_\alpha = \sum_\beta \frac{(-1)^{\sum_{i=0}^r (m_i - \beta_i)}}{\prod_{i=0}^r \beta_i! (m_i - \beta_i)!} \times \int_a^b P^{m_0 - \beta_0 + \beta_1}(x_1)q(x_1)dx_1 \dots \int_a^{x_{r-1}} P^{m_{r-1} - \beta_{r-1} + \beta_r}(x_r)q(x_r)dx_r.$$

In particular, for  $\alpha$  with exactly two indices 2 appearing after  $m_0$  and  $m_1$  indices 1 respectively,

$$\begin{aligned} I_\alpha &= \sum_{i=0}^{m_1} \frac{(-1)^{m_0+m_1-i}}{m_0!m_2!i!(m_1-i)!} \int_a^b P^{m_0+i}(x)q(x)dx \int_a^x P^{m_1+m_2-i}(t)q(t)dt \\ &= \sum_{i=0}^{m_1} \frac{(-1)^{m_0+m_1-i}}{m_0!m_2!i!(m_1-i)!} im_{m_0+i, m_1+m_2-i}(p, q), \end{aligned}$$

where the “iterated one-sided moments”  $im_{i,j}(p, q)$  are given by the equation

$$(2.13) \quad im_{i,j}(p, q, a, b) = im_{i,j}(p, q) = \int_a^b P^i(x)q(x)dx \left( \int_a^x P^j(t)q(t)dt \right).$$

*Proof.* By induction. We omit the detailed computations. □

2.2.1. *Center equations as polynomials in  $p, q$ .* In this section we refer to the Poincaré coefficients  $\psi_k$ , iterated integrals  $I_\alpha$ , moments etc. as computed between any two fixed points  $a, b$  in  $\mathbb{C}$ . Consequently, we sometimes omit the integration limits in the corresponding formulae.

For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_s)$  denote by  $i(\alpha)$  (resp.  $j(\alpha)$ ) the number of  $\alpha_l = 1$  (resp.  $\alpha_l = 2$ ), i.e. the number of appearances of  $p$  (resp. of  $q$ ) in the iterated integral  $I_\alpha$ .

**PROPOSITION 2.5.** *For each  $k \geq 2$  the Poincaré coefficient  $\psi_k$  is a weighted quasi-homogeneous polynomial of degree  $k - 1$  in  $(\lambda, \mu)$ , for the weights of  $(\lambda, \mu)$  equal to 1, 2, respectively. The degree of  $\psi_k$  as a polynomial in  $\lambda$  alone is  $k - 1$ . Its degree as a polynomial in  $\mu$  alone is  $\lceil \frac{k-1}{2} \rceil$ .*

*Proof.* By Proposition 2.4, for each  $k \geq 2$  the Poincaré coefficient  $\psi_k$  is given as the integer linear combination of the iterated integrals of  $p$  and  $q$ :

$$(2.14) \quad \psi_k = \sum n_\alpha I_\alpha,$$

with the sum running over all the multi-indices  $\alpha = (\alpha_1, \dots, \alpha_s)$  for which  $\sum_1^s \alpha_m = i(\alpha) + 2j(\alpha) = k - 1$ . Now, the polynomials  $p(x), q(x)$  are linear forms in their coefficients  $(\lambda, \mu)$ . Therefore, for each iterated integral  $I_\alpha$  the common degree of

$I_\alpha$  with respect to both sets of variables  $(\lambda, \mu)$  is  $i(\alpha) + j(\alpha)$ . The degree of  $I_\alpha$  with respect to the set of variables  $\lambda$  separately is  $i(\alpha)$ , and its degree with respect to  $\mu$  separately is  $j(\alpha)$ .

Thus,  $i(\alpha) + 2j(\alpha)$  is exactly the weighted degree of  $I_\alpha$ . As stated above, for each  $I_\alpha$  entering  $\psi_k$  this weighted degree is  $k - 1$ . This proves that  $\psi_k$  is a weighted quasi-homogeneous polynomial of degree  $k - 1$ . To compute the degrees of  $\psi_k$  in  $\lambda$  and  $\mu$  separately, it remains to notice that the maximal values of  $i(\alpha)$  and  $j(\alpha)$ , for the iterated integrals  $I_\alpha$  in  $\psi_k$ , are  $k - 1$  and  $\lfloor \frac{k-1}{2} \rfloor$ , respectively.  $\square$

The structure of the algebraic equation at infinity is determined by the highest degree, homogeneous term of this equation. Accordingly, we have to describe these highest degree homogeneous terms in the Poincaré coefficients  $\psi_k(\lambda, \mu)$ . To avoid too long statements, we split this description in two different results:

**THEOREM 2.3.** *For each  $k \geq 2$  the term of the highest degree  $k - 1$  in both  $\lambda$  and  $\mu$  in the polynomial  $\psi_k(\lambda, \mu)$  is  $I_{1\dots 1}$  with the coefficient  $(-1)^k k!$ . The term of the degree  $k - 2$  is given by an integer linear combination of the iterated integrals  $I_\alpha$  with exactly one appearance of  $q$ , which is reduced to the moment  $-m_{k-3}(p, q) = -\int P^{k-3}q$ . Exactly the same are the terms of the highest degrees  $k - 1$  and  $k - 3$  with respect to  $\lambda$  alone.*

*Proof.* The degree of each  $I_\alpha$  with respect to both sets of variables  $(\lambda, \mu)$  is  $i(\alpha) + j(\alpha)$  which is equal to  $k - 1 - j(\alpha)$  under the condition  $i(\alpha) + 2j(\alpha) = k - 1$ . Hence the highest degree terms with respect to both sets of variables  $(\lambda, \mu)$  are those with the minimal number  $j(\alpha)$  of the appearances of  $q$ . (Of course, the same is true for the highest degree with respect to  $\lambda$  alone, which is equal to  $k - 1 - 2j(\alpha)$ .) For no appearances of  $q$  do we get  $I_{1\dots 1}$  with the coefficient  $(-1)^k k!$ . For exactly one appearance of  $q$  we get an integer linear combination of the iterated integrals  $I_\alpha$  with exactly one appearance of  $q$ , which by Theorem 2.1 above is reduced to the integer multiple of the moment  $m_{k-3}(p, q) = \int P^{k-3}q$ . It remains to show that this integral multiple is  $-1$ . This will follow from Proposition 2.7 below.  $\square$

Notice that in our setting where  $P$  and  $Q$  vanish on the points of  $A$ , the iterated integral  $I_{1\dots 1}$  is identically zero. Indeed, it can be explicitly integrated to a difference of values at two points of  $A$  of a certain polynomial in  $P$ .

The second theorem describes the highest degree terms of  $\psi_k(\lambda, \mu)$  considered as a polynomial in  $\mu$  alone. In this setting we use below only the equations  $\psi_k = 0$  for  $k$  even; so the statement of the theorem is restricted only to this case.

**THEOREM 2.4.** *For  $k$  even the homogeneous part of  $\psi_k(\lambda, \mu)$  of the highest degree  $l = \lfloor \frac{k-1}{2} \rfloor = \frac{k}{2} - 1$  in  $\mu$  is given by the integer linear combination of the iterated integrals  $I_\alpha$  with exactly one appearance of  $p$ , which is reduced to the moment  $C m_l(q, p) = \int Q^l p$ . Here  $C = \frac{1 \cdot 3 \dots (2l-3)}{l!}$ .*

*Proof.* All the statements of Theorem 2.4 are immediate, except the exact value of the coefficient  $C$  with which the moment  $m_l(q, p)$  enters the equation

$\psi_k(\mu) = 0$ . Clearly, knowing the exact value of this coefficient is essential for our approach (in particular, the fact that  $C \neq 0$ ). However, its “combinatorial” calculation turns out to be rather tricky. Instead, we compute it using some naturally arising “generating functions” in Proposition 2.6 below.  $\square$

2.2.2. *Generating functions.* In this section we complete the proof of Theorems 2.3 and 2.4. In order to compute in closed form the sums of iterated integrals with the same number of appearances of  $p$  ( $q$ ) in the Poincaré coefficients  $\psi_k$ , we introduce an auxiliary parameter  $\varepsilon$  into equation (1.1), as a multiple either of  $p$  or of  $q$ :

$$(2.15) \quad \frac{dy}{dx} = \varepsilon p(x)y^2 + q(x)y^3,$$

$$(2.16) \quad \frac{dy}{dx} = p(x)y^2 + \varepsilon q(x)y^3.$$

It turns out that the derivatives with respect to  $\varepsilon$  of the Poincaré mappings for (2.15) and (2.16) are exactly the generating functions for the moments sums we are interested in. Below we compute these derivatives solving certain linear differential equations. Notice that by a different method the higher order derivatives with respect to  $\varepsilon$  of the Poincaré mapping for (2.16) have been computed in [36] (see also [35]).

For  $\varepsilon = 0$  in (2.15), (2.16) we get the unperturbed equations

$$(2.17) \quad \frac{dy}{dx} = q(x)y^3,$$

$$(2.18) \quad \frac{dy}{dx} = p(x)y^2.$$

The solutions of the unperturbed equation (2.17) are given by  $y = \frac{y_a}{\sqrt{1-2y_a^2 Q(x)}}$ , with the integration constant in  $Q = \int q$  chosen in such a way that  $Q(a) = 0$ , for the solutions starting at  $a$  and satisfying  $y(a) = y_a$ . If we take  $b$  as the initial point and consider solutions  $y$  with  $y(b) = y_b$ , then  $y = \frac{y_b}{\sqrt{1-2y_b^2 Q(x)}}$ , with the integration constant in  $Q = \int q$  chosen in such a way that  $Q(b) = 0$ .

The solutions of the unperturbed equation (2.18) are given by  $y = \frac{y_a}{1-y_a P(x)}$ , with the integration constant in  $P = \int p$  chosen in such a way that  $P(a) = 0$ , for the solutions starting at  $a$  and satisfying  $y(a) = y_a$ . If we take  $b$  as the initial point and consider solutions  $y$  with  $y(b) = y_b$ , then  $y = \frac{y_b}{1-y_b P(x)}$ , with the integration constant in  $P = \int p$  chosen in such a way that  $P(b) = 0$ .

In each of these two integrable cases, the necessary and sufficient condition for the center is  $Q(a) = Q(b) = 0$  (resp.  $P(a) = P(b) = 0$ ). As stated above, we always assume both these conditions to be satisfied. Now for  $\varepsilon$  small we can consider (2.15), (2.16) as perturbations of the integrable equations (2.17), (2.18).

Let us return now to the expressions of Proposition 2.4 above, representing the Poincaré coefficients  $\psi_k$  as the sums of iterated integrals. If we write these expressions for the parametric equation (2.15) (resp. (2.16)), the factor  $\varepsilon^l$  appears before each term, with  $l$  being equal to the number of appearances of the polynomial  $p$  (resp., polynomial  $q$ ) in the corresponding iterated integral.

In particular, with the first power of  $\varepsilon$ , there appear the sums of the iterated integrals with exactly one appearance of  $p$  (resp. of  $q$ ). Let us denote this sum in the  $k^{\text{th}}$  Poincaré coefficient  $\psi_k$  for the parametric equations (2.15) (resp. (2.16)), by  $s_k^1(p, q)$  (resp.  $s_k^2(p, q)$ ).

