Elementary and Darboux first integrals for planar polynomial vector fields

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Plan

We consider the polynomial (differential) system in $\mathbf{C}^2$ defined by

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \tag{1}$$

with $P, Q \in \mathbf{C}[x,y]$. The polynomial vector field is

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y},$$

Definition

We call a function

$$f_1^{\lambda_1} \cdots f_r^{\lambda_r} \exp(g/(f_1^{n_1} \cdots f_r^{n_r})) = \exp(R) \cdot \prod S_i^{c_i},$$

with rational functions R and S_i and complex constants c_i, a *Darboux function*.

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Consider $\mu = f_1^{-d_1} \cdots f_r^{-d_r}$. Define

$$Z_g = \text{ Hamiltonian vector field of } g/\left(f_1^{d_1-1} \cdots f_r^{d_r-1}\right).$$

Then, the trivial vector field

$$f_1^{d_1} \cdots f_r^{d_r} \cdot Z_g = f \cdot X_g - \sum_{i=1}^{r} (d_i - 1) g \frac{f}{f_i} \cdot X_{f_i} \tag{2}$$

is polynomial and admits $\mu = f_1^{-d_1} \cdots f_r^{-d_r}$.

Under conditions: $\mathcal{F} = \mathcal{F}^0$ (Christopher,Llibre, Pantazi and Walcher)

Proposition If $d_1, \ldots, d_r$ are rational numbers then every element of $\mathcal{F}^0$ admits a rational (hence elementary) first integral.

Chavariga, Giacomini,Gine, Llibre and also Llibre,Pantazi

Let

$$H(x,y) = f_1^{\lambda_1} \cdots f_r^{\lambda_r} \exp(g/(f_1^{n_1} \cdots f_r^{n_r}))$$

be a Darboux function with $\lambda_1, \cdots, \lambda_r \in \mathbf{C}$, $n_1, \cdots, n_r \in \mathbf{N} \cup \{0\}$ and $g \in \mathbf{C}[x,y]$ coprime with f_i whenever $n_i \neq 0$. Then H is a first integral of the polynomial vector field

$$\widehat{X} = \prod_{k=1}^{r} f_k^{n_k+1} \cdot \left(\sum_{k=1}^{r} \lambda_k X_{f_k}/f_k + Z_g^{(n_1+1,\ldots,n_r+1)} \right)$$

which, in turn, admits the integrating factor $\prod_{k=1}^{r} f_k^{-(n_k+1)}$. Moreover, any polynomial vector field admitting the first integral H admits a rational integrating factor.

Vector fields with Rational first integral are special ones.

If X admits a Darboux first integral but not a rational first integral then it admits (up to constant multiples) a unique integrating factor, and this integrating factor is rational.

but also the inverse is true

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Let $\mathbb{K} = \mathbf{C}(x, y)$ with derivations $\partial/\partial x$ and $\partial/\partial y$.
Every derivation of $\mathbb{K}$ has the form $Y = R\frac{\partial}{\partial x} + S\frac{\partial}{\partial y}$ with rational R and S.

- An extension field $\mathbb{L}$ of $\mathbb{K}$ is called elementary if there is a finite tower of extension fields $\mathbb{K} = \mathbb{L}_0 \subset \mathbb{L}_1 \subset \cdots \subset \mathbb{L}_n = \mathbb{L}$ such that $\mathbb{L}_{i+1}$ is obtained from $\mathbb{L}_i$ by adjoining an algebraic element, an exponential (i.e. an element w such that $Y(w)/w \in \mathbb{L}_i$ for some derivation Y) or a logarithm (i.e. an element w such that $Z(w) = Z(a)/a$ for some $a \in \mathbb{L}_i$ and some derivation Z).
- An extension field $\mathbb{L}$ is called *Liouvillian* over $\mathbb{K}$ if there is a finite tower of extension fields $\mathbb{K} = \mathbb{L}_0 \subset \mathbb{L}_1 \subset \cdots \subset \mathbb{L}_n = \mathbb{L}$ such that $\mathbb{L}_{i+1}$ is obtained from $\mathbb{L}_i$ by adjoining an algebraic element, an integral element (i.e. an element w such that $Z(w) = a$ for some $a \in \mathbb{L}_i$), or an exponential. Every element of is called a *Liouvillian function of two variables*.

