Subject: Elementary integration and x^x

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I really really really ought to polish this up for FAQ inclusion. I am combining two old articles of mine, the first giving and sketching the general Liouville theory, the second applying this theory to x^x:

We give a fairly complete sketch of the proof that certain functions, including the asked for one, are not integrable in elementary terms. The central theorem is due to Liouville in 1835. His proof was analytic. The sketch below is mostly algebraic and is due to Maxwell Rosenlicht. See his papers in the *Pacific Journal of Mathematics*, 54, (1968) 153–161 and 65, (1976), 485–492.

WARNING: Prerequisites for understanding the proof is a first year graduate course in algebra, and a little complex analysis. No deep results are used, but I cannot take the time to explain standard notions or all the deductions.

Notation: $\mathbb{C}$ is the complex numbers, for fields $\mathbb{F}$, $\mathbb{F}[x]$ is the ring of polynomials in x OR an algebraic extension of $\mathbb{F}$, $\mathbb{F}(x)$ is the field of rational functions in a transcendental x, $\mathbb{M}$ is the field of meromorphic functions in one variable. If f is a complex function, $\int(f)$ will denote an antiderivative of f.

A differential field is a field $\mathbb{F}$ of characteristic 0 with a derivation. Thus, in addition to the field operations $+$ and $\cdot$, there is a derivative mapping $' : \mathbb{F} \to \mathbb{F}$ such that $(a+b)' = a' + b'$ and $(ab)' = a'b + ab'$. Two standard examples are $\mathbb{C}(z)$ and $\mathbb{M}$ with the usual derivative map. Notice a basic identity (logarithmic differentiation) holds:

$$\frac{[a_1^{k_1} \cdots a_n^{k_n}]'}{a_1^{k_1} \cdots a_n^{k_n}} = k_1 \frac{a_1'}{a_1} + \cdots + k_n \frac{a_n'}{a_n}$$

The usual rules like the quotient rule also hold. If a in $\mathbb{F}$ satisfies $a' = 0$, we call a a constant of $\mathbb{F}$. The set of constants of $\mathbb{F}$ is called $\mathrm{Con}(\mathbb{F})$, and forms a subfield of $\mathbb{F}$.

The basic idea in showing something has no elementary integral is to reduce the problem to a sequence of differential fields $\mathbb{F}_0$, $\mathbb{F}_1$, etc., where $\mathbb{F}_0 = \mathbb{C}(z)$, and $\mathbb{F}_{i+1}$ is obtained from

$\mathbb{F}_i$ by adjoining one new element t. t is obtained either algebraically, because t satisfies some polynomial equation $p(t) = 0$, or exponentially, because $t'/t = s'$ for some s in $\mathbb{F}_i$, or logarithmically, because $t' = s'/s$ is in $\mathbb{F}_i$. Notice that we don't actually take exponentials or logarithms, but only attach abstract elements that have the appropriate derivatives. Thus a function f is integrable in elementary terms iff such a sequence exists starting with $\mathbb{C}(z)$.

Just so there is no confusion, there is no notion of "composition" involved here. If you want to take $\log s$, you adjoin a transcendental t with the relation $t' = s'/s$. There is no log function running around, for example, except as motivation, until we reach actual examples.

We need some easy lemmas. Throughout the lemmas $\mathbb{F}$ is a differential field, and t is transcendental over $\mathbb{F}$.

Lemma 1: If $\mathbb{K}$ is an algebraic extension field of $\mathbb{F}$, then there exists a unique way to extend the derivation map from $\mathbb{F}$ to $\mathbb{K}$ so as to make $\mathbb{K}$ into a differential field.

Lemma 2: If $\mathbb{K} = \mathbb{F}(t)$ is a differential field with derivation extending $\mathbb{F}$'s, and t' is in $\mathbb{F}$, then for any polynomial $f(t)$ in $\mathbb{F}[t]$, $f(t)'$ is a polynomial in $\mathbb{F}[t]$ of the same degree (if the leading coefficient is not in $\mathrm{Con}(\mathbb{F})$) or of degree one less (if the leading coefficient is in $\mathrm{Con}(\mathbb{F})$).

Lemma 3: If $\mathbb{K} = \mathbb{F}(t)$ is a differential field with derivation extending $\mathbb{F}$'s, and t'/t is in $\mathbb{F}$, then for any a in $\mathbb{F}$, n a positive integer, there exists h in $\mathbb{F}$ such that $(a \cdot t^n)' = h \cdot t^n$. More generally, if $f(t)$ is any polynomial in $\mathbb{F}[t]$, then $f(t)'$ is of the same degree as $f(t)$, and is a multiple of $f(t)$ iff $f(t)$ is a monomial.

These are all fairly elementary. For example, $(a \cdot t^n)' = (a' + at'/t) \cdot t^n$ in lemma 3. The final 'iff' in lemma 3 is where transcendence of t comes in. Lemma 1 in the usual case of subfields of $\mathbb{M}$ can be proven analytically using the implicit function theorem.

