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DIPARTIMENTO DI INFORMATICA E SISTEMISTICA

Nonlinear Model Predictive Control

Stability, Robustness and Applications

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A Giovanni, Elena ed Alessandro

A winner is a dreamer who never quits

N. Mandela

Preface

This thesis is the product of three years of research that I have carried out in the Department of Informatica and Sistemistica at the University of Pavia.

First of all I would like to thank my supervisor, Prof. Lalo Magni, for his continuous support and encouragement. I am really in debt to him for all the opportunities he gave me and all he teached me. In these years he was not only a supervisor, he was a friend.

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Contents

Chapter 1

Introduction

Model Predictive Control (MPC), also known as moving horizon control or receding horizon control, refers to a class of algorithms which make explicit use of a process model to optimize the future predicted behavior of a plant. During the past 30 years, MPC has proved enormously successful in industry mainly due to the ease with which constraints can be included in the controller formulation. It is worth to note that this control technique, has achieved great popularity in spite of the original lack of theoretical results concerning some crucial points such as stability and robustness. In fact, a theoretical basis for this technique started to emerge more than 15 years after it appeared in industry. Originally developed to cope with the control needs of power plants and petroleum refineries, it is currently successfully used in a wide range of applications, not only in the process industry but also other processes ranging from automotive to clinical anaesthesia. Several recent publications provide a good introduction to theoretical and practical issues associated with MPC technology (see e.g. the books [Maciejowski 2002, Rossiter 2003, Camacho & Bordons 2004] and the survey papers [Morari & H. Lee 1999, Mayne *et al.* 2000, Rawlings 2000, Qin & Badgwell 2003, Findeisen *et al.* 2003, Magni & Scattolini 2007]).

One of the reasons for the success of MPC algorithms consists in the intuitive way of addressing the control problem. Predictive control uses a model of the system to obtain an estimate (prediction) of its future behavior. The main items in the design of a predictive controller are:

- the process model
- a performance index reflecting the reference tracking error and the control action
- an optimization algorithm to compute a sequence of future control

signals that minimizes the performance index subject to a given set of constraints

- the receding horizon strategy, according to which only the first element of the optimal control sequence is applied on-line.

At each sampling time t, a finite horizon optimal control problem is solved over a prediction horizon N, using the current state x of the process as the initial state. The on-line optimization problem takes account of system dynamics, constraints and control objectives. The optimization yields an optimal control sequence, and only the control action for the current time is applied while the rest of the calculated sequence is discarded. At the next time instant the horizon is shifted one sample and the optimization is restarted with the information of the new measurements. Figure 1.1 depicts the basic principle of model predictive control.

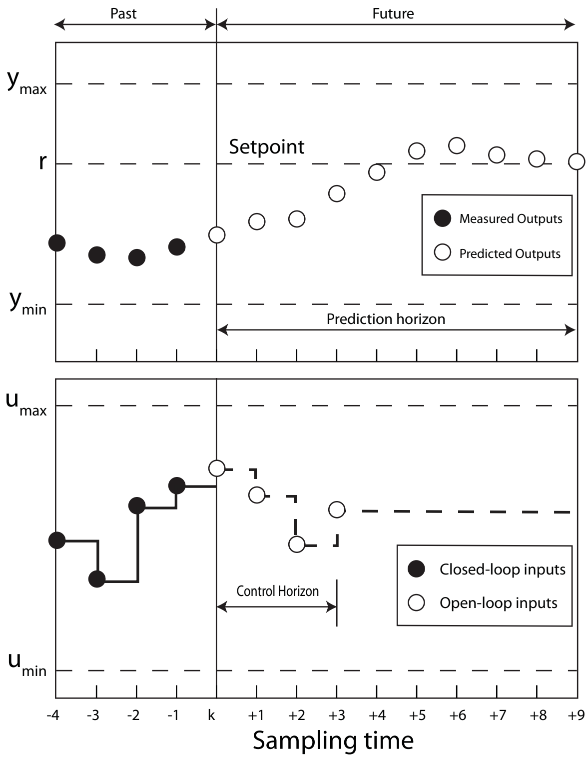


Figure 1.1: A graphical illustration of Model Predictive Control

In general, linear and nonlinear MPC are distinguished. Linear MPC refers to a family of MPC schemes in which linear models are used to predict the system dynamics. Linear MPC approaches have found successful applications, especially in the process industries [Qin & Badgwell 2000, Qin & Badgwell 2003]. By now, linear MPC theory is fairly mature (see [Morari & H. Lee 1999] and the reference therein). However many systems are inherently nonlinear. In this case, the results obtained by using a LMPC are poor in term of performance and often not sufficient in order to cope with the process requirements. Clearly the use of a more accurate model, i.e. a nonlinear model, will provide quite better results. On the other hand, the use of a nonlinear model implies higher difficulties in the calculation of the control law and in the stability analysis of the obtained closed-loop system.

Model predictive control for nonlinear systems (NMPC) has received considerable attention over the past years. Many theoretical and practical issues have been addressed [Allgöwer *et al.* 1999, Mayne *et al.* 2000, Rawlings 2000, De Nicolao *et al.* 2000]. By now there are several predictive control schemes with guaranteed stability for nonlinear systems: nonlinear MPC with zero state terminal equality constraint [Chen & Shaw 1982, Keerthi & Gilbert 1988, Mayne & Michalska 1990], dual-mode nonlinear MPC [Michalska & Mayne 1993], nonlinear MPC with a suitable terminal state penalty and terminal inequality state constraint [Parisini & Zoppoli 1995, Chen & Allgöwer 1998, De Nicolao *et al.* 1998b, Magni *et al.* 2001a], and nonlinear MPC without terminal constraint [Jadbabaie & Hauser 2001, Grimm *et al.* 2005, Limon *et al.* 2006b].

In [Keerthi & Gilbert 1988, Mayne & Michalska 1990] the value function (of a finite horizon optimal control problem) was first employed as a Lyapunov function for establishing stability of model predictive control of time-varying, constrained, nonlinear, discrete-time systems (when a terminal equality constraint is employed). Nowadays the value function is universally employed as a Lyapunov function for stability analysis of MPC.

In practical applications, the assumption that system is identical to the model used for prediction is unrealistic. In fact, model/plant mismatch or unknown disturbances are always present. The introduction of uncertainty in the system description raises the question of robustness, i.e. the maintenance of certain properties such as stability and performance in the presence of uncertainty.

Input-to-State Stability (ISS) is one of the most important tools to study the dependence of state trajectories of nonlinear continuous and discrete time systems on the magnitude of inputs, which can represent control variables or disturbances. The concept of ISS was first introduced in [Sontag 1989] and then further exploited by many authors in view of its equivalent characterization in terms of robust stability, dissipativity and input-output stability, see e.g. [Jiang *et al.* 1994], [Angeli *et al.* 2000], [Jiang & Wang 2001], [Nesić & Laila 2002], [Huang *et al.* 2005]. Now, several variants of ISS equivalent to the original one have been developed and applied in different contexts (see e.g. [Sontag & Wang 1995a], [Sontag & Wang 1996], [Gao & Lin 2000], [Huang *et al.* 2005]).

In MPC, it is well known that asymptotic stability of the controlled system does not suffice to ensure robustness. As studied in [Grimm *et al.* 2004], a nominal stabilizing MPC may exhibit zero-robustness. Then further conditions must be considered.

When the plant is unconstrained, the resulting MPC is known to be robust under certain variations of the gain of inputs [Glad 1987, Geromel & Da Cruz 1987, De Nicolao *et al.* 1996]. Inherent robustness of unconstrained MPC can also be derived from the inverse optimality of the controller [Magni & Sepulchre 1997].

In the case of constrained MPC, in [Scokaert *et al.* 1997], the authors demonstrate that Lipschitz continuity of the control law provides some robustness under decaying uncertainties. In [De Nicolao *et al.* 1998a], similar results are obtained from the continuity of the optimal cost function. Recently [Limon *et al.* 2002b], [Kellett & Teel 2004] and [Grimm *et al.* 2004] have demonstrated that continuity of the optimal cost function plays an important role in the robustness of nominal MPC.

Continuity of the optimal cost is a property difficult to be proved in general. In [Meadows *et al.* 1995] it was shown that MPC could generate discontinuous feedback control law with respect to the state variable x even if the dynamic system is continuous. This is due to the fact that the feedback law comes from the solution of a constrained optimization problem (when constraints, as for example the terminal constraint, are considered). Figure 1.2 shows this fact by recalling the example of [Meadows *et al.* 1995]: the trajectories (depicted with triangles and daggers respectively) obtained starting from two close initial states (depicted inside a circle) are completely different. Note that in the example just the terminal constraint is present. Therefore, MPC control law could be a priori discontinuous and

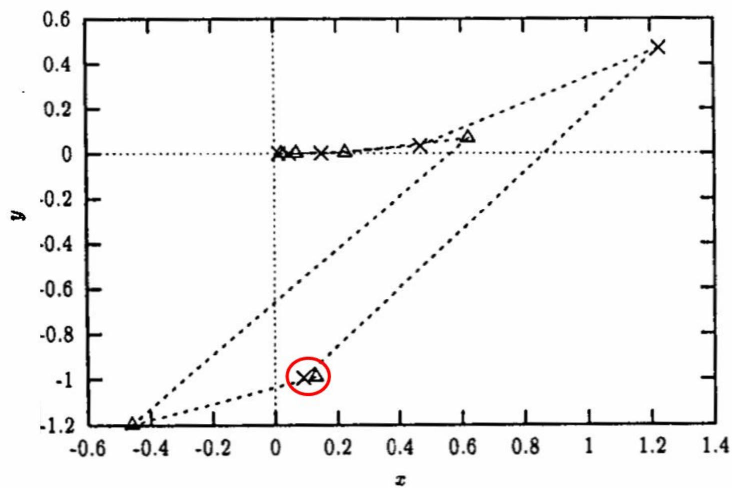


Figure 1.2: Closed-loop state trajectories of example proposed in [Meadows *et al.* 1995]

consequently also the closed-loop system dynamics and the value function. This could be a problem since continuity of the value function is required in most of the literature in order to prove robust stability of MPC (see [Mayne *et al.* 2000]).

When the plant is unconstrained and the terminal constraint is not active [Jadbabaie & Hauser 2001], or when only constraints on the inputs are present [Limon *et al.* 2006b], discontinuity of the control law is avoided. Another relevant case is provided for linear systems with polytopic constraints in [Grimm *et al.* 2004].

When the model function is discontinuous (as for instance the case of hybrid systems) these results can not be used and then, this problem remains open [Lazar 2006].

Robust MPC stems from the consideration of the uncertainties explicitly in the design of the controller. To this aim, firstly, satisfaction of the constrained for any possible uncertainty, namely the robust constraint satisfaction, must be ensured. This further requirement adds complexity to the MPC synthesis. Several results have been proposed concerning robust MPC (see e.g. [Mayne *et al.* 2000, Magni & Scattolini 2007] for a survey, the reference therein, and some of the most recent papers [Grimm *et al.* 2004, Grimm *et al.* 2007, Mhaskar *et al.* 2005, DeHaan & Guay 2007, Limon *et al.* 2005, Raković *et al.* 2006b]).

