How far can we push the AdS/Ricci-flat correspondence?

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STAG RESEARCH CENTRE

MARIE CURIE ACTIONS

UNIVERSITY OF Southampton

- ✧ **Gravity is believed to be holographic:** it should be described by a non-gravitational theory in one dimension less

  't Hooft '93, **Susskind** '94

- ✧ **This is well understood for asymptotically anti-de Sitter spacetimes:** AdS/CFT correspondence

  Maldacena '97, **Gubser Klebanov Polyakov** '98, **Witten** '98, ...

- ✧ **Original arguments for holography are insensitive to asymptotics**

- ✧ **Decoupling argument extends to nonconformal branes** (non-trivial dilaton & non-AdS asymptotics)

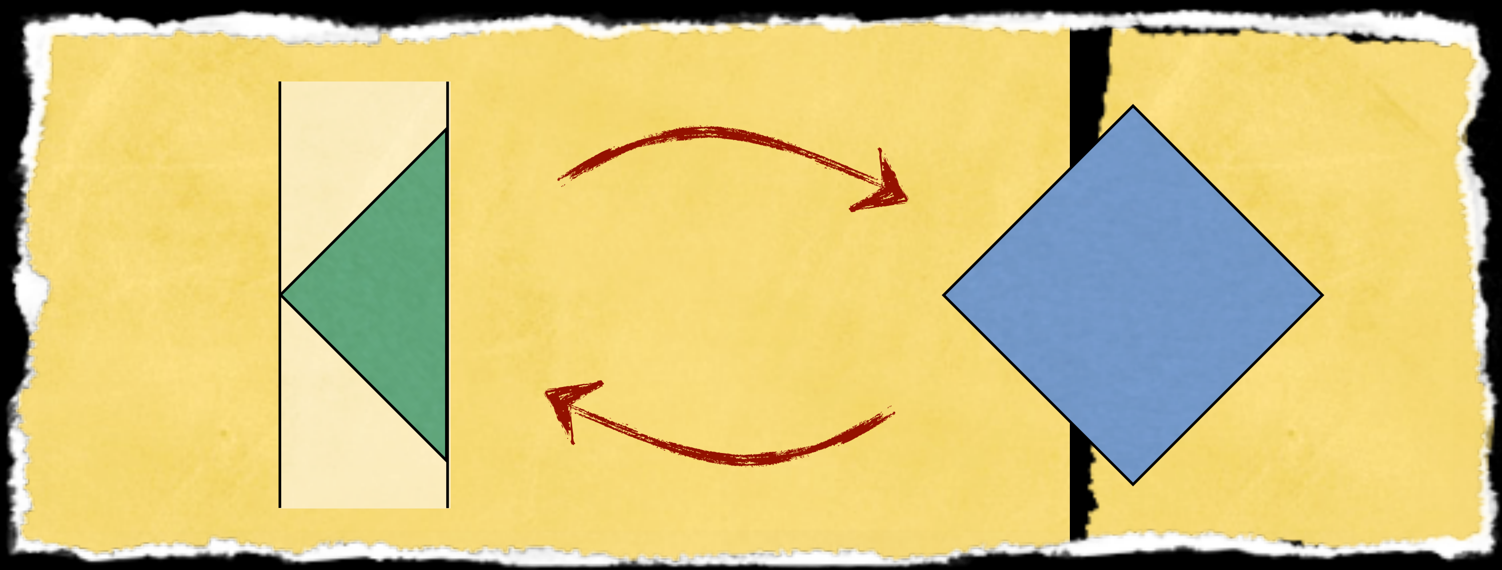
  Kanitscheider et al '08
  Wiseman & Withers '08

  - obtained from AdS via a ***generalized dimensional reduction***
  - Holographic dictionary ***inherited*** *from AdS*

  Kanitscheider & **Skenderis** '09

I will discuss a generalized dimensional reduction linking Ricci-flat and AdS solutions, aiming at formulating holography for AF spacetimes.

AdS/Ricci-flat correspondence



~ a map linking AdS gravity and vacuum Einstein gravity ~

A map relating AdS and Ricci-flat solutions

MC, Camps, Goutéraux & Skenderis '12

1. Solutions to **AdS gravity** in d+1 dimensions of the form:

$$ds_\Lambda^2 = d\hat{s}_{p+2}^2(x) + e^{\frac{2\phi(x)}{d-p-1}}\, d\vec{y}_{d-p-1}^{\,2}$$

unit $\mathcal{T}^{d-p-1}$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

$$\Lambda = -\frac{d(d-1)}{2\ell^2}$$

2. Extract (p+2)-dim metric $\hat{g}(x)$ and the scalar $\phi(x)$

3. Substitute $d \to -n$ in $\hat{g}(x)$ and $\phi(x)$

unit $\mathcal{S}^{n+1}$

4. Insert back in $ds_0^2 = e^{\frac{2\phi(x)}{n+p+1}} \left(d\hat{s}_{p+2}^2(x) + \ell^2\, d\Omega_{n+1}^2 \right)$

Then, the metric ds_0^2 is **Ricci-flat** $\tilde{R}_{\mu\nu} = 0$

It solves **vacuum Einstein** equations in (n+p+3) dimensions

Trading curvatures: from AdS to Ricci-flat

AdS gravity

$D = d + 1$

$$S = \frac{1}{16\pi G_N} \int_{\mathcal{M}} d^{d+1}x \, \sqrt{-g} \, (R - 2\Lambda)$$

Reduction on $\mathcal{T}^{d-p-1}$

$$ds^2_\Lambda = d\hat{s}^{\,2}_{p+2} + e^{\frac{2\phi(x)}{d-p-1}} \, d\vec{y}^{\,2}_{d-p-1}$$

$\hat{D} = p + 2$

$$\alpha = \frac{d-p-2}{d-p-1}$$

$$\beta = -2\Lambda$$

$$\hat{S} = \frac{1}{16\pi \hat{G}_N} \int_{\mathcal{M}} d^{p+2}x \, \sqrt{-\hat{g}} \, e^{\phi} \left(\hat{R} + \alpha \, (\partial\phi)^2 + \beta \right)$$

Trading curvatures: from AdS to Ricci-flat

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Vacuum Einstein gravity

$\tilde{D} = n + p + 3$

$$\tilde{S} = \frac{1}{16\pi \tilde{G}_N} \int_{\mathcal{M}} d^{n+p+3}x \sqrt{-\tilde{g}}\,\tilde{R}$$

Reduction on $\mathcal{T}^{d-p-1}$

$$ds^2_\Lambda = d\hat{s}^2_{p+2} + e^{\frac{2\phi(x)}{d-p-1}}\, d\vec{y}^{\,2}_{d-p-1}$$

Reduction on $\mathcal{S}^{n+1}$

$$ds^2_0 = e^{\frac{2\phi(x)}{n+p+1}} \left(d\hat{s}^2_{p+2} + \ell^2\, d\Omega^2_{n+1}\right)$$

$\hat{D} = p + 2$

$$\hat{S} = \frac{1}{16\pi \hat{G}_N} \int_{\mathcal{M}} d^{p+2}x \sqrt{-\hat{g}}\, e^{\phi} \left(\hat{R} + \alpha\,(\partial\phi)^2 + \beta\right)$$

$$\alpha = \frac{d-p-2}{d-p-1}$$

$$\beta = -2\Lambda$$

$$\alpha = \frac{n+p+2}{n+p+1}$$

$$\beta = \mathcal{R}_{\mathcal{S}^{n+1}}$$

Trading curvatures: from AdS to Ricci-flat

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Reduction on $\mathcal{T}^{d-p-1}$

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Reduction on $\mathcal{S}^{n+1}$

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$$\alpha = \frac{d-p-2}{d-p-1} \qquad \beta = -2\Lambda$$

$$\alpha = \frac{n+p+2}{n+p+1} \qquad \beta = \mathcal{R}_{\mathcal{S}^{n+1}}$$

$d \leftrightarrow -n \qquad -2\Lambda \leftrightarrow \mathcal{R}_{\mathcal{S}^{\tilde{n}+1}}$

Dimension ***d*** (and ***n***) enters analytically as a parameter in the equations of motion

Some remarks

1. Requires knowing the solution for any *d* (or *n*): we are mapping families of AdS solutions to families of Ricci-flat solutions

2. Analytical continuation $d \to -n$ on the lower dimensional theory: *d* and *n* should not be thought of as spacetime dimensions