Now, denoting the Poincaré mapping for (2.15) (resp. (2.16)) by  $G_1$  (resp.  $G_2$ ) we see that the derivative of the inverse Poincaré mapping  $G_i^{-1}(y, \varepsilon)$ ,  $i = 1, 2$ , with respect to  $\varepsilon$ , at  $\varepsilon = 0$ , is given by the sum of the series (2.5) where the coefficients  $\psi_k(p, q)$  are replaced with  $s_k^i(p, q)$ . Let us state this last result separately:

LEMMA 2.1. *For the inverse Poincaré mapping  $G_i^{-1}(y, \varepsilon)$ ,  $i = 1, 2$ , of the parametric equations (2.15) (resp. (2.16)), we have*

$$(2.19) \quad \frac{dy_a}{d\varepsilon} \Big|_{\varepsilon=0} = \frac{dG_i^{-1}(y_b, \varepsilon)}{d\varepsilon} \Big|_{\varepsilon=0} = \sum_{k=0}^{\infty} s_k^i(p, q) y_b^k, \quad i = 1, 2.$$

In other words,  $\frac{dG_i^{-1}(y_b, \varepsilon)}{d\varepsilon} \Big|_{\varepsilon=0}$  is the generating function for the sum  $s_k^i(p, q)$ ,  $i = 1, 2$ .

It remains to compute directly the derivative of the inverse Poincaré mapping  $G_i^{-1}(y, \varepsilon)$  with respect to  $\varepsilon$ , at  $\varepsilon = 0$ . We summarize this computation in two separate propositions.

PROPOSITION 2.6. *The derivative of the inverse Poincaré mapping  $G_1^{-1}(y_b, \varepsilon)$  with respect to  $\varepsilon$  at  $\varepsilon = 0$ , is given by*

$$(2.20) \quad \frac{d}{d\varepsilon} G_1^{-1}(y_b, \varepsilon) \Big|_{\varepsilon=0} = -y_b^2 \int_a^b p \sqrt{1 - 2y_b^2 Q} dx,$$

or, explicitly, by

$$(2.21) \quad \frac{d}{d\varepsilon} G_1^{-1}(y, \varepsilon) \Big|_{\varepsilon=0} = y_b^2 \sum_{l=1}^{\infty} \frac{1 \cdot 3 \dots (2l - 3)}{l!} m_l(q, p) y_b^{2l}.$$

In particular, we have  $s_{2k+2}^1(p, q) = \frac{1 \cdot 3 \dots (2k - 3)}{k!} m_k(q, p) = \frac{1 \cdot 3 \dots (2k - 3)}{k!} \int_a^b Q^k p$ , and  $s_{2k+1}^1(p, q) = 0$ .

PROPOSITION 2.7. *The derivative of the inverse Poincaré mapping  $G_2^{-1}(y_b, \varepsilon)$  with respect to  $\varepsilon$  at  $\varepsilon = 0$ , is given by*

$$(2.22) \quad \frac{d}{d\varepsilon} G_2^{-1}(y_b, \varepsilon) \Big|_{\varepsilon=0} = - \int_a^b \frac{y_b^3 q(x)}{1 - y_b P(x)} dx = - \sum_{k=3}^{\infty} m_{k-3}(p, q) y_b^k.$$

In particular,  $s_k^2(p, q) = -m_{k-3}(p, q) = - \int_a^b P^{k-3} q$ .

*Proof of Proposition 2.6.* We consider equation (2.15):  $y' = \varepsilon py^2 + qy^3$ . Observe that we always assume that  $P(a) = P(b) = Q(a) = Q(b) = 0$ . Let us write the expansion in  $\varepsilon$  of the solution  $y(x, \varepsilon)$  of (2.15), satisfying the initial condition  $y(b) = y_b$ :

$$(2.23) \quad y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots$$

Substituting (2.23) in (2.15) we obtain:

$$(y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots)' = \varepsilon p(x)(y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots)^2 + q(x)(y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots)^3.$$

Taking here  $\varepsilon = 0$  we see that  $y_0(x)$  is the solution of equation (2.17), satisfying  $y_0(b) = y_b$ . Now,  $y_0(x) = \frac{y_b}{\sqrt{1-2y_b^2 Q(x)}}$  and we note that  $y_0(a) = y_b$ .

Comparing the first powers of  $\varepsilon$  in the above equation, we obtain the following linear differential equation for  $y_1(x)$ :

$$(2.24) \quad y_1' = 3qy_0^2 y_1 + py_0^2, \quad y_1(b) = 0.$$

Solving (2.24) we obtain

$$(2.25) \quad y_1(x) = e^{3 \int_b^x q(s)y_0^2(s)ds} \int_b^x p(s)y_0^2(s)e^{-3 \int_b^s q(\xi)y_0^2(\xi)d\xi} ds.$$

Substituting into the interior integrals in (2.25) the explicit expression for  $y_0$  given above, we find

$$\int_b^x qy_0^2 ds = \int_b^x \frac{y_b^2 q ds}{1-2y_b^2 Q} = -\frac{1}{2} \ln(1-2y_b^2 Q(x)).$$

Therefore

$$e^{-3 \int_b^x qy_0^2 ds} = e^{\frac{3}{2} \ln(1-2y_b^2 Q)} = (1-2y_b^2 Q)^{\frac{3}{2}}.$$

Finally,

$$y_1(x) = y_b^2 (1-2y_b^2 Q(x))^{-3/2} \int_b^x p \sqrt{1-2y_b^2 Q} ds.$$

In particular,

$$(2.26) \quad \frac{d}{d\varepsilon} G_1^{-1}(y_b, \varepsilon)|_{\varepsilon=0} = y_1(a) = y_b^2 \int_b^a p \sqrt{1-2y_b^2 Q} dx.$$

Expanding  $\sqrt{1-2y_b^2 Q}$  with respect to the powers of  $2y_b^2 Q$ , we obtain

$$\sqrt{1-2y_b^2 Q} = 1 + \sum_{l=1}^{\infty} \frac{(\frac{1}{2}) \cdot (-\frac{1}{2}) \cdot (-\frac{3}{2}) \dots (\frac{3-2l}{2})}{l!} (-2y_b^2 Q)^l,$$

which gives

$$(2.27) \quad \frac{d}{d\varepsilon} G_1^{-1}(y_b, \varepsilon)|_{\varepsilon=0} = -y_b^2 \sum_{l=1}^{\infty} \left( \frac{1 \cdot 3 \dots (2l-3)}{l!} \int_b^a p Q^l dx \right) y_b^{2l}.$$

Comparing the coefficients of the powers of  $y_b$  in (2.19) and (2.27) completes the proof of the proposition.  $\square$

*Proof of Proposition 2.7.* It is similar to the proof of Proposition 2.6 above. We consider equation (2.16):  $y' = py^2 + \varepsilon qy^3$ . As usual,  $P(a) = P(b) = Q(a) = Q(b) = 0$ . Write the expansion in  $\varepsilon$  of the solution  $y(x, \varepsilon)$  of (2.16) satisfying the initial condition  $y(b) = y_b$ :

$$(2.28) \quad y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots$$

Substituting (2.28) in (2.16) we obtain:

$$(y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots)' = p(x)(y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots)^2 + \varepsilon q(x)(y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots)^3.$$

Taking here  $\varepsilon = 0$  we see that  $y_0(x)$  is the solution of the equation (2.18), satisfying  $y_0(b) = y_b$ . So  $y_0(x) = \frac{y_b}{1 - y_b P(x)}$  and we note that  $y_0(a) = y_b$ .

Comparing the first powers of  $\varepsilon$  in the above equation, we obtain the following linear differential equation for  $y_1(x)$ :

$$(2.29) \quad y_1' = 2py_0^2 y_1 + qy_0^3, \quad y_1(b) = 0.$$

Solving (2.29) we obtain

$$y_1(x) = C e^{2 \int_b^x p(s)y_0(s)ds} + e^{2 \int_b^x p(s)y_0(s)ds} \int_b^x q(s)y_0^3(s) e^{-2 \int_b^s p(\xi)y_0(\xi)d\xi} ds.$$

We have  $y_1(b) = C$ , hence  $C = 0$ , and we come to

$$(2.30) \quad y_1(a) = \int_b^a qy_0^3 e^{-2 \int_b^s p y_0(\xi)d\xi} ds,$$

since  $\int_b^a p(s)y_0(s)ds = \int_b^a y_0(P(s))dP(s) = 0$ .

It remains to substitute into (2.30) the explicit expression  $y_0(x) = \frac{y_b}{1 - y_b P(x)}$ . We get

$$\begin{aligned} \int_a^s p(\xi)y_0(\xi)d\xi &= \int_a^s \frac{y_a P(\xi)}{1 - y_a P(\xi)} d\xi \\ &= - \int_a^s \frac{d(1 - y_a P)}{1 - y_a P} d\xi = -\ln(1 - y_a P(s)). \end{aligned}$$

Thus  $e^{-2 \int_a^s p y_0(\xi)d\xi} = (1 - y_a P(s))^2$ , and finally we obtain

$$(2.31) \quad \frac{d}{d\varepsilon} G_2^{-1}(y_b, \varepsilon)|_{\varepsilon=0} = y_1(b) = y_a^3 \int_b^a \frac{q dx}{1 - y_a P(x)}.$$

Comparing the coefficients of the powers of  $y_b$  in (2.19) and (2.31) completes the proof of the proposition.  $\square$

### 3. Center equations at infinity

We consider three different versions of the Center-Focus problem for the Abel differential equation (1.1)  $y' = py^2 + qy^3$ :

- (A) *Full version.* Here we take both  $p$  and  $q$  in the Abel equation (1.1) as unknowns.
- (B) *First restricted version.* Here we fix the polynomial  $q$  and consider the Abel equation (1.1) with the polynomial  $p$  free.
- (C) *Second restricted version.* Here we fix the polynomial  $p$  and consider the Abel equation (1.1) with the polynomial  $q$  free.

As usual, the set  $A = \{a_1, \dots, a_r\}$ ,  $r \geq 2$ , is fixed. Let us denote by  $V_d(A)$  the set of polynomials  $v(x)$  of degree  $d$  such that  $V(x) = \int_{a_1}^x v(t)dt$  vanishes at the points  $a_1, \dots, a_r$ .  $V_d(A)$  is a linear space of dimension  $d + 1 - r$  and its elements can be represented in the form  $v(x) = V'(x)$  for  $V(x) = \hat{V}(x)V_0(x)$ , with  $V_0(x) = (x - a_1) \cdots (x - a_r)$  and  $\hat{V}(x)$  an arbitrary polynomial of degree  $d + 2 - r$ .

So for the degrees  $d_1$  and  $d_2$  of  $p$  and  $q$  fixed we consider as the free parameters of the problem  $(p, q) \in V_{d_1, d_2}(A) = V_{d_1}(A) \times V_{d_2}(A)$  in the setting (A),  $p \in V_{d_1}(A)$  in the setting (B), and  $q \in V_{d_2}(A)$  in the setting (C).

Let us denote in each case the resulting complex vector space of the free parameters by  $V$ . In order to define the Center equations and the Center set at infinity, in each case we consider the standard projectivization  $PV$  of the space  $V$ , given by the following construction: if the affine coordinates in  $V$  are, say,  $(x_0, x_1, \dots, x_r)$  then the homogeneous coordinates in  $PV$  are  $(x_0 : x_1 : \cdots : x_r : \nu)$  where  $\nu$  is an auxiliary complex coordinate. So  $PV$  consists of all the  $r + 2$ -tuples  $(x_0 : x_1 : \cdots : x_r : \nu)$  with not all the coordinates zero, where the proportional  $r + 2$ -tuples are identified. The infinite hyperplane  $H = PV^\infty$  is defined in  $PV$  by the equation  $\nu = 0$ . The affine part  $V \subset PV$  can be identified with the original space  $V$  by setting  $\nu = 1$ . Of course, up to an isomorphism, the projectivization  $PV$  does not depend on the specific choice of the coordinates  $(x_0, x_1, \dots, x_r)$ .

The affine polynomials  $R(x)$  in  $x = (x_0, \dots, x_r)$  are transformed into the homogeneous polynomials  $R(x, \nu)$  on  $PV$  via multiplying each term by the complementary (to the total degree of  $R$ ) power of  $\nu$ .