In particular two different approaches have been followed so far to derive robust MPC algorithms:

- open-loop MPC formulations with restricted constraints, see for example [Limon *et al.* 2002a], [Grimm *et al.* 2003]
- min-max open and closed-loop formulations, see for example [Chen *et al.* 1998, Magni *et al.* 2001b, Gyurkovics 2002, Magni *et al.* 2003, Magni & Scattolini 2005, Limon *et al.* 2006a].

The first method for the design of robust MPC consists in minimizing a nominal performance index while imposing the fulfillment of constraints for each admissible disturbance. This calls for the inclusion in the problem formulation of tighter state and terminal constraints. The idea was introduced in [Chisci *et al.* 2001] for linear systems and applied to nonlinear systems in [Limon *et al.* 2002a] [Raimondo & Magni 2006]. The main drawback of this open-loop strategy is the large spread of trajectories along the optimization horizon due to the effect of the disturbances and leads to very conservative solutions or even to unfeasible problems.

With a significant increase of the computational burden, an alternative approach is the second one, that consists in solving a min-max optimization problem, originally proposed in the context of robust receding horizon control in [Witsenhausen 1968]. Specifically, in an open-loop formulation, the performance index is minimized with respect to the control sequence for the worst case, i.e. the disturbance sequence over the optimization horizon which maximizes the performance index. However, this solution may still be unsatisfactory, since the minimization with respect to a single control profile may produce a very little domain of attraction and a poor performance of the closed-loop systems. As demonstrated in [Chen *et al.* 1997, Bemporad 1998] if some feedback is added by pre-stabilizing the plant, then the conservativeness is reduced. Recently, a closed-loop formulation of the min-max controller has been proposed to reduce this conservativeness [Magni *et al.* 2001b, Scokaert & Mayne 1998, Mayne *et al.* 2000]. In this predictive control technique, a vector of feedback control policies is considered in the minimization of the cost in the worst disturbance case. This allows to take into account the reaction to the effect of the uncertainty in the predictions at expense of a practically untractable optimization problem. In this context robust stability issues have been recently studied and some novel contributions on this topic have appeared in the literature [Scokaert & Mayne 1998, Gyurkovics 2002, Kerrigan & Mayne 2002, Magni *et al.* 2003, Gyurkovics & Takacs 2003, Fontes & Magni 2003, Löfberg 2003, Kerrigan & Maciejowski 2004, Magni & Scattolini 2005, Lazar 2006, Limon *et al.* 2006a, Lazar *et al.* 2008].

The ISS property has been recently introduced in the study of nonlinear perturbed discrete-time systems controlled with MPC (see for example [Goulart *et al.* 2006, Limon *et al.* 2006a, Kim *et al.* 2006, Grimm *et al.* 2007, Lazar *et al.* 2008]), Note that, in order to apply the ISS property to MPC closed-loop systems global results are in general not useful due to the presence of state and input constraints. On the other hand local results, see e.g. [Jiang & Wang 2001, Jiang *et al.* 2004], do not allow the analysis of the properties of the predictive control law in terms of its region of attraction.

Thesis overview

In this thesis, the ISS tool is used in order to study the stability properties of nonlinear perturbed discrete-time systems controlled with MPC. As previously stated, since global and local results do not allow the analysis of the properties of MPC, the notion of regional ISS will be introduced. The equivalence between the ISS property and the existence of a suitable possibly discontinuous ISS-Lyapunov function will be established. As discussed before, since MPC could provide a discontinuous control law with respect to the state x, the value function, used as ISS-Lyapunov function, should be possible discontinuous. Just the continuity of the Lyapunov function at the equilibrium point is required.

The developed tool will be used in order to analyze and synthesize robust MPC controllers. Both open-loop and min-max robust MPC are considered. In particular it will be shown that open-loop MPC algorithms can guarantee ISS of the closed-loop systems while min-max MPC algorithms with standard stage cost can only guarantee Input-to-State practical Stability (ISpS). Different stage cost and dual-mode strategy will be used in order to recover the ISS for closed-loop system. Moreover, since the min-max problem is very computational demanding, a relaxed formulation, in which the max stage is replaced by a simple suitable choice of an uncertain realization, will be presented. Then, in order to efficiently consider state dependent disturbances and improve existent algorithms, a new open-loop robust MPC algorithm will be presented.

Subsequently the problems of decentralized and cooperative NMPC control will be addressed and stability properties will be stated by using the concepts of ISS and small-gain theorem.

Finally, the problem of glucose control in type 1 diabetic patients will be coped with by using linear and nonlinear model predictive controllers.

Thesis structure

Chapter 2: Regional ISS for NMPC controllers. In this chapter, regional input-to-state stability is introduced and studied in order to analyze the domain of attraction of nonlinear constrained systems with disturbances. ISS is derived by means of a possible non smooth ISS-Lyapunov function with an upper bound guaranteed only in a sub-region of the domain of attraction. These results are used to study the ISS properties of nonlinear model predictive control algorithms.

Chapter 2 contains results published in:

- [Magni *et al.* 2006b]: L. Magni, D. M. Raimondo and R. Scattolini. Input-to-State Stability for Nonlinear Model Predictive Control. In Proceedings of 45th IEEE CDC, pages 4836–4841, 2006.
- [Magni *et al.* 2006a]: L. Magni, D. M. Raimondo and R. Scattolini. Regional Input-to-State Stability for Nonlinear Model Predictive Control. IEEE Transactions on Automatic Control, vol. 51, no. 9, pages 1548–1553, 2006.

Chapter 3: Min-Max NMPC: an overview on stability. Min-Max MPC is one of the few techniques suitable for robust stabilization of uncertain nonlinear systems subject to constraints. Stability issues as well as robustness have been recently studied and some novel contributions on this topic have appeared in the literature. In this chapter, a general framework for synthesizing min-max MPC schemes with an a priori robust stability guarantee is distilled. Firstly, a general prediction model that covers a wide class of uncertainties, which includes bounded disturbances as well as state and input dependent disturbances (uncertainties) is introduced. Secondly, the notion of regional Input-to-State Stability (ISS) is extended in order to fit the considered class of uncertainties. Then, it is established that only the standard min-max approach can only guarantee practical stability. Two different solutions for solving this problem are proposed. The first one is based on a particular design of the stage cost of the performance index, which leads to a H_∞ strategy, while the second one is based on a dual-mode strategy. Under fairly mild assumptions both controllers guarantee Input-to-State Stability of the resulting closed-loop system. Moreover, it is shown that the nonlinear auxiliary control law introduced in [Magni *et al.* 2003] to solve the H_∞ problem can be used, for nonlinear systems affine in control, in all the proposed min-max schemes and also in presence of state independent disturbances. A simulation example illustrates the techniques surveyed in this article.

The results presented in Chapter 3 are published in:

- [Raimondo *et al.* 2007b] D. M. Raimondo, D. Limón, M. Lazar, L. Magni and E. F. Camacho. Regional input-to-state stability of min-max model predictive control. In Proceedings of IFAC Symp. on Nonlinear Control Systems, 2007.

- [Raimondo *et al.* 2008b]: D. M. Raimondo, D. Limón, M. Lazar, L. Magni and E. F. Camacho. Min-max model predictive control of nonlinear systems: A unifying overview on stability. To appear in European Journal of Control, 2008.

Chapter 4: Min-Max Nonlinear Model Predictive Control: a min formulation with guaranteed robust stability.

This chapter presents a relaxed formulation of the min-max MPC for constrained nonlinear systems. In the proposed solution, the maximization problem is replaced by the simple evaluation of an appropriate sequence of disturbances. This reduces dramatically the computational burden of the optimization problem and produces a solution that does not differ much from the one obtained with the original min-max problem. Moreover, the proposed predictive control inherits the convergence and the domain of attraction of the standard min-max strategy.

The results of Chapter 4 are presented in:

- [Raimondo *et al.* 2007a]: D. M. Raimondo, T. Alamo, D. Limón and E. F. Camacho. Towards the practical implementation of min-max nonlinear Model Predictive Control. In Proceedings of 46th IEEE CDC, pages 1257–1262, 2007.
- [Raimondo *et al.* 2008a] D. M. Raimondo, T. Alamo, D. Limón and E. F. Camacho. Min-Max Nonlinear Model Predictive Control: a Min formulation with guaranteed robust stability. Submitted to a journal, 2008.

Chapter 5 Robust MPC of Nonlinear Systems with Bounded and State-Dependent Uncertainties.

In this chapter, a robust model predictive control scheme for constrained discrete-time nonlinear systems affected by bounded disturbances and state-dependent uncertainties is presented. Two main ideas are used in order to improve the performance and reduce the conservatism of some existing robust open-loop algorithms. In order to guarantee the robust satisfaction of the state constraints, restricted constraint sets are introduced in the optimization problem, by exploiting the state-dependent nature of the considered class of uncertainties. Moreover, different control and prediction horizons are used. Unlike the nominal model predictive control algorithm, a stabilizing state constraint is imposed at the end of the control horizon in place of the usual terminal constraint posed at the end prediction horizon. The regional input-to-state stability of the closed-loop system is analyzed. A simulation example shows the effectiveness of the proposed approach.

Chapter 5 is based on:

- [Pin *et al.* 2008a]: G. Pin, L. Magni, T. Parisini and D. M. Raimondo. Robust Receding - Horizon Control of Nonlinear Systems with State Dependent Uncertainties: an Input-to-State Stability Approach. In Proceedings of American Control Conference, Seattle, Washington, USA, June 11-13, pages 1667-1672, 2008.
- [Pin *et al.* 2008b] G. Pin, D. M. Raimondo, L. Magni and T. Parisini. Robust Model Predictive Control of Nonlinear Systems with Bounded

and State-Dependent Uncertainties. Provisionally accepted for publication in IEEE Transaction on Automatic Control, 2008.

Chapter 6 Decentralized NMPC: an ISS approach.

This chapter presents stabilizing decentralized model predictive control algorithms for discrete-time nonlinear systems. The overall system under control is composed by a number of subsystems, each one locally controlled with an MPC algorithm guaranteeing the ISS property. Then, the main stability result is derived by considering the effect of interconnections as perturbation terms and by showing that also the overall system is ISS. Both open-loop and closed-loop min-max formulations of robust MPC are considered.

The results in Chapter 6 are appeared in:

- [Raimondo *et al.* 2007c] D. M. Raimondo L. Magni and R. Scattolini. A decentralized MPC algorithm for nonlinear systems. In Proceedings of IFAC Symp. on Nonlinear Control Systems, 2007.

- [Raimondo *et al.* 2007e] D. M. Raimondo L. Magni and R. Scattolini. A decentralized MPC algorithm for nonlinear systems. In Proceedings of IFAC Symp. on Nonlinear Control Systems, 2007.

- [Raimondo *et al.* 2007d]: D. M. Raimondo L. Magni and R. Scattolini. Decentralized MPC of nonlinear systems: An input-to-state stability approach. International Journal of Robust and Nonlinear Control, vol. 17, pages 1651-1667, 2007.

Chapter 7 Cooperative NMPC for Distributed Agents.