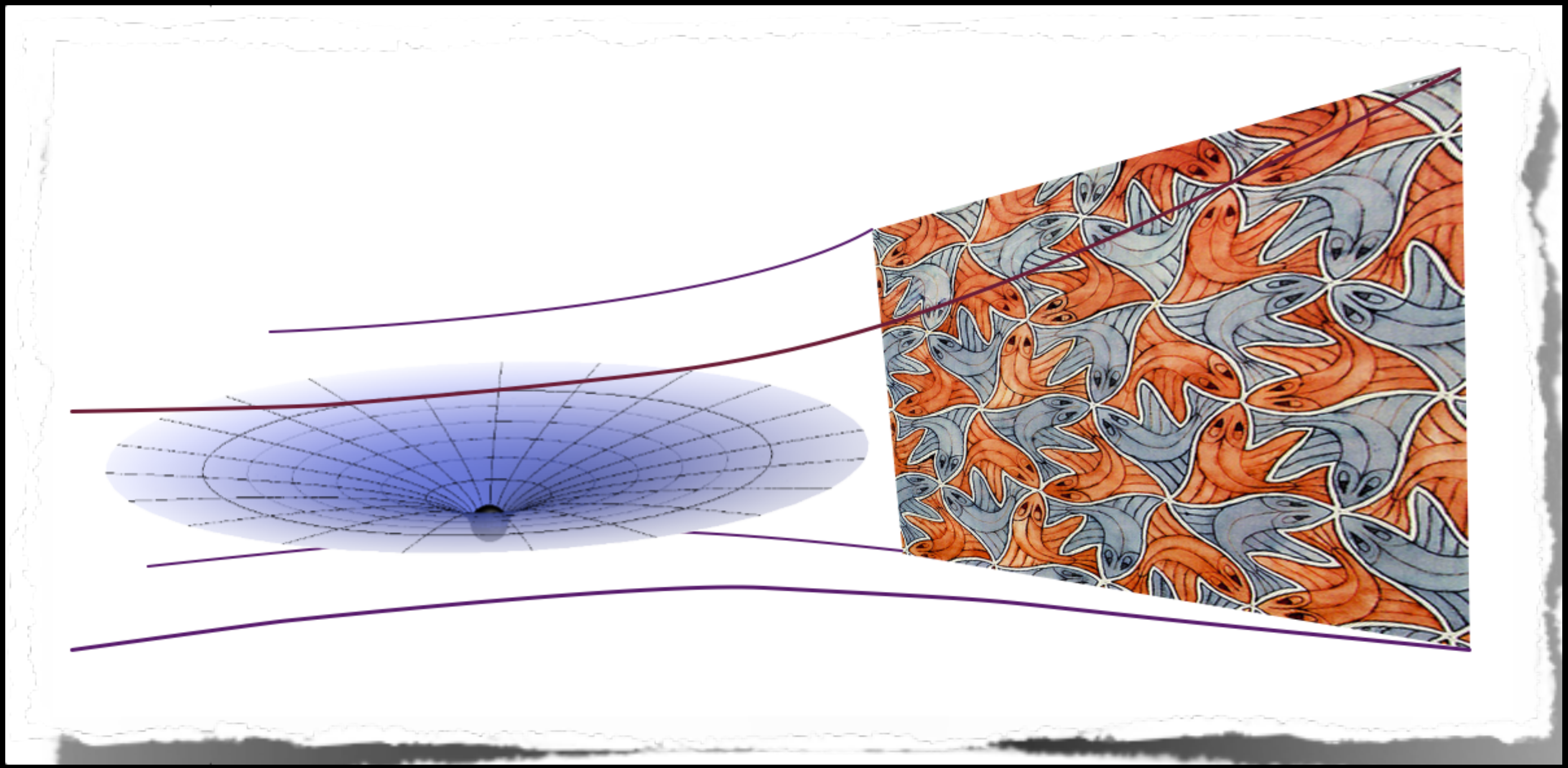
3. This is an example of *Generalized Dimensional Reduction*
   Kanitscheider & Skenderis '09 - Goutéraux, Smolic, Smolic, Skenderis & Taylor '11 - Goutéraux & Kiritsis '11

4. We are trading the curvature of AdS with the curvature of the sphere $(-2\Lambda \leftrightarrow \mathcal{R}_{\mathcal{S}^{\tilde{n}+1}})$

5. Extensions with other compactifications/cosmological constants
   e.g. **AdS/dS** correspondence Di Dato & Fröb '14

The resulting Ricci-flat class of solutions has an underlying holographic structure and hidden conformal symmetry inherited from the locally asymptotically AdS class of solutions.

Some simple examples



~ what happens to simple known solutions under this map? ~

First example: **AdS$_{d+1}$ on a Torus**

1. AdS spacetime in *d*+1 dimensions:

$\mathcal{T}^{d-p-1}$

$$ds^2_\Lambda = \frac{\ell^2}{r^2}\left(\mathrm{d}r^2 + \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b + \mathrm{d}\vec{y}^2\right)$$

2. Extract the metric and scalar:

$$ds^2_\Lambda = d\hat{s}^2_{p+2} + e^{\frac{2\phi}{d-p-1}}\, d\vec{y}^2_{d-p-1} \Rightarrow \begin{cases} d\hat{s}^2_{p+2} = \dfrac{\ell^2}{r^2}\left(dr^2 + \eta_{ab}\, dx^a dx^b\right) \\ \phi(x) = -(d-p-1)\ln\dfrac{r}{\ell} \end{cases}$$

3. Substitute $d \to -n$

$$\Rightarrow \begin{cases} d\hat{s}^2_{p+2} = \dfrac{\ell^2}{r^2}\left(dr^2 + \eta_{ab}\, dx^a dx^b\right) \\ \phi(x) = (n+p+1)\ln\dfrac{r}{\ell} \end{cases}$$

$\mathcal{S}^{n+1}$

4. Lift to *n*+*p*+3 dimensions:

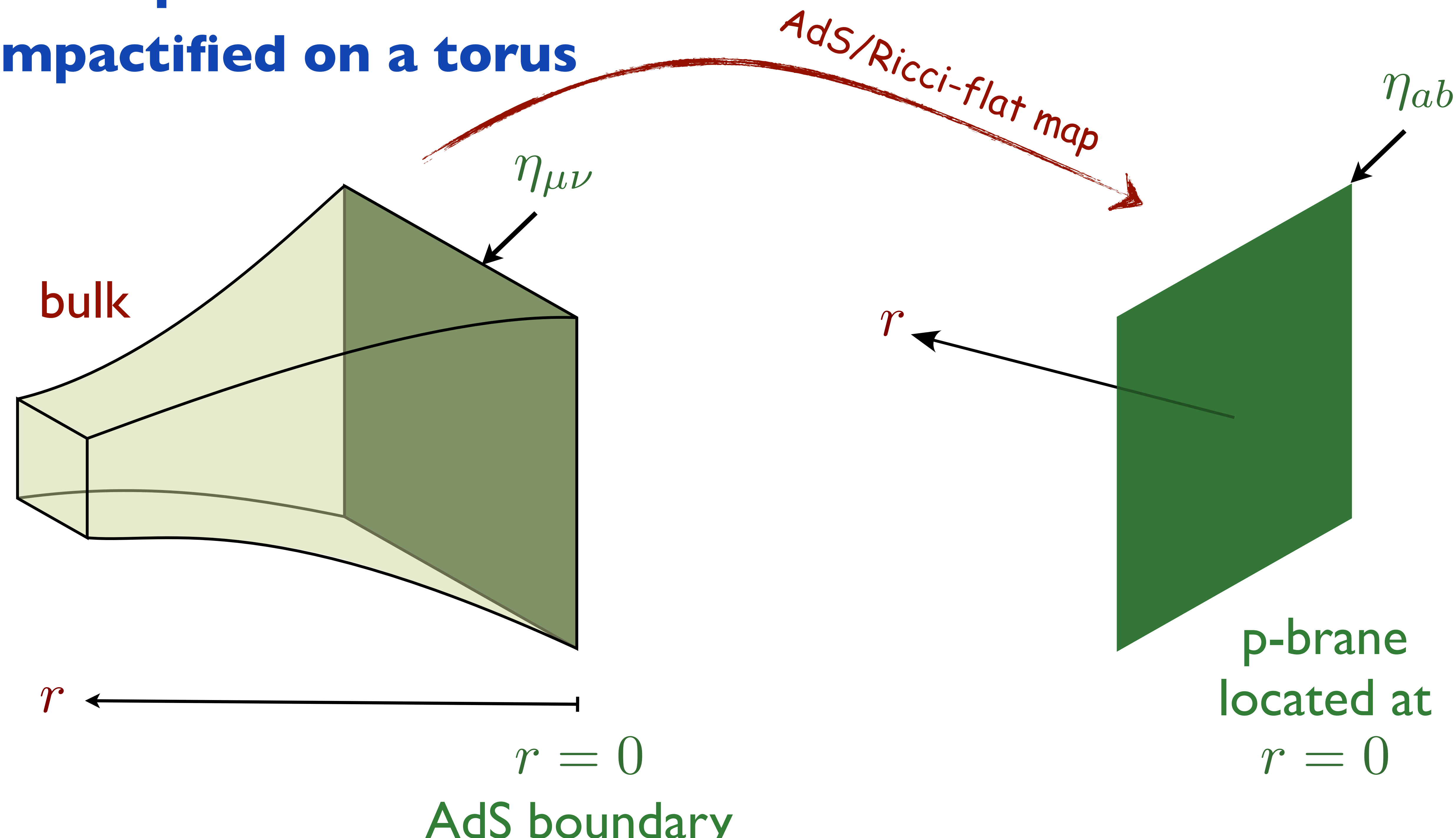
$$ds^2_0 = e^{\frac{2\phi}{n+p+1}}\left(d\hat{s}^2_{p+2} + \ell^2\, d\Omega^2_{n+1}\right)$$

$$\Rightarrow \quad ds^2_0 = \underbrace{\eta_{ab}\mathrm{d}x^a\mathrm{d}x^b}_{\mathbb{R}^{1,p}} + \underbrace{\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{n+1}}_{\mathbb{R}^{n+2}}$$

Minkowski in *n*+*p*+3 dim.