Of course, the homogenization of polynomials on  $V$  can be described in an invariant way, without coordinates, by using instead linear, bi-linear, and so on, functionals on  $V$ . Notice that all the iterated integrals above, in particular, the Center equations, as well as the Moment ones, are given in such an invariant, coordinate free form. An important advantage of this setting, in contrast with the usual coordinate representation of the Center equations, is that it can be automatically extended to arbitrary  $p$  and  $q$ , not only polynomials (see [22]–[25] where a similar generalized setting has been successively pushed forward).



Below we write our equations only in the coordinates free form.

To see the structure of the homogenized Center equations in each of the settings chosen, we reformulate in a more convenient form Proposition 2.4 above.

Notice that for a multi-index  $\alpha = (\alpha_1, \dots, \alpha_s)$  we've denoted by  $i(\alpha)$  (respectively,  $j(\alpha)$ ) the number of  $\alpha_l = 1$  (respectively,  $\alpha_l = 2$ ), i.e., the number of appearances of  $p$  (resp. of  $q$ ) in the iterated integral  $I_\alpha$ .

PROPOSITION 3.1. *For each  $k \geq 2$  the Poincaré coefficient  $\psi_k(p, q)$  is given as the integer linear combination of the iterated integrals of  $p$  and  $q$ :*

$$(3.1) \quad \psi_k = \sum n_\alpha I_\alpha,$$

with the sum running over all the multi-indices  $\alpha = (\alpha_1, \dots, \alpha_s)$  for which  $\sum_1^s \alpha_m = i(\alpha) + 2j(\alpha) = k - 1$ . The common degree of  $I_\alpha$  with respect to both the sets of variables  $(p, q)$  is  $i(\alpha) + j(\alpha)$ . The degree of  $I_\alpha$  with respect to the set of variables  $p$  separately is  $i(\alpha)$ , and its degree with respect to  $q$  separately is  $j(\alpha)$ .

Let us denote by  $\hat{\psi}_k(p, q)$  the ‘‘homogenization’’ of the Poincaré coefficient  $\psi_k$  with respect to both sets of variables  $(p, q)$ . This corresponds to the setting (A) of the Center-Focus problem, described above.

LEMMA 3.1. *For each  $k \geq 2$  the homogeneous Poincaré coefficient  $\hat{\psi}_k(p, q, \nu)$  is given by*

$$(3.2) \quad \hat{\psi}_k = \sum n_\alpha \nu^{j(\alpha)-1} I_\alpha,$$

with the sum running over all the multi-indices  $\alpha = (\alpha_1, \dots, \alpha_s)$  for which  $\sum_1^s \alpha_m = i(\alpha) + 2j(\alpha) = k - 1$ ,  $i(\alpha), j(\alpha) \geq 1$ . Its degree is  $k - 2$ . The terms not containing  $\nu$  are the sums of all the iterated integrals with exactly one appearance of  $q$ .

*Proof.* The degree  $i(\alpha) + j(\alpha)$  of the iterated integrals in  $\psi_k$  attains its maximum (under the condition  $i(\alpha) + 2j(\alpha) = k - 1$ ) for  $j(\alpha) = 0$ . However, the corresponding iterated integral contains only  $p$ ; so it is zero, since by our assumptions  $P = \int p$  vanishes on  $A$ . Hence the highest degree nonzero terms are those iterated integrals with  $j(\alpha) = 1$ . Their degree is  $k - 2$ , so that the powers of  $\nu$  chosen in (3.2) make  $\hat{\psi}_k(p, q, \nu)$  homogeneous of degree  $k - 2$ .  $\square$

Now we denote by  $\tilde{\psi}_k$  the homogenization of the Poincaré coefficient  $\psi_k$  with respect to the variable  $p$ . This correspond to the setting (B) of the problem, where we assume the polynomial  $q$  to be fixed.

LEMMA 3.2. *For each  $k \geq 2$  the homogeneous Poincaré coefficient  $\tilde{\psi}_k(p, \nu)$  is given by*

$$(3.3) \quad \tilde{\psi}_k = \sum n_\alpha \nu^{2j(\alpha)-2} I_\alpha,$$

with the sum running over all the multi-indices  $\alpha = (\alpha_1, \dots, \alpha_s)$  for which  $\sum_1^s \alpha_m = i(\alpha) + 2j(\alpha) = k - 1$ ,  $i(\alpha), j(\alpha) \geq 1$ . Its degree is  $k - 3$ . The terms not containing  $\nu$  are the sums of all the iterated integrals with exactly one appearance of  $q$ .

*Proof.* As above, the degree  $i(\alpha)$  of the nonzero iterated integrals in  $\psi_k$  with respect to  $\lambda$  attains its maximum (under the condition  $i(\alpha) + 2j(\alpha) = k - 1$ ) for  $j(\alpha) = 1$ . It is equal to  $k - 3$ , and the powers of  $\nu$  chosen in (3.3) make  $\widehat{\psi}_k(p, \nu)$  homogeneous of degree  $k - 3$ .  $\square$

Finally, let us denote by  $\bar{\psi}_k$  the homogenization of the Poincaré coefficient  $\psi_k$  with respect to the variable  $q$ . This correspond to the setting (C) of the problem where we assume the polynomial  $p$  to be fixed.

LEMMA 3.3. *For each  $k \geq 2$  the homogeneous Poincaré coefficient  $\bar{\psi}_k(q, \nu)$  is given by*

$$(3.4) \quad \bar{\psi}_k = \sum n_\alpha \nu^{l(\alpha)} I_\alpha,$$

where  $l(\alpha) = \frac{1}{2}(i(\alpha) - 1)$  for  $k$  even, and  $l(\alpha) = \frac{1}{2}(i(\alpha) - 2)$  for  $k$  odd. The sum runs over all the multi-indices  $\alpha = (\alpha_1, \dots, \alpha_s)$  for which  $\sum_1^s \alpha_m = i(\alpha) + 2j(\alpha) = k - 1$ ,  $i(\alpha), j(\alpha) \geq 1$ . The degree of  $\bar{\psi}_k(q, \nu)$  is  $[\frac{k-2}{2}]$ .

For  $k$  even, the terms not containing  $\nu$  are the sums of all the iterated integrals with exactly one appearance of  $p$ .

For  $k$  odd, the terms not containing  $\nu$  are the sums of all the iterated integrals with exactly two appearances of  $p$ .

*Proof.* In the considered case of  $p$  fixed and only  $q$  variable, the maximal degree  $j(\alpha)$  of the nonzero iterated integrals in  $\psi_k$  with respect to  $\mu$  depends on the parity of  $k$ . For  $k$  even it corresponds to  $i(\alpha) = 1, j(\alpha) = \frac{k-2}{2}$ . The terms of this degree are all the iterated integrals with exactly one appearance of  $p$ .

For  $k$  odd this corresponds to  $i(\alpha) = 2, j(\alpha) = \frac{k-3}{2}$ . The terms of this degree are all the iterated integrals with exactly two appearances of  $p$ .

In each case this maximal degree is  $[\frac{k-2}{2}]$ . The powers of  $\nu$  chosen in (3.4) make  $\bar{\psi}_k(q, \nu)$  homogeneous of this degree  $[\frac{k-2}{2}]$ .  $\square$

*Some examples.* According to the description above, the homogenization of the few initial Center equations given on page 450 with respect to both  $p$  and  $q$  is the following (we take into account that each iterated integral with only  $p$  or only  $q$  is zero):

$$\widehat{\psi}_2 = - \int_a^b p = I_1 = 0,$$

$$\widehat{\psi}_3 = 2I_{11} - \nu I_2 = 0,$$

$$\widehat{\psi}_4 = 3I_{12} + 2I_{21} = - \int_a^b Pq,$$

$$\widehat{\psi}_5 = -12I_{112} - 8I_{121} - 6I_{211} = - \int_a^b P^2q,$$

$$\begin{aligned}\widehat{\psi}_6 &= 60I_{1112} + 40I_{1121} + 30I_{1211} + 24I_{2111} \\ &\quad - v(15I_{122} + 12I_{212} + 8I_{221}) = - \int_a^b P^3 q + \frac{1}{2}v \int_a^b Q^2 p, \\ \widehat{\psi}_7 &= - \int_a^b P^4 q - 2v \int_a^b P p Q^2, \\ \widehat{\psi}_8 &= - \int_a^b P^5 q + v \int_a^b P^3 Q q - 320v \int_a^b P^2(t)q(t)dt \int_a^t P q \\ &\quad + 185v \int_a^b P(t)q(t)dt \int_a^t P^2 q + \frac{1}{2}v^2 \int_a^b Q^3 p.\end{aligned}$$

Performing the homogenization with respect to  $p$  (while  $q$  is fixed) we get essentially the same answer (but with a different interpretation) for  $k \leq 5$ , but a slightly different one for  $k = 6, 7, 8$ :

$$\begin{aligned}\tilde{\psi}_6 &= - \int_a^b P^3 q + \frac{1}{2}v^2 \int_a^b Q^2 p, \\ \tilde{\psi}_7 &= - \int_a^b P^4 q - 2v^2 \int_a^b P p Q^2, \\ \tilde{\psi}_8 &= - \int_a^b P^5 q + v^2 \int_a^b P^3 Q q - 320v^2 \int_a^b P^2(t)q(t)dt \int_a^t P q \\ &\quad + 185v^2 \int_a^b P(t)q(t)dt \int_a^t P^2 q + \frac{1}{2}v^4 \int_a^b Q^3 p.\end{aligned}$$

Finally, performing the homogenization with respect to  $q$  (while  $p$  is fixed) we obtain

$$\begin{aligned}\bar{\psi}_6 &= -v \int_a^b P^3 q + \frac{1}{2} \int_a^b Q^2 p, \\ \bar{\psi}_7 &= -v \int_a^b P^4 q - 2 \int_a^b P p Q^2, \\ \bar{\psi}_8 &= -v^2 \int_a^b P^5 q + v \int_a^b P^3 Q q - 320v \int_a^b P^2(t)q(t)dt \int_a^t P q \\ &\quad + 185v \int_a^b P(t)q(t)dt \int_a^t P^2 q + \frac{1}{2} \int_a^b Q^3 p.\end{aligned}$$

#### 4. Proof of Theorem 1.5

Now we are ready to describe, in each of the settings (A), (B) and (C), the Center equations at infinity. In particular, we prove Theorem 1.5 stated in the introduction. Notice that we consider the Center problem for the Abel equation (1.1) on the set of points  $A = \{a_1, \dots, a_r\}$ . As usual, we assume that the polynomials  $p = P'$  and  $q = Q'$ , considered, satisfy condition (1.4):  $P(a_j) = Q(a_j) = 0$ ,  $j = 1, \dots, r$ .

With the definitions given in the introduction in mind, we let, as above,  $V_d(A)$  denote the space of complex polynomials of degree at most  $d$ , satisfying (1.4). We assume that the coefficients  $p, q$  of the Abel equation (1.1) are polynomials of the degrees  $d_1$  and  $d_2$ , respectively:  $p \in V_{d_1}(A), q \in V_{d_2}(A)$ .

The Center set of the Abel equation (1.1) on  $A = \{a_1, \dots, a_r\}$  has been defined as the set  $\mathcal{C} \subset V_{d_1, d_2}(A) = V_{d_1}(A) \times V_{d_2}(A)$  of  $(p, q) \in V_{d_1, d_2}(A)$  for which equation (1.1) has a center on  $A$ . In a similar way we define the Composition and the Moment sets.