This chapter addresses the problem of cooperative control of a team of distributed agents, with decoupled nonlinear discrete-time dynamics, which operate in a common environment and exchange-delayed information between them. Each agent is assumed to evolve in discrete-time, based on locally computed control laws, which are computed by exchanging delayed state information with a subset of neighboring agents. The cooperative control problem is formulated in a receding-horizon framework, where the control laws depend on the local state variables (feedback action) and on delayed information gathered from cooperating neighboring agents (feedforward action). A rigorous stability analysis exploiting the input-to-state stability properties of the receding-horizon local control laws is carried out. The stability of the team of agents is then proved by utilizing small-gain theorem results.

The results in Chapter 7 are published in:

- [Franco *et al.* 2008]: E. Franco, L. Magni, T. Parisini, M. Polycarpou and D. M. Raimondo. Cooperative Constrained Control of Distributed Agents With Nonlinear Dynamics and Delayed Information Exchange: A Stabilizing Receding-Horizon Approach. IEEE Trans. on Automatic Control, vol. 53, no. 1, pages 324–328, 2008.

Chapter 8 MPC of Glycaemia in Type 1 Diabetic Patients.

In this chapter, the feedback control of glucose concentration in type 1 diabetic patients using subcutaneous insulin delivery and subcutaneous continuous glucose monitoring is considered. A recently developed in-silico model of glucose metabolism is employed to generate virtual patients on which control algorithms can be validated against interindividual variability. An in silico trial consisting of 100 patients is used to assess the performances of a linear and a nonlinear state feedback model predictive controller designed on the basis of the in-silico model. The simulation experiments highlight the increased effectiveness of the meal announcement signal with respect to the linear MPC due to a more accurate nonlinear model. Moreover, one of the main advantages of a nonlinear approach is the possibility to use a nonlinear cost function based on the risk index defined in [Kovatchev *et al.* 2005]. The obtained results encourage a deeper investigation along this direction. It is worth to note that currently, experiments using the developed linear model predictive controller are in progress at the Charlottesville and Padova hospitals.

Chapter 8 contains results based on:

- [Dalla Man *et al.* 2007a]: C. Dalla Man, D. M. Raimondo, R. A. Rizza, C. Cobelli. GIM, Simulation Software of Meal Glucose-Insulin Model. Journal of Diabetes Science and Technology, vol. 2, no. 4, pages 323–330, 2007.

- [Magni *et al.* 2007b]: L. Magni, D. M. Raimondo, L. Bossi, C. Dalla Man, G. De Nicolao, B. Kovatchev and C. Cobelli. Model Predictive

Control of Type 1 Diabetes: An in Silico Trial. Journal of Diabetes Science and Technology, vol. 1, no. 6, pages 804–812, 2007.

- [Magni *et al.* 2007a]: L. Magni, G. De Nicolao, D. M. Raimondo. Model Predictive Control Based Method for Closed-Loop Control of Insulin Delivery in Diabetes Using Continuous Glucose Sensing. U.S. Provisional Application Patent. Serial No. 60/984,956. Filed 11/02/2007.

- [Magni *et al.* 2008b]: L. Magni, D. M. Raimondo, C. Dalla Man, G. De Nicolao, B. Kovatchev and C. Cobelli. Model Predictive Control of glucose concentration in subjects with type 1 diabetes: an in silico trial. In Proceedings of 17th IFAC World Congress, Seoul, Korea, July 6-11, pages 4246–4251, 2008.

- [Kovatchev *et al.* 2008]: B. Kovatchev, D. M. Raimondo, M. Breton, S. Patek and C. Cobelli. In Silico Testing and in Vivo Experiments with Closed-Loop Control of Blood Glucose in Diabetes. In Proceedings of 17th IFAC World Congress, Seoul, Korea, July 6-11, pages 4234-4239, 2008.

- [Magni *et al.* 2008a]: L. Magni, D. M. Raimondo, C. Dalla Man, M. Breton, S. Patek, G. de Nicolao, C. Cobelli and B. Kovatchev. Evaluating the efficacy of closed-loop glucose regulation via control-variability grid analysis (CVGA). Journal of Diabetes Science and Technology, vol. 2, no. 4, pages 630–635, 2008.

Each chapter has an appendix that contains the proofs of all lemmas and theorems. Moreover, the notations and basic definitions used in the thesis are are gathered in the Chapter 9, Appendix of the Thesis.

Chapter 2

Regional ISS for NMPC controllers

Contents

2.1 Introduction

Input-to-state stability (ISS) is one of the most important tools to study the dependence of state trajectories of nonlinear continuous and discrete time systems on the magnitude of inputs, which can represent control variables or disturbances. In order to apply the ISS property to MPC, global results are in general not useful in view of the presence of state and input constraints. On the other hand, local results, do not allow to analyze the properties of the predictive control law in terms of its region of attraction. Then, in this chapter, the notion of regional ISS is initially introduced, see also [Nesić & Laila 2002], and the equivalence between the ISS property and the existence of a suitable Lyapunov function is established. Notably, this

Lyapunov function is not required to be smooth nor to be upper bounded in the whole region of attraction. An estimation of the region where the state of the system converges asymptotically is also given.

The achieved results are used to derive the ISS properties of two families of MPC algorithms for nonlinear systems. The first one relies on an open-loop formulation for the nominal system, where state and terminal constraints are modified to improve robustness, see also [Limon *et al.* 2002a]. The second algorithm resorts to the closed-loop min-max formulation already proposed in [Magni *et al.* 2003]. No continuity assumptions are required on the value function or on the resulting MPC control law, which indeed are difficult and not immediately verifiable hypothesis.

2.2 Problem statement

Assume that the plant to be controlled is described by discrete-time nonlinear dynamic model

$$x_{k+1} = \mathcal{F}(x_k, u_k, d_k),\ \ k \geq 0 \tag{2.1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the current control vector, $d_k \in \mathbb{R}^q$ is the disturbance term. Given the system (2.1), let $f(x_k, u_k)$ denote the nominal model, such that

$$x_{k+1} = f(x_k, u_k) + w_k,\ \ k \geq 0 \tag{2.2}$$

where $w_k \triangleq \mathcal{F}(x_k, u_k, d_k) - f(x_k, u_k)$ denote the additive uncertainty.

The system is supposed to fulfill the following assumption.

Assumption 2.1

1. *For simplicity of notation, it is assumed that the origin is an equilibrium point, i.e.* $f(0, 0) = 0$.

2. *The disturbance* w *is such that*

$$w \in \mathcal{W} \tag{2.3}$$

where $\mathcal{W}$ *is a compact set containing the origin, with* $\mathcal{W}^{sup}$ *known.*

3. *The state and the control variables are restricted to fulfill the following constraints*

$$x \in \mathcal{X} \tag{2.4}$$

$$u \in \mathcal{U} \tag{2.5}$$

where $\mathcal{X}$ and $\mathcal{U}$ are compact sets, both containing the origin as an interior point.

4. *The map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz in x in the domain $\mathcal{X} \times \mathcal{U}$, i.e. there exists a positive constant $\mathcal{L}_f$ such that*

$$|f(a, u) - f(b, u)| \leq \mathcal{L}_f |a - b| \tag{2.6}$$

for all $a, b \in \mathcal{X}$ and all $u \in \mathcal{U}$.

5. *The state of the plant x_k can be measured at each sample time.*

$\square$

The control objective consists in designing a control law $u = \kappa(x)$ such that it steers the system to (a neighborhood of) the origin fulfilling the constraints on the input and the state along the system evolution for any possible disturbance and yielding an optimal closed-loop performance according to certain performance index.

In the following section it is presented a suitable framework for the analysis of stability: the regional ISS.

2.3 Regional Input-to-State Stability

Consider a discrete-time autonomous nonlinear dynamic system described by

$$x_{k+1} = F(x_k, w_k), \; k \geq 0, \tag{2.7}$$

where $F : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ is a nonlinear possibly discontinuous function, $x_k \in \mathbb{R}^n$ is the state and $w_k \in \mathbb{R}^q$ is an unknown disturbance. The transient of the system (2.7) with initial state $x_0 = \bar{x}$ and disturbance sequence $\mathbf{w}$ is denoted by $x(k, \bar{x}, \mathbf{w})$. This system is supposed to fulfill the following assumptions.

Assumption 2.2

1. *The origin of the system is an equilibrium point, i.e.* $F(0,0)=0$.

2. *The disturbance* w *is such that*

$$w \in \mathcal{W} \tag{2.8}$$

where $\mathcal{W}$ *is a compact set containing the origin, with* $\mathcal{W}^{sup}$ *known.*

Assumption 2.3 *The solution of (2.7) is continuous at* $\bar{x}=0$ *and* $\mathbf{w}=0$ *with respect to disturbances and initial conditions.* □

Let introduce the following definitions.

Definition 2.1 (UAG in Ξ) *Given a compact set* $\Xi \subset \mathcal{R}^n$ *including the origin as an interior point, the system (2.7) with* $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$ *satisfies the UAG (Uniform Asymptotic Gain) property in* Ξ, *if* Ξ *is a RPI set for system (2.7) and if there exists a* $\mathcal{K}$*-function* γ *such that for each* $\varepsilon > 0$ *and* $\nu > 0$, $\exists T = T(\varepsilon, \nu)$ *such that*

$$|x(k, \bar{x}, \mathbf{w})| \leq \gamma(\|\mathbf{w}\|) + \varepsilon \tag{2.9}$$

for all $\bar{x} \in \Xi$ *with* $|\bar{x}| \leq \nu$, *and all* $k \geq T$.

Definition 2.2 (LS) *The system (2.7) with* $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$ *satisfies the LS (Local Stability) property if, for each* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that*

$$|x(k, \bar{x}, \mathbf{w})| \leq \varepsilon, \ \forall k \geq 0 \tag{2.10}$$

for all $|\bar{x}| \leq \delta$ *and all* $|w_k| \leq \delta$.

Definition 2.3 (ISS in Ξ) *Given a compact set* $\Xi \subset \mathcal{R}^n$ *including the origin as an interior point, the system (2.7) with* $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$, *is said to be ISS (Input-to-State Stable) in* Ξ *if* Ξ *is a RPI set for system (2.7) and if there exist a* $\mathcal{KL}$*-function* β *and a* $\mathcal{K}$*-function* γ *such that*

$$|x(k, \bar{x}, \mathbf{w})| \leq \beta(|\bar{x}|, k) + \gamma(\|\mathbf{w}\|), \ \forall k \geq 0, \ \forall \bar{x} \in \Xi. \tag{2.11}$$

□

Note that, by causality, the same definition of ISS in Ξ would result if one would replace (2.11) by

$$|x(k, \bar{x}, \mathbf{w})| \leq \beta(|\bar{x}|, k) + \gamma(\|\mathbf{w}_{[k-1]}\|), \ \forall k \geq 0, \ \forall \bar{x} \in \Xi. \tag{2.12}$$

Recall that $w_{[k-1]}$ denotes the truncation of w at $k-1$.