First example: **AdS$_{d+1}$ on a Torus**

AdS spacetime compactified on a torus



Minkowski spacetime

Second example: **Excitations on top of AdS**

1. Fefferman-Graham coordinates for Einstein-AdS solutions: $(\rho = r^2)$

$$\mathrm{d}s_\Lambda^2 = \frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho}\Big(\eta_{\mu\nu} + \rho^{d/2} g_{(d)\mu\nu} + \cdots\Big)\,\mathrm{d}z^\mu \mathrm{d}z^\nu$$

flat boundary metric

normalizable perturbation $T_{\mu\nu} \propto g_{(d)\mu\nu}$

2. Reduced theory: non conformal branes with dual stress tensor $\hat{T}_{ab}$

$$\partial^a \hat{T}_{ab} = 0\,, \qquad \hat{T}_a{}^a = (d-p-1)\hat{\mathcal{O}}_\phi$$

Kanitscheider & Skenderis '09

the expectation value of the scalar operator breaks conformal invariance

3. & 4. Analytical continuation and uplift to $n+p+3$ dimensions:

$$\mathrm{d}s_0^2 = (\eta_{AB} + h_{AB} + \ldots)\,\mathrm{d}x^A \mathrm{d}x^B$$

$$\Box \bar{h}_{AB} = -16\pi \tilde{G}_N \tilde{T}_{AB}$$

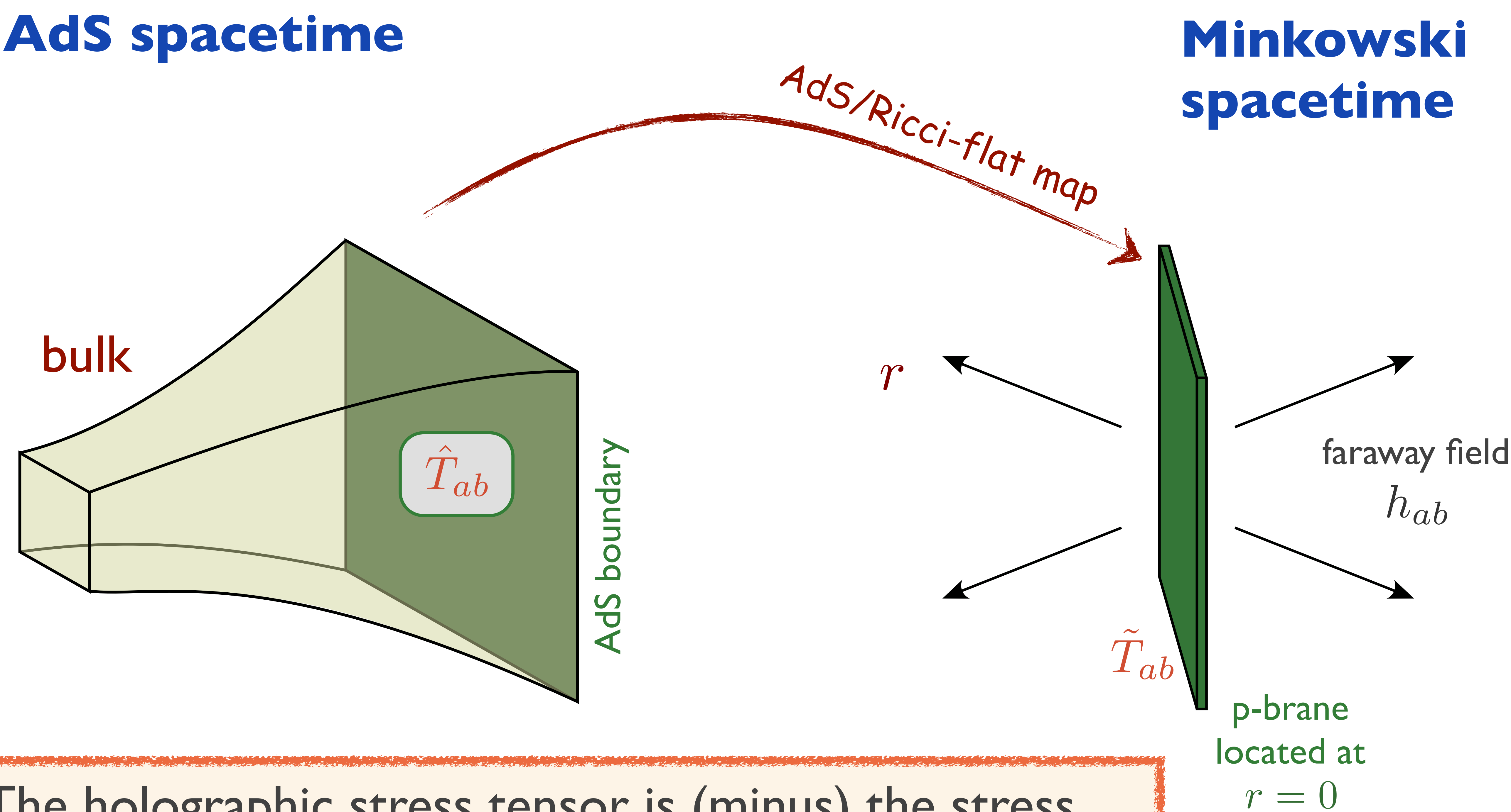
$$\bar{h}_{AB} = h_{AB} - \tfrac{h}{2}\eta_{AB}$$

$$\tilde{T}_{ab} = -\frac{\hat{G}_N}{\tilde{G}_N}\Omega_{n+1}\hat{T}_{ab}\delta^{n+2}(r)$$

(stress tensor of a p-brane located at $r=0$)

The holographic stress tensor sources the faraway gravitational field

Second example: **Excitations on top of AdS**



The holographic stress tensor is (minus) the stress tensor of a brane, located at the origin of Minkowski, that sources the linearized gravitational field h_{ab}

[NB: A similar picture arises in the AdS/dS correspondence Di Dato & Fröb '14]

Second example: **Correlation functions**

- On AdS, we find **regular** linear transverse traceless fluctuation satisfying **Dirichlet** boundary conditions

- This translates on the Ricci-flat side into a **choice of a metric at the location of a p-brane**

- At linear order, the holographic stress energy tensor becomes the **stress energy tensor due to this p-brane**, that sources the linearized gravitational field

- The regularity in the bulk of AdS becomes the requirement that **the Ricci-flat perturbation preserves asymptotic flatness**

A coherent picture is emerging, hinting at holography for asymptotically flat spacetimes

Third example: **black branes**

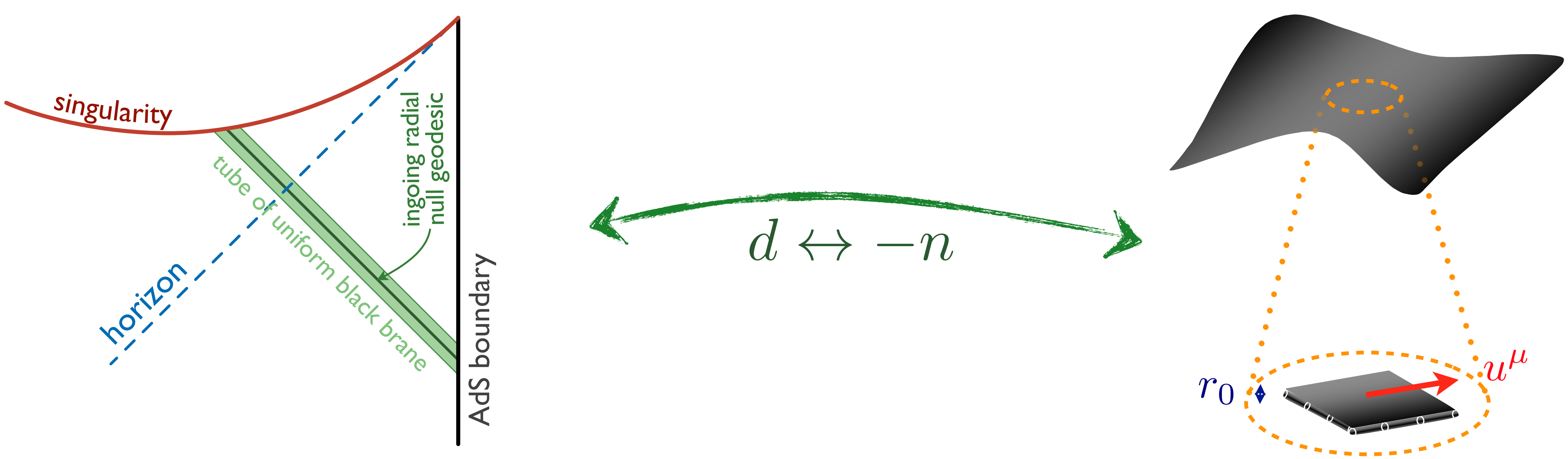
⋆ AdS/RF maps **planar AdS black holes** to **Schwarzschild black branes**

$$ds_\Lambda^2 = z^2(-f(z)d\tau^2 + d\vec{x}^2 + d\vec{y}^2) + \frac{dz^2}{z^2 f(z)}, \qquad f(z) = 1 - \frac{1}{(bz)^d}$$

$$d \leftrightarrow -n \qquad \updownarrow \qquad z = 1/r, \quad b = r_0$$

$$ds_0^2 = \underbrace{-f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n+1}^2}_{\text{Schwarzschild}} + d\vec{x}^2, \qquad f(r) = 1 - \frac{r_0^n}{r^n}$$

⋆ It also maps **AdS fluid/gravity metric** to **blackfolds**



AdS/RF summary

* **AdS/Ricci-flat correspondence** maps asymptotically locally **AdS** solutions on a torus to **Ricci-flat** spacetimes