For a fixed polynomial  $q$  (resp.  $p$ ) let us denote by  $\mathcal{C}_q \subset V_{d_1}$  (resp.  $\mathcal{C}_p \subset V_{d_2}$ ) the set consisting of those  $p$  (resp.  $q$ ) for which the Abel equation (1.1) has a center on  $A$ . Similarly we define the Composition sets  $\mathcal{L}_q$  and  $\mathcal{L}_p$  and the Moment set  $\mathcal{M}_q$ . In fact, these are fibers of the projections of  $\mathcal{C}$  (resp. of  $\mathcal{L}$  and  $\mathcal{M}$ ) onto the components  $p$  and  $q$ :  $\mathcal{C}_p = \mathcal{C} \cap (\{p\} \times V_{d_2})$  (resp.  $\mathcal{C}_q = \mathcal{C} \cap (V_{d_1} \times \{q\})$ ). In the same way  $\mathcal{L}_p = \mathcal{L} \cap (\{p\} \times V_{d_2}), \mathcal{L}_q = \mathcal{L} \cap (V_{d_1} \times \{q\}), \mathcal{M}_q = \mathcal{M} \cap (V_{d_1} \times \{q\})$ .

However, the Moment set  $\mathcal{M}_p$  is defined in a slightly different way (because in setting (C) the roles of  $p$  and  $q$  in the Moment equations are interchanged):  $\mathcal{M}_p$  is the set of  $q \in V_{d_2}(A)$  for which  $p, q$  satisfy the second Moment vanishing condition

$$(4.1) \quad m_k(q, p, a_1, a_s) = \int_{a_1}^{a_s} Q^k(x)p(x)dx = 0, \quad k = 0, 1, \dots, s = 2, \dots, r.$$

Now let us reformulate (and extend, according to settings (A), (B) and (C) of the problem), Theorem 1.5, stated in the introduction:

**THEOREM 4.1.** *The Center set at infinity  $\mathcal{C}_\infty$  in setting (A) coincides with the Moment set  $\mathcal{M}$  defined by the first Moment vanishing condition.*

*The Center set at infinity in setting (B) coincides with the Moment set  $\mathcal{M}_q$  defined by the first Moment vanishing condition.*

*The Center set at infinity in setting (C) is contained in the Moment set  $\mathcal{M}_p$  defined by the second Moment vanishing condition. The inclusion may be strict.*

*Proof.* For an algebraic set  $\mathcal{A}$  given in an affine space  $V$  by a system of polynomial equations  $\eta_k = 0$  its projectivization  $P\mathcal{A} \subset PV$  is defined in  $PV$  by a system of homogeneous equations  $\hat{\eta}_k = 0$ , where the homogeneous polynomials  $\hat{\eta}_k$  are obtained from  $\eta_k$  by the homogenization procedure described above. The infinite hyperplane  $H$  is given in  $PV$  by the equation  $v = 0$ . Consequently, the intersection  $\mathcal{A}_\infty = \mathcal{A} \cap H$  is defined in  $H$  by the equations  $\hat{\hat{\eta}}_k = 0$ , where  $\hat{\hat{\eta}}_k$  is the homogeneous part of the highest degree in the polynomial  $\eta_k$ .

Lemmas 3.1–3.3 describe this homogeneous part of the highest degree in the Center equations for settings (A), (B), (C). In setting (A), by Lemma 3.1, it is the sum of the iterated integrals  $I_\alpha$  in  $\phi_k$  with exactly one appearance of  $q$ . By Lemma 3.2, the same is true in setting (B). By Theorem 2.3 this sum is equal to the moment  $-m_{k-3}(p, q)$ . This proves first two statements of Theorem 4.1.

Now let us consider setting (C). By Lemma 3.3 for  $p$  fixed and  $q$  free, the Center equations at infinity take the following form: for  $k = 2l$  even we get the sum of the iterated integrals in  $\psi_k$  with exactly one appearance of  $p$ . By Theorem 2.4, this sum is equal to the moment  $m_l(q, p)$  with a nonzero coefficient  $C$ . Hence we get the second system of the Moment equations, and an additional infinite set of equations (for  $k$  odd), provided by certain rational linear combinations of the iterated integrals with exactly two appearances of  $p$ . Therefore, in this last case the Center set at infinity is a subset of the Moment set. It remains to show that the Center set may be strictly smaller than the Moment one. This happens in the following example:

*Example 4.1.* Let  $A = \{a, b\} = \{-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\}$  and let  $p = (T_2 + T_3)'$ , where  $T_n$ , is the  $n^{\text{th}}$  Chebyshev polynomial. Consider  $q \in V_5(A)$ . Then the Moment set  $\mathcal{M}_p \subset V_5(A)$  contains (at least) two components: the composition component  $\mathcal{L}_p = \{q = Q', Q = R(T_2 + T_3)\}$ , with  $R$  any polynomial of degree 2, and the non-Composition component  $\mathcal{T} = \{q = \alpha T'_6, \alpha \in \mathbb{C}\}$ . Indeed, by [49] the second system of the Moment equations is satisfied on each of these two components, while  $p = (T_2 + T_3)'$  and  $q = \alpha T'_6$  do not satisfy the Composition Condition. Hence the Moment set contains both the composition and the non-Composition components:  $\mathcal{L}_p \cup \mathcal{T} \subset \mathcal{M}_p$ . (In Remark 4.1 below we prove that in fact  $\mathcal{M}_p$  is exactly  $\mathcal{L}_p \cup \mathcal{T}$ .)

Now we use the second set of the Center equations at infinity: those provided by the iterated integrals with exactly two appearances of  $p$ . We consider the first nontrivial such equation  $\Psi_5 = \int_a^b P^2 q = 0$  (see the list of the initial Center equations in §2). The computation given below shows that for  $P = T_2 + T_3$  and  $q = \alpha T'_6$  this integral does not vanish. Hence in our example the Center set at infinity is the Composition set  $\mathcal{L}_p = \{q = Q', Q = R(T_2 + T_3)\}$ , and it is strictly smaller than the part  $\mathcal{L}_p \cup \mathcal{T}$  of the Moment set. This completes the proof of Theorem 4.1.  $\square$

*Computing  $\Psi_5(p, q)$ .* It is convenient to rescale the interval  $[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$  to the interval  $[0, 1]$ , and to shift the free terms in the Chebyshev polynomials. So let  $p = P' = (S_2 + S_3)'$ , where  $S_2(u) = u(u - 1)$ ,  $S_3(u) = u(u - 1)(2u - 1)$ . Consider now

$$(4.2) \quad Q = S_6 = S_3^2 = 4S_2^3 + S_2^2.$$

( $S_6$  differs from rescaled  $T_6$  by a certain “composition” term.) Now for  $\Psi_5(p, q) = \int_0^1 P^2 q$  we get

$$(4.3) \quad \Psi_5(p, q) = \int_0^1 (S_2 + S_3)^2 dS_6 = \int_0^1 S_2^2 dS_6 + \int_0^1 S_3^2 dS_6 + \int_0^1 S_2 S_3 dS_6.$$

The integrals of  $S_2^2$  and  $S_3^2$  in (4.3) vanish, since each couple  $S_2, S_6$  and  $S_3, S_6$  separately satisfy Composition condition on  $[0, 1]$  by (4.2). Now, writing  $S_3 =$

$S_2(2u - 1)$ ,  $dS_6 = (12S_2^2 + 2S_2)dS_2$ , we have

$$\begin{aligned} \Psi_5 &= \int_0^1 S_2^2(12S_2^2 + 2S_2)(2u - 1)dS_2 = \left(\frac{24}{5}S_2^5 + S_2^4\right)(2u - 1)\Big|_0^1 \\ &\quad - \int_0^1 \left(\frac{24}{5}S_2^5 + S_2^4\right)du = - \int_0^1 \left(\frac{24}{5}S_2^5 + S_2^4\right)du < 0, \end{aligned}$$

since  $S_2$  is positive on  $(0, 1)$ . Consequently,  $\Psi_5 \neq 0$  on all the “non-Composition stratum”  $Q = \alpha S_6$ , except the origin.

*Remark 4.1.* One can show that in this example the Moment set  $\mathcal{M}_p$  is exactly  $\mathcal{L}_p \cup \mathcal{T}$ . Since this result is somewhat beyond the scope of the present paper, we give only a sketch of the proof.

We call a polynomial  $Q$  *definite on  $A$*  if the vanishing of the moments  $m_k(q, p)$  on  $A$  implies the Composition condition for  $q, p$  on  $A$ . By the results of [52], [53] the only nondefinite polynomials of the degree at most 9 are the compositions of  $T_6$  with the first degree polynomials on both sides. Among all these compositions only the polynomials of the form  $\alpha T_6$  vanish on  $A = \{a, b\} = \left\{-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right\}$ . Therefore, for any  $Q$  of degree at most 6 not of the form  $\alpha T_6$ ,  $Q$  is definite on  $A$ , and the second system of the Moment equations implies the Composition Condition (PCC) for  $Q$  and  $P = T_2 + T_3$ . But  $P = T_2 + T_3$  is indecomposable (see, for example, [49]), and (PCC) takes a form  $Q = R(T_2 + T_3)$ . Thus we get  $\mathcal{M}_p = \mathcal{L}_p \cup \mathcal{T}$ .

*Remark 4.2.* Let us stress once more that in each case the system of Moment equations, defining the Center set at infinity, is nonlinear: in setting (A), system (1.5) of the moment equations is linear with respect to  $q$  and nonlinear with respect to  $p$ . In setting (B) we fix  $q$  and system (1.5) remains nonlinear with respect to the free variable  $p$ .

In setting (C) we fix  $p$  but system (1.5) is replaced by a symmetric system (4.1), and it remains nonlinear with respect to the free variable  $q$ .

*Remark 4.3.* It would be interesting to compare the results of this section with the results of [24] where a (nonexplicit) moment representation for the Center equations is obtained.

### 5. Moments set versus Composition set

This section and Section 7 present the most essential part of our approach, as the *affine* Center problem is concerned. In the present section we apply the results of [15], [18], [13], [14], [49]–[53], [54], and [69] in order to show that under appropriate assumptions the Moment set at infinity coincides with the Composition one. In Section 7 we extend this coincidence to the entire neighborhood of the infinity, using the method of [11]. It remains to apply simple algebraic-geometric considerations in order to extend our description of the Center set to the entire affine part.

In a sense, the coincidence of the Center and the Composition sets at infinity is the key point of our approach. Indeed, both our “local” and “global” algebraic-geometric tools (i.e. Nakayama lemma used in [11] and in Section 7, on one side, and a comparison of the affine and infinite parts of algebraic sets in the proof of affine results, on the other), heavily rely upon the fact that *Composition condition (PCC) implies vanishing of each term in the Center equations*. We do not know any other such “integrability condition” in the case of a polynomial Abel equation.

In contrast, in the case of a *trigonometric* Abel equation, there are strong indications that our approach can work equally well not only with Composition condition (PCC), but also with some other known integrability conditions. We plan to present some results in this direction separately.

Returning to our polynomial case, we see that if the Moment set contains non-Composition components (as may happen, according to [49]; see also Example 4.1 above, Theorem 1.6, and Theorem 5.2 below) we cannot apply our methods *directly* to these components. Indeed, the higher degree terms in the Center equations do not necessarily vanish on these components, as Example 4.1 shows. Accordingly, we restrict the main “affine” results of the present paper only to the cases where the Moment set coincides with the Composition one.

Now we are ready to prove the main results of this section, which include, in particular, Theorems 1.6 and 1.7 stated in the introduction. We consider the Center problem for the Abel equation (1.1)  $y' = py^2 + qy^3$  on a set  $A = \{a_1, \dots, a_r\}$ ,  $r \geq 2$ . Let the coefficients  $p, q$  of equation (1.1) be polynomials of degrees  $d_1$  and  $d_2$ , respectively,  $p \in V_{d_1}(A)$ ,  $q \in V_{d_2}(A)$ . The following result implies, in particular, Theorem 1.7 of the introduction:

**THEOREM 5.1.** 1. *In setting (A), where both  $p$  and  $q$  are free, the Center set at infinity  $\mathcal{C}_\infty \subset H \subset \mathbb{P}V_{d_1, d_2}(A)$  coincides with the Composition set  $\mathcal{L}$  in the following cases:*

- (a)  $d_1 \leq 2r - 2$ .
- (b)  $r = 2$  and  $d_1 \leq 4$ .