In [Sontag & Wang 1995a] and in [Gao & Lin 2000], it was shown that, for continuous-time and discrete-time systems respectively (with $F(\cdot,\cdot)$ continuously differentiable), the ISS property is equivalent to the conjunction of UAG and LS. By examinating the proof of Lemma 2.7 in [Sontag & Wang 1995a] carefully, one can see that it also applies to discontinuous systems (both continuous and discrete-time), if a RPI compact set Ξ is considered. In fact, in the proof, the continuity is necessary just in order to use Proposition 5.1 in [Lin *et al.* 1996], to prove that given $\varepsilon, r, s > 0$ and time $T = T(\varepsilon, s) > 0$ as in Definition 2.1, *there is an* $L > 0$ *such that* $|x(t, \bar{x}, \mathbf{w})| \leq L$ *for all* $0 \leq t \leq T$, *all* $|\bar{x}| \leq s$ *and all* $||\mathbf{w}|| \leq r$. But if Ξ is a RPI compact set, this is always satisfied, since all the possible trajectories are bounded. Hence, the following lemma can be stated.

Lemma 2.1 *Suppose that Assumption 2.2 holds. System (2.7) is ISS in Ξ if and only if UAG in Ξ and LS hold.* □

If also Assumption 2.3 holds, it turns out that LS property is redundant. In fact, the following proposition holds.

Proposition 2.1 *Consider system (2.7). If Assumptions 2.2 and 2.3 hold, UAG in Ξ implies LS.* □

The proof is equal to the one given in [Sontag & Wang 1995a] and [Gao & Lin 2000], for continuous-time and discrete-time case respectively, because the continuity of function $F(\cdot,\cdot)$ is not used.

Lemma 2.2 *Suppose that Assumptions 2.2 and 2.3 hold. System (2.7) is ISS in Ξ if and only if UAG in Ξ holds.* □

Note that, by Definition 2.3, Assumption 2.3 is necessary in order to have ISS. In fact, if the solution of (2.7) is not continuous at $\bar{x} = 0$ and

$\mathbf{w} = 0$ with respect to disturbances and initial conditions, ISS does not hold.

The ISS property is now related to the existence of a suitable a-priori non smooth Lyapunov function defined as follows. In order to clarify the relation between the sets introduced in the definition see Figure 2.1.

Definition 2.4 (ISS-Lyapunov function in Ξ)
A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an ISS-Lyapunov function in Ξ for system (2.7), if

1. *Ξ is a compact RPI set including the origin as an interior point*
2. *there exist a compact set $\Omega \subseteq \Xi$ (including the origin as an interior point), and a pair of suitable $\mathcal{K}_\infty$-functions α_1, α_2 such that*
$$V(x) \geq \alpha_1(|x|), \forall x \in \Xi \tag{2.13}$$
$$V(x) \leq \alpha_2(|x|), \forall x \in \Omega \tag{2.14}$$
3. *there exist a suitable $\mathcal{K}_\infty$-function α_3 and a $\mathcal{K}$-function σ such that*
$$\Delta V(x) \triangleq V(F(x,w)) - V(x) \leq -\alpha_3(|x|) + \sigma(|w|) \tag{2.15}$$
for all $x \in \Xi$, and all $w \in \mathcal{W}$
4. *there exist a suitable $\mathcal{K}_\infty$-function ρ (with ρ such that $(id - \rho)$ is a $\mathcal{K}_\infty$-function) and a suitable constant $c_\theta > 0$, such that, given a disturbance sequence $\mathbf{w} \in \mathcal{M}_\mathcal{W}$, there exists a nonempty compact set $\Theta_\mathbf{w} \subseteq I\Omega \triangleq \{x : x \in \Omega, |x|_{\delta\Omega} > c_\theta\}$ (including the origin as an interior point) defined as follows*
$$\Theta_\mathbf{w} \quad \triangleq \quad \{x : V(x) \leq b(||\mathbf{w}||)\} \tag{2.16}$$
where $b \triangleq \alpha_4^{-1} \circ \rho^{-1} \circ \sigma$, with $\alpha_4 \triangleq \alpha_3 \circ \alpha_2^{-1}$. □

Note that, by (2.13) and (2.14), function V is continuous at the origin.

Remark 2.1 *Note that, in order to verify that $\Theta_\mathbf{w} \subseteq I\Omega$ for all $\mathbf{w} \in \mathcal{M}_\mathcal{W}$, one has to verify that*
$$\Theta \quad \triangleq \quad \{x : V(x) \leq b(\mathcal{W}^{sup})\} \subseteq I\Omega \tag{2.17}$$

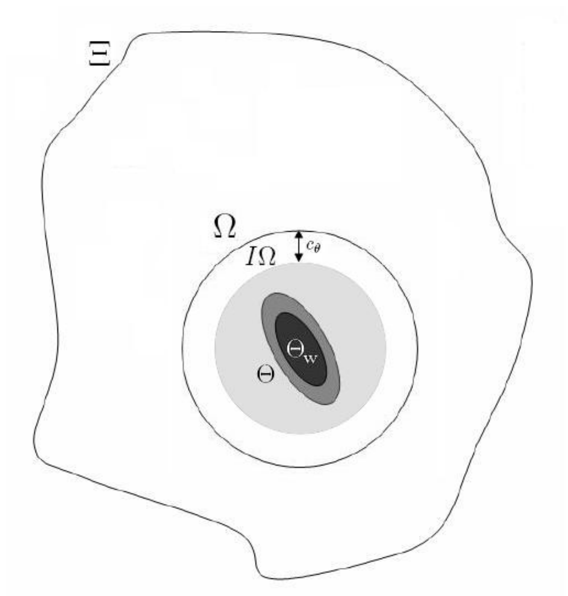


Figure 2.1: Example of sets satisfying Definition 2.4

□

Now, the following sufficient condition for regional ISS of system (2.7) can be stated.

Theorem 2.1 *Suppose that Assumptions 2.2 and 2.3 hold. If system (2.7) admits an ISS-Lyapunov function in* Ξ, *then it is ISS in* Ξ *and, for all disturbance sequences* $\mathbf{w} \in \mathcal{M}_\mathcal{W}$, $\lim_{k\to\infty} |x(k,\bar{x},\mathbf{w})|_{\Theta_\mathbf{w}} = 0$. ■

Remark 2.2 *Theorem 2.1 gives an estimation of the region* $\Theta_\mathbf{w}$ *where the state of the system converges asymptotically. In some cases, as for example in the MPC, the function* V *is not known in explicit form. Hence, in order to verify that point 4 of Definition 2.4 is satisfied, considering also Remark 6.1, one has to verify that*

$$\bar{\Theta} \triangleq \{x : |x| \leq \alpha_1^{-1} \circ b(\mathcal{W}^{sup})\} \subseteq I\Omega \tag{2.18}$$

In fact, in view of (2.13) and Remark 6.1, $\bar{\Theta} \supseteq \Theta$.

This region depends on the bound on w *through* σ, *as well as on* α_1, α_2, α_3 *and* ρ. *Note that, since the size of* $\bar{\Theta}$ *(as well as the size of* Θ*)*

is directly related to α_2*, an accurate estimation of the upper bound* α_2 *is useful in order to compute a smaller region of attraction* Θ*. Moreover, if* $\mathcal{W}^{sup} = 0$*, then* $\sigma(\mathcal{W}^{sup}) = 0$ *so that asymptotic stability is guaranteed (for this particular case see [Lazar et al. 2005] that copes also discontinuous Lyapunov function). In order to clarify the relation between the sets, see Figure 2.2.* □

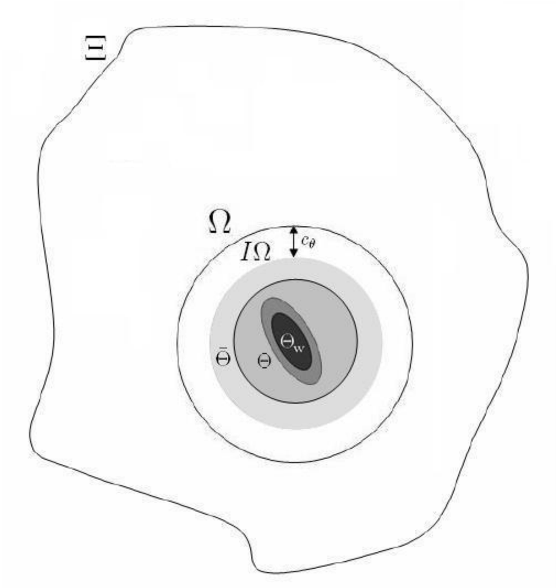


Figure 2.2: Relation between sets

2.4 Nonlinear Model Predictive Control

In this section, the results derived in Theorem 2.1 are used to analyze under which assumptions open-loop and closed-loop min-max formulations of stabilizing MPC for system (2.2) fulfill the ISS property.

In the following, since not necessary, the regularities of the value function or of the resulting control law are not assumed.

2.4.1 Open-loop formulation

In order to introduce the MPC algorithm formulated according to an open-loop strategy, first let $\mathbf{u}_{[t_2,t_3|t_1]} \triangleq [u_{t_2|t_1}\ u_{t_2+1|t_1} \dots u_{t_3|t_1}]$, with $t_1 \leq t_2 \leq t_3$, a control sequence. Moreover, given $k \geq 0$, $j \geq 1$, let $\hat{x}_{k+j|k}$ be the predicted state at $k+j$ obtained with the nominal model $f(x_k, u_k)$, with initial condition x_k and input $\mathbf{u}_{[k,k+j-1|k]}$.

Then, the following finite-horizon optimization problem can be stated.

Definition 2.5 (FHOCP) *Consider system (2.2) with $x_t = \bar{x}$. Given the positive integer N, the stage cost l, the terminal penalty V_f and the terminal set $\mathcal{X}_f$, the Finite Horizon Optimal Control Problem (FHOCP) consists in minimizing, with respect to $\mathbf{u}_{[t,t+N-1|t]}$, the performance index*

$$J(\bar{x}, \mathbf{u}_{[t,t+N-1|t]}, N) \triangleq \sum_{k=t}^{t+N-1} l(\hat{x}_{k|t}, u_{k|t}) + V_f(\hat{x}_{t+N|t})$$

subject to

1. *the nominal state dynamics $\hat{x}_{k+1} = f(\hat{x}_k, u_k)$, with $\hat{x}_t = \bar{x}$*
2. *the constraints (2.4), (2.5), $k \in [t, t+N-1]$*
3. *the terminal state constraints $\hat{x}_{t+N|t} \in \mathcal{X}_f$.* □

The stage cost defines the performance index to optimize and satisfies the following assumption.

Assumption 2.4 *The stage cost $l(x,u)$ is such that $l(0,0) = 0$ and $l(x,u) \geq \alpha_l(|x|)$ where α_l is a $\mathcal{K}_\infty$-function. Moreover, $l(x,u)$ is Lipschitz in x, in the domain $\mathcal{X} \times \mathcal{U}$, i.e. there exists a positive constant $\mathcal{L}_l$ such that*

$$|l(a,u) - l(b,u)| \leq \mathcal{L}_l |a-b|$$

for all $a, b \in \mathcal{X}$ and all $u \in \mathcal{U}$. □

It is now possible to define a "prototype" of a nonlinear MPC algorithm: at every time instants t, given $x_t = \bar{x}$, find the optimal control sequence

$\mathbf{u}^o_{[t,t+N-1|t]}$ by solving the FHOCP. Then, according to the Receding Horizon (RH) strategy, define

$$\kappa^{MPC}(\bar{x}) \triangleq u^o_{t|t}(\bar{x})$$

where $u^o_{t|t}(\bar{x})$ is the first column of $\mathbf{u}^o_{[t,t+N-1|t]}$, and apply the control law

$$u = \kappa^{MPC}(x). \tag{2.19}$$

Recall that, since $\mathbf{u}^o_{[t,t+N-1|t]}$ is the solution of a constrained optimization, (2.19) could be discontinuous w.r.t. x.