These Ricci-flat spaces inherit the holographic properties of AdS

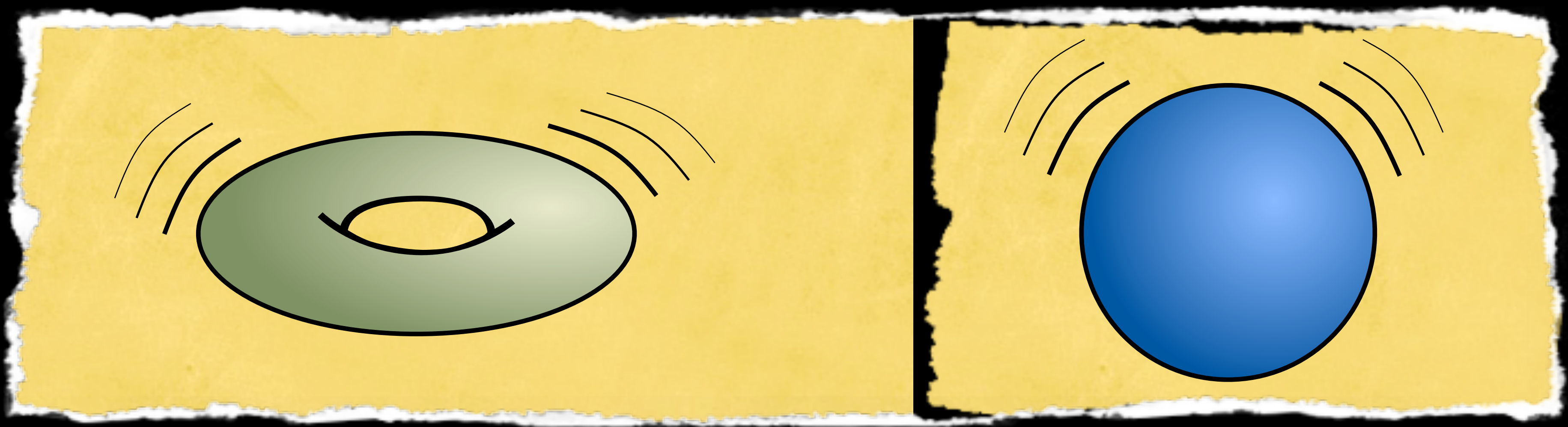
* **Holography for asymptotically flat spacetimes?**

- The map captures all the dynamics in the extended directions along the brane *(correlation functions, hydro mode, transport coeffs)*
- The transverse sphere is **frozen**! These are critical modes to understand asymptotically flat holography:
  we are killing all the dynamics of the Schwarzschild black hole
- However non extremal BHs near horizon geometry is Rindler, that we can link to AdS_2; hints to an effective chiral CFT

* **Extensions of the AdS/RF map** Di Dato, Gath & Vigand Pedersen '15

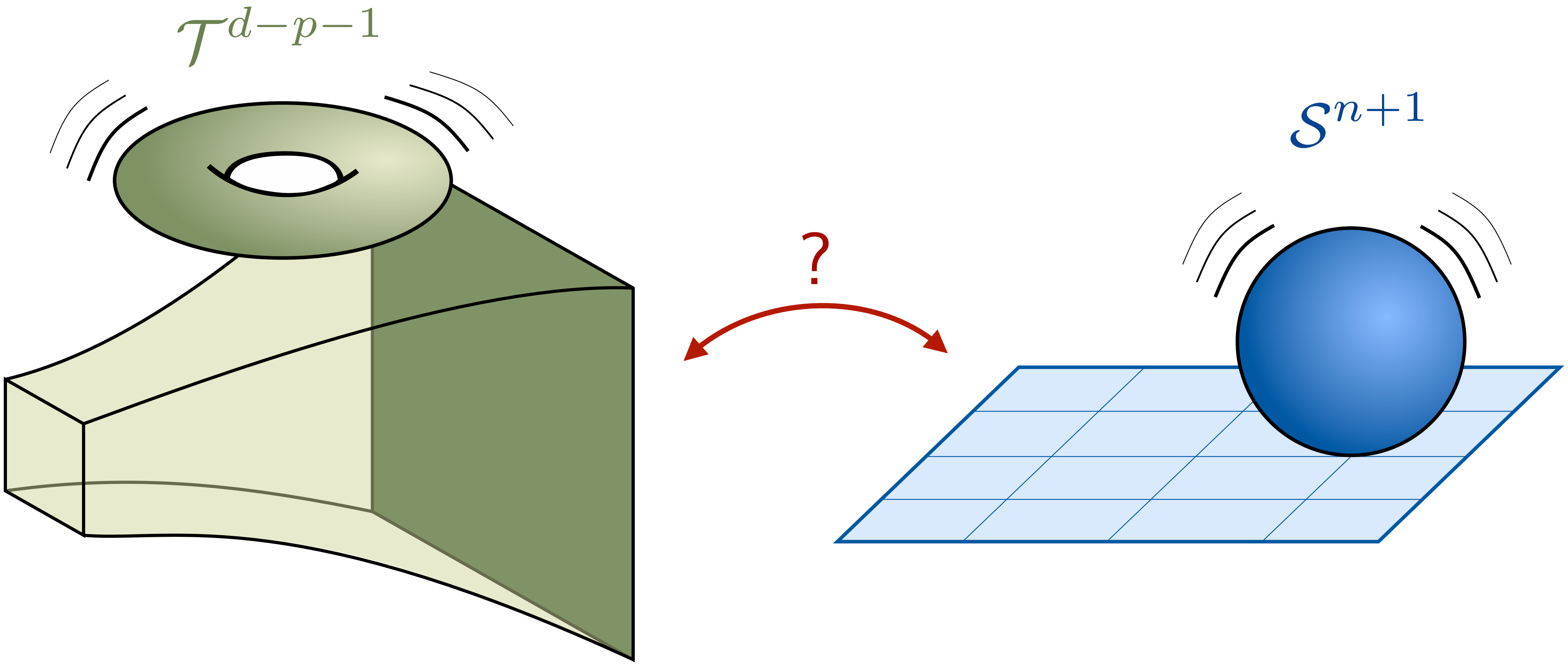
- Auxiliary reduction: extension with gauge fields (EMD)
  but the sphere remains frozen, so it does not help us

How far can we push the AdS/RF correspondence?



~ Unfreezing the torus & the sphere ~

Beyond AdS/RF: perturbations



- General linearized perturbation of AdS and Minkowski
- Full Kaluza-Klein reduction down to p+2 dimensions
- Compare the resulting modes
- ???

Linearized perturbations of Minkowski

$$ds^2_{n+p+3} = \underbrace{\eta_{\mu\nu}\, dx^\mu dx^\nu + dr^2}_{p+2} + r^2 \underbrace{\sigma_{ij}\, d\theta^i d\theta^j}_{\mathcal{S}^{n+1}} + \underbrace{h_{AB}\, dX^A dX^B}_{\text{perturbation}}$$

Field expansion in $SO(n+2)$ representations:

$$h_{ab} = h^{I_\mathsf{s}}_{ab}(x,r)\mathbb{S}^{I_\mathsf{s}}(\theta),$$

$$h_{ai} = B^{I_\mathsf{v}}_{(\mathsf{v})a}(x,r)\mathbb{V}^{I_\mathsf{v}}_i(\theta) + B^{I_\mathsf{s}}_{(\mathsf{s})a}(x,r)\mathcal{D}_i\mathbb{S}^{I_\mathsf{s}}(\theta),$$

$$h_{(ij)} = \hat{\phi}^{I_\mathsf{t}}_\mathsf{t}(x,r)\mathbb{T}^{I_\mathsf{t}}_{(ij)}(\theta) + \phi^{I_\mathsf{v}}_\mathsf{v}(x,r)\mathcal{D}_{(i}\mathbb{V}^{I_\mathsf{v}}_{j)}(\theta) + \phi^{I_\mathsf{s}}_\mathsf{s}(x,r)\mathcal{D}_{(i}\mathcal{D}_{j)}\mathbb{S}^{I_\mathsf{s}}(\theta),$$

$$h^i{}_i \equiv \sigma^{ij}h_{ij} = \pi^{I_\mathsf{s}}(x,r)\mathbb{S}^{I_\mathsf{s}}(\theta)$$

	Fields	Spherical harmonics	
Scalars	$h^{I_\mathsf{s}}_{ab}$, $B^{I_\mathsf{s}}_{(\mathsf{s})a}$, $\phi^{I_\mathsf{s}}_\mathsf{s}$, π^{I_s}	$\mathbb{S}^{I_\mathsf{s}}(\theta)$	Λ^{I_s}
Vectors	$B^{I_\mathsf{v}}_{(\mathsf{v})a}$, $\phi^{I_\mathsf{v}}_\mathsf{v}$	$\mathbb{V}^{I_\mathsf{v}}_i(\theta)$	Λ^{I_v}
Tensor	$\hat{\phi}^{I_\mathsf{t}}_\mathsf{t}$	$\mathbb{T}^{I_\mathsf{t}}_{(ij)}(\theta)$	Λ^{I_t}

Gauge invariant variables

Skenderis & Taylor '05

Some modes are diffeomorphic to each other, or to the background solution. Consider $X^{A'} = X^A - \xi^A$ with

$$\xi_a = \xi_a^{I_\mathsf{s}}(x,r)\,\mathbb{S}^{I_\mathsf{s}}, \qquad \xi_i = \xi_\mathsf{v}^{I_\mathsf{v}}(x,r)\,\mathbb{V}_i^{I_\mathsf{v}} + \xi_\mathsf{s}^{I_\mathsf{s}}(x,r)\,\mathcal{D}_i\mathbb{S}^{I_\mathsf{s}}$$

Define $\hat{B}^{I_\mathsf{s}}_{(\mathsf{s})a} = B^{I_\mathsf{s}}_{(\mathsf{s})a} - \frac{1}{2}\partial_a\phi_\mathsf{s}^{I_\mathsf{s}} + \frac{1}{r}\delta^r{}_a\phi_\mathsf{s}^{I_\mathsf{s}}$,

$$\boxed{\delta\phi_\mathsf{s}^{I_\mathsf{s}} = 2r^2\xi_\mathsf{s}^{I_\mathsf{s}}, \qquad \delta\phi_\mathsf{v}^{I_\mathsf{v}} = 2r^2\xi_\mathsf{v}^{I_\mathsf{v}}, \qquad \delta\hat{B}^{I_\mathsf{s}}_{(\mathsf{s})a} = \xi_a^{I_\mathsf{s}}}$$

pure gauge!