2. *In setting (B), where  $q$  of degree  $d_2$  is fixed, the Center set at infinity  $\mathcal{C}_{q, \infty} \subset H \subset \mathbb{P}V_{d_1}(A)$  coincides with the Composition set  $\mathcal{L}_q$  in the following cases:*

- (a)  $d_1 \leq 2r - 2$ .
- (b)  $r = 2$  and  $d_1 \leq 4$ .
- (c)  $r = 2$  and the polynomial  $p$  runs over the space

$$\Omega_{m,n}(A) = \left\{ p(x) = \sum_{i=m}^n \alpha_i x^i, P|_A = 0 \right\}.$$

*The interval  $[m + 1, n + 1]$  does not contain any nontrivial multiples of the prime divisors of  $d_2 + 1$  (but possibly contains some of these prime divisors themselves).*

3. In setting (C), where  $p$  of degree  $d_1$  is fixed, the Center set at infinity  $\mathcal{C}_{p,\infty} \subset H \subset PV_{d_2}(A)$  coincides with the Composition set  $\mathcal{L}_p$  in the following cases:

- (a)  $d_2 \leq 2r - 2$ .
- (b)  $r = 2$  and  $d_2 \leq 4$ .
- (c)  $r = 2$  and the polynomial  $q$  runs over the space

$$\Omega_{m,n}(A) = \{q(x) = \sum_{i=m}^n \alpha_i x^i, Q|_A = 0\}.$$

The interval  $[m + 1, n + 1]$  does not contain any nontrivial multiples of the prime divisors of  $d_1 + 1$  (but possibly contains some of these prime divisors themselves).

*Proof.* Let us first formulate the results of [13]–[15], [18], [49]–[53], [54] that we need. Let the set  $A = \{a_1, \dots, a_r\}$  as above be given.

**PROPOSITION 5.1.** *Let  $\deg P \leq 2r - 1$ . Then for any polynomial  $q$  the first Moment vanishing condition (1.5) on the set  $A = \{a_1, \dots, a_r\}$  implies Composition condition (PCC) on  $A$ .*

This result is proved in [17], [18]. See also [12]–[14]. □

In an important special case  $r = 2$  this result can be improved:

**PROPOSITION 5.2.** *Let  $\deg P \leq 5$ . Then for any polynomial  $q$  the first Moment vanishing condition (1.5) on the set  $\{a, b\}$  implies Composition condition (PCC) on  $\{a, b\}$ .*

The case  $\deg P \leq 3$  of this result follows from Proposition 5.1 above. For  $\deg P = 4, 5$  the result follows from [51], [52].

Finally, we need the following result of [50]:

**PROPOSITION 5.3.** *The first Moment vanishing condition (1.5) on the set  $\{a, b\}$  implies that the degrees of  $P$  and  $Q$  are not relatively prime.*

*If the polynomial  $P$  is indecomposable (in particular, if the degree of  $P$  is prime), then for any  $q$  the first Moment vanishing condition (1.5) on the set  $\{a, b\}$  implies Composition Condition (PCC) on  $\{a, b\}$ .*

Of course, in each of the results above we can interchange  $p$  and  $q$  while replacing the first Moment vanishing condition (1.5) with the second Moment vanishing condition (4.1).

Now statement 1 of Theorem 5.1 follows directly from Theorem 4.1 and Propositions 5.1 and 5.2. The same is true for statements 2(a), (b) and 3(a), (b). (In the statement 3 we should remember that the Center equations  $\psi_k = 0$  for odd  $k$ , that we do not consider, vanish on the Composition set.) It remains to prove the statements 2(c) and 3(c).



To prove statement 2(c), we proceed as follows: for a fixed polynomial  $q = Q'$  the Moment vanishing condition (1.5) implies, via Proposition 5.3, that the degree  $d_2 + 1$  of  $Q$  cannot be relatively prime with the degree of  $P$ . But  $p \in \Omega_{m,n}$ , so that, unless  $p \equiv 0$ , its degree  $d$  is between  $m$  and  $n$ , and hence  $\deg P = d + 1 \in [m + 1, n + 1]$ . By assumptions, the interval  $[m + 1, n + 1]$  does not contain any nontrivial multiples of the prime divisors of  $d_2 + 1$  (but possibly contains some of these prime divisors themselves). Therefore,  $\deg P = d + 1$  cannot have common divisors with  $\deg Q = d_2 + 1$ , unless  $d + 1$  is prime. But then the polynomial  $P$  is indecomposable, having a prime degree. Once more, by Proposition 5.3, we conclude that in this case the Moment vanishing condition (1.5) implies that  $p, q$  satisfies the Composition Condition (PCC) on  $A = \{a, b\}$ . This completes the proof of statements 2 of Theorem 5.1. The proof of statement 3(c), is exactly the same as above, with  $p$  and  $q$  interchanged.

Now we prove Theorem 1.6. Let us reformulate it, taking into account our separation of the Center problem into different cases:

**THEOREM 5.2.** *For  $A = \{a, b\}$  and for any fixed degrees  $d_1 \geq 5$  and  $d_2 \geq 2$ , the Center set at infinity (in setting  $A$ )  $\mathcal{C}_\infty \subset H \subset PV_{d_1, d_2}(A)$  is strictly larger than the Composition set  $\mathcal{L}_\infty \subset H \subset PV_{d_1, d_2}(A)$ .*

*Proof.* By rescaling and shift we can assume that  $A = \{a, b\} = \left\{-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right\}$ . Take  $p_0 = T'_6$  and  $q_0 = (T_2 + T_3)'$ , where  $T_n$  is the  $n^{\text{th}}$  Chebyshev polynomial. By [49] the first system of the Moment equations (1.5) is satisfied for the couple  $p_0, q_0$  on  $A$ , while  $p_0, q_0$  violate the Composition condition on  $A$ . Notice that  $\deg p_0 = 5$ ,  $\deg q_0 = 2$ . Hence for each  $d_1 \geq 5$  and  $d_2 \geq 2$  the couple  $p_0, q_0$  belongs to the Center set at infinity (defined by the Moment equations (1.5)) while it does not belong to the Composition set. This completes the proof.  $\square$

### 6. The structure of the Composition set

In the previous section we showed that in many cases the Center set at infinity coincides with the Composition set. However, from the point of view, taken in Section 5, the Composition set arises as the set of zeroes of a complicated nonlinear system of the Moment equations. This system by itself does not say much about the structure of its solutions.

However, to proceed to the description of the Center set in a *neighborhood of infinity*, we need rather detailed information on the geometric structure of the Composition set, in particular, on its singularities.

To get this information, in the present section we provide a description of the Composition set *ad hoc*, using the information on the algebraic structure of polynomial compositions, provided by the original results of Ritt ([57]). The description we get turns out to be rather simple (in the cases we consider, the Composition set is a union of certain linear subspaces).

In comparison with the classical theory (see, for example, [57], [62]), we are interested in what we call below  $A$ -compositions, i.e. compositions of polynomials under the requirement that some factors take equal values on all the points of  $A = \{a_1, \dots, a_r\}$ . We do not try here to give a detailed presentation of the theory of polynomial  $A$ -compositions, providing only the simplest definitions and results, sufficient for our purposes.

We formulate and prove our results below only for  $q$  fixed. The case of  $p$  fixed is considered in exactly the same way. (In the description of the Composition set, in contrast with the Center and the Moment ones, the setting is formally symmetric with respect to  $p$  and  $q$ .)

Assume that a set of points  $A$  is given, consisting of at least two points:  $A = \{a_1, \dots, a_r\}$ ,  $a_j \in \mathbb{C}$ ,  $a_i \neq a_l$  for  $i \neq l$ ,  $r \geq 2$ . We want to take into account the fact that the right composition factors we are interested in take equal values at the points of  $A$ .

*Definition 6.1.* Let  $A = \{a_1, \dots, a_r\}$  as above and let a polynomial  $Q$  satisfying  $Q(a_1) = Q(a_2) = \dots = Q(a_r)$  be given. We call polynomial  $W$  a right  $A$ -factor of  $Q$  if  $Q = \tilde{Q} \circ W$  and  $W(a_1) = W(a_2) = \dots = W(a_r)$ .  $Q$  is called  $A$ -prime, if it does not have nontrivial right  $A$ -factors.

We have the following initial result:

**PROPOSITION 6.1.** *Up to a composition from the left with a ‘‘unit’’  $\lambda(x) = ax + b$ , there are finitely many  $A$ -prime right  $A$ -factors  $W_j$ ,  $j = 1, \dots, s$ , of  $P$ . Each right  $A$ -factor  $W$  of  $P$  can be represented as  $W = \tilde{W}(W_j)$  for some  $j = 1, \dots, s$ .*

This proposition follows in a straightforward way from the results of Ritt (see [57], [62]).

Composing  $W_j$ , if necessary, with a linear ‘unit’  $\lambda(x) = ax + b$  on the left, we can assume that  $W_j(a_1) = \dots = W_j(a_r) = 0$ .

The structure of the Composition set  $\mathcal{L}_q$  is described by the following proposition:

**PROPOSITION 6.2.** *Let  $A = \{a_1, \dots, a_r\}$  and a polynomial  $Q$  be given. Let  $W_j$ ,  $j = 1, \dots, s$ , be all the prime right  $A$ -factors of  $Q$ . Then the Composition set  $\mathcal{L}_q$  in the space  $V_{d_1}(A)$  is the union of the linear subspaces  $\mathcal{L}_j \subset V_{d_1}(A)$ ,  $j = 1, \dots, s$ , where  $\mathcal{L}_j$  consists of all the polynomials  $p \in V_{d_1}(A)$  for which  $P = \int p$  is representable as  $P = \tilde{P}(W_j)$ ,  $j = 1, \dots, s$ , for a certain polynomial  $\tilde{P}$ .*

*Proof.* By definition,  $\mathcal{L}_q$  consists of all  $p \in V_{d_1}(A)$  for which  $P = \int p$  is representable as  $P = \tilde{P}(W)$  for a certain polynomial  $\tilde{P}$  and for  $W$  a right composition  $A$ -factor of  $Q$ . Hence each of the linear subspaces  $\mathcal{L}_j$  is contained in  $\mathcal{L}_q$ . In the opposite direction, if  $W$  is a right composition  $A$ -factor of  $Q$  then by Proposition 6.1 above  $W = \tilde{W}(W_j)$  for some  $j = 1, \dots, s$ . Therefore  $P = \tilde{P}(W) = \tilde{P}(\tilde{W}(W_j))$  belongs to  $\mathcal{L}_j$ . □

Now we apply Proposition 6.2 to get an explicit description of the Composition sets arising in this paper. As usual, we assume that both  $P$  and  $Q$  are equal to zero on  $A$ .

Let us consider our specific cases:

1.  $A = \{a_1, \dots, a_r\}$ ,  $d_1 \leq 2r - 2$ ,  $q$  is fixed. Let  $W_l$ ,  $l = 1, \dots, n$ , be all the prime right  $A$ -factors of  $Q = \int q$  satisfying  $\deg W_l \leq d_1 + 1$ .

LEMMA 6.1. *The Composition set  $\mathcal{L}_q \in V_{d_1}(A)$  is a union of  $n$  one-dimensional subspaces  $\mathcal{L}_l = \{\alpha W_l, \alpha \in \mathbb{C}\} \subset V_{d_1}(A)$ .*

*Proof.* This follows directly from Proposition 6.2. Indeed, for each prime  $A$ -factor  $W_j$  of  $Q$  we have  $\deg W_j \geq r$  since each of these polynomials takes equal values at the points of  $A$ . On the other hand,  $\deg P \leq d_1 + 1 \leq 2r - 1$  by the assumptions. Hence if  $P = \tilde{P}(W_j)$  then in fact  $P = \alpha W_j$ ,  $\alpha \in \mathbb{C}$ . In particular,  $\deg W_j \leq d_1 + 1$ , and hence  $W_j$  is one of the polynomials  $W_l$  defined above.  $\square$

2.  $A = \{a, b\}$ ,  $d_1 \leq 4$ ,  $q$  is fixed. Let  $W_l$ ,  $l = 1, \dots, n$ , be all the prime right  $A$ -factors of  $Q = \int q$  of the degrees 2, 3, 4 and 5.