MAIN THEOREM. Let $\mathbb{F}$,$\mathbb{G}$ be differential fields, let a be in $\mathbb{F}$, let y be in $\mathbb{G}$, and suppose $y' = a$ and $\mathbb{G}$ is an elementary differential extension field of $\mathbb{F}$, and $\mathrm{Con}(\mathbb{F}) = \mathrm{Con}(\mathbb{G})$. Then there exist $c_1, \ldots, c_n \in \mathrm{Con}(\mathbb{F})$, $u_1, \ldots, u_n, v \in \mathbb{F}$ such that

$$a = c_1 \frac{u_1'}{u_1} + \cdots + c_n \frac{u_n'}{u_n} + v' \qquad (*)$$

In other words, the only functions that have elementary anti-derivatives are the ones that have this very specific form.

This is a very useful theorem for proving non-integrability. In the usual case, $\mathbb{F},\mathbb{G}$ are subfields of $\mathbb{M}$, so $\mathrm{Con}(\mathbb{F}) = \mathrm{Con}(\mathbb{G}) = \mathbb{C}$ always holds.

Proof: By assumption there exists a finite chain of fields connecting $\mathbb{F}$ to $\mathbb{G}$ such that the extension from one field to the next is given by performing an algebraic, logarithmic, or exponential extension. We show that if the form (*) can be satisfied with values in $\mathbb{F}_2$, and $\mathbb{F}_2$ is one of the three kinds of allowable extensions of $\mathbb{F}_1$, then the form (*) can be satisfied in $\mathbb{F}_1$. The form (*) is obviously satisfied in $\mathbb{G}$: let all the c's be 0, the u's be 1, and let v be the original y for which $y' = a$. Thus, if the form (*) can be pulled down one field, we will be able to pull it down to $\mathbb{F}$, and the theorem holds.

So we may assume without loss of generality that $\mathbb{G} = \mathbb{F}(t)$.

Case 1: t is algebraic over $\mathbb{F}$. Say t is of degree k. Then there are polynomials U_i and V such that $U_i(t) = u_i$ and $V(t) = v$. So we have

$$a = c_1 \frac{U_1(t)'}{U_1(t)} + \cdots + c_n \frac{U_n(t)'}{U_n(t)} + V(t)'$$

Now, by the uniqueness of extensions of derivatives in the algebraic case, we may replace t by any of its conjugates $t_1, \ldots, t_k$, and the same equation holds. In other words, because a is in $\mathbb{F}$, it is fixed under the Galois automorphisms. Summing up over the conjugates, and converting the U'/U terms into products using logarithmic differentiation, we have

$$ka = c_1 \frac{[U_1(t_1) \cdots U_1(t_k)]'}{U_1(t) \cdots U_n(t_k)} + \cdots + c_n \frac{U_n(t)'}{U_n(t)} + [V(t_1) + \cdots + V(t_k)]'$$

But the expressions in [...] are symmetric polynomials in t_i, and as they are polynomials with coefficients in $\mathbb{F}$, the resulting expressions are in $\mathbb{F}$. So dividing by k gives us (*) holding in $\mathbb{F}$.

Case 2: t is logarithmic over $\mathbb{F}$. Because of logarithmic differentiation we may assume that the u's are monic and irreducible in t and distinct. Furthermore, we may assume v has been decomposed into partial fractions. The fractions can only be of the form f/g^j, where $\deg(f) < \deg(g)$ and g is monic irreducible. The fact that no terms outside of $\mathbb{F}$ appear on the left hand side of (*), namely just a appears, means a lot of cancellation must be occuring.

Let $t' = s'/s$, for some s in $\mathbb{F}$. If $f(t)$ is monic in $\mathbb{F}[t]$, then $f(t)'$ is also in $\mathbb{F}[t]$, of one less degree. Thus $f(t)$ does not divide $f(t)'$. In particular, all the u'/u terms are in lowest terms already. In the f/g^j terms in v, we have a g^{j+1} denominator contribution in v' of the form $-jfg'/g^{j+1}$. But g doesn't divide fg', so no cancellation occurs. But no u'/u term can cancel, as the u's are irreducible, and no $(xx)/g^{j+1}$ term appears in a, because a is a member of $\mathbb{F}$. Thus no f/g^j term occurs at all in v. But then none of the u's can be outside of $\mathbb{F}$, since nothing can cancel them. (Remember the u's are distinct, monic, and irreducible.) Thus each of the u's is in $\mathbb{F}$ already, and v is a polynomial. But $v' = a -$ expression in u's, so v' is in $\mathbb{F}$ also. Thus $v = bt + c$ for some $b \in \text{Con}(\mathbb{F})$, $c \in \mathbb{F}$, by lemma 2. Then

$$a = c_1\frac{u_1'}{u_1} + \cdots + c_n\frac{u_n'}{u_n} + b\frac{s'}{s} + c'$$

is the desired form. So case 2 holds.

