2.3 The Dot\Inner\Scallar Product (9.3)

Def: The dot product is an operation between two vectors that results in a scalar:

$$\vec{a}=\langle a_1,a_2\rangle,\vec{b}=\langle b_1,b_2\rangle: \qquad \vec{a}\cdot\vec{b}=a_1b_1+a_2b_2$$

$$\vec{c}=\langle c_1,c_2,c_3\rangle,\vec{d}=\langle d_1,d_2,d_3\rangle: \quad \vec{c}\cdot\vec{d}=c_1d_1+c_2d_2+c_3d_3$$

If we consider vectors as starts at origin then the components of the 2D vector can be considered x,y coordinates. Let write it in the polar form, thus:

$$\vec{a}=\langle a_1,a_2\rangle=\langle |a|\cos\theta,|a|\sin\theta\rangle$$

$$b_1=\langle b_1,b_2\rangle=\langle |b|\cos\varphi,|b|\sin\varphi\rangle$$

Lets now review dot product

$$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2=|a|\cos\theta|b|\cos\varphi+|a|\sin\theta|b|\sin\varphi=$$

$$=|a||b|(\cos\theta\cos\varphi+\sin\theta\sin\varphi)=|a||b|\cos(\varphi-\theta)$$

Thus we just found:

Def: The geometric definition of the dot product between 2 vectors **u,v** (in either 2D or 3D) is given by $\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}||\mathbf{v}|\cos\alpha$, where α is the angle between the vectors **u** and **v**.

Note: In order to understand why it works in 3D, consider the plane where both vectors lies.

Ex 5. Let $\vec{a}=\langle 1,\sqrt{3}\rangle,\vec{b}=\langle 1,1\rangle$, find angle and magnitude and calculate dot product using both definition, to show they give the same result.

$$|\vec{a}|=\sqrt{1+3}=2,\theta_a=\frac{\pi}{3} \qquad |\vec{b}|=\sqrt{1+1}=\sqrt{2},\theta_b=\frac{\pi}{4}$$

$$\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}=\frac{1}{2}\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}=\frac{1+\sqrt{3}}{2\sqrt{2}}$$

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|=2\sqrt{2}\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=2\sqrt{2}\cdot\frac{1+\sqrt{3}}{2\sqrt{2}}=1+\sqrt{3}$$

$$\vec{a}\cdot\vec{b}=1\cdot1+1\cdot\sqrt{3}=1+\sqrt{3}$$

Def: An angle between 2 vectors defined $\alpha=\arccos\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$

Def: Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal iff $\vec{a}\cdot\vec{b}=0$

Properties of the Dot Product:

1) $\mathbf{a}\cdot\mathbf{a}=a_1^2+a_2^2+a_3^2=|\mathbf{a}|^2$

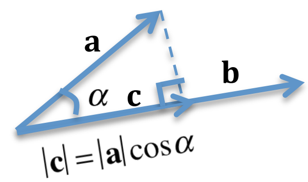
2) $\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3=b_1a_1+b_2a_2+b_3a_3=\mathbf{b}\cdot\mathbf{a}$

3) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\langle b_1+c_1,b_2+c_2,b_3+c_3\rangle=\langle a_1(b_1+c_1),a_2(b_2+c_2),a_3(b_3+c_3)\rangle=$

$=\langle a_1b_1+a_1c_1,a_2b_2+a_2c_2,a_3b_3+a_3c_3\rangle=\langle a_1b_1,a_2b_2,a_3b_3\rangle+\langle a_1c_1,a_2c_2,a_3c_3\rangle=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$

$$4)\ (c\,\mathbf{a})\cdot\mathbf{b}=(ca_1)b_1+(ca_2)b_2+(ca_3)b_3=(ca_1)b_1+(ca_2)b_2+(ca_3)b_3=$$
$$=c(a_1b_1)+c(a_2b_2)+c(a_3b_3)=c(\mathbf{a}\cdot\mathbf{b})$$
$$=a_1(cb_1)+a_2(cb_2)+(ca_3)b_3=\mathbf{a}\cdot(c\,\mathbf{b})$$
$$5)\ 0\cdot\mathbf{a}=0$$

2.3.1 Projections

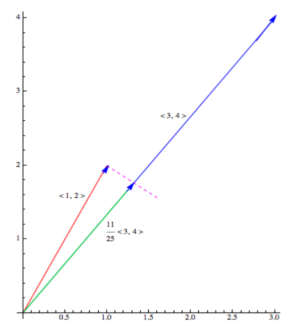


The example of projection given on following figure, it shows a projection vector **c** of vector **a** on vector **b**. The magnitude of **c**, by geometrical considerations, appears to be $|\mathbf{c}|=|\mathbf{a}|\cos\alpha$. Since we know the angle between vectors is given by $\alpha=\arccos\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$ we get $|\mathbf{c}|=\left||\mathbf{a}|\cos\alpha\right|=\left||\mathbf{a}|\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right|=\left|\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right|$. The quantity $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$ has several names: a signed magnitude, a scalar projection, and a component of the projection of **a** on **b**.

Def: A **Scalar Projection** of vector **a** on vector **b** denoted as $\operatorname{comp}_{\mathbf{b}}\mathbf{a}=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$

Vector Projection: of vector **a** on vector **b** $\operatorname{proj}_{\mathbf{b}}\mathbf{a}=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\frac{\mathbf{b}}{|\mathbf{b}|}=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$

Ex 6. Let $\mathbf{u}=\langle 1,2\rangle$, $\mathbf{v}=\langle 3,4\rangle$, compute the component and the projection of **u** on **v**



$$\operatorname{comp}_{\mathbf{v}}\mathbf{u}=\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}=\frac{1\cdot 3+2\cdot 4}{\sqrt{3^2+4^2}}=\frac{11}{\sqrt{25}}=\frac{11}{5}$$

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u}=\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}=\frac{1\cdot 3+2\cdot 4}{3^2+4^2}\mathbf{v}=\frac{11}{25}\mathbf{v}=\frac{\langle 33,44\rangle}{25}$$

Note: The common mistake is to mix up the vectors, therefore