Recall: Equivalent conditions of a differential characteristic set (Theorem 2.2.13): Let $\mathcal{A}$ be an autoreduced set of a proper differential ideal $I \subseteq K\{y_1, \ldots, y_n\}$. Then

$\mathcal{A}$ is a characteristic set of I

$\iff \forall f \in I$, δ-rem$(f, \mathcal{A})$=0

$\iff I$ doesn't contain a nonzero differential polynomial reduced with respect to $\mathcal{A}$.

And we learnt from Theorem 2.3.4 that given a differential field K of characteristic 0, the differential polynomial ring $K\{y_1, \ldots, y_n\}$ is Ritt-Noetherian (i.e, each radical differential ideal is finitely generated as radical differential ideals).

Theorem 2.3.5. *Let R be a differential ring which is Ritt-Noetherian and $\mathbb{Q} \subseteq R$. Then for every radical differential ideal $I \subsetneqq R$, there exist a finite number of prime differential ideals $P_1, \ldots, P_l$ s.t.*

$$I = \bigcap_{i=1}^{l} P_i. \tag{2.1}$$

Moreover, if (2.1) is irredundant ($\forall\, i$, $\bigcap_{j\neq i} P_j \nsubseteq P_i$), then this set of prime ideals is unique. In this case, $P_1, \ldots, P_l$ are called prime components *of I.*

Proof. Suppose the statement is false, i.e., the set $U = \{I \mid I \subsetneqq K\{y_1, \ldots, y_n\}$ is a radical differential ideal and I is not a finite intersection of prime differential ideals$\}$ is not empty. Since R is Ritt-Noetherian, every ascending chain of radical differential ideals has an upper bound in U. By Zorn's Lemma, U has a maximal element $J \in U$. Clearly, J is not prime. So $\exists\, a, b \notin J$ but $ab \in J$. Thus, $\{J, a\} \supsetneqq J$ and $\{J, b\} \supsetneqq J$. Also, $\{J, a\} \neq R$. Indeed, if not, then $1 \in \{J, a\}$. Since $\mathbb{Q} \subseteq R, 1 \in [J, a]$ and $1 = f + \sum * \delta^k(a)$, where $f \in J$. By $ab \in J$ and J is radical, $b\delta^k(a) \in J\, \forall\, k \in \mathbb{N}$. So $b = fb + \sum * b\delta^k(a) \in J$, contradicting to $b \notin J$. Similarly, $\{J, b\} \neq R$ could be shown.

By the maximality of J, $\exists\, P_1^a, \ldots, P_l^a, P_{l+1}^b, \ldots, P_{l+t}^b$ prime differential ideals in R s.t.

$$\{J, a\} = P_1^a \cap \cdots \cap P_l^a \text{ and}$$
$$\{J, b\} = P_{l+1}^b \cap \cdots \cap P_{l+t}^b.$$

Now show $J = \{J, a\} \cap \{J, b\}$. Indeed, let $f \in \{J, a\} \cap \{J, b\}$, then $f^2 \in \{J, a\} \cdot \{J, b\} \subseteq \{J, ab\} \subseteq J \Rightarrow f \in J$. Thus, $J = \{J, a\} \cap \{J, b\} = P_1^a \cap \cdots \cap P_l^a \cap P_{l+1}^b \cap \cdots \cap P_{l+t}^b$, contradicting to the hypothesis $J \in U$. So every radical differential ideal is a finite intersection of prime differential ideals.

Uniqueness. Suppose $I = \bigcap_{i=1}^{l} P_i = \bigcap_{j=1}^{t} Q_j$ be irredundant intersections. For each $j = 1, \ldots, t$, $\bigcap_{i=1}^{l} P_i \subseteq Q_j$. Then $\exists\, k_j \in \{1, \ldots, l\}$ s.t. $P_{k_j} \subseteq Q_j$. Indeed, suppose the contrary, then $\exists\, f_i \in P_i \backslash Q_j$ for each $i = 1, \ldots, l$. Thus, $f_1 f_2 \cdots f_l \in \bigcap_{i=1}^{l} P_i \subseteq Q_j$, which yields a contradiction. Similarly, $\exists\, j' \in \{1, \ldots, t\}$ s.t. $Q_{j'} \subseteq P_{k_j} \subseteq Q_j$. Since $I = \bigcap_{j=1}^{t} Q_j$ is irredundant, $j' = j$ and $P_{k_j} = Q_j$. Thus, $l = t$ and $\exists$ a permutation $\sigma \in S_l$ s.t. $P_i = Q_{\sigma(i)}$. $\square$

Corollary 2.3.6. *Every proper radical differential ideal $I \subsetneqq K\{y_1, \ldots, y_n\}$ ($\mathrm{char}(K) = 0$) can be written as a finite intersection of prime differential ideals. If $I = \bigcap_{i=1}^{l} P_i$ is irredundant, P_i are called prime components of I.*

Example: $I = \{y'^2 - 4y\} \subseteq \mathbb{Q}\{y\}$. Then $I = \{y'^2 - 4y, y'' - 2\} \cap \{y\}$ (see Chapter 3).