LEMMA 6.2. *The Composition set  $\mathcal{L}_q \subset V_{d_1}(A)$  is a union of one-dimensional subspaces  $\mathcal{L}_l = \{\alpha W_l\} \subset V_{d_1}(A)$ , and of at most one two-dimensional subspace  $\hat{\mathcal{L}}_{l_0} = \{\alpha W_{l_0}^2 + \beta W_{l_0}\}$  for  $\deg W_{l_0} = 2$ . Each couple of these subspaces intersects only at the origin.*

*Proof.* The form of the subspaces  $\mathcal{L}_l$  follows directly from Proposition 6.2 and from the assumption  $\deg P \leq 5$ . There may be at most one (up to a scalar factor) right factor  $W_{l_0}$  of  $Q$  of degree 2 vanishing on  $A = \{a, b\}$ . Indeed, any two quadratic polynomials vanishing on  $A = \{a, b\}$  are proportional. Now the lines  $\{\alpha W_j\} \subset V_{d_1}(A)$  intersect only at zero, since different right  $A$ -factors  $W_l$  cannot be proportional. No one of these lines can be inside the two-dimensional subspace  $\hat{\mathcal{L}}_{l_0}$  of  $\mathcal{L}_q$  - otherwise one of the  $W_l$  would be a quadratic polynomial of  $W_{l_0}$ .  $\square$

3.  $A = \{a, b\}$ ,  $q = Q'$  is fixed with  $\deg Q = d + 1$ . Also,  $p$  runs over the space  $\Omega_{m,n}(A)$  of all the polynomials of the form  $p(x) = \sum_{i=m}^n \alpha_i x^i$ , such that  $P = \int_a p$  vanishes at  $b$ . We assume that the interval  $[m + 1, n + 1]$  does not contain any nontrivial multiples of the prime divisors of  $d + 1$  (but possibly contains some of these prime divisors themselves).

Let  $W_l$ ,  $l = 1, \dots, t$ , be all the prime right  $A$ -factors of  $Q$  with the following property: the degree  $d_l$  of  $W_l$  is a prime number, belonging to the interval  $[m + 1, n + 1]$ , and  $W_l'$  itself belongs to  $\Omega_{m,n}$ .

LEMMA 6.3. *The Composition set  $\mathcal{L}_q \subset \Omega_{m,n}(A)$  is a union of one-dimensional subspaces  $\cup_{l=1}^t \{p = \alpha W_l'\} \subset \Omega_{m,n}(A)$ .*

*Proof.* Let  $p \in \Omega_{m,n}(A)$  belong to the Composition set  $\mathcal{L}_q$ . By definition,  $P = \int p = \tilde{P}(W)$ ,  $Q = \tilde{Q}(W)$ , with  $W(a) = W(b)$ . The degree  $s$  of  $W$  divides  $d + 1 = \deg Q$ . But the degree of  $P$  belongs to the interval  $[m + 1, n + 1]$  and

it is equal to  $\deg \tilde{P} \cdot s$ . By our assumption no nontrivial multiples of the prime divisors of  $d + 1$  belong to the interval  $[m + 1, n + 1]$ , so the last equality is only possible if  $s$  itself is prime and  $\deg \tilde{P} = 1$ . In particular,  $W$ , having a prime degree, is indecomposable, and hence, up to a composition with a linear polynomial on the left,  $W$  coincides with one of the prime right  $A$ -factors  $W_j$  of  $Q$ , provided by Proposition 6.1. Now we can write  $\tilde{P}(w) = \alpha w + \beta$ . But since both  $P$  and  $W_j$  vanish at the points of  $A$ , we conclude that  $\beta = 0$ . Therefore,  $P = \alpha W_j$  and  $p = P' = \alpha W'_j$ . This implies that  $W'_j$  itself belongs to  $\Omega_{m,n}$ , so in fact  $W_j$  is one of the prime right  $A$ -factors  $W_l$  of  $Q$  defined above. This completes the proof of the lemma.  $\square$

**COROLLARY 6.1.** *In each of the three cases above the Composition set at infinity is nonsingular.*

*Proof.* Indeed, in each case this set consists of isolated points and possibly one straight line, not passing through these points.  $\square$

*Remark 6.1.* In general, starting with  $\deg P \geq 6$ , this last conclusion is not true anymore. For example, we fix  $Q = T_6$  and  $(a, b) = \left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$ . Then by Proposition 6.2 the set  $\mathcal{L}_q \subset V_5$  is a union of two linear subspaces  $\mathcal{L}_1 = \{R(T_2), \deg R = 3\}$  and  $\mathcal{L}_2 = \{S(T_3), \deg S = 2\}$ . These subspaces intersect along a straight line  $\{\alpha T_6\}$ . So at infinity the Composition set consists of two linear subspaces intersecting at one point.

*Remark 6.2.* As we shall see in the next section, our method for the extension of the coincidence of the Center and Composition sets from infinity to a neighborhood of infinity works at present only for these sets being nonsingular. Moreover, we need the Moment equations to define these sets in a nondegenerate way. This assumption significantly restricts the range of our results.

### 7. Neighborhood of infinity

In the previous sections we have shown that in all the main situations treated in this paper, the Center set at infinity  $\mathcal{C}_\infty$  coincides with the Composition set  $\mathcal{L}_\infty$ . In this section we extend this coincidence to the entire neighborhood of the infinite hyperplane  $H$ . However, we do this only for  $q$  or  $p$  fixed, i.e. for settings (B) and (C) of the Center-Focus problem, and from now on we exclude from consideration the most general setting (A). The main reason is that in setting (A) the Composition set at infinity, or, at least, the Moment equations defining this set, have singularities.

Let us concentrate on setting (B), with  $q$  fixed and  $p$  free. Setting (C) is treated exactly in the same way, just by interchanging  $p$  and  $q$ . The exact statement of our result is as follows: assume, as usual, that the set  $A = \{a_1, \dots, a_r\}$  is fixed.

**THEOREM 7.1.** *Let the polynomial  $q$  of degree  $d_2$  be fixed, while  $p \in V_{d_1}(A)$  is free. Then in all the cases considered above there exists a neighborhood  $U$  of*

the infinite hyperplane  $H \subset PV_{d_1}(A)$  such that the Center and the Composition sets coincide in  $U$ , i.e.  $\mathcal{C}_q \cap U = \mathcal{L}_q \cap U$ .

We consider the following cases:

- a.  $d_1 \leq 2r - 2$ .
- b.  $r = 2$  and  $d_1 \leq 4$ .
- c.  $r = 2$  and the polynomial  $p$  runs over the space

$$\Omega_{m,n}(A) = \left\{ p(x) = \sum_{i=m}^n \alpha_i x^i, P|_A = 0 \right\}.$$

The interval  $[m + 1, n + 1]$  does not contain any nontrivial multiples of the prime divisors of  $d_2 + 1$  (but possibly contains some of these prime divisors themselves).

The rest of this section is devoted to the proof of Theorem 7.1 . We have to show that the Center set  $\mathcal{C}_q$  coincides with the Composition set  $\mathcal{L}_q$  not only on the infinite hyperplane  $H$  but *inside the entire neighborhood of the infinity*. This extension will be based on the following result, which generalizes the main result of [11], and which we prove at the end of this section:

**THEOREM 7.2.** *Let the point  $p_0 \in PV_{d_1}(A)$  belong to the Composition set  $\mathcal{L}_q$ . Assume that the linear parts of the Center equations at the point  $p_0$  have as their zero set (locally) the Composition set  $\mathcal{L}_q$ . Then there exists a neighborhood  $U_{p_0}$  of the point  $p_0$  such that  $\mathcal{C}_q \cap U_{p_0} = \mathcal{L}_q \cap U_{p_0}$ .*

Theorem 7.2 is applicable, in particular, to the points  $p_0$  at infinity. We get the following corollary:

**COROLLARY 7.1.** *Assume that in one of the settings (considered above) of the Center-Focus problem we have  $\mathcal{C}_\infty = \mathcal{L}_\infty$ . Assume in addition that at each point of these sets the linear parts of the Center equations have as their zero set the Composition set  $\mathcal{L}$ . Then there exists a neighborhood  $U$  of the infinite hyperplane  $H$  such that  $\mathcal{C} \cap U = \mathcal{L} \cap U$ .*

*Proof.* For each point  $h \in H$  belonging to  $\mathcal{C}_\infty = \mathcal{L}_\infty$  a neighborhood  $U_h$  of  $h$  with the required property is provided by Theorem 7.2 . For each point  $h \in H$  not belonging to this set, we can find a neighborhood  $U_h$  not intersecting  $\mathcal{C}$  and  $\mathcal{L}$  at all. The required neighborhood  $U$  is the union  $\cup_{h \in H} U_h$ . □

*Remark.* Theorem 7.2 implies that both the Center set and the Composition set are *nonsingular* at  $p_0$  (being locally the set of zeroes of a system of linear equations), and the Center equations are nondegenerate at  $p_0$ . Let us stress once more that this fact restricts the applicability of our approach in its present form only to the situations where the Center and Moment sets at infinity coincide with the Composition set and are nonsingular there, as well as the Moment and Center equations.

According to Corollary 7.1 , in order to obtain a desired extension of the coincidence of the Center and Composition sets from the infinite hyperplane  $H$  to its neighborhood, it is enough to check that at each point  $p_0 \in \mathcal{C}_\infty = \mathcal{L}_\infty$  the vanishing of the first differentials (or of the linear parts) of the Center equations implies the Composition condition.

First of all, we have to describe the structure of the Center equations, as expanded around a certain given point  $p_0$ . Let us start with the expansion of one iterated integral. As usual, the set  $A = \{a_1, \dots, a_r\}$  is fixed, and  $(a, b)$  denotes any of the couples  $(a_1, a_i)$ .

Consider an iterated integral

$$(7.1) \quad I_\alpha = \int_a^b h_{\alpha_1}(x_1)dx_1 \left( \int_a^{x_1} h_{\alpha_2}(x_2)dx_2 \dots \left( \int_a^{x_{s-1}} h_{\alpha_s}(x_s)dx_s \right) \dots \right).$$

Here, as above,  $\alpha$  are the multi-indices  $\alpha = (\alpha_1, \dots, \alpha_s)$  with  $\alpha_j = 1$  or  $2$ , and  $h_1 = p, h_2 = q$ . For  $l = i(\alpha)$  being the number of appearances of  $p$  in  $I_\alpha$  consider the multi-indices  $\beta = (\beta_1, \dots, \beta_l), \beta_j = 0, 1$ .

**PROPOSITION 7.1.** *Let  $p_0 \in V_{d_1}(A)$  be given. Put  $p = p_0 + p_1$ , with  $p_1 \in V_{d_1}(A)$  a new variable polynomial. Then the integral  $I_\alpha$  takes the following form in the variable  $p_1$ :*

$$(7.2) \quad I_\alpha = \sum_{\beta} I_{\alpha,\beta}, \quad \beta = (\beta_1, \dots, \beta_l), \quad \beta_j = 0, 1.$$

Here the iterated integral  $I_{\alpha,\beta}$  is obtained from  $I_\alpha$  by replacing  $p$  at its  $j^{\text{th}}$  entrance ( $j = 1, \dots, l$ ) with  $p_0$  (resp.  $p_1$ ) according to  $\beta_j = 0$  or  $1$ .