$\Rightarrow$ Use $\hat{B}^{I_\mathsf{s}}_{(\mathsf{s})a}$, $\phi_\mathsf{s}^{I_\mathsf{s}}$, $\phi_\mathsf{v}^{I_\mathsf{v}}$ to compensate the variations generated by the gauge parameters $\xi_a^{I_\mathsf{s}}$, $\xi_\mathsf{v}^{I_\mathsf{v}}$, $\xi_\mathsf{s}^{I_\mathsf{s}}$

$$\delta h_{ab}^{I_\mathsf{s}} = \partial_a\xi_b^{I_\mathsf{s}} + \partial_b\xi_a^{I_\mathsf{s}} \qquad\qquad \delta B^{I_\mathsf{v}}_{(\mathsf{v})a} = r^2\partial_a\xi_\mathsf{v}^{I_\mathsf{v}}$$

$$\delta\pi^{I_\mathsf{s}} = 2r^2\Lambda^{I_\mathsf{s}}\xi_\mathsf{s}^{I_\mathsf{s}} + 2(n+1)r\xi_r^{I_\mathsf{s}} \qquad\qquad \delta\hat{\phi}_\mathsf{t}^{I_\mathsf{t}} = 0$$

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$$\delta\phi_\mathsf{s}^{I_\mathsf{s}} = 2r^2\xi_\mathsf{s}^{I_\mathsf{s}}, \qquad \delta\phi_\mathsf{v}^{I_\mathsf{v}} = 2r^2\xi_\mathsf{v}^{I_\mathsf{v}}, \qquad \delta\hat{B}^{I_\mathsf{s}}_{(\mathsf{s})a} = \xi_a^{I_\mathsf{s}}$$

Define the hatted, gauge-invariant fields

$$\hat{h}^{I_\mathsf{s}}_{ab} = h^{I_\mathsf{s}}_{ab} - \partial_a\hat{B}^{I_\mathsf{s}}_{(\mathsf{s})b} - \partial_b\hat{B}^{I_\mathsf{s}}_{(\mathsf{s})a}$$

$$\hat{B}^{I_\mathsf{v}}_{(\mathsf{v})a} = B^{I_\mathsf{v}}_{(\mathsf{v})a} - \frac{1}{2}\partial_a\phi_\mathsf{v}^{I_\mathsf{v}} + \frac{1}{r}\delta^r{}_a\phi_\mathsf{v}^{I_\mathsf{v}}$$

$$\hat{\pi}^{I_\mathsf{s}} = \pi^{I_\mathsf{s}} - \Lambda^{I_\mathsf{s}}\phi_\mathsf{s}^{I_\mathsf{s}} - 2(n+1)r\hat{B}^{I_\mathsf{s}}_{(\mathsf{s})r}$$

$$\delta\hat{h}^{I_\mathsf{s}}_{ab} = 0$$

$$\delta\hat{B}^{I_\mathsf{v}}_{(\mathsf{v})a} = 0$$

$$\delta\hat{\pi}^{I_\mathsf{s}} = 0$$

$$\delta\hat{\phi}_\mathsf{t}^{I_\mathsf{t}} = 0$$

gauge-invariant fields

Linearized field equations

$$E^{(0)}_{CD} \equiv \bar{g}^{AB}(h_{BC|DA} - h_{AB|CD} + h_{BD|CA} - h_{CD|BA}) = 0$$

Decompose & project onto harmonics:

$$E^{(0)}_{ab}\Big|_{\mathbb{S}^{I_\mathsf{s}}} = 0$$

$$E^{(0)}_{ai}\Big|_{\mathbb{V}_i^{I_\mathsf{v}}} = 0 \quad E^{(0)}_{ai}\Big|_{\mathcal{D}_i\mathbb{S}^{I_\mathsf{s}}} = 0$$

$$E^{(0)}_{ij}\Big|_{\mathbb{T}_{ij}^{I_\mathsf{t}}} = 0 \quad E^{(0)}_{ij}\Big|_{\mathcal{D}_{(i}\mathbb{V}_{j)}^{I_\mathsf{v}}} = 0 \quad E^{(0)}_{ij}\Big|_{\mathcal{D}_{(i}\mathcal{D}_{j)}\mathbb{S}^{I_\mathsf{s}}} = 0 \quad E^{(0)}_{ij}\Big|_{\sigma_{ij}\mathbb{S}^{I_\mathsf{s}}} = 0$$

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→ Decoupled equations for $B^{I_\mathsf{v}}_{(\mathsf{v})a}$ and $\hat{\phi}^{I_\mathsf{t}}_\mathsf{t}$

$$\begin{cases} \Box \hat{B}^{I_\mathsf{v}}_{(\mathsf{v})a} + \dfrac{n-1}{r}\partial_r \hat{B}^{I_\mathsf{v}}_{(\mathsf{v})a} - \dfrac{2}{r}\partial_a \hat{B}^{I_\mathsf{v}}_{(\mathsf{v})r} - \dfrac{n-3}{r^2}\delta_a{}^r \hat{B}^{I_\mathsf{v}}_{(\mathsf{v})r} + \dfrac{\Lambda^{I_\mathsf{v}} - n}{r^2}\hat{B}^{I_\mathsf{v}}_{(\mathsf{v})a} = 0 \\ \Box \hat{\phi}^{I_\mathsf{t}}_\mathsf{t} + \dfrac{n-3}{r}\partial_r \hat{\phi}^{I_\mathsf{t}}_\mathsf{t} + \dfrac{1}{r^2}\left(\Lambda^{I_\mathsf{t}} - 2n + 2\right)\hat{\phi}^{I_\mathsf{t}}_\mathsf{t} = 0 \end{cases}$$

Linearized field equations

$$E^{(0)}_{CD} \equiv \bar{g}^{AB}(h_{BC|DA} - h_{AB|CD} + h_{BD|CA} - h_{CD|BA}) = 0$$

Coupled equations for $\hat{h}^{I_\mathsf{s}}_{ab}$ and $\hat{\pi}^{I_\mathsf{s}}$

$$\boxed{E^{(0)}_{ab}\Big|_{\mathbb{S}^{I_\mathsf{s}}} = 0}$$

$$E^{(0)}_{ai}\Big|_{\mathbb{V}_i^{I_\mathsf{v}}} = 0 \qquad E^{(0)}_{ai}\Big|_{\mathcal{D}_i\mathbb{S}^{I_\mathsf{s}}} = 0$$

$$E^{(0)}_{ij}\Big|_{\mathbb{T}_{ij}^{I_\mathsf{t}}} = 0 \qquad E^{(0)}_{ij}\Big|_{\mathcal{D}_{(i}\mathbb{V}_{j)}^{I_\mathsf{v}}} = 0 \qquad E^{(0)}_{ij}\Big|_{\mathcal{D}_{(i}\mathcal{D}_{j)}\mathbb{S}^{I_\mathsf{s}}} = 0 \qquad \boxed{E^{(0)}_{ij}\Big|_{\sigma_{ij}\mathbb{S}^{I_\mathsf{s}}} = 0}$$

Decoupled equations for $B^{I_\mathsf{v}}_{(\mathsf{v})a}$ and $\hat{\phi}^{I_\mathsf{t}}_\mathsf{t}$

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Linearized field equations

$$E^{(0)}_{CD} \equiv \bar{g}^{AB}(h_{BC|DA} - h_{AB|CD} + h_{BD|CA} - h_{CD|BA}) = 0$$

Coupled equations for $\hat{h}^{I_{\mathsf{s}}}_{ab}$ and $\hat{\pi}^{I_{\mathsf{s}}}$

$E^{(0)}_{ab}\Big|_{\mathbb{S}^{I_{\mathsf{s}}}} = 0$

Follow from other equations
(gauge freedom to choose $B^{I_{\mathsf{s}}}_{(\mathsf{s})a}$, $\phi^{I_{\mathsf{s}}}_{\mathsf{s}}$, $\phi^{I_{\mathsf{v}}}_{\mathsf{v}}$)

$E^{(0)}_{ai}\Big|_{\mathbb{V}^{I_{\mathsf{v}}}_i} = 0$

$E^{(0)}_{ij}\Big|_{\mathbb{T}^{I_{\mathsf{t}}}_{ij}} = 0$

$E^{(0)}_{ai}\Big|_{\mathcal{D}_i\mathbb{S}^{I_{\mathsf{s}}}} = 0$

$E^{(0)}_{ij}\Big|_{\mathcal{D}_{(i}\mathbb{V}^{I_{\mathsf{v}}}_{j)}} = 0 \quad E^{(0)}_{ij}\Big|_{\mathcal{D}_{(i}\mathcal{D}_{j)}\mathbb{S}^{I_{\mathsf{s}}}} = 0$

$E^{(0)}_{ij}\Big|_{\sigma_{ij}\mathbb{S}^{I_{\mathsf{s}}}} = 0$

Decoupled equations for $B^{I_{\mathsf{v}}}_{(\mathsf{v})a}$ and $\hat{\phi}^{I_{\mathsf{t}}}_{\mathsf{t}}$

$$\begin{cases} \Box\hat{B}^{I_{\mathsf{v}}}_{(\mathsf{v})a} + \dfrac{n-1}{r}\partial_r\hat{B}^{I_{\mathsf{v}}}_{(\mathsf{v})a} - \dfrac{2}{r}\partial_a\hat{B}^{I_{\mathsf{v}}}_{(\mathsf{v})r} - \dfrac{n-3}{r^2}\delta_a{}^r\hat{B}^{I_{\mathsf{v}}}_{(\mathsf{v})r} + \dfrac{\Lambda^{I_{\mathsf{v}}} - n}{r^2}\hat{B}^{I_{\mathsf{v}}}_{(\mathsf{v})a} = 0 \\ \Box\hat{\phi}^{I_{\mathsf{t}}}_{\mathsf{t}} + \dfrac{n-3}{r}\partial_r\hat{\phi}^{I_{\mathsf{t}}}_{\mathsf{t}} + \dfrac{1}{r^2}\left(\Lambda^{I_{\mathsf{t}}} - 2n + 2\right)\hat{\phi}^{I_{\mathsf{t}}}_{\mathsf{t}} = 0 \end{cases}$$