Chapter 3

The Differential Algebra-Geometry Dictionary

Let (K, δ) be a differential field of characteristic 0. Let $K\{Y\} = K\{y_1, \ldots, y_n\}$ be the differential polynomial ring in the differential variables $y_1, \ldots, y_n$ over K. Any $\Sigma \subseteq K\{Y\}$ defines a system of algebraic differential equations $\Sigma(Y) = 0$. The main objective of differential algebra is to study the solutions of such system (i.e., differential varieties, our main protagonists).

3.1 Ideal-Variety correspondence in differential algebra

In algebraic geometry, we consider algebraic varieties in affine/projective spaces with coordinates taken from algebraically closed fields. In differential algebra, we have similar concepts of differentially closed fields introduced by the model theorist A. Robinson in 1950s. Robinson also proved the fundamental theorem that every differential field can be extended to a differentially closed field.

For $f \in K\{Y\} = K\{y_1, \ldots, y_n\}$ and $\eta = (\eta_1, \ldots, \eta_n) \in L^n$ with $(L, \delta) \supseteq (K, \delta)$, η is a differential zero of f if $f(\eta) = 0$. Here, $f(\eta)$ means replacing $\delta^k y_i$ by $\delta^k \eta_i$ in f.

Definition 3.1.1. *A differential field (E, Δ) is said to be* **differentially closed** *if for every finite system of algebraic differential equations and inequations*

$$f_1(y_1, \ldots, y_n) = \cdots = f_s(y_1, \ldots, y_n) = 0 \wedge g(y_1, \ldots, y_n) \neq 0$$

with coefficients in E, whenever the system has a solution in some differential extension field of E, it has a solution in E.

Blum gives a simple axiom defining a differentially closed ordinary differential field and shows that Robinson's definition is equivalent to it:

A differential field (E, δ) is said to be *differentially closed* if for any $f, g \in E\{y\}$ with $g \neq 0$ and $\mathrm{ord}(g) < \mathrm{ord}(f)$, there exists $\alpha \in E$ such that $f(\alpha) = 0$ and $g(\alpha) \neq 0$.

Remark. Note that the beauty of the axiom is that it reduces the question of the solvability of differential equations in several unknows to the case of a single unknown. Unfortunately, there does not exist such a simple, elegant axiom characterizing differentially closed partial differential fields.

Let $(K, \delta) \subseteq (E, \delta)$. (E, δ) is called a differential closure of (K, δ) if

1) (E, δ) is differentially closed, and

2) for every differentially closed field $(M,\delta) \supseteq (K,\delta)$, there is a differential embedding $\varphi : E \hookrightarrow M$ with $\varphi\mid_K = \mathrm{id}_K$.

Throughout this chapter, $(E,\delta) \supseteq (K,\delta)$ is a fixed differentially closed field. By a differential affine space, we mean any E^n for $n \in \mathbb{N}$. An element $(\eta_1,\ldots,\eta_n) \in E^n$ is called a point.

Definition 3.1.2. *A set $V \subseteq E^n$ is called a* ***differential variety*** *over K if $\exists\, \Sigma \subseteq K\{Y\}$ such that*

$$V = \mathbb{V}(\Sigma) \triangleq \{\eta \in E^n \mid f(\eta) = 0, \forall\, f \in \Sigma\}.$$

Let $\Pi = \{$differential varieties in E^n over $K\}$. Then Π satisfies:

1) $\emptyset, E^n \in \Pi$;

2) If $V_1, V_2 \in \Pi, V_1 \cup V_2 \in \Pi$;

3) Any intersection of elements of Π is an element of Π.

So Π is a topology on E^n, called the Kolchin topology, as compared to the Zariski topology in algebraic geometry. For this reason, a differential variety is also called a Kolchin-closed set. For a set $S \subseteq E^n$, the smallest differential variety (with respect to inclusion) containing S is called the Kolchin closure of S, denoted by S^{Kol}.

For a subset $S \subseteq E^n$, define $\mathbb{I}(S) = \{f \in K\{y_1,\ldots,y_n\} \mid \forall\, \eta \in S, f(\eta) = 0\}$. It is easy to show that $\mathbb{I}(S)$ is a radical differential ideal in $K\{Y\}$, called the vanishing differential ideal of S.