*Proof.* We just open the parenthesis and use the multi-linearity of the iterated integral with respect to its entries. □

Notice that the iterated integrals  $I_{\alpha,\beta}$  in (7.2) differ from the iterated integrals  $I_\alpha$  of (7.1) in the following essential feature:  $I_\alpha$  has only two different entrances:  $p$  and  $q$ , while  $I_{\alpha,\beta}$  has three:  $p_0, q$ , and  $p_1$ .

Now let us compute the linear parts of the Center equations at infinity. Recall that in Section 3 we denoted by  $\tilde{\psi}_k$  the homogenization of the Poincaré coefficient  $\psi_k$  with respect to the variable  $p$ . This corresponds to the present section setting (B) of the Center-Focus problem, where we assume the polynomial  $q$  to be fixed. The following description of  $\tilde{\psi}_k$  was given in Lemma 3.1:

For each  $k \geq 2$  the homogeneous Poincaré coefficient  $\tilde{\psi}_k(p, v)$  is given by

$$(7.3) \quad \tilde{\psi}_k = \sum n_\alpha v^{2j(\alpha)-2} I_\alpha,$$

with the sum running over all the multi-indices  $\alpha = (\alpha_1, \dots, \alpha_s)$  for which  $\sum_1^s \alpha_m = i(\alpha) + 2j(\alpha) = k - 1, i(\alpha), j(\alpha) \geq 1$ . Its degree is  $k - 3$ . The terms not containing  $v$  are the sums of all the iterated integrals with exactly one appearance of  $q$ .

Notice that *a priori* we cannot restrict ourselves to the Moment equations (1.5) only, but have to consider the full set of homogeneous Center equations. Nevertheless, as the following proposition shows, we get a nonzero contribution at infinity only from the Moment equations.

**PROPOSITION 7.2.** *Let the point  $(p_0, 0) \in PV_{d_1}(A)$ , belonging to the Composition set at infinity  $\mathcal{L}_\infty$ , be given. Put  $p = p_0 + p_1$ , with  $p_1 \in V_{d_1}(A)$  a new variable polynomial. Consider the expansion of  $\tilde{\psi}_k(p, \nu)$  around the point  $(p_0, 0)$  in variables  $p_1, \nu$ . Then the only nonzero linear terms in this expansion are the following linear functionals in  $p_1$ :*

$$(7.4) \quad -(k - 3) \int_a^b P_0^{k-4}(x) P_1(x) q(x) dx, \quad k = 4, 5, \dots$$

Here  $P_1 = \int_a p_1$ .

*Proof.* First of all, the coefficient of the linear term containing  $\nu$  vanishes at  $(p_0, 0)$ . Indeed, by Lemma 3.1 this coefficient is a sum of the usual iterated integrals  $I_\alpha$  of  $p_0$  and  $q$ . But since by assumptions  $p_0$  is in a Composition set, all these integrals vanish by Proposition 2.3. Second, only the part of  $\tilde{\psi}_k(p, \nu)$  not containing  $\nu$  contributes to the linear in  $p_1$  part of the expansion of  $\tilde{\psi}_k(p, \nu)$  around the point  $(p_0, 0)$ . By Lemma 3.1 the terms not containing  $\nu$  are the sums of all the iterated integrals with exactly one appearance of  $q$ , which by Theorem 2.3 are the moments  $-m_{k-3}(p, q) = -\int_a^b P^{k-3}(x) q(x) dx$ . Substituting here  $P = P_0 + P_1$  we get as the linear terms in  $P_1$  exactly the expressions (7.4). This completes the proof of Proposition 7.2. □

The next step in the proof of Theorem 7.1 is the following lemma:

**LEMMA 7.1.** *In each of the cases considered in Theorem 7.1 for each point  $(p_0, 0) \in PV_{d_1}(A)$ , belonging to the Composition set at infinity  $\mathcal{L}_\infty$ , the vanishing of the expressions (7.4) implies that  $P_0$  and  $H = \int P_1 q$  satisfy Composition Condition on  $A$ .*

*Proof.* We use a description of the Composition set in each of the considered cases, provided by Lemmas 6.1–6.3, as well as by Propositions 5.1–5.3.

In case (a), where  $d_1 \leq 2r - 2$ , we, in fact, do not need the description of the Composition set. It is enough to notice that  $P_0$  is a polynomial of degree  $d_1 + 1 \leq 2r - 1$ , and therefore the Moment equations (7.4) imply the Composition condition for  $P_0$  and  $H$  by Proposition 5.1. The same is true in case (b) where  $r = 2$  and  $d_1 \leq 4$ . Here  $\deg P_0 \leq 5$  and we use Proposition 5.2. So let us consider case (c). Here the polynomial  $p$  runs over the space  $\Omega_{m,n}(A) = \{p(x) = \sum_{i=m}^n \alpha_i x^i, P|_A = 0\}$ , while the interval  $[m + 1, n + 1]$  does not contain any nontrivial multiples of the prime divisors of  $d_2 + 1$  (but possibly contains some of these prime divisors themselves). By Lemma 6.3 the Composition set in this case consists of scalar multiples of the prime composition right divisors  $W_l$  of  $Q$ , having prime degrees,

belonging to the interval  $[m + 1, n + 1]$ . In particular, each  $P_0$  in the Composition set has a prime degree. Once more, the conclusion is that the Moment equations (7.4) imply the Composition condition for  $P_0$  and  $H$  via Proposition 5.3. This completes the proof of the lemma.  $\square$

Now we almost have the required conclusion: we know that the vanishing of the linear parts of the Center equations implies the Composition condition for the couple  $(P_1, H)$ . We want the Composition condition for the couple  $(P_1, Q)$  (and thus for  $(P = P_0 + P_1, Q)$ ). We obtain this in two steps, first for the couple  $(Q, H)$ , and finally for the couple  $(P_1, Q)$ , insisting that the right composition factor remains the same. The first of these two steps is provided by the following lemma:

**LEMMA 7.2.** *In each of the cases considered in Theorem 7.1 for each point  $(p_0, 0) \in PV_{d_1}(A)$ , belonging to the Composition set at infinity  $\mathcal{L}_\infty$ , the equations (7.4) imply that  $Q$  and  $H = \int P_1 q$  satisfy a composition condition on  $A$ .*

*Proof.* In case (a), where  $d_1 \leq 2r - 2$ , we notice that  $P_0$ , being a polynomial of degree  $d_1 + 1 \leq 2r - 1$ , is  $A$ -prime. Indeed, each polynomial  $W$  taking equal values on  $A = \{a_1, \dots, a_r\}$  has degree at least  $r$ . Since the equality  $P_0 = \tilde{P}_0 \circ W$  implies  $\deg P_0 = \deg \tilde{P}_0 \cdot \deg W$ , the only possibility is that  $\deg \tilde{P}_0 = 1$ . Now, the polynomial  $P_0$  satisfies the Composition condition on  $A$  with the polynomial  $Q$  (since the point  $(p_0, 0) \in PV_{d_1}(A)$ , belongs to the Composition set by the assumptions), and with the polynomial  $H$ , by Lemma 7.1. Being  $A$ -prime,  $P_0$  must therefore be a right composition  $A$ -factor of  $Q$  and of  $H$ . But this implies that the couple  $(Q, H)$ , having a common composition  $A$ -factor, satisfies the Composition condition on  $A$ .

Exactly the same argument works in case (c), where  $P_0$  has a prime degree, and hence is also  $A$ -prime. It remains to consider case (b) where  $r = 2$  and  $d_1 \leq 4$ . Here  $\deg P_0 \leq 5$ . If  $\deg P_0 = 2, 3, 5$ , this polynomial is  $A$ -prime, and the result follows in the same way as above. If  $\deg P_0 = 4$ , there are two possibilities: either  $P_0$  is  $A$ -prime (and the proof is completed as above) or  $P_0 = \tilde{P}_0 \circ W_0$ , with  $\deg \tilde{P}_0 = 2, \deg W_0 = 2$ . In this second case, let us show that the only right composition  $A$ -factors of  $P_0$  are  $P_0$  itself and  $W_0$ . Indeed, the right composition factors  $W$  of  $P_0$  must have degree either 2 or 4. Since  $\deg P_0 = 4$ , in the first case  $W = \alpha P_0$ . In the second case  $W = \beta W_0$ , since, up to a scalar factor, there is only one quadratic polynomial vanishing on  $A = \{a_1, a_2\}$ . Hence, the only right composition  $A$ -factors of  $P_0$  are  $P_0$  itself and  $W_0$ .

Consequently, the fact that the polynomials  $P_0$  and  $Q$  satisfy the Composition condition on  $A$ , i.e. that they have a common right composition  $A$ -factor, implies that  $W_0$  is a prime  $A$ -factor of  $Q$ . Exactly in the same way, from the fact that the polynomials  $P_0$  and  $H$  satisfy Composition condition on  $A$  it follows that  $W_0$  is a prime  $A$ -factor of  $H$ . Thus  $W_0$  is a common prime  $A$ -factor of  $Q$  and  $H$ . This completes the proof of Lemma 7.2.  $\square$



Notice that the common factor  $W_0$  of  $Q$  and  $H$  is also a right  $A$ -factor of  $P_0$ . Finally, from the fact that  $Q$  and  $H = \int P_1 q$  satisfy the Composition condition on  $A$  we conclude that the same is true for  $Q$  and  $P_1$ . This last step is provided by the following lemma, which we prove in a slightly more general form than required:

LEMMA 7.3. *Let polynomials  $P_1(x)$  and  $q(x) = Q'(x)$  be given. Assume that there are polynomials  $\tilde{Q}$ ,  $T$  and  $W$  such that the following composition identities are satisfied:*

$$(7.5) \quad Q(x) = \tilde{Q}(W(x)), \quad H(x) = \int_a^x P_1(t)q(t)dt = T(W(x)).$$

Then there exists a polynomial  $\tilde{P}_1$  such that

$$P_1(x) = \tilde{P}_1(W(x)).$$

*Proof.* By differentiating the equations (7.5) we obtain

$$P_1(x)q(x) = P_1(x)\tilde{Q}'(W(x))W'(x) = T'(W(x))W'(x)$$

which implies

$$(7.6) \quad P_1(x)\tilde{Q}'(W(x)) = T'(W(x)).$$

This implies  $P_1(x) = \frac{T'}{\tilde{Q}'}(W(x))$ , and using the approach of [57] we can show that by a linear transformation the rational function  $\frac{T'}{\tilde{Q}'}$  can be made a polynomial. Instead we give a direct proof of existence of  $\tilde{P}_1$  such that  $P_1(x) = \tilde{P}_1(W(x))$ .

Let  $\deg P_1 = d$ ,  $\deg W = k$ ,  $\deg Q = k_1$  and let  $V_W \subset V_{d+k_1}$  be the subspace formed by all the polynomials in  $W$ .  $V_W$  is a linear subspace of  $V_{d+k_1}$  with the basis  $\{1, W, W^2, \dots\}$ . Let  $\bar{V}_W$  be the complementary subspace of  $V_W$  in  $V_{d+k_1}$ , with the basis

$$\left\{ W^{(k-1)}, \dots, W', WW^{(k-1)}, \dots, WW', W^2W^{(k-1)}, \dots, W^2W', \dots \right\}$$

formed by the products of the powers of  $W$  and its successive derivatives. Together the bases of  $V_W$  and  $\bar{V}_W$  form a new basis  $B$  of  $V_{d+k_1}$ .

Via identity (7.6) above, it is enough to prove for any two polynomials  $P_1 \in V_d$  and  $R \in V_W$  that if  $P_1R \in V_W$  then  $P_1 \in V_W$ . Write

$$P_1(x) = \sum_{i=1}^{v_1} \alpha_i W^i + \sum_{i=0}^{v_1-1} W^i \sum_{j=1}^{k-1} \beta_{ij} W^{(k-j)},$$

$$R(x) = \sum_{i=1}^{v_2} \gamma_i W^i.$$

We shall prove that  $\beta_{ij} = 0$ .

