Lecture 22

Review

$D \subseteq uv$ plane, $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ parametrized surface.

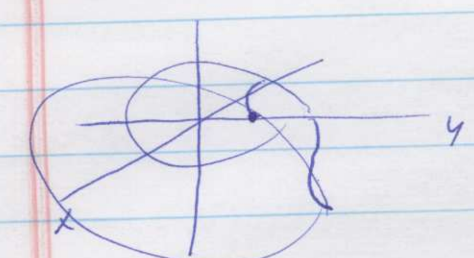
Examples

1. Graph of $z = f(x,y)$ $\quad \vec{r}(u,v) = (u, v, f(u,v))$

2. Plane through point (x_0, y_0, z_0) containing nonparallel vectors $\vec{a} = (a_1, a_2, a_3)$ & $\vec{b} = (b_1, b_2, b_3)$

$$\vec{r}(s,t) = (x_0, y_0, z_0) + s(a_1, a_2, a_3) + t(b_1, b_2, b_3)$$
$$= (x_0 + sa_1 + tb_1,\ y_0 + sa_2 + tb_2,\ z_0 + sa_3 + tb_3)$$

3 Rotate curve $y = f(x)$ $a \le x \le b$ around x axis, $\theta = \angle$ of rotation



$x = x \quad y = f(x)\cos\theta \quad z = f(x)\sin\theta$

$\vec{r}(x,\theta) = (x, f(x)\cos\theta, f(x)\sin\theta)$

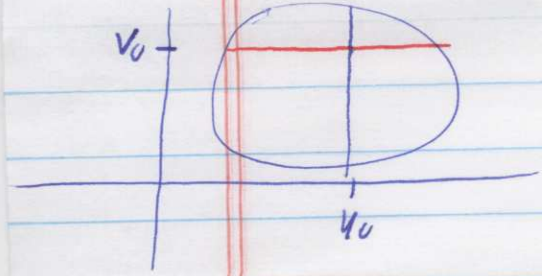
Tangent Planes

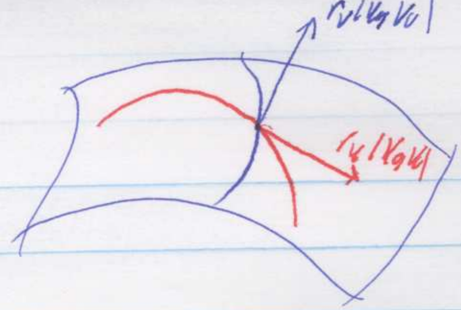
Review last class $\quad \vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ then

$\vec{r}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$ at a point (u_0, v_0) is tang vector to "grid curve" fixing $v = v_0$.

Similarly

$\vec{r}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$ tangent to curve fixing $u = u_0$





<u>Fact</u> The tangent plane at point $\vec{r}(u_0,v_0)$ contains tangent vectors $\vec{r}_u(u_0,v_0)$ and $\vec{r}_v(u_0,v_0)$, so has $\vec{r}_u \times \vec{r}_v$ as normal vector.

<u>Examples</u>

1. $\vec{r}(u,v) = (u, v, f(u,v))$ $\quad r_u = (1, 0, \frac{\partial f}{\partial u})$ $\quad r_v = (0, 1, \frac{\partial f}{\partial v})$

$$r_u \times r_v = \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right)$$

Already did this in 14.4

2. Find tangent plane to $\vec{r}(u,v) = (u^2+v, uv, u^2-v^2)$ at point $(2,1,0)$.

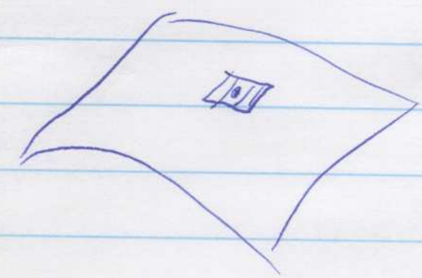
3. Find par. equation for the part of the part of the elliptic paraboloid $x+y^2+2z^2=4$ that lies in front of the plane $x=0$. Then find the tangent plane at $(1,1,1)$.