Proposition 3.1.3. *1) If $S_1 \subseteq S_2 \subseteq E^n$, then $\mathbb{I}(S_2) \subseteq \mathbb{I}(S_1)$.*

2) If $P_1 \subseteq P_2 \subseteq K\{Y\}$, then $\mathbb{V}(P_2) \subseteq \mathbb{V}(P_1)$.

3) If $S \subseteq E^n$, then $V = \mathbb{V}(\mathbb{I}(S))$ is the Kolchin closure of S and $\mathbb{I}(V) = \mathbb{I}(S)$.

Proof. 1) and 2) are straightforward.

To show 3): Let $S^{\mathrm{Kol}} = \mathbb{V}(\Sigma)$ for $\Sigma \subseteq K\{Y\}$. For every $f \in \Sigma, f\mid_S \equiv 0 \Rightarrow f \in \mathbb{I}(S)$. So $\Sigma \subseteq \mathbb{I}(S)$. Thus, $V = \mathbb{V}(\mathbb{I}(S)) \subseteq \mathbb{V}(\Sigma) = S^{\mathrm{Kol}}$. Hence, $S^{\mathrm{Kol}} = V$.

$S \subseteq V \Rightarrow \mathbb{I}(V) \subseteq \mathbb{I}(S)$. If $\exists\, f \in \mathbb{I}(S)\backslash\mathbb{I}(V)$, then $\exists\, \eta \in V$ s.t. $f(\eta) \neq 0$. Set $\Sigma_1 = \mathbb{I}(V) \cup \{f\}$. Then $\Sigma_1 \subseteq \mathbb{I}(S) \Rightarrow \mathbb{V}(\Sigma_1) \supseteq \mathbb{V}(\mathbb{I}(S)) = V$. Since $\eta \in V, \eta \in \mathbb{V}(\Sigma_1)$. So $f(\eta) = 0$, which yields a contradiction. Hence, $\mathbb{I}(V) = \mathbb{I}(S)$. ☐

Now we have two maps between Π and the set of radical δ-ideals in $K\{Y\} = K\{y_1,\ldots,y_n\}$:

$$\begin{array}{rccc} \mathbb{I}: & \{\delta\text{-varieties in } E^n \text{ over } K\} & \longrightarrow & \{\text{ radical } \delta\text{-ideals in } K\{Y\}\} \\ & V & & \mathbb{I}(V) \end{array}$$

and

$$\begin{array}{rccc} \mathbb{V}: & \{\text{ radical } \delta\text{-ideals in } K\{Y\}\} & \longrightarrow & \{\delta\text{-varieties in } E^n \text{ over } K\} \\ & J & & \mathbb{V}(J) \end{array}$$

Corollary 3.1.4. *For every δ-variety V, $\mathbb{V}(\mathbb{I}(V)) = V$. Hence $\mathbb{I}$ is injective and $\mathbb{V}$ is surjective.*

Proof. By Proposition 3.1.3 3), $\mathbb{V}(\mathbb{I}(V)) = V^{\mathrm{Kol}} = V$. ☐

Recall the notion of generic point introduced in Section 2.1 (the paragraph below Definition 2.1.6): A point $\eta \in L^n$ ($L \supseteq K$ a differential extension field) is a *generic zero* of a differential ideal I if $I = \mathbb{I}(\eta)$. Lemma 2.1.7 tells that a differential ideal is prime if and only if it has a generic zero.

Next section, we will give the differential Nullstellensatz theorem (both the weak and strong analogues of the Hilbert's Nullstellensatz theorem). Following Corollary 3.1.4, we will show $\mathbb{I}$ and $\mathbb{V}$ are both inclusion-reversing bijective maps. For the content in this section, all the results are valid even if E is not differentially closed. But for the differential Nullstellensatz theorem to be valid, E is required to be differentially closed.

3.2 Differential Nullstellensatz

The Hilbert Nullstellensatz in algebraic geometry has two versions: Given $F \subseteq K[x_1, \ldots, x_n]$,

Weak Nullstellensatz $\mathcal{V}(F) = \{\eta \in \bar{K}^n \mid F(\eta) = 0\} = \emptyset \Leftrightarrow 1 \in (F)$.

Strong Nullstellensatz Let $f \in K[x_1, \ldots, x_n]$. If $f \mid_{\mathcal{V}(F)} \equiv 0$, then $f \in \sqrt{(F)}$.

Below, we show the differential analogues of Hilbert's Nullstellensatz hold in differential algebra.