Action for fluctuations on Minkowski

$$S_0 = \int d^{p+2}x\, r^{n+1} \Big\{ \frac{1}{2}\partial_c \hat{h}^{I_\mathsf{s}}_{ab}\partial^c \hat{h}^{I_\mathsf{s}\,ab} - \frac{1}{4}\partial_a \hat{h}^{I_\mathsf{s}}_{bc}\partial^a \hat{h}^{I_\mathsf{s}\,bc} + \frac{\Lambda^{I_\mathsf{s}}}{4r^2}\hat{h}^{I_\mathsf{s}}_{ab}\hat{h}^{I_\mathsf{s}\,ab} + \frac{1}{4}\partial^a \hat{H}^{I_\mathsf{s}}\partial_a \hat{H}^{I_\mathsf{s}}$$
$$- \frac{\Lambda^{I_\mathsf{s}}}{4r^2}\hat{H}^{I_\mathsf{s}}\hat{H}^{I_\mathsf{s}} - \frac{1}{2}\partial^a \hat{H}^{I_\mathsf{s}}\partial^b \hat{h}^{I_\mathsf{s}}_{ab} - \frac{n+1}{2r}\hat{h}^{I_\mathsf{s}}_{ra}\partial^a \hat{H}^{I_\mathsf{s}} + \frac{1}{4r^4}\frac{n}{n+1}\partial_a \hat{\pi}^{I_\mathsf{s}}\partial^a \hat{\pi}^{I_\mathsf{s}}$$
$$- \frac{1}{4r^6}\left(\frac{n(n-1)}{(n+1)^2}\Lambda^{I_\mathsf{s}} - \frac{4}{n+1}\right)\hat{\pi}^{I_\mathsf{s}}\hat{\pi}^{I_\mathsf{s}} - \frac{1}{2r^2}\partial^a \hat{\pi}^{I_\mathsf{s}}\partial^b \hat{h}^{I_\mathsf{s}}_{ab} - \frac{n-1}{2r^3}\hat{h}^{I_\mathsf{s}}_{ar}\partial^a \hat{\pi}^{I_\mathsf{s}} + \frac{1}{r^3}\hat{\pi}^{I_\mathsf{s}}\partial^a \hat{h}^{I_\mathsf{s}}_{ar}$$
$$+ \frac{n-1}{r^4}\hat{\pi}^{I_\mathsf{s}}\hat{h}^{I_\mathsf{s}}_{rr} + \frac{1}{2r^2}\partial_a \hat{H}^{I_\mathsf{s}}\partial^a \hat{\pi}^{I_\mathsf{s}} - \frac{1}{2r^3}\hat{\pi}^{I_\mathsf{s}}\partial_r \hat{H}^{I_\mathsf{s}} - \frac{\Lambda^{I_\mathsf{s}}}{2r^4}\frac{n}{n+1}\hat{H}^{I_\mathsf{s}}\hat{\pi}^{I_\mathsf{s}} - \frac{1}{2r^5}\hat{\pi}^{I_\mathsf{s}}\partial_r \hat{\pi}^{I_\mathsf{s}} \Big\} S^{I_\mathsf{s}}_\mathsf{s}$$
$$+ \int d^{p+2}x\, r^{n-1}\Big\{ -\frac{1}{4}\hat{G}^{I_\mathsf{v}}_{(\mathsf{v})ab}\hat{G}^{I_\mathsf{v}\,ab}_{(\mathsf{v})} + \frac{\Lambda^{I_\mathsf{v}} - n}{2r^2}\hat{B}^{I_\mathsf{v}}_{(\mathsf{v})a}\hat{B}^{I_\mathsf{v}\,a}_{(\mathsf{v})} - \frac{2}{r}\hat{B}^{I_\mathsf{v}\,a}_{(\mathsf{v})}\partial_a \hat{B}^{I_\mathsf{v}}_{(\mathsf{v})r} + \frac{2}{r^2}\hat{B}^{I_\mathsf{v}}_{(\mathsf{v})r}\hat{B}^{I_\mathsf{v}}_{(\mathsf{v})r}\Big\} V^{I_\mathsf{v}}_\mathsf{s}$$
$$+ \int d^{p+2}x\, r^{n-3}\Big\{ -\frac{1}{4}\partial_a \hat{\phi}^{I_\mathsf{t}}_\mathsf{t}\partial^a \hat{\phi}^{I_\mathsf{t}}_\mathsf{t} + \frac{\Lambda^{I_\mathsf{t}} - 2n + 2}{4r^2}\hat{\phi}^{I_\mathsf{t}}_\mathsf{t}\hat{\phi}^{I_\mathsf{t}}_\mathsf{t}\Big\} T^{I_\mathsf{t}}_\mathsf{s}$$

Variations:

$\hat{h}^{I_\mathsf{s}}_{ab}$ $\hat{\pi}^{I_\mathsf{s}}$ $\hat{B}^{I_\mathsf{v}}_{(\mathsf{v})a}$ $\hat{\phi}^{I_\mathsf{t}}_\mathsf{t}$ $\Rightarrow$ independent field equations

$B^{I_\mathsf{s}}_{(\mathsf{s})a}$ $\phi^{I_\mathsf{s}}_\mathsf{s}$ $\phi^{I_\mathsf{v}}_\mathsf{v}$ $\Rightarrow$ linear comb. of the field eqns & ∇

Field equations for $\hat{\phi}^{I_\mathsf{t}}_\mathsf{t}$ and $\hat{B}^{I_\mathsf{v}}_{(\mathsf{v})a}$ are decoupled

Linearized perturbations of AdS

$$ds^2_{d+1} = \frac{\ell^2}{r^2}\Big(\underbrace{\eta_{\mu\nu}\,dx^\mu dx^\nu + dr^2}_{p+2} + \underbrace{\delta_{ij}\,d\chi^i d\chi^j}_{\mathcal{T}^{d-p-1}}\Big) + \underbrace{h_{AB}\,dX^A dX^B}_{\text{perturbation}}$$

Field expansion in Fourier modes on the torus:

$$h_{ab} = h^{\mathbf{m}_\mathsf{s}}_{ab}(y)\mathbb{S}^{\mathbf{m}_\mathsf{s}}$$

$$h_{ai} = C^{(k,\mathbf{m}_\mathsf{v})}_{(\mathsf{v})a}(y)\mathbb{V}^{(k,\mathbf{m}_\mathsf{v})}_i + C^{\mathbf{m}_\mathsf{s}}_{(\mathsf{s})a}(y)\partial_i\mathbb{S}^{\mathbf{m}_\mathsf{s}}$$

$$h_{(ij)} = \hat{\psi}^{(k,l,\mathbf{m}_\mathsf{t})}_\mathsf{t}(y)\mathbb{T}^{(k,l,\mathbf{m}_\mathsf{t})}_{(ij)} + \psi^{(k,\mathbf{m}_\mathsf{v})}_\mathsf{v}(y)\partial_{(i}\mathbb{V}^{(k,\mathbf{m}_\mathsf{v})}_{j)} + \psi^{\mathbf{m}_\mathsf{s}}_\mathsf{s}(y)\partial_{(i}\partial_{j)}\mathbb{S}^{\mathbf{m}_\mathsf{s}}$$

$$h^i{}_i \equiv \delta^{ij}h_{ij} = \varpi^{\mathbf{m}_\mathsf{s}}(y)\mathbb{S}^{\mathbf{m}_\mathsf{s}}$$

	Fields				Harmonics
Scalars	$h^{\mathbf{m}_\mathsf{s}}_{ab}$	$C^{\mathbf{m}_\mathsf{s}}_{(\mathsf{s})a}$	$\psi^{\mathbf{m}_\mathsf{s}}_\mathsf{s}$	$\varpi^{\mathbf{m}_\mathsf{s}}$	$\mathbb{S}^{\mathbf{m}_\mathsf{s}}(\chi)$
Vectors	$C^{(k,\mathbf{m}_\mathsf{v})}_{(\mathsf{v})a}$	$\psi^{(k,\mathbf{m}_\mathsf{v})}_\mathsf{v}$			$\mathbb{V}^{(k,\mathbf{m}_\mathsf{v})}(\chi)$
Tensor	$\hat{\psi}^{(k,l,\mathbf{m}_\mathsf{t})}_\mathsf{t}$				$\mathbb{T}^{(k,l,\mathbf{m}_\mathsf{t})}(\chi)$

Gauge invariant variables (AdS)

Again, some modes are diffeomorphic to each other, or to the background solution. Consider $X^{A'} = X^A - \xi^A$ with

$$\xi_a = \xi_a^{\mathbf{m}_\mathsf{s}}(y)\,\mathbb{S}^{\mathbf{m}_\mathsf{s}}(\chi), \qquad \xi_i = \xi_\mathsf{v}^{\mathbf{m}_\mathsf{v}}(y)\,\mathbb{V}_i^{\mathbf{m}_\mathsf{v}}(\chi) + \xi_\mathsf{s}^{\mathbf{m}_\mathsf{s}}(y)\,\partial_i\mathbb{S}^{\mathbf{m}_\mathsf{s}}(\chi)$$