Surface Area

Review $\vec{r}(t)$ $a \le t \le b$ To get arc length we split curve into segments, each of length $|\vec{r}'(t_i)|$, and

$$A.L. = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{|r'(t_i)|}_{\text{speed}} \underbrace{\Delta t}_{\text{time}} = \int_a^b |\vec{r}'(t)|\, dt$$

For a surface, the corr. problem is surface area



Estimated area = area of parallelogram in tang plane
$= |r_u \times r_v|$.

Def / Thm Suppose $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ is a smooth parametric surface covered once as $(u,v) \in D$. Then the surface Area is:

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA.$$

Ex Sphere radius a $(a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi)$
$0 \le \phi \le \pi$ $0 \le \theta \le 2\pi$

Check $r_\phi \times r_\theta = (a^2 \sin^2\phi \cos\theta, a^2 \sin^2\phi \sin\theta, a^2 \sin\phi \cos\phi)$

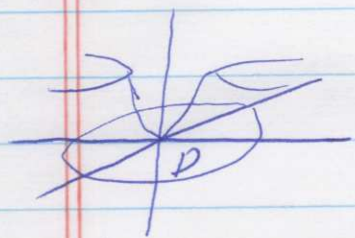
Ex Graph of a function $\vec{r}(x,y) = (x, y, f(x,y))$

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

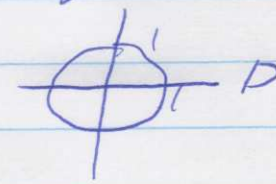
compare to A.L.

$$= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Ex Find S.A. of portion of saddle $z = x^2 - y^2$ inside cylinder $x^2 + y^2 = 1$.



Par. as $(x, y, x^2 - y^2)$ $(x,y) \in D$



$r_x = (1, 0, 2x)$ $r_y = (0, 1, -2y)$

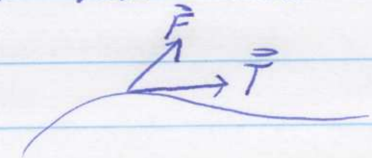
$r_x \times r_y = (-2x, 2y, 1)$ $|r_x \times r_y| = \sqrt{1 + 4x^2 + 4y^2}$

$$\iint_D \sqrt{1+4x^2+4y^2}\, dx\, dy = \int_0^{2\pi} \int_0^1 \sqrt{1+4r^2}\, r\, dr\, d\theta$$

Ex Find area of $\vec{r}(u,v) = (u\cos v, u\sin v, v)$
$0 \le u \le 1$ $0 \le v \le \pi$

spiral ramp

Review Line integral of vector field over curve:



At each point we add up $\vec{F} \cdot \vec{T}$

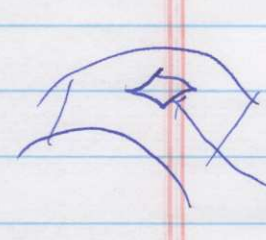
$$\int_C \vec{F} \cdot \vec{T}\, ds = \boxed{\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt}$$

Integral of a function

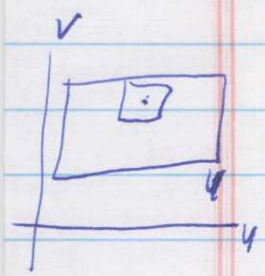
$$\int_C f(x,y)\, ds = \int f(r(t))\, |r'(t)\, dt|$$

Surface Integrals

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}$$



area $\approx |r_u \times r_v|\,\Delta u\,\Delta v$



Def The surface integral of a function $f(x,y,z)$ over a surface parametrized by

$$\vec{r}(u,v) : D \longrightarrow \mathbb{R}^3 \quad \text{is}$$

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,|r_u \times r_v|\,dA}$$

Compare to

$$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

Ex $\iint_S xy\,dS$ $\quad S$ is plane $x+y+z=1$ in 1^{st} octant.