Theorem 3.2.1 (Weak Differential Nullstellensatz)**.** *Let $F \subseteq K\{y_1, \ldots, y_n\}$ and $(E, \delta) \supseteq (K, \delta)$ a differentially closed field. Then $\mathbb{V}(F) = \{\eta \in E^n \mid F(\eta) = 0\} = \emptyset$ if and only if $1 \in [F]$.*

Proof. It suffices to show that if $[F] \neq K\{y_1, \ldots, y_n\}$, then $\exists\, \eta \in E^n$ s.t. $f(\eta) = 0$ for all $f \in F$. Since $1 \notin [F]$, $\sqrt{[F]} \neq K\{y_1, \ldots, y_n\}$. Let $\sqrt{[F]} = \cap_{i=1}^{l} P_i$ be the minimal prime decomposition. Let $M = \mathrm{Frac}(K\{y_1, \ldots, y_n\}/P_1)$. Then M is a differential extension field of K and $(\bar{y}_1, \ldots, \bar{y}_n) \in M^n$ is a generic zero of P_1. $F \subseteq P_1$ implies that $(\bar{y}_1, \ldots, \bar{y}_n)$ is a differential zero of F. Since $E \supseteq K$ is differentially closed, there exists $\eta = (\eta_1, \ldots, \eta_n) \in E^n$ s.t. $\forall\, f \in F, f(\eta) = 0$. □

Theorem 3.2.2 (Differential Nullstellensatz)**.**

- *Let $F \subseteq K\{y_1, \ldots, y_n\}$ and $f \in K\{y_1, \ldots, y_n\}$. If f vanishes at every differential zero of F in E^n, then $f \in \{F\}$.*
- $\mathbb{I}(\mathbb{V}(F)) = \{F\}$.

Proof. (Use Rabinowitsch's trick for the case $f \neq 0$)

Intruduce a new differential indeterminate t and consider the new differential polynomial set $F, 1 - fz$ in $K\{y_1, \ldots, y_n, z\}$. Since f vanishes at every differential zero in E^n of F, $\mathbb{V}(F, 1 - fz) \subseteq E^{n+1}$ is the emptyset. By the weak differential Nullstellensatz, $1 \in [F, 1 - fz] \subseteq K\{y_1, \ldots, y_n, z\}$. Hence, $\exists\, A_i, B_i \in K\{y_1, \ldots, y_n, z\}$ and $s \in \mathbb{N}$ s.t.

$$1 = \sum_{i=0}^{s} A_i F^{(i)} + \sum_{j=0}^{s} B_j (1 - fz)^{(j)}.$$

Since $f \neq 0$, replace z by $\frac{1}{f}$ at both sides, then we have

$$1 = \sum_{i=0}^{s} A_i(y_1, \ldots, y_n, \tfrac{1}{f}) F^{(i)}.$$

There exists $m \in \mathbb{N}$ s.t. $f^m \sum_{i=0}^{s} A_i(y_1, \ldots, y_n, \frac{1}{f}) F^{(i)} \in K\{y_1, \ldots, y_n\}$. So we have $f^m \in [F]$. □

Remark: As above, we give an abstract proof for the weak differential Nullstellensatz following Ritt. The first constructive proof was given by Seidenberg using elimination theory.

The differential Nullstellensatz and Corollary 3.1.2 show that the two maps $\mathbb{I}$ and $\mathbb{V}$ are bijections.

Theorem 3.2.3. *The maps $V \to \mathbb{I}(V)$ and $I \to \mathbb{V}(I)$ define inclusion reversing bijections between the set of all differential varieties in E^n over K and the set of all radical differential ideals in $K\{y_1, \dots, y_n\}$.*

Proof. By Corollary 3.1.4, for each differential variety V, $\mathbb{V}(\mathbb{I}(V)) = V$. And for each radical differential ideal I, $\mathbb{I}(\mathbb{V}(I)) = \{I\} = I$ by Theorem 3.2.2. □

In this chapter, all differential varieties are assumed over K unless otherwise indicated.

Definition 3.2.4. *Let $V \subseteq E^n$ be a differential variety. Then the differential ring*

$$K\{V\} := K\{y_1, \dots, y_n\}/\mathbb{I}(V)$$

is called the differential coordinate ring of V.[1]

$W \subseteq E^n$ is called a differential subvariety of V if $W \subseteq V$ and W is a differential variety in E^n.

Theorem 3.2.3 can be generalized to arbitrary differential varieties in place of $\mathbb{A}^n = E^n$.