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Liouvillian first integrals came from Darboux functions

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Prelle and Singer's Theorem

(a) If the polynomial vector field X admits an elementary first integral then there exist an integer $m \geq 0$, algebraic functions $v, u_1, \ldots, u_m$ over $\mathbb{K}$ and nonzero constants $c_1, \ldots, c_m \in \mathbb{C}$ such that

$$X(v) + \sum_{i=1}^{m} c_i \frac{X(u_i)}{u_i} = X\left(v + \sum_{i=1}^{m} c_i \log(u_i)\right) = 0, \tag{3}$$

but $v + \sum_{i=1}^{m} c_i \log(u_i)$ is not constant. The c_i may be chosen linearly independent over the rational numbers $\mathbb{Q}$. $\prod {u_i}^{c_i} \exp v$.

(b) If the vector field X admits an elementary first integral then it admits an integrating factor of the special form

$$\mu = f_1^{-d_1} \cdots f_r^{-d_r}, \tag{4}$$

with irreducible and pairwise relatively prime polynomials $f_1, \ldots, f_r$, and exponents $d_1, \ldots, d_r \in \mathbb{Q}$.

Given an algebraic function w, we denote its *minimal polynomial* by

$$M_w(T) := T^d + \sum_{i=1}^{d} g_i T^{d-i} \in \mathbb{K}[T].$$

Here $-g_1$ is the *trace* of w
$(-1)^d g_d$ is the *norm* of w.

The other zeros of M_w (in a suitable extension field $\mathbb{F}$ of $\mathbb{K}$) are called the *conjugates* of w.

Theorem: Llibre,Pantazi and Walcher

Let the polynomial vector field X admit the elementary first integral $v + \sum_{i=1}^{m} c_i \log(u_i)$, where $m > 0$ and v, $u_1, \ldots, u_m$ are nonconstant algebraic functions over $\mathbb{K} = \mathbb{C}(x, y)$, and furthermore $c_1, \ldots, c_m$ are complex constants linearly independent over $\mathbb{Q}$. If X does not admit a rational first integral then the following hold.

(a) If some u_j has non-constant norm, or if v has non-constant trace, then X admits a Darboux first integral.

(b) Moreover, if $\mu = f_1^{-d_1} \cdots f_r^{-d_r}$ is an integrating factor for X, with some u_j having non-constant norm, then it is uniquely determined (up to a nonzero complex factor) and all d_i are nonnegative integers.

It remains exceptional cases' admitting an elementary first integral

$$v + \sum_{i=1}^{m} c_i \log(u_i)$$

with all u_i of constant norm
and v of constant trace.

Assume $m = 1$. Then take $c_1 = 1$. Then $u \exp v$ or $v + logu$ So we have

$$X(v) + \frac{X(u)}{u} = 0, \quad X(v + logu) = 0 \qquad (5)$$

with u and v non-constant algebraic functions over $\mathbb{K}$ (but not both in $\mathbb{K}$),
with u of constant norm
and v of constant trace.

Theorem for Quadratic Extensions

Assume $X(v) + \frac{X(u)}{u} = 0$ with u of constant norm and v of constant trace, both contained in a degree two extension of $\mathbb{K}$, but neither contained in $\mathbb{K}$. Then, with no loss of generality, one may take u and v to satisfy

$$u^2 + 2g \cdot u + 1 = 0 \text{ and } v = b(g + u)$$

with a non-constant rational function g and a nonzero rational function b.