Define $\hat{C}^{\mathbf{m}_\mathsf{s}}_{(\mathsf{s})a} = C^{\mathbf{m}_\mathsf{s}}_{(\mathsf{s})a} - \frac{1}{2}\partial_a\psi^{\mathbf{m}_\mathsf{s}}_\mathsf{s} - \frac{1}{r}\delta^r{}_a\psi^{\mathbf{m}_\mathsf{s}}_\mathsf{s}$,

$$\delta\psi^{\mathbf{m}_\mathsf{s}}_\mathsf{s} = 2\xi^{\mathbf{m}_\mathsf{s}}_\mathsf{s}, \qquad \delta\psi^{\mathbf{m}_\mathsf{v}}_\mathsf{v} = 2\xi^{\mathbf{m}_\mathsf{v}}_\mathsf{v}, \qquad \delta\hat{C}^{\mathbf{m}_\mathsf{s}}_{(\mathsf{s})a} = \xi^{\mathbf{m}_\mathsf{s}}_a$$

Define the hatted, gauge-invariant fields

$$\hat{h}^{\mathbf{m}_\mathsf{s}}_{ab} = h^{\mathbf{m}_\mathsf{s}}_{ab} - \partial_a\hat{C}^{\mathbf{m}_\mathsf{s}}_{(\mathsf{s})b} - \partial_b\hat{C}^{\mathbf{m}_\mathsf{s}}_{(\mathsf{s})a} + \frac{2}{r}\left(\eta_{ab}\hat{C}^{\mathbf{m}_\mathsf{s}}_{(\mathsf{s})r} - \delta_a{}^r\hat{C}^{\mathbf{m}_\mathsf{s}}_{(\mathsf{s})b} - \delta_b{}^r\hat{C}^{\mathbf{m}_\mathsf{s}}_{(\mathsf{s})a}\right)$$

$$\hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})a} = C^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})a} - \frac{1}{2}\partial_a\psi^{\mathbf{m}_\mathsf{v}}_\mathsf{v} - \frac{1}{r}\delta^r{}_a\psi^{\mathbf{m}_\mathsf{v}}_\mathsf{v}$$

$$\hat{\varpi}^{\mathbf{m}_\mathsf{s}} = \varpi^{\mathbf{m}_\mathsf{s}} + \mathbf{m}_\mathsf{s}^2\psi^{\mathbf{m}_\mathsf{s}}_\mathsf{s} + \frac{2}{r}(d-p-1)\hat{C}^{\mathbf{m}_\mathsf{s}}_{(\mathsf{s})r}$$

$$\delta\hat{h}^{\mathbf{m}_\mathsf{s}}_{ab} = 0$$

$$\delta\hat{\varpi}^{\mathbf{m}_\mathsf{s}} = 0$$

$$\delta\hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})a} = 0$$

$$\delta\hat{\psi}^{\mathbf{m}_\mathsf{t}}_\mathsf{t} = 0$$

gauge-invariant fields

Linearized field equations with Λ

$$E^{(\Lambda)}_{MN} \equiv \delta R_{MN} + \frac{d}{\ell^2} h_{MN} = 0$$

Decompose & expand in Fourier modes:

$$E^{(\Lambda)}_{ab}\Big|_{\mathbb{S}^{\mathbf{m}_\mathsf{s}}} = 0$$

$$\boxed{\begin{aligned} E^{(\Lambda)}_{ai}\Big|_{\mathbb{V}^{\mathbf{m}_\mathsf{v}}_i} &= 0 \\ E^{(\Lambda)}_{ij}\Big|_{\mathbb{T}^{\mathbf{m}_\mathsf{t}}_{(ij)}} &= 0 \end{aligned}} \qquad \begin{aligned} &E^{(\Lambda)}_{ai}\Big|_{\partial_i \mathbb{S}^{\mathbf{m}_\mathsf{s}}} = 0 \\ &E^{(\Lambda)}_{ij}\Big|_{\partial_{(i} \mathbb{V}^{\mathbf{m}_\mathsf{v}}_{j)}} = 0 \quad E^{(\Lambda)}_{ij}\Big|_{\partial_{(i}\partial_{j)} \mathbb{S}^{\mathbf{m}_\mathsf{s}}} = 0 \quad E^{(\Lambda)}_{ij}\Big|_{\delta_{ij} \mathbb{S}^{\mathbf{m}_\mathsf{s}}} = 0 \end{aligned}$$

Decoupled equations for $C^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})a}$ and $\hat{\psi}^{\mathbf{m}_\mathsf{t}}_\mathsf{t}$

$$\begin{cases} \Box \hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})a} - \dfrac{d-5}{r} \partial_r \hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})a} + \dfrac{d-1}{r^2} \delta_a{}^r \hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})r} - \left(\mathbf{m}^2_\mathsf{v} + \dfrac{2(d-2)}{r^2} \right) \hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})a} = 0 \\ \Box \hat{\psi}^{\mathbf{m}_\mathsf{t}}_\mathsf{t} - \mathbf{m}^2_\mathsf{t} \hat{\psi}^{\mathbf{m}_\mathsf{t}}_\mathsf{t} - \dfrac{d-5}{r} \partial_r \hat{\psi}^{\mathbf{m}_\mathsf{t}}_\mathsf{t} - \dfrac{2(d-2)}{r^2} \hat{\psi}^{\mathbf{m}_\mathsf{t}}_\mathsf{t} = 0 \end{cases}$$

Linearized field equations with Λ

$$E^{(\Lambda)}_{MN} \equiv \delta R_{MN} + \frac{d}{\ell^2} h_{MN} = 0$$

Coupled equations for $\hat{h}^{\mathbf{m}_\mathsf{s}}_{ab}$ and $\hat{\varpi}^{\mathbf{m}_\mathsf{s}}$

$$E^{(\Lambda)}_{ab}\Big|_{\mathbb{S}^{\mathbf{m}_\mathsf{s}}} = 0$$

Follow from other equations

(gauge freedom to choose $C^{\mathbf{m}_\mathsf{s}}_{(\mathsf{s})a}$, $\psi^{\mathbf{m}_\mathsf{s}}_{\mathsf{s}}$, $\psi^{\mathbf{m}_\mathsf{v}}_{\mathsf{v}}$)

$$E^{(\Lambda)}_{ai}\Big|_{\mathbb{V}^{\mathbf{m}_\mathsf{v}}_i} = 0$$

$$E^{(\Lambda)}_{ij}\Big|_{\mathbb{T}^{\mathbf{m}_\mathsf{t}}_{(ij)}} = 0$$

$$E^{(\Lambda)}_{ai}\Big|_{\partial_i \mathbb{S}^{\mathbf{m}_\mathsf{s}}} = 0$$

$$E^{(\Lambda)}_{ij}\Big|_{\partial_{(i}\mathbb{V}^{\mathbf{m}_\mathsf{v}}_{j)}} = 0 \qquad E^{(\Lambda)}_{ij}\Big|_{\partial_{(i}\partial_{j)}\mathbb{S}^{\mathbf{m}_\mathsf{s}}} = 0$$

$$E^{(\Lambda)}_{ij}\Big|_{\delta_{ij}\mathbb{S}^{\mathbf{m}_\mathsf{s}}} = 0$$

Decoupled equations for $C^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})a}$ and $\hat{\psi}^{\mathbf{m}_\mathsf{t}}_{\mathsf{t}}$

$$\begin{cases} \Box \hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})a} - \dfrac{d-5}{r}\partial_r \hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})a} + \dfrac{d-1}{r^2}\delta_a{}^r \hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})r} - \left(\mathbf{m}^2_\mathsf{v} + \dfrac{2(d-2)}{r^2}\right)\hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})a} = 0 \\ \Box \hat{\psi}^{\mathbf{m}_\mathsf{t}}_{\mathsf{t}} - \mathbf{m}^2_\mathsf{t}\hat{\psi}^{\mathbf{m}_\mathsf{t}}_{\mathsf{t}} - \dfrac{d-5}{r}\partial_r \hat{\psi}^{\mathbf{m}_\mathsf{t}}_{\mathsf{t}} - \dfrac{2(d-2)}{r^2}\hat{\psi}^{\mathbf{m}_\mathsf{t}}_{\mathsf{t}} = 0 \end{cases}$$