Although the FHOCP has been stated for nominal conditions, under suitable assumptions and by choosing accurately the terminal cost function V_f and the terminal constraint $\mathcal{X}_f$, it is possible to guarantee the ISS property of the closed-loop system formed by (2.2) and (2.19), subject to constraints (2.3)-(2.5).

Assumption 2.5 *The solution of closed-loop system formed by (2.2), (2.19) is continuous at $\bar{x} = 0$ and $\mathbf{w} = 0$ with respect to disturbances and initial conditions.*

Assumption 2.6 *The design parameters V_f and $\mathcal{X}_f$ are such that, given an auxiliary control law κ_f,*

1. *$\mathcal{X}_f \subseteq \mathcal{X}$, $\mathcal{X}_f$ closed, $0 \in \mathcal{X}_f$*
2. *$\kappa_f(x) \in \mathcal{U}$, for all $x \in \mathcal{X}_f$*
3. *$f(x, \kappa_f(x)) \in \mathcal{X}_f$, for all $x \in \mathcal{X}_f$*
4. *there exist a pair of suitable $\mathcal{K}_\infty$-functions α_{V_f} and β_{V_f} such that $\alpha_{V_f} < \beta_{V_f}$ and*
$$\alpha_{V_f}(|x|) \leq V_f(x) \leq \beta_{V_f}(|x|)$$
5. *$V_f(f(x, \kappa_f(x))) - V_f(x) \leq -l(x, \kappa_f(x))$, for all $x \in \mathcal{X}_f$*
6. *V_f is Lipschitz in $\mathcal{X}_f$ with a Lipschitz constant $\mathcal{L}_{V_f}$.* □

Assumption 2.6 implies that the closed-loop system formed by the nominal system $f(x_k, \kappa_f(x_k))$ is asymptotically stable in $\mathcal{X}_f$ (V_f is a Lyapunov function in $\mathcal{X}_f$ for the nominal system).

In the following, let $\mathcal{X}^{MPC}(N)$ denote the set of states for which a solution of the FHOCP problem exists.

Assumption 2.7 *Consider the closed-loop system (2.2) and (2.19). For each* $x_t \in \mathcal{X}^{MPC}(N)$, $\tilde{\mathbf{u}}_{[t+1,t+N|t+1]} \triangleq [\mathbf{u}^o_{[t+1,t+N-1|t]}\ \kappa_f(\hat{x}_{t+N|t+1})]$ *is an admissible, possible suboptimal, control sequence for the FHOCP at time* $t+1$*, for all possible* $w \in \mathcal{W}$. $\square$

Note that Assumption 2.7 implies that $\mathcal{X}^{MPC}(N)$ is a RPIA set for the closed-loop system (2.2) and (2.19).

In what follows, the optimal value of the performance index, i.e.

$$V(x) \triangleq J(x, \mathbf{u}^o_{[t,t+N-1|t]}, N) \tag{2.20}$$

is employed as an ISS-Lyapunov function for the closed-loop system formed by (2.2) and (2.19).

Assumption 2.8 *Let*

- $\Xi = \mathcal{X}^{MPC}$
- $\Omega = \mathcal{X}_f$
- $\alpha_1 = \alpha_l$
- $\alpha_2 = \beta_{V_f}$
- $\alpha_3 = \alpha_l$
- $\sigma = \mathcal{L}_J$*, where* $\mathcal{L}_J \triangleq \mathcal{L}_{V_f}\mathcal{L}_f^{N-1} + \mathcal{L}_l \frac{\mathcal{L}_f^{N-1}-1}{\mathcal{L}_f-1}$.

The set $\mathcal{W}$ *is such that the set* Θ *(depending from* $\mathcal{W}^{sup}$*), defined in (2.17), with function* V *given by (2.20), is contained in* $I\Omega$.

Remark 2.3 *Many methods have been proposed in the literature to compute* V_f, $\mathcal{X}_f$ *satisfying Assumption 2.6 (see e.g [Mayne et al. 2000]). On the contrary, with the MPC based on the FHOCP defined above, Assumption 2.7 is not a-priori satisfied. A way to fulfill it is shown in [Limon et al. 2002a] by properly restricting the state constraints 2 and 3 in the formulation of the FHOCP.* $\square$

The main stability result can now be stated.

Theorem 2.2 *Under Assumptions 2.1, 2.4-2.8 the closed-loop system formed by (2.2) and (2.19) subject to constraints (2.3)-(2.5) is ISS with RPIA set $\mathcal{X}^{MPC}(N)$.* ■

2.4.2 Closed-loop formulation

As underlined in Remark 2.3, the robust invariance of the feasible set $\mathcal{X}^{MPC}(N)$ in a standard open-loop MPC formulation can be achieved through a wise choice of the state constraints in the FHOCP. However, this solution can be extremely conservative and can provide a small RPIA set, so that a less stringent approach explicitly accounting for the intrinsic feedback nature of any RH implementation has been proposed, see e.g. [Magni *et al.* 2003, Magni *et al.* 2001b, Chen *et al.* 1998, Gyurkovics 2002, Limon *et al.* 2006a]. In the following, it is shown that the ISS result of the previous section is also useful to derive the ISS property of min-max *MPC*. In this framework, at any time instant the controller chooses the input u as a function of the current state x, so as to guarantee that the effect of the disturbance w is compensated for any choice made by the "nature". Hence, instead of optimizing with respect to a control sequence, at any time t the controller has to choose a vector of feedback control policies $\kappa_{[t,t+N-1]} \triangleq [\kappa_0(x_t)\ \kappa_1(x_{t+1})\ \ldots\ \kappa_{N-1}(x_{t+N-1})]$ minimizing the cost in the worst disturbance case. Then, the following optimal min-max problem can be stated.

Definition 2.6 (FHCLG) *Consider system (2.2) with $x_t = \bar{x}$. Given the positive integer N, the stage cost $l - l_w$, the terminal penalty V_f and the terminal set $\mathcal{X}_f$, the Finite Horizon Closed-Loop Game (FHCLG) problem consists in minimizing, with respect to $\kappa_{[t,t+N-1]}$ and maximizing with respect to $\mathbf{w}_{[t,t+N-1]}$ the cost function*

$$J(\bar{x}, \kappa_{[t,t+N-1]}, \mathbf{w}_{[t,t+N-1]}, N) \triangleq \sum_{k=t}^{t+N-1} \{l(x_k, u_k) - l_w(w_k)\} + V_f(x_{t+N}) \tag{2.21}$$

subject to:

1. *the state dynamics (2.2)*

2. *the constraints (2.3)-(2.5),* $k \in [t, t+N-1]$

3. *the terminal state constraints* $x_{t+N} \in \mathcal{X}_f$. $\square$

Letting $\kappa^o_{[t,t+N-1]}$, $\mathbf{w}^o_{[t,t+N-1]}$ be the solution of the FHCLG, according to the RH paradigm, the feedback control law $u = \kappa^{MPC}(x)$ is obtained by setting

$$\kappa^{MPC}(x) = \kappa^o_0(x) \tag{2.22}$$

where $\kappa^o_0(x)$ is the first element of $\kappa^o_{[t,t+N-1]}$.

Recall that, since $\kappa^o_{[t,t+N-1]}$ is the solution of a constrained optimization, (2.19) could be discontinuous w.r.t. x.

In order to derive the stability and performance properties associated to the solution of FHCLG, the following assumptions are introduced.

Assumption 2.9 *The solution of closed-loop system formed by (2.2), (2.22) is continuous at* $\bar{x} = 0$ *and* $\mathbf{w} = 0$ *with respect to disturbances and initial conditions.*

Assumption 2.10 $l_w(w)$ *is such that* $\alpha_w(|w|) \leq l_w(w) \leq \beta_w(|w|)$, *where* α_w *and* β_w *are* $\mathcal{K}_\infty$*-functions.*

Assumption 2.11 *The design parameters* V_f *and* $\mathcal{X}_f$ *are such that, given an auxiliary law* κ_f,

1. $\mathcal{X}_f \subseteq \mathcal{X}$, $\mathcal{X}_f$ *closed,* $0 \in \mathcal{X}_f$

2. $\kappa_f(x) \in \mathcal{U}$, *for all* $x \in \mathcal{X}_f$

3. $f(x, \kappa_f(x)) + w \in \mathcal{X}_f$, *for all* $x \in \mathcal{X}_f$, *and all* $w \in \mathcal{W}$

4. *there exist a pair of suitable* $\mathcal{K}_\infty$*-functions* α_{V_f} *and* β_{V_f} *such that* $\alpha_{V_f} < \beta_{V_f}$ *and*
$$\alpha_{V_f}(|x|) \leq V_f(x) \leq \beta_{V_f}(|x|)$$
for all $x \in \mathcal{X}_f$

5. $V_f(f(x, \kappa_f(x)) + w) - V_f(x) \leq -l(x, \kappa_f(x)) + l_w(w)$, *for all* $x \in \mathcal{X}_f$, *and all* $w \in \mathcal{W}$.

□

Assumption 2.11 implies that the closed-loop system formed by the system (2.2) and $u = \kappa_f(x_k)$ is ISS in $\mathcal{X}_f$ (V_f is an ISS-Lyapunov function in $\mathcal{X}_f$).

In what follows, the optimal value of the performance index, i.e.

$$V(x) \triangleq J(x, \kappa^o_{[t,t+N-1]}, \mathbf{w}^o_{[t,t+N-1]}, N) \tag{2.23}$$

is employed as an ISS-Lyapunov function for the closed-loop system formed by (2.2) and (2.22).

Assumption 2.12 *Let*

- $\Xi = \mathcal{X}^{MPC}$
- $\Omega = \mathcal{X}_f$
- $\alpha_1 = \alpha_l$
- $\alpha_2 = \beta_{V_f}$
- $\alpha_3 = \alpha_l$
- $\sigma = \beta_w$.

The set $\mathcal{W}$ is such that the set Θ (depending from $\mathcal{W}^{sup}$), defined in (2.17), with function V given by (2.23), is contained in $I\Omega$. □

The main result can now be stated.

Theorem 2.3 *Under Assumptions 2.1, 2.4, 2.9-2.12 the closed-loop system formed by (2.2) and (2.22) subject to constraints (2.3)-(2.5) is ISS with RPIA set $\mathcal{X}^{MPC}(N)$.* ■

Note that, in this case, in order to prove the ISS, $f(x,u)$, l, l_w and V_f are not required to be Lipschitz.