(a) There is a rational function s such that

$$\begin{aligned} P = & \ -s \cdot \left((g^2 - 1)b_y + (bg - 1)g_y\right), \\ Q = & \ s \cdot \left((g^2 - 1)b_x + (bg - 1)g_x\right), \end{aligned}$$

defines a polynomial vector field. If one requires P and Q to have relatively prime entries then s is unique up to a factor in $\mathbb{C}^*$.

(b) This vector field admits the elementary first integral $v + \log u$ and the integrating factor $(\sqrt{g^2 - 1} \cdot s)^{-1}$.

Theorem for Quadratic Extensions

- If this vector field admits a Darboux first integral then it admits a rational first integral.

Example

This is known Prelle and Singer: Let $g = x$ and $b = y$. The vector field

$$\widehat{X} = \begin{pmatrix} 1 - x^2 \\ xy - 1 \end{pmatrix}$$

admits the integrating factor $1/\sqrt{1 - x^2}$ and the elementary first integral

$$H(x, y) = y \cdot \sqrt{x^2 - 1} + \log\left(-x + \sqrt{x^2 - 1}\right).$$

Prelle and Singer showed by differential-algebraic arguments that no rational first integral exists.

Example

The polynomial differential system

$$\dot{x} = 1 - x^2,$$
$$\dot{y} = 1 - x^2 - xy,$$

admits the integrating factor $(x^2 - 1)^{-3/2}$ and the elementary first integral

$$H = \frac{e^{\frac{y}{\sqrt{x^2-1}}}}{x + \sqrt{x^2 - 1}}$$

(or equivalently $\log H = \frac{y}{\sqrt{x^2-1}} - \log(x + \sqrt{x^2 - 1})$).
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Example

$$\dot{x} = 1 - x^2,$$
$$\dot{y} = 1 - x^2 - xy,$$

Stationary points: $(1, 0)$ and $(-1, 0)$,
$x \pm 1 = 0$ are invariant algebraic curves with cofactors $-x \pm 1$,
NO other invariant algebraic curves exist.
The only possible rational FI: $(x-1)^m(x+1)^n$, with integers m and n.

$$m \cdot (-x+1) + n \cdot (-x-1) = 0$$

m and n must be both zero.

Consider: $\mathbb{F} = \mathbb{K}[u] = \mathbb{K}[v]$ be a degree three extension of $\mathbb{K}$, with u of norm one and v of trace zero and X admitting the elementary first integral

$$X(v) + \frac{X(u)}{u} = 0, \quad X(v + logu) = 0$$

Proposition Let μ be the integrating factor of X (which is unique up to a nonzero scalar). Then $\mathbb{F} = \mathbb{K}[\mu]$ and there exists $g \in \mathbb{K}$ such that

$$\mu^3 - g = 0. \tag{6}$$

A general element of $\mathbb{F} = \mathbb{K}[\mu]$ has the form

$$w := a + b \cdot \mu + c \cdot \mu^2; \quad a,\ b,\ c \in \mathbb{K}, \tag{7}$$

and by Hilbert's Theorem 90 we have $u = \frac{\sigma(w)}{w}$, for some nonzero w, while the trace zero element has the form

$$v = r \cdot \mu + s \cdot \mu^2; \quad r, s \in \mathbb{K}.$$

Lemma

Let $w \in \mathbb{F}^*$, $w := a + b \cdot \mu + c \cdot \mu^2$, and $u = \sigma(w)/w$. Moreover let $r, s \in \mathbb{K}$ and

$$v := r \cdot \mu + s \cdot \mu^2 \in \mathbb{F}$$

an element of trace zero. The elementary function

$$H := v + \log u$$

is a first integral of a nonzero rational vector field with integrating factor μ if and only if the identities

$$\begin{aligned} s_x + \tfrac{2}{3} s g^{-1} g_x + R_{12}/N(w) &= 0, \\ s_y + \tfrac{2}{3} s g^{-1} g_y + R_{22}/N(w) &= 0, \end{aligned}$$

are satisfied. R_{12}, R_{22} are rational functions.

$$\begin{aligned} R_{12} &= bc^2g \cdot \left(c_x/c - b_x/b + \tfrac{1}{3}g_x/g)\right) \\ &+ ab^2 \cdot \left(b_x/b - a_x/a + \tfrac{1}{3}g_x/g\right) \\ &+ a^2c \cdot \left(a_x/a - c_x/c - \tfrac{2}{3}g_x/g\right). \end{aligned}$$

In general, no rational solution exists for the system of partially differential equations.