Case 3: t is exponential over $\mathbb{F}$. So let $t'/t = s'$ for some $s \in \mathbb{F}$. As in case 2 above, we may assume all the u's are monic, irreducible, and distinct and put v in partial fraction decomposition form. Indeed the argument is identical as in case 2 until we try to conclude what form v is. Here lemma 3 tells us that v is a finite sum of terms $b \cdot t^j$ where each coefficient is in $\mathbb{F}$. Each of the u's is also in $\mathbb{F}$, with the possible exception that one of them may be t. Thus every u'/u term is in $\mathbb{F}$, so again we conclude v' is in $\mathbb{F}$. By lemma 3, v is in $\mathbb{F}$. So if every u is in $\mathbb{F}$, a is in the desired form. Otherwise, one of the u's, say u_n, is actually t, then

$$a = c_1\frac{u_1'}{u_1} + \cdots + (c_n s + v)'$$

is the desired form. So case 3 holds.

This proof, by the way, is a LOT easier than it looks. Just work out some examples, and you'll see what's going on. (If this were a real expository paper, such examples would be provided. Maybe it's better this way. Indeed, if anybody out there takes the time to work some out and post them, I would be much obliged.)

So how to you actually go about using this theorem? Suppose you want to integrate fe^g for f,g in $\mathbb{C}(z)$, g non zero. [This isn't yet the asked for problem.] Let $t = e^g$, so $t'/t = g'$. Let $\mathbb{F} = \mathbb{C}(z)(t)$, $\mathbb{G}$ =any differential extension field containing an antiderivative of $f \cdot t$. [Note that t is in fact transcendental over $\mathbb{C}(z)$: g is rational and non-zero, so it has a pole (possibly at infinity) and so t has an essential singularity and can't be algebraic over $\mathbb{C}(z)$.] Is $\mathbb{G}$ an elementary extension? If so, then

$$f \cdot t = c_1\frac{u_1'}{u_1} + \cdots + c_n\frac{u_n'}{u_n} + v'$$

where the c_i, u_i, and v are in $\mathbb{F}$. Now the left hand side can be viewed as a polynomial in $\mathbb{C}(z)[t]$ with exactly one term. We must identify the coefficient of t in the right hand side and get an equation for f. But the first n terms can be factored until the u_i's are linear (using the logarithmic differentiation identity to preserve the abstract from). As for the v' term, long divide and use partial fractions to conclude v is a sum of monomials: if v had a linear denominator other than t, raised to some power in its partial fraction decomposition, its derivative would be one higher power, and so cannot be cancelled with anything from the u_i terms. (As in the proof.) If w is the coefficient of t in v, we have $f = w' + wg'$ with w in $\mathbb{C}(z)$. Solving this first order ODE, we find that $w = e^{-g} \int (fe^g)$. In other words, if an elementary antiderivative can be found for fe^g, where f,g are rational functions, then it is of the form we^g for some rational function w. [Notice that the conclusion would fail for g equal to 0!]

For example, consider $f = 1$ and $g = -z^2$. Now $e^{z^2} \int e^{-z^2}$ has no poles (except perhaps at infinity), so if it is a rational function, it must be a polynomial. So let $(p(z)e^{-z^2})' = e^{-z^2}$. One quickly verifies that $p' - 2zp = 1$. But the only solution to that ODE is the error function $\int e^{-z^2}$ itself (within an additive constant somewhere)! And the error function is NOT a polynomial! (Proof? OK, for one thing, its Taylor series obtained by termwise integration is infinite. For another, its derivative is an exponential.)

As an exercise, prove that $\int (e^z/z)$ is not elementary. Conclude that neither is $\int e^{e^z}$ and $\int (1/\log(z))$.

For a slightly harder exercise, prove that $\int (\sin(z)/z)$ is not elementary. Conclude that neither is $\int (\sin e^z)$.

Finally, we consider the case $\int (z^z)$.

So this time, let $\mathbb{F} = \mathbb{C}(z, l)(t)$, the field of rational functions in z,l,t, where $l = \log z$ and $t = e^{zl} = z^z$. Note that z,l,t are algebraically independent. (Choose some appropriate domain of definition.) Then $t' = (1 + l)t$, so for $a = t$ in the above situation, the partial fraction analysis (of the sort done in the previous posts) shows that the only possibility is for $v = wt + \cdots$ to be the source of the t term on the left, with w in $\mathbb{C}(z, l)$.

So this means, equating t coefficients, $1 = w' + (l + 1)w$. This is a first order ODE, whose solution is $w = \int (z^z)/z^z$. So we must prove that no such w exists in $\mathbb{C}(z, l)$. So suppose (as in one of Ray Steiner's posts) $w = P/Q$, with P,Q in $\mathbb{C}[z, l]$ and having no common factors. Then

$$z^z = (z^z \cdot P/Q)' = z^z \cdot [(1 + l)PQ + P'Q - PQ']/Q^2$$

or $Q^2 = (1 + l)PQ + P'Q - PQ'$. So $Q|Q'$, meaning Q is a constant, which we may assume to be one. So we have it down to $P' + P + lP = 1$.

Let $P = \sum_{i=0}^{n} P_i l^i$, with $P_i \in \mathbb{C}[z]$. But then in our equation, there's a dangling $P_n l^{n+1}$ term, a contradiction.