Remark 2.4 *The computation of the auxiliary control law, of the terminal penalty and of the terminal inequality constraint satisfying Assumption 2.11, is not trivial at all. In this regard, a solution for affine system will be discuss in Chapter 3, where it will shown how to compute a nonlinear auxiliary control law based on the solution of a suitable H_∞ problem for the linearized system under control.*

2.5 Remarks

Remark 2.5 *Following the result in [Scokaert et al. 1999] for standard NMPC, assuming that an initial feasible solution of the FHOCP (or of the FHCLG) is available, it is possible to show that it is not necessary to obtain the global optimum solution of the FHOCP (or of the FHCLG) in order to guarantee the ISS of the closed-loop system. In fact, the control sequence $\tilde{\mathbf{u}}_{[t+1,t+N|t+1]} \triangleq [\tilde{\mathbf{u}}_{[t+1,t+N-1|t]}\ \kappa_f(\hat{x}_{t+N|t+1})]$ (or the vector of feedback control policies $\tilde{\kappa}_{[1,N]} \triangleq [\tilde{\kappa}_{[0,N-1]}\ \kappa_f]$), where $\tilde{\mathbf{u}}_{[t+1,t+N-1|t]}$ (or $\tilde{\kappa}_{[0,N-1]}$) is the possible sub-optimal solution obtained at the previous step, is an available feasible solution that guarantees ISS. Indeed this solution is such that the value function satisfies (2.4). The only requirement on the possible sub-optimal solution is to be not worst than $\tilde{\mathbf{u}}_{[t+1,t+N|t+1]}$ (or than $\tilde{\kappa}_{[1,N]}$).* $\square$

Remark 2.6 *The usual way to derive the upper bound for the value function V in $\mathcal{X}^{MPC}(N)$ requires the assumption that the solutions $\mathbf{u}^o_{[t,t+N-1]}$ of the FHOCP and $\kappa^o_{[t,t+N-1]}$ of the FHCLG are Lipschitz in $\mathcal{X}^{MPC}(N)$. On the contrary, Theorem 2.1 gives the possibility to find the upper bound for the ISS-Lyapunov function only in a subset of the RPIA set. This can be derived in $\mathcal{X}_f$, without assuming any regularity of the control strategies, by using the monotonicity properties (2.30) and (2.34) respectively. However, in order to enlarge the set $\mathcal{W}$ that satisfies Assumptions 2.8 and 2.12 for the FHOCP and FHCLG respectively, it could be useful to find an upper bound $\bar{\alpha}_2$ of V in a region $\Omega_1 \supseteq \Omega$. To this regard, define*

$$\bar{\alpha}_2 = \max\left(\frac{\bar{V}}{\alpha_2(r)}, 1\right)\alpha_2$$

where $\bar{V} = max_{x\in\Omega_1}(V(x))$ and r is such that $B_r = \{x \in \mathcal{R}^n : |x| \leq r\} \subseteq \Omega$, as suggested in [Limon et al. 2006a]. This idea can either enlarge or restrict the set $\mathcal{W}$ since $\Omega_1 \supseteq \Omega$ but $\bar{\alpha}_2 \geq \alpha_2$.

Remark 2.7 *Following the results reported in [Magni et al. 2001a] for the open-loop and in [Magni et al. 2003] for the closed-loop min-max MPC formulations, it is easy to show that the robust output admissible sets guaranteed by the NMPC control law include the terminal region* $\mathcal{X}_f$ *used in the optimization problem. Moreover the robust output admissible set guaranteed with a longer optimization horizon includes the one obtained with a shorter horizon, i.e.* $\mathcal{X}^{MPC}(N+1) \supseteq \mathcal{X}^{MPC}(N) \supseteq \mathcal{X}_f$.

2.6 Conclusions

Using a suitable, non necessarily continuous, Lyapunov function, regional Input-to-State Stability for discrete-time nonlinear constrained systems has been established. Moreover an estimation of the region where the state of the system converges asymptotically is given. This result has been used to study the robustness characteristics of Model Predictive Control algorithms derived according to open-loop and closed-loop min-max formulations. No continuity assumptions on the optimal Receding Horizon control law have been required. It is believed that the elements provided here can be used to improve the analysis of existing MPC algorithms as well as to develop new synthesis methods with enhanced robustness properties. The next chapters will be based on the fundamental results here presented.

2.7 Appendix

Proof of Theorem 2.1: the proof will be carried out in three steps.

Step 1: first, it is going to be shown that $\Theta_{\mathbf{w}}$ is a RPI set for system (2.7). To this aim, assume that there exists a finite time t such that $x(t,\bar{x},\mathbf{w}) \in \Theta_{\mathbf{w}}$. Then $V(x(t,\bar{x},\mathbf{w})) \leq b(||\mathbf{w}||)$; this implies $\rho \circ \alpha_4(V(x(t,\bar{x},\mathbf{w})) \leq \sigma(||\mathbf{w}||)$. Using (2.14), (2.15) can be rewritten as

$$\Delta V(x) \leq -\alpha_4(V(x)) + \sigma(|w|),\ \forall x \in \Omega,\ \forall w \in \mathcal{W}$$

where $\alpha_4 = \alpha_3 \circ \alpha_2^{-1}$. Without loss of generality, assume that $(id - \alpha_4)$ is a $\mathcal{K}_\infty$-function, otherwise take a bigger α_2 so that $\alpha_3 < \alpha_2$. Then

$$\begin{aligned} V(x(t+1,\bar{x},\mathbf{w})) &\leq (id-\alpha_4)(V(x(t,\bar{x},\mathbf{w}))) + \sigma(|w|) \\ &\leq (id-\alpha_4)\circ b(||\mathbf{w}||) + \sigma(||\mathbf{w}||) \\ &= -(id-\rho)\circ\alpha_4\circ b(||\mathbf{w}||) + b(||\mathbf{w}||) \\ &\quad -\rho\circ\alpha_4\circ b(||\mathbf{w}||) + \sigma(||\mathbf{w}||). \end{aligned}$$

Considering that $\rho\circ\alpha_4\circ b(||\mathbf{w}||) = \sigma(||\mathbf{w}||)$ and $(id-\rho)$ is a $\mathcal{K}_\infty$-function, one has

$$V(x(t+1,\bar{x},\mathbf{w})) \leq -(id-\rho)\circ\alpha_4\circ b(||\mathbf{w}||) + b(||\mathbf{w}||) \leq b(||\mathbf{w}||).$$

By induction one can show that $V(x(t+j,\bar{x},\mathbf{w})) \leq b(||\mathbf{w}||)$ for all $j\in\mathbf{Z}_{\geq 0}$, that is $x(k,\bar{x},\mathbf{w})\in\Theta_{\mathbf{w}}$ for all $k\geq t$. Hence $\Theta_{\mathbf{w}}$ is a RPI set for system (2.7).

Step 2: now it is shown that, starting from $\Xi\setminus\Theta_{\mathbf{w}}$, the state tends asymptotically to $\Theta_{\mathbf{w}}$. Firstly, since $\alpha_4 = \alpha_3\circ\alpha_2^{-1}$, if $x\in\Omega\setminus\Theta_{\mathbf{w}}$ then

$$\rho\circ\alpha_3\circ\alpha_2^{-1}(V(x(k,\bar{x},\mathbf{w}))) > \sigma(||\mathbf{w}||).$$

By the fact that $\alpha_2^{-1}(V(x(k,\bar{x},\mathbf{w}))) \leq |x(k,\bar{x},\mathbf{w})|$, one has

$$\rho\circ\alpha_3(|x(k,\bar{x},\mathbf{w})|) > \sigma(||\mathbf{w}||).$$

Considering that $(id-\rho)$ is a $\mathcal{K}_\infty$-function

$$id(s) > \rho(s),\ \forall s>0$$

then

$$\alpha_3(x(k,\bar{x},\mathbf{w})) > \rho\circ\alpha_3(|x(k,\bar{x},\mathbf{w})|) > \sigma(||\mathbf{w}||),\ \forall x\in\Omega\setminus\Theta_{\mathbf{w}}$$

which implies

$$-\alpha_3(|x(k,\bar{x},\mathbf{w})|) + \sigma(||\mathbf{w}||) < 0,\ \forall x\in\Omega\setminus\Theta_{\mathbf{w}}. \tag{2.24}$$

Moreover, by definition of $\Theta_{\mathbf{w}}$ (see point 4 of Definition 2.4), there exists $\bar{c}>0$ such that for all $x\in\Xi\setminus\Omega$, there exists $\tilde{x}\in\Omega\setminus\Theta_{\mathbf{w}}$ such that

$\alpha_3(|\tilde{x}|) \leq \alpha_3(|x|) - \bar{c}$. Then from (2.24) it follows that

$$-\alpha_3(|x|) + \bar{c} \leq -\alpha_3(|\tilde{x}|) < -\sigma(||\mathbf{w}||), \ \forall x \in \Xi \setminus \Omega.$$

Then

$$\Delta V(x(k, \bar{x}, \mathbf{w})) \leq -\alpha_3(|x(k, \bar{x}, \mathbf{w})|) + \sigma(||\mathbf{w}||) < -\bar{c}, \ \forall x \in \Xi \setminus \Omega$$

so that, considering that $|x| \leq \alpha_1^{-1}(V(x))$, there exists T_1 such that

$$x(T_1, \bar{x}, \mathbf{w}) \in \Omega.$$

Therefore, starting from Ξ, the state will reach the region Ω in a finite time. If in particular $x(T_1, \bar{x}, \mathbf{w}) \in \Theta_{\mathbf{w}}$, then the previous result states that the region $\Theta_{\mathbf{w}}$ is achieved in a finite time. Since $\Theta_{\mathbf{w}}$ is a RPI set, it is true that $\lim_{k\to\infty} |x(k, \bar{x}, \mathbf{w})|_{\Theta_{\mathbf{w}}} = 0$. Otherwise, if $x(T_1, \bar{x}, \mathbf{w}) \notin \Theta_{\mathbf{w}}$, $\rho \circ \alpha_4(V(x(T_1, \bar{x}, \mathbf{w}))) > \sigma(||\mathbf{w}||)$ and

$$\begin{aligned}
\Delta V(x(T_1, \bar{x}, \mathbf{w})) &\leq -\alpha_4(V(x(T_1, \bar{x}, \mathbf{w}))) + \sigma(||\mathbf{w}||) \qquad (2.25)\\
&= -(id - \rho) \circ \alpha_4(V(x(T_1, \bar{x}, \mathbf{w}))) \\
&\quad -\rho \circ \alpha_4(V(x(T_1, \bar{x}, \mathbf{w}))) + \sigma(||\mathbf{w}||) \\
&\leq -(id - \rho) \circ \alpha_4(V(x(T_1, \bar{x}, \mathbf{w}))) \\
&\leq -(id - \rho) \circ \alpha_4 \circ \alpha_1(|x(T_1, \bar{x}, \mathbf{w})|)
\end{aligned}$$

where the last step is obtained using (2.13). Then, $\forall \varepsilon' > 0$, $\exists T_2(\varepsilon') \geq T_1$ such that

$$V(x(T_2, \bar{x}, \mathbf{w})) \leq \varepsilon' + b(||\mathbf{w}||). \qquad (2.26)$$

Therefore, starting from Ξ, the state will arrive close to $\Theta_{\mathbf{w}}$ in a finite time and in $\Theta_{\mathbf{w}}$ asymptotically. Hence $\lim_{k\to\infty} |x(k, \bar{x}, \mathbf{w})|_{\Theta_{\mathbf{w}}} = 0$.
Note that if $||\mathbf{w}|| \neq 0$, there exists a $\tilde{c} > 0$ such that $\Theta_{\mathbf{w}}^{sup} = \tilde{c}$. Then, for $x(T_1, \bar{x}, \mathbf{w}) \notin \Theta_{\mathbf{w}}$, there is $|x(T_1, \bar{x}, \mathbf{w})| > \tilde{c}$. Hence, there exists a $c_2 > 0$ such that, for $x(T_1, \bar{x}, \mathbf{w}) \notin \Theta$, $\Delta V(x(T_1, \bar{x}, \mathbf{w})) < -c_2$, that means, the state arrives in $\Theta_{\mathbf{w}}$ in a finite time.