Corollary 3.2.5. *Let $V \subseteq E^n$ be a differential variety. The map*

$$W \longmapsto \{f \in K\{V\} \mid f(a) = 0 \; \forall \, a \in W\}$$

is an inclusion reversing bijection between the set of differential subvarieties of V and the set of radical differential ideals in $K\{V\}$.

Effective Hilbert Nullstellensatz and Effective differential Nullstellensatz

Effective Nullstellensatz

Let $P_1, \dots, P_m \in \mathbb{C}[X] := \mathbb{C}[x_1, \dots, x_n]$ have degree at most $d \geq 1$. If $P_1, \dots, P_m$ have no common zero in $\mathbb{C}^n$, then there are polynomials $A_1, \dots, A_m \in \mathbb{C}[X]$ of degree bounded by $\mathrm{B}(d, n, m)$ s.t. $1 = A_1 P_1 + \cdots + A_m P_m$. If such a degree bound $\mathrm{B}(d, n, m)$ for A_i exists, to decide whether $P_1 = \cdots = P_m = 0$ has a zero is reduced to solve linear equations.

- $\deg(A_i) \leq 2(2d)^{2^{n-1}}$ (Hermann, Math. Ann., 1926)
- lower bound: $\deg(A_i) \geq d^n - d^{n-1}$ (Masser-Philippon)
- $\deg(A_i) \leq \mu n d^\mu + \mu d$ for $\mu = \min\{m, n\}$
  $\leq 2n^2 d^\mu$ (Brownawell, Ann. Math., 1987)
- $\deg(A_i P_i) \leq \begin{cases} d_1 d_2 \cdots d_m & \text{if } m \leq n \\ d_1 \cdots d_{n-1} d_m & \text{if } m > n > 1 \\ d_1 + d_m - 1 & \text{if } m > n = 1 \end{cases}$ (Kollar, J. Amer. Math. Soc., 1988)

  Here $\deg(P_i) = d_i$ and assume $d_1 \geq d_2 \geq \cdots \geq d_m > 2$.
- $\deg(A_i P_i) \leq \begin{cases} N'(d_1, \dots, d_m; n) & \text{if } m \leq n \\ 2N'(d_1, \dots, d_m; n) - 1 & \text{if } m > n \end{cases}$ (Jelonek, Invent. Math., 2005. New Proof)

[1] Since for any $a \in V$, $\bar{f}_1 = \bar{f}_2$ implies $f_1(a) = f_2(a)$. So $K\{V\}$ could be regarded as a ring of differential functions on V.

Subsequent work on sharper bounds or new proofs.

Effective Differential Nullstellensatz

If $F_1, \ldots, F_k \in K\{y_1, \ldots, y_n\}$ have no common differential zeros in E^n, then $\exists\, s \in \mathbb{N}$ and $A_{ij} \in K\{y_1, \ldots, y_n\}$ s.t. $1 = \sum_{i=1}^{k} \sum_{j=0}^{s} A_{ij} F_i^{(j)}$.

Effective differential Nullstellensatz is to give a bound for s in terms of the order h, degree d and the number of derivation operators m and the number of differential variables n. If such a computable bound is given, to decide whether $\mathbb{V}(F_1, \ldots, F_k) = \emptyset$ or not is reduced to an algebraic problem and then results about effective Hilbert Nullstellensatz could be applied here.

Uniform bounds for s in the ordinary differential case:

- $s \le A(q, \max\{n, h, d\})$ (Golubitsky et al., J. Algebra, 2009), where $A(\cdot, \cdot)$ is the Ackermann function recursively defined by
$$\begin{cases} A(0, n) = n + 1 \\ A(m + 1, 0) = A(m, 1) \\ A(m + 1, n + 1) = A(m, A(m + 1, n)). \end{cases}$$
- K: constant differential field. $s \le (n(h + 1)d)^{2^{c(n(e+1))^3}}$ for a universal constant $c > 0$. (D' Alfonso et al., J. complexity, 2014)
- $s \le (nTd)^{2^{O(n^3(T+1)^3)}}$ (Gustavson et al., Adv. Math., 2016)
- $s \le \begin{cases} d^{nh-p+1} 2^{p+1} & \text{if } D \ge 2 \\ p+1 & \text{if } D = 1. \end{cases}$ Here $p = \dim((F))$ in $K[y_i^{(j)} : j \le h]$. (Ovchinnikov et al., International Mathematics Research Notices, 2021)

Example: $F = \{y_1^2, y_1 - y_2^2, \ldots, y_{n-1} - y_n^2, 1 - y_n'\}, \mathbb{V}(F) = \emptyset$. Since $1 \notin (F, \ldots, F^{(2^n-1)})$ and $1 \in (F, \ldots, F^{(2^n)})$, we have $s \ge 2^n$.