Proposition

Let w, u and v be as above ($w := a + b \cdot \mu + c \cdot \mu^2$), and denote by $N(w) = c^3 \cdot g^2 - 3abc \cdot g + b^3 \cdot g + a^3$ the norm of w.

(a) For every derivation Y of $\mathbb{K}$ one has

$$\frac{Y(u)}{\mu \cdot u} = \frac{1}{N(w)} A_Y + \frac{\mu}{N(w)} B_Y, \quad A_Y, B_Y \text{rationals}$$

(b) There exists a nontrivial polynomial vector field X which admits the first integral $H := v + \log u$ if and only if the identity

$$Y(s) + \frac{2}{3} s \cdot \frac{Y(g)}{g} + \frac{1}{N(w)} B_Y = 0$$

holds for all derivations Y of $\mathbb{K}$.

(c) Given any $u \in \mathbb{F}$ with norm one, there exists at most one rational function s such that H (with $v = r \cdot \mu + s \cdot \mu^2$ and arbitrary $r \in \mathbb{K}$) is a first integral for some nontrivial polynomial vector field.

Example

We keep notation and terminology from above. Let g be arbitrary with $g_x \neq 0$, $a \in \mathbb{C}^*$ and set

$$w := a + \mu, \quad N(w) = g + a^3, \quad R_{12} = \frac{a}{3} \cdot \frac{g_x}{g}.$$

$$s_x + \frac{2}{3} s g^{-1} g_x + R_{12}/N(w) = 0.$$

Fix $y = y_0$. We show there exists no rational function $\widehat{s}$ such that

$$\widehat{s}' + \frac{2}{3}\widehat{s} \cdot \frac{\widehat{g}'}{\widehat{g}} + \frac{a}{3} \cdot \frac{\widehat{g}'}{\widehat{g} \cdot (\widehat{g} + a^3)} = 0.$$

The general solution of this differential equation is given by

$$S = \widehat{g}^{-2/3} \cdot q, \quad q' = -\frac{a}{3} \cdot \frac{\widehat{g}^{2/3} \cdot \widehat{g}'}{\widehat{g} \cdot (\widehat{g} + a^3)} = -\frac{a}{3} \cdot \frac{\widehat{g}'}{\widehat{g}^{1/3} \cdot (\widehat{g} + a^3)}.$$

Example

A substitution leads to the indefinite integration problem

$$-\frac{a}{3}\cdot\int\frac{dt}{t^{1/3}(t+a^3)}=-a\cdot\int\frac{z}{z^3+a^3}\,dz$$

with one more substitution $z=t^{1/3}$, $t=z^3$ and $dt=3z^2\,dz$. With ζ a primitive third root of unity, one has

$$-a\cdot\frac{z}{z^3+a^3}=\frac{1}{3}\left(\frac{1}{z+a}+\frac{\zeta^2}{z+\zeta a}+\frac{\zeta}{1+\zeta^2 a}\right),$$

and therefore (upon back-substitution) one has a local representation

$$q=\frac{1}{3}\cdot\left(\log(\widehat{g}^{1/3}+a)+\zeta^2\log(\widehat{g}^{1/3}+\zeta a)+\zeta\log(\widehat{g}^{1/3}+\zeta^2 a)\right)+\text{const.}$$

Since q is a transcendental function for any value of the constant, we are done.

Representative references far to be complete

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