Step 3: finally it is shown that system (2.7) is regional ISS in Ξ. By (2.26) and (2.13) one has

$$|x(T_2, \bar{x}, \mathbf{w})| \leq \alpha_1^{-1}(\varepsilon' + b(||\mathbf{w}||)).$$

Noting that, given a $\mathcal{K}_\infty$-function θ_1, $\theta_1(s_1+s_2) \leq \theta_1(2s_1)+\theta_1(2s_2)$, see [Limon *et al.* 2006a], it follows that

$$|x(T_2,\bar{x},\mathbf{w})| \leq \alpha_1^{-1}(2\varepsilon') + \alpha_1^{-1}(2b(||\mathbf{w}||)).$$

Now, letting $\varepsilon = \alpha_1^{-1}(2\varepsilon')$ and $\gamma(||\mathbf{w}||) = \alpha_1^{-1}(2b(||\mathbf{w}||))$ the UAG property in Ξ is proven. In view of Assumptions 2.2 and 2.3 and Lemma 2.2, the system is ISS in Ξ with respect to w. ∎

Proof of Theorem 2.2: by Theorem 2.1, if system admits an ISS-Lyapunov function in $\mathcal{X}^{MPC}(N)$, then it is ISS in $\mathcal{X}^{MPC}(N)$.
In the following, it will be shown that the function $V(x)$ defined in (2.20) is an ISS-Lyapunov function in $\mathcal{X}^{MPC}(N)$.

First, the lower bound is easily obtained using Assumption 2.4 and considering that

$$V(x,N) \geq l(x,\kappa^{MPC}(x)) \geq \alpha_l(|x|),\ \forall x \in X^{MPC}(N). \tag{2.27}$$

Moreover, in view of Assumption 2.6

$$\bar{\mathbf{u}}_{[t,t+N|t]} \triangleq [\mathbf{u}^o_{[t,t+N-1|t]}\ \ \kappa_f(\hat{x}_{t+N|t})]$$

is an admissible, possible suboptimal, control sequence for the FHOCP with horizon $N+1$ at time t with cost

$$\begin{aligned} J(x,\bar{\mathbf{u}}_{[t,t+N|t]},N+1) &= V(x,N) + V_f(f(\hat{x}_{t+N|t},\kappa_f(\hat{x}_{t+N|t}))) \\ &\quad -V_f(\hat{x}_{t+N|t}) + l(\hat{x}_{t+N|t},\kappa_f(\hat{x}_{t+N|t})). \end{aligned}$$

Using point 5 of Assumption 2.6, it follows that

$$J(x,\bar{\mathbf{u}}_{[t,t+N|t]},N+1) \leq V(x,N).$$

Since $\bar{\mathbf{u}}_{[t,t+N|t]}$ is a suboptimal sequence

$$V(x,N+1) \leq J(x,\bar{\mathbf{u}}_{[t,t+N|t]},N+1). \tag{2.28}$$

Then

$$V(x,N+1) \leq V(x,N),\ \forall x \in \mathcal{X}^{MPC}(N) \tag{2.29}$$

with $V(x,0) = V_f(x)$, $\forall x \in \mathcal{X}_f$. Therefore, using point 4 of Assumption

2.6, the upper bound is obtained

$$V(x,N) \le V(x,N-1) \le V(x,0) = V_f(x) \le \beta_{V_f}(|x|)\,,\ \forall x \in \mathcal{X}_f. \quad (2.30)$$

Now, let $\tilde{\mathbf{u}}_{[t+1,t+N|t+1]}$ be as defined in Assumption 2.7. Denote $\tilde{x}_{k|t+1}$ the state obtained at time k applying $\tilde{\mathbf{u}}_{[t+1,t+N|t+1]}$ to the nominal model, starting from the real state x_{t+1} at time $t+1$. Define ΔJ as

$$\begin{aligned}
\Delta J \;&\triangleq\; J(x_{t+1}, \tilde{\mathbf{u}}_{[t+1,t+N|t+1]}, N) - J(x(t), \bar{\mathbf{u}}_{[t,t+N|t]}, N+1)\\
&= -l(x_t, u^o_{t|t}) + \sum_{k=t+1}^{t+N-1} \{ l(\tilde{x}_{k|t+1}, \tilde{u}_{k|t+1}) - l(\hat{x}_{k|t}, \bar{u}_{k|t}) \}\\
&\quad + l(\tilde{x}_{t+N|t+1}, \kappa_f(\tilde{x}_{t+N|t+1})) - l(\hat{x}_{t+N|t}, \kappa_f(\hat{x}_{t+N|t}))\\
&\quad + V_f(f(\tilde{x}_{t+N|t+1}, \kappa_f(\tilde{x}_{t+N|t+1}))) - V_f(f(\hat{x}_{t+N|t}, \kappa_f(\hat{x}_{t+N|t}))).
\end{aligned}$$

From the definition of $\tilde{\mathbf{u}}$, $\tilde{u}_{k|t+1} = \bar{u}_{k|t}$, for $k \in [t+1, t+N-1]$, and hence, by Assumptions 2.1 and 2.4

$$|l(\tilde{x}_{k|t+1}, \tilde{u}_{k|t+1}) - l(\hat{x}_{k|t}, u^o_{k|t})| \le \mathcal{L}_l \mathcal{L}_f^{k-t-1} |w_t|.$$

Then, by applying also point 5 of Assumption 2.6, at both $\tilde{x}$ and $\hat{x}$

$$\begin{aligned}
\Delta J \;&\le\; -l(x_t, u^o_{t|t}) + \sum_{k=t+1}^{t+N-1} \mathcal{L}_l \mathcal{L}_f^{k-t-1} |w_t|\\
&\quad + V_f(\tilde{x}_{t+N|t+1}) - V_f(\hat{x}_{t+N|t})
\end{aligned}$$

Moreover, by using point 6 of Assumptions 2.1 and 2.6

$$|V_f(\tilde{x}_{t+N|t+1}) - V_f(\hat{x}_{t+N|t})| \;\le\; \mathcal{L}_v \mathcal{L}_f^{N-1} |w_t|.$$

Substituting these expressions in ΔJ, and by Assumption 2.4, there is

$$\Delta J \le -l(x_t, u^o_{t|t}) + \mathcal{L}_J |w_t| \le -\alpha_l(|x_t|) + \mathcal{L}_J |w_t|$$

where $\mathcal{L}_J \triangleq \mathcal{L}_v \mathcal{L}_f^{N-1} + \mathcal{L}_l \frac{\mathcal{L}_f^{N-1} - 1}{\mathcal{L}_f - 1}$. Finally, by using (2.28) and (2.29) with sequence $\bar{u}$ and (2.29) with sequence $\tilde{u}$, the bound on the decrease of V is obtained

$$V(x_{t+1}, N) - V(x_t, N) \le \Delta J \le -\alpha_l(|x_t|) + \mathcal{L}_J |w_t| \quad (2.31)$$

for all $x \in \mathcal{X}^{MPC}(N)$ and all $w \in \mathcal{W}$.

Therefore, by (2.27), (2.30), (2.31) the optimal cost $J(x, \mathbf{u}^o_{[t,t+N-1|t]}, N)$ is an ISS-Lyapunov function for the closed-loop system in $\mathcal{X}^{MPC}(N)$ and hence, considering also Assumption 2.5, the closed-loop system (2.2), (2.19) is ISS with RPIA set $\mathcal{X}^{MPC}(N)$. ■

Proof of Theorem 2.3: by Theorem 2.1, if system admits an ISS-Lyapunov function in $\mathcal{X}^{MPC}(N)$, then it is ISS in $\mathcal{X}^{MPC}(N)$. In the following it will be shown that the function $V(x)$ defined in (2.23) is an ISS-Lyapunov function for the closed-loop system (2.2), (2.22) in $\mathcal{X}^{MPC}(N)$.

First, the robust invariance of $\mathcal{X}^{MPC}(N)$ is easily derived from Assumption 2.11 by taking $\bar{\kappa}_{[t+1,t+N]} \triangleq [\kappa^o_{[t+1,t+N-1|t]} \;\; \kappa_f(x_{t+N})]$ as admissible policy vector at time $t+1$ starting from the optimal sequence $\kappa^o_{[t,t+N-1]}$ at time t. Then, the lower bound is easily obtained

$$\begin{aligned} V(x,N) &= J(x, \kappa^o_{[t,t+N-1]}, \mathbf{w}^o_{[t,t+N-1]}, N) \\ &\geq \min_{\kappa_{[t,t+N-1]}} J(\bar{x}, \kappa_{[t,t+N-1]}, 0, N) \\ &\geq l(x, \kappa_0(x)) \\ &\geq \alpha_l(|x|) \end{aligned} \tag{2.32}$$

for all $x \in \mathcal{X}^{MPC}(N)$.
In order to derive the upper bound, consider the following policy vector $\tilde{\kappa}_{[t,t+N]} \triangleq [\kappa^o_{[t,t+N-1]} \;\; \kappa_f(x_{t+N})]$ as admissible policy vector for the FHCLG at time t with horizon $N+1$. Then

$$\begin{aligned} J(x, \tilde{\kappa}_{[t,t+N]}, \mathbf{w}_{[t,t+N]}, N+1) \;&= \sum_{k=t}^{t+N-1} \{l(x_k, u_k) - l_w(w_k)\} \\ &\quad + V_f(x_{t+N}) - V_f(x_{t+N}) + V_f(x_{t+N+1}) \\ &\quad + l(x_{t+N}, u_{t+N}) - l_w(w_{t+N}). \end{aligned}$$

In view of Assumption 2.11

$$J(x, \tilde{\kappa}_{[t,t+N]}, \mathbf{w}_{[t,t+N]}, N+1) \leq \sum_{k=t}^{t+N-1} \{l(x_k, u_k) - l_w(w_k)\} + V_f(x_{t+N})$$

which implies

$$\begin{aligned}
V(x,N+1) &\leq \max_{\mathbf{w}\in\mathcal{M}_{\mathcal{W}}} J(x,\tilde{\kappa}_{[t,t+N]},\mathbf{w}_{[t,t+N]},N+1) \\
&\leq \max_{\mathbf{w}\in\mathcal{M}_{\mathcal{W}}} \sum_{k=t}^{t+N-1}\{l(x_k,u_k)-l_w(w_k)\}+V_f(x_{t+N}) \\
&= V(x,N)
\end{aligned} \tag{2.33}$$

which holds for all $x\in\mathcal{X}^{MPC}(N)$ and all $\mathbf{w}\in\mathcal{M}_{\mathcal{W}}$.