Action for fluctuations on AdS

$$S_\Lambda = \int d^{p+2}x \frac{\ell^{d-5}}{r^{d-5}} \left\{ \left[-\frac{1}{4}\partial_a \hat{\psi}_{\mathsf{t}}^{\mathbf{m}_\mathsf{t}} \partial^a \hat{\psi}_{\mathsf{t}}^{\mathbf{m}_\mathsf{t}} - \frac{1}{4}\left(\mathbf{m}_\mathsf{t}^2 + \frac{2(d-2)}{r^2}\right)(\hat{\psi}_{\mathsf{t}}^{\mathbf{m}_\mathsf{t}})^2 \right] T_{\mathsf{t}}^{\mathbf{m}_\mathsf{t}} \right.$$
$$+ \left[-\frac{1}{4}\hat{F}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})ab}\hat{F}^{\mathbf{m}_\mathsf{v}ab}_{(\mathsf{v})} + \frac{2}{r}\hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})a}\partial^a \hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})r} - \frac{1}{2}\left(\mathbf{m}_\mathsf{v}^2 + \frac{2(d-2)}{r^2}\right)\hat{C}^{\mathbf{m}_\mathsf{v}a}_{(\mathsf{v})}\hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})a} + \frac{2}{r^2}\hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})r}\hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})r} \right] V_{\mathsf{t}}^{\mathbf{m}_\mathsf{v}}$$
$$+ \left[\frac{1}{2}\partial_a \hat{h}^{\mathbf{m}_\mathsf{s}}_{bc}\partial^b \hat{h}^{\mathbf{m}_\mathsf{s}ac} - \frac{1}{4}\partial_a \hat{h}^{\mathbf{m}_\mathsf{s}}_{bc}\partial^a \hat{h}^{\mathbf{m}_\mathsf{s}bc} + \frac{2}{r}\hat{h}^{\mathbf{m}_\mathsf{s}ab}\partial_a \hat{h}^{\mathbf{m}_\mathsf{s}}_{br} - \frac{1}{4}\left(\mathbf{m}_\mathsf{s}^2 + \frac{2(d-2)}{r^2}\right)\hat{h}^{\mathbf{m}_\mathsf{s}}_{ab}\hat{h}^{\mathbf{m}_\mathsf{s}ab} \right.$$
$$+ \frac{2}{r^2}\hat{h}^{\mathbf{m}_\mathsf{s}a}{}_r \hat{h}^{\mathbf{m}_\mathsf{s}}_{ar} - \frac{1}{2}\partial^b \hat{h}^{\mathbf{m}_\mathsf{s}}_{ab}\partial^a\left(\hat{H}^{\mathbf{m}_\mathsf{s}} + \hat{\varpi}^{\mathbf{m}_\mathsf{s}}\right) + \frac{d-3}{2r}\hat{h}^{\mathbf{m}_\mathsf{s}}_{ar}\partial^a\left(\hat{H}^{\mathbf{m}_\mathsf{s}} + \hat{\varpi}^{\mathbf{m}_\mathsf{s}}\right) - \frac{1}{r}\partial^a \hat{h}^{\mathbf{m}_\mathsf{s}}_{ar}\left(\hat{H}^{\mathbf{m}_\mathsf{s}} + \hat{\varpi}^{\mathbf{m}_\mathsf{s}}\right)$$
$$+ \frac{d-3}{r^2}\hat{h}^{\mathbf{m}_\mathsf{s}}_{rr}\left(\hat{H}^{\mathbf{m}_\mathsf{s}} + \hat{\varpi}^{\mathbf{m}_\mathsf{s}}\right) + \frac{1}{4}\frac{d-p-2}{d-p-1}\partial_a \hat{\varpi}^{\mathbf{m}_\mathsf{s}}\partial^a \hat{\varpi}^{\mathbf{m}_\mathsf{s}} + \frac{1}{2r}\hat{\varpi}^{\mathbf{m}_\mathsf{s}}\partial_r \hat{\varpi}^{\mathbf{m}_\mathsf{s}} + \frac{1}{4}\left(\mathbf{m}_\mathsf{s}^2 + \frac{d}{r^2}\right)\hat{H}^{\mathbf{m}_\mathsf{s}}\hat{H}^{\mathbf{m}_\mathsf{s}}$$
$$+ \frac{1}{4}\partial_a \hat{H}^{\mathbf{m}_\mathsf{s}}\partial^a \hat{H}^{\mathbf{m}_\mathsf{s}} + \frac{1}{2r}\hat{H}^{\mathbf{m}_\mathsf{s}}\partial_r \hat{H}^{\mathbf{m}_\mathsf{s}} + \frac{1}{4}\left(\frac{(d-p-2)(d-p-3)}{(d-p-1)^2}\mathbf{m}_\mathsf{s}^2 - \frac{1}{r^2}\left(\frac{2(d-2)}{d-p-1} - d\right)\right)\hat{\varpi}^{\mathbf{m}_\mathsf{s}}\hat{\varpi}^{\mathbf{m}_\mathsf{s}}$$
$$\left. \left. + \frac{1}{2}\partial^a \hat{H}^{\mathbf{m}_\mathsf{s}}\partial_a \hat{\varpi}^{\mathbf{m}_\mathsf{s}} + \frac{1}{2}\left(\frac{d-p-2}{d-p-1}\mathbf{m}_\mathsf{s}^2 + \frac{d}{r^2}\right)\hat{H}^{\mathbf{m}_\mathsf{s}}\hat{\varpi}^{\mathbf{m}_\mathsf{s}} + \frac{1}{2r}\hat{H}^{\mathbf{m}_\mathsf{s}}\partial_r \hat{\varpi}^{\mathbf{m}_\mathsf{s}} + \frac{1}{2r}\hat{\varpi}^{\mathbf{m}_\mathsf{s}}\partial_r \hat{H}^{\mathbf{m}_\mathsf{s}} \right] S_{\mathsf{t}}^{\mathbf{m}_\mathsf{s}} \right\}$$

Variations:

$\hat{h}^{\mathbf{m}_\mathsf{s}}_{ab}$ $\quad \hat{\varpi}^{\mathbf{m}_\mathsf{s}}$ $\quad \hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})a}$ $\quad \hat{\psi}^{\mathbf{m}_\mathsf{t}}_{\mathsf{t}}$ $\quad \Rightarrow$ independent field equations

$C^{\mathbf{m}_\mathsf{s}}_{(\mathsf{s})a}$ $\quad \psi^{\mathbf{m}_\mathsf{s}}_{\mathsf{s}}$ $\quad \psi^{\mathbf{m}_\mathsf{v}}_{\mathsf{v}}$ $\quad \Rightarrow$ linear comb. of the field eqns & ∇

Field equations for $\hat{\psi}^{\mathbf{m}_\mathsf{t}}_{\mathsf{t}}$ and $\hat{C}^{\mathbf{m}_\mathsf{v}}_{(\mathsf{v})a}$ are decoupled

Comparing AdS and Minkowski modes

⋆ The zero-modes h^0_{ab}, $\pi(x;n)$ and h^Λ_{ab}, $\varpi(x;n)$ are mapped into each other.

$$h^0_{(ab)}(x;n) = \frac{r^2}{\ell^2} h^\Lambda_{(ab)}(x;-n) \qquad \pi(x;n) = \frac{r^4}{\ell^2}\frac{n+1}{n+p+1}\varpi(x;-n)$$

$$H^0(x;n) = \frac{r^2}{\ell^2}\left[H^\Lambda(x;-n) + \frac{p+2}{n+p+1}\varpi(x;-n)\right]$$

⋆ Naive map of all modes (scalar sector, up to total derivatives),

$$\ell^{d-1}\tilde{S}_0 - S_{AdS} = \int d^{p+2}x \left(\frac{\ell}{r}\right)^{d-5} \left\{ \frac{1}{4}\left(\mathbf{m}^2 + \frac{\Lambda}{r^2}\right)\left(\hat{h}_{ab}\hat{h}^{ab} - \hat{H}^2\right) - \frac{1}{4}\frac{d-p-2}{d-p-1}\left(\frac{d-p-3}{d-p-1}\mathbf{m}^2 + \frac{\Lambda}{r^2}\right)\hat{\varpi}^2 - \frac{1}{2}\left(\frac{d-p-2}{d-p-1}\mathbf{m}^2 + \frac{\Lambda}{r^2}\right)\hat{H}\hat{\varpi}\right\}$$

- match for $\Lambda^{I_\mathsf{s}} = 0$ and $\mathbf{m}^2_\mathsf{s} = 0$
- the two actions are very similar
- however, qualitatively different behaviour for non-zero modes
- vector and tensor modes follow the same pattern

Comparing AdS and Minkowski modes

- ⋆ Solve explicitely for all AdS perturbation modes, in terms of Bessel functions,

$$r^{\frac{d}{2}-2} J_\nu(k_r r) e^{i\mathbf{k}\cdot\mathbf{x}} \qquad r^{\frac{d}{2}-2} Y_\nu(k_r r) e^{i\mathbf{k}\cdot\mathbf{x}} \qquad \text{with } \nu = \frac{d}{2}, \quad \frac{d}{2}-1, \quad \frac{d}{2}+2$$

$$k^a k_a = k_r^2 + \mathbf{k}^2 = -\mathbf{m}_\mathsf{s}^2 \qquad \text{(resp. } -\mathbf{m}_\mathsf{v}^2 \ / \ -\mathbf{m}_\mathsf{t}^2 \text{ for vector and tensor pert.)}$$

- ⋆ Minkowski perturbation modes also in terms of Bessel functions (+part. sol),

$$r^q J_{l+\frac{n}{2}}(k_r r) e^{i\mathbf{k}\cdot\mathbf{x}} \qquad r^q Y_{l+\frac{n}{2}}(k_r r) e^{i\mathbf{k}\cdot\mathbf{x}} \qquad \text{with } q = -\frac{n}{2}, \quad 1-\frac{n}{2}, \quad 2-\frac{n}{2}$$

$$k^a k_a = k_r^2 + \mathbf{k}^2 = 0$$

- ⋆ A rescaling by a factor r^l together with a mode-dependent analytical continuation of the dimensions can tackle the radial part, but the wavevector is **timelike** for AdS and **null** for Minkowski

- ⋆ There are however similarities and structures appearing, it remains to understand how to exploit them at best to extract information on asymptotically flat holography.

Conclusions

~ ***How far can we push the AdS/RF correspondence?*** ~

* **AdS/Ricci-flat correspondence** maps asymptotically locally **AdS** solutions on a torus to **Ricci-flat** spacetimes

These Ricci-flat spaces inherit the holographic properties of AdS

* **Unfreezing the sphere and torus is not straightforward**
  - The modes on the sphere and the torus are not mapped into each other, at least naively
  - There are many similarities though, and maybe it will be possible to exploit them to learn more about AF holography

* **An alternative path:**
  - Keep all KK modes from the sphere reduction of Minkowski
  - Uplift them to AdS, and combine them into extra matter fields living in AdS
  - Holographically renormalize and translate to the original AF fields

~ Thank you! ~