Multiplying these expressions we get a representation of  $P_1 R$  in the basis  $B$ . The terms

$$\gamma_{v_2} W^{v_1-1} W^{v_2} \sum_{j=1}^{k-1} \beta_{v_1-1,j} W^{(k-j)}$$

are the highest degree non-Composition terms in this representation and hence they cannot cancel with any other term. Since by the assumption  $P_1 R \in V_W$ , we necessarily have  $\beta_{v_1-1,j} = 0, j = 1, \dots, k - 1$ . Therefore

$$P_1(x) = \sum_{i=1}^{v_1} \alpha_i W^i + \sum_{i=0}^{v_1-2} W^i \sum_{j=1}^{k-2} \beta_{ij} W^{(k-j)}.$$

By the same reason as above we obtain once more  $\beta_{v_1-2,j} = 0, j = 1, \dots, k - 1$ , and so on. This completes the proof of Lemma 7.3 and of Theorem 7.1.  $\square$

*Proof of Theorem 7.2.* Following [11], we use the next result which is essentially a version of the ‘‘Nakayama Lemma’’ in commutative algebra (see for example [43, Ch. 4 and Lemma 3.4]) adapted to our situation:

LEMMA 7.4. *Let  $f_1, \dots, f_m$  be polynomials in  $n$  complex variables. Let  $f_i = f_i^1 + f_i^2, i = 1, \dots, m$ , with all the  $f_i^1$  homogeneous of degree  $d_1$  and  $f_i^2$  having all the terms of degrees greater than  $d_1$ .*

*Let  $C = \{f_1 = 0, \dots, f_m = 0\}, C^1 = \{f_1^1 = 0, \dots, f_m^1 = 0\}$ . Assume in addition that  $f_1^1, \dots, f_m^1$  generate the ideal  $I_1$  of the set  $C^1$  and that each  $f_i^2$  vanishes on  $C^1$ .*

*Then there exists  $\varepsilon > 0$  such that for the ball  $B_\varepsilon$  in  $\mathbb{C}^m$ ,*

1.  $C \cap B_\varepsilon = C^1 \cap B_\varepsilon$ .
2. *In the ring of holomorphic functions on  $B_\varepsilon$  the ideals  $I = \{f_1, \dots, f_m\}$  and  $I_1 = \{f_1^1, \dots, f_m^1\}$  coincide.*

The proof of this specific version of the Nakayama lemma can be found, for example, in [11].

We apply Lemma 7.4 in our situation as follows:  $f_i$  are the full Center equations,  $f_i^1$  their linear parts at  $p_0$  and  $f_i^2$  their higher order terms. By assumption, the zero set of the linear equations  $f_i^1$  is the Composition set  $\mathcal{L}_q$ . Being linear, the equations  $f_i^1$  generate also the local ideal of this set.

We have to show that all the higher order parts  $f_i^2$  vanish on the Composition set  $\mathcal{L}_q$  near  $p_0$ . The problem is that the Proposition 7.1 above describes these higher order terms as sums of iterated integrals with three different entrances:  $p_0, q$ , and  $p_1$ . While by assumption  $p_0, q$  satisfy the composition Condition, as well as  $p_1, q$ , *a priori* the composition factorization for these two couples may be different. Then the change of variables, applied in Proposition 2.3 to show that all the iterated integrals vanish under the composition condition, would be not possible anymore. However, Lemmas 7.1, 7.2, and 7.3 show, in fact, that in our specific cases  $P_0, P_1$ , and  $Q$  have *the same* right composition  $A$ -factor  $W$ .

Notice also that a description of  $\mathcal{L}_q$  given in Proposition 6.2 above allows us to show in a general situation that  $P_0, P_1$ , and  $Q$  have *the same* right composition  $A$ -factor  $W$ . Indeed, by Proposition 6.2,  $\mathcal{L}_q$  is a union of the linear subspaces  $\mathcal{L}_j$  with  $\mathcal{L}_j = \{p = P', P = \tilde{P}(W_j)\}$ , where  $W_j, j = 1, \dots, n$  are all the prime right  $A$ -factors of  $Q$ . The points  $p_0$  and  $p = p_0 + p_1$ , belonging to  $\mathcal{L}_q$ , necessarily belong to *the same linear subspace*  $\mathcal{L}_j$  (notice that  $p_1$  is small). Hence  $p_1$  also belongs to  $\mathcal{L}_j$ . This is true in each case: either for  $p_0$  belonging to the regular part of  $\mathcal{L}_q$ , or for  $p_0$  belonging to the singular part, consisting of intersections of different  $\mathcal{L}_j$ . Therefore, we have  $P_0 = \tilde{P}_0(W_j), P_1 = \tilde{P}_1(W_j)$  and  $Q = \tilde{Q}(W_j)$ , and the change of variables applied in Proposition 2.3 works also in this case.

This completes the proof of the vanishing of the high-order terms on the Composition set  $\mathcal{L}_q$ . Now application of Lemma 7.4 completes the proof of Theorem 7.2. □

### 8. Proof of Theorems 1.1–1.4

By Theorem 7.1 we know that in all the situations covered by Theorems 1.1–1.3 the Center set coincides with the Composition set in the entire neighborhood of the infinite hyperplane. It remains to apply the following elementary fact of complex Algebraic Geometry: Let  $\mathcal{A}$  and  $\mathcal{A}'$  be two complex algebraic sets in  $\mathbb{C}P^n$ . Let  $U$  denote a certain open neighborhood of the infinite hyperplane  $H = \mathbb{C}P_\infty^{n-1} \subset \mathbb{C}P^n$ .

**THEOREM 8.1.** *If  $U \cap \mathcal{A} \subset U \cap \mathcal{A}'$  then  $\mathcal{A} \subset \mathcal{A}' \cup \{b_1, \dots, b_s\}$ , where  $b_j, j = 1, \dots, s$ , are certain isolated points in  $\mathbb{C}P^n$ . The number  $s$  is bounded in terms of the number and the degrees of the equations defining  $\mathcal{A}$ .*

*Proof.* Let  $\mathcal{A} = \cup_j \mathcal{A}_j$  be the decomposition of  $\mathcal{A}$  into the union of its irreducible components. Consider those  $\mathcal{A}_j$  with  $\dim \mathcal{A}_j > 0$ . For any such irreducible component if  $U \cap \mathcal{A}_j \subset U \cap \mathcal{A}'$  then everywhere in  $\mathbb{C}P^n$  we have  $\mathcal{A}_j \subset \mathcal{A}'$ . Indeed, for any polynomial  $h$  vanishing on  $\mathcal{A}'$ ,  $h$  vanishes on the intersection of  $\mathcal{A}_j$  with  $U$ . Since  $\dim \mathcal{A}_j > 0$ , this intersection is a *nonempty* open subset of  $\mathcal{A}_j$  (otherwise the affine part of  $\mathcal{A}_j$  would be a compact set, which is impossible). But for an irreducible set  $\mathcal{A}_j$  this implies that  $h \equiv 0$  on the entire  $\mathcal{A}_j$ . This is true for any  $h$  from the ideal  $I(\mathcal{A}')$  of the set  $\mathcal{A}'$ . Hence everywhere in  $\mathbb{C}P^n, \mathcal{A}_j \subset \mathcal{A}'$ . Therefore only zero-dimensional irreducible components  $\mathcal{A}_j$  of  $\mathcal{A}$ , i.e. its isolated points, may lie outside of  $\mathcal{A}'$ . The number  $s$  of these points is bounded in terms of the number and the degrees of the equations defining  $\mathcal{A}$  through the Bézout theorem. □

Now, in order to get Theorems 1.1–1.3 *without explicit bounds on the number of the required Center equations and on the number  $s$  of the isolated points not in  $\mathcal{L}$*  it remains to apply Theorem 8.1 to  $\mathcal{A} = \mathcal{C}$  and  $\mathcal{A}' = \mathcal{L}$ . Indeed, in the situations covered by these theorems, the required inclusion in a neighborhood of infinity is provided by Theorem 7.1. This completes the proof of Theorems 1.1–1.3. □

*Proof of Theorem 1.4.* Let us show first that in *all the cases covered in the present paper* the problem of bounding explicitly the necessary number  $N = N(d_1, d_2, r)$  of the Center equations can be reduced to a similar problem for the Moments.

To bound explicitly this number  $N$ , as well as the number  $s$  of the isolated points which may appear in  $\mathcal{C}$  outside of  $\mathcal{L}$ , we proceed as follows: denote by  $N_1(d_1, d_2, r)$  the number of the *Moment equations* at infinity which imply Composition condition. The Center equations at infinity become the Moment equations, so that it is enough to take  $N_1(d_1, d_2, r)$  of the Center equations to get the Center set (equal to the Composition set) at infinity.

As shown in Section 7, to guarantee this coincidence also in a neighborhood of infinity, it is enough that the Moment-like equations (7.4) (which present the first differential of the Center equations at infinity) imply the Composition condition. Denote by  $N_2(d_1, d_2, r)$  the number of the equations in (7.4) sufficient to imply the Composition condition. Now we define  $N = N(d_1, d_2, r)$  as  $N = \max(N_1, N_2)$ . Let us show that the first  $N$  Center equations is enough.

Take as  $\mathcal{A}$  the set defined by the first  $N$  Center equations. Clearly,  $\mathcal{C} \subseteq \mathcal{A}$  and  $\mathcal{A}' \subseteq \mathcal{C}$  where as above we take  $\mathcal{A}' = \mathcal{L}$ . By Theorem 7.1 we have  $\mathcal{A} = \mathcal{A}'$  in a certain neighborhood of infinity.

Now application of Theorem 8.1 provides  $\mathcal{L} \subseteq \mathcal{C} \subseteq \mathcal{A} \subseteq \mathcal{L} \cup \{p_1, \dots, p_s\}$ . To get an explicit (but certainly not sharp) bound on  $s$  in terms of  $N, d_1, d_2, r$  we use the (simplified) inequality on the number  $S$  of bounded connected components of a set  $\{g(x) \leq 0\} \subset \mathbb{R}^n$ , given by [70, Th. 4.8]:  $S \leq \frac{1}{2}d^n$ , where  $d$  is the degree of a polynomial  $g$ .

We take  $g$  to be the sum of squares of the real and imaginary parts of the Center equations. By Proposition 2.5 the degree of the  $k^{\text{th}}$  center equation in  $p$  is  $\lfloor \frac{k}{r-1} \rfloor$ , so that the degree  $d$  of  $g$  is  $2\lfloor \frac{N}{r-1} \rfloor$ . The real dimension  $n$  of the space  $V_{d_1}(A)$  of the polynomials  $p$  is  $2(d_1 - r + 1)$ . We get

$$(8.1) \quad s \leq \frac{1}{2} \left( 2 \left\lfloor \frac{N}{r-1} \right\rfloor \right)^{2(d_1-r+1)} .$$

Finally we return to the situation of Theorem 4.1:  $A = \{a_1, \dots, a_r\}$ ,  $r \geq 3$ , a polynomial  $q$  of an arbitrary degree  $d_2$  is fixed, and  $p$  varies in the set  $V_{d_1}(A)$  with  $d_1 = 2r - 2$ . It was shown in [12]–[14] (see also [11, Th. 5.1]) that the vanishing of  $N_1 = (r - 1)(d_2 - r + 1)$  (or of  $r - 1$ , if  $d_2 \leq r$ ) first moment equations implies composition. The same result applied to system (7.4) gives

$$N_2 = (r - 1)(d_2 + 2r - 2 - r + 1) = (r - 1)(d_2 + r - 1).$$

Therefore

$$N = \max(N_1, N_2) = N_2 = (r - 1)(d_2 + r - 1).$$

Substituting this into (8.1) we get  $s \leq \frac{1}{2} 4^{(r-1)} (d_2 + r - 1)^{2(r-1)}$ . This completes the proof of Theorem 1.4. □

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