Therefore, using Assumption 2.11, the upper bound is obtained

$$V(x,N)\leq V(x,N-1)\leq\ldots\leq V(x,0)=V_f(x)\leq\beta_{V_f}(|x|) \tag{2.34}$$

for all $x\in\mathcal{X}_f$.
From the monotonicity property (2.33) and by sub-optimality there is

$$\begin{aligned}
&V(f(x,\kappa^{MPC}(x))+w,N)-V(x,N)\\
&\quad\leq V(f(x,\kappa^{MPC}(x))+w,N-1)-V(x,N)\\
&\quad\leq J(f(x,\kappa^{MPC}(x))+w,\kappa^{o}_{[t+1,t+N-1|t]},\mathbf{w}^{o}_{[t+1,t+N-1|t+1]},N-1)\\
&\qquad-J(x,\kappa^{o}_{[t,t+N-1|t]},[w\ \mathbf{w}^{o}_{[t+1,t+N-1|t+1]}],N)\\
&\quad\leq -l(x,\kappa^{MPC}(x))+l_w(w)
\end{aligned}$$

so that, by applying Assumptions 2.4 and 2.10, the bound on the decrease of V is obtained

$$V(f(x,\kappa^{MPC}(x))+w,N)-V(x,N)\leq-\alpha_l(|x|)+\beta_w(|w|) \tag{2.35}$$

for all $x\in\mathcal{X}^{MPC}(N)$, and all $w\in\mathcal{W}$. Therefore, by (2.32), (2.34), (2.35), the optimal cost $V(x)$ is an ISS-Lyapunov function for the closed-loop system in $\mathcal{X}^{MPC}(N)$ and hence, considering also Assumption 2.9, the closed-loop system (2.2), (2.22) is ISS with RPIA set $\mathcal{X}^{MPC}(N)$. ■

CHAPTER 3

Min-Max NMPC: an overview on stability

Contents

3.1 Introduction

Min-Max model predictive control (MPC) is one of the few techniques suitable for robust stabilization of uncertain nonlinear systems subject to constraints. Stability issues as well as robustness have been recently studied and some novel contributions on this topic have appeared in the literature [Scokaert & Mayne 1998, Gyurkovics 2002, Kerrigan & Mayne 2002, Magni *et al.* 2003, Gyurkovics & Takacs 2003, Fontes & Magni 2003, Magni & Scattolini 2005, Lazar 2006, Lazar *et al.* 2008, Limon *et al.* 2006a, Magni *et al.* 2006a, Löfberg 2003, Kerrigan & Maciejowski 2004]. In this chapter, a general framework for synthesizing min-max MPC schemes with an a priori robust stability guarantee is distilled starting from an extensive literature. To this aim, a general prediction model that covers a wide class of uncertainty modeling that includes bounded disturbances as well as state (and input) dependent disturbances (uncertainties) is introduced. This requires that the regional Input-to-State Stability (ISS) and Input-to-State practical Stability (ISpS) results are extended in order to cover both state dependent and state independent uncertainties. Moreover, Lyapunov-type sufficient conditions for the regional ISS and ISpS are presented for the considered class of systems; this constitutes the base of the stability analysis of the min-max MPC for the generalized prediction model.

Using the ISS tool, it is proven that if the auxiliary control law ensures ISS of the system, with respect to the state independent part of the disturbance (i.e. the state dependent part of the disturbance remains in a certain stability margin) then the min-max MPC ensures that the closed loop system is ISpS maintaining the same stability margin. The practical nature of the stability is a consequence of the worst-case approach of the control action and causes the system to be ultimately bounded even if the real disturbances vanish.

In order to avoid this problem, two different possible solutions are considered: the first one is based on an $\mathcal{H}_\infty$ like cost of the performance index (see e.g. [Magni *et al.* 2001b, Magni *et al.* 2003, Magni *et al.* 2006a, Gyurkovics & Takacs 2003]), while the second one is based on a dual-mode strategy [Michalska & Mayne 1993, Chisci *et al.* 1996, Scokaert *et al.* 1999, Scokaert & Mayne 1998, Kerrigan & Mayne 2002, Lazar *et al.* 2008]. These solutions are extended to a general prediction model and it is shown that under fairly mild assumptions both controllers guarantee Input-to-State Stability.

Moreover, a nonlinear auxiliary control law, based on the one presented in [Magni *et al.* 2003], is proposed for the case of nonlinear systems affine in control (which are very usual). It is shown that a nonlinear auxiliary control law and the terminal penalty can be derived from the solution of the discrete-time $\mathcal{H}_\infty$ algebraic Riccati equation for the linearized system.

3.2 Problem statement

Assume that the plant to be controlled is described by discrete-time nonlinear model

$$x_{k+1} = f(x_k, u_k, d_{1_k}, d_{2_k}), \ k \geq 0 \tag{3.1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$ is a nonlinear, possibly discontinuous function, $x_k \in \mathbb{R}^n$ is the system state, $u_k \in \mathbb{R}^m$ is the current control vector and d_{1_k} and d_{2_k} are disturbances which model the uncertainties present in the model. This partition on the disturbance signals stems from its nature: $d_{1_k} \in \mathbb{R}^p$ models a class of uncertainty which depends on the state and the control input while $d_{2_k} \in \mathbb{R}^q$ models a class of uncertainty that does not depend neither on the state nor on the input signal. The transient of the system (3.1) with initial state $x_0 = \bar{x}$ and disturbance sequences $\mathbf{d}_1$ and $\mathbf{d}_2$ is denoted by $x(k, \bar{x}, \mathbf{d}_1, \mathbf{d}_2)$.

Most of the models of nonlinear systems considers the uncertainty as bounded disturbances, that is, the only knowledge of the model mismatches is a bounded set where the error lies in. However, this representation may lead to conservative results when, as usually occurs, there exists a relationship between the model mismatch bounds and the state and input of the plant. In this case, this conservativeness would be reduced if this information were considered in the model of the plant by means of the proposed partition of the disturbance model.

In the following assumption, the considered structure of such models is formally presented.

Assumption 3.1

1. *The system has an equilibrium point at the origin, that is* $f(0,0,0,0) = 0$.
2. *The uncertainty* d_1 *is such that*

$$d_1 = d_{1\eta}\eta(|(x,u)|) \tag{3.2}$$

where η *is a known* $\mathcal{K}$*-function and* $d_{1\eta} \in \mathbb{R}^p$ *is modeled as confined in a compact set*

$$\mathcal{D}_{1\eta} \subset \mathbb{R}^p \tag{3.3}$$

not necessarily including the origin with $\mathcal{D}_{1\eta}^{inf}$ *and* $\mathcal{D}_{1\eta}^{sup}$ *known.*

3. *The uncertainty* d_2 *is such that*

$$d_2 \in \mathcal{D}_2 \tag{3.4}$$

where $\mathcal{D}_2 \subset \mathrm{IR}^q$ *is a compact set containing the origin with* $\mathcal{D}_2^{sup}$ *known.*

4. *The state and control of the plant must fulfill the following constraints on the state and the input:*

$$x \in \mathcal{X} \tag{3.5}$$

$$u \in \mathcal{U} \tag{3.6}$$

where $\mathcal{X}$ *and* $\mathcal{U}$ *are compact sets, both of them containing the origin as an interior point.*

5. *The state of the plant* x_k *can be measured at each sample time.*

□

The control objective consists in designing a control law $u = \kappa(x)$ such that it steers the system to (a neighborhood of) the origin fulfilling the constraints on the input and the state along the system evolution for any possible disturbance and yielding an optimal closed-loop performance according to certain performance index.

This control problem is well studied in the literature and there exist a number of control techniques that could be used. However, among the existing solutions, one of the most successfully used control technique is the model predictive control in its min-max approach. This is due to its optimal formulation, its capability to ensure the robust constraint satisfaction and its stabilizing design [Mayne *et al.* 2000].

In the min-max MPC controllers, the control law is based on the solution of a finite horizon game, where u is the input of the minimizing player (the controller), and $d_{1\eta}, d_2$ are the maximizing player (the 'nature'). More precisely, the controller chooses the input u_k as a function of the current state x_k so as to ensure constraint satisfaction along the predicted trajectory of the plant for any possible uncertainty, minimizing at the same time the worst case performance index of the predicted evolution of the system.

In the *open-loop min-max MPC* strategy, the performance index is minimized with respect to a sequence of control action, a sequence which belongs to a finite-dimensional space. It is well known that it is not convenient to consider open-loop control strategies since open-loop control would not account for changes in the state due to unpredictable inputs played by 'nature' (see also [Scokaert & Mayne 1998]).

If a control law is considered as decision variable in the optimization problem (instead of a control action), the solution results to be less conservative since the predicted controlled system reacts to the effect of the disturbance. The predictive controllers derived from this approach are called *closed-loop min-max MPC controllers* and can provide larger domain of attraction and a better closed-loop performance index. However, the optimization problem may be difficult (or even impossible) to be solved even for linear prediction models.

A practical solution, located between the open-loop and the closed-loop approach, is the so-called *semi-feedback* formulation of the problem. In this case, control policies are considered as decision variables, but forcing a given structure of the control law. Thus, the decision variable of each control law is its set of defining parameters, yielding to an optimization problem similar to the open-loop case one.

In this chapter, the considered control law is derived from a closed-loop min-max MPC formulation $\kappa(x) = \kappa^{MPC}(x)$. Although, from a practical point of view, the control law is difficult to calculate, from a theoretical point of view makes sense since the closed-loop approach includes the open-loop and semi-feedback controllers, and these can be considered as particular cases. Thus, the stability results derived in the following for closed-loop min-max MPC will be valid for the rest of formulations. It is worth remarking that this control law might be a discontinuous function of the state.

The resulting closed-loop system is given by

$$x_{k+1} = f(x_k, \kappa^{MPC}(x_k), d_{1_k}, d_{2_k}),\ k \geq 0$$

where the disturbance d_{1_k} is such that $d_{1_k} = d_{1\eta_k}\eta(|(x_k, \kappa^{MPC}(x_k))|)$ with $d_{1\eta} \in \mathcal{D}_{1\eta}$, and the disturbance d_{2_k} is such that $d_{2_k} \in \mathcal{D}_2$. The control law should ensure that if the initial state is in a certain RPIA set, i.e. $x_t \in \mathcal{X}^{MPC}$, then the resulting evolution of the system fulfills the constraints, that is $x_k \in \mathcal{X}$ and $\kappa^{MPC}(x_k) \in \mathcal{U}$ for all $k \geq t$, for any possible evolution of the disturbance signals.

In the following section it is presented a suitable framework for the analysis of stability of such class of closed loop systems: the regional ISpS.

3.3 Regional Input-to-State practical Stability

In this section the ISpS framework for discrete-time autonomous nonlinear systems is presented and Lyapunov-like sufficient conditions are provided. This will be employed in this chapter to study the behavior of perturbed nonlinear systems in closed-loop with min-max MPC controllers.

Differently from section 2.3 in Chapter 2, this tool copes also with systems that, even in absence of disturbances, does not converge to the origin but just to a neighborhood of it. As it will be explained, this is for example the case of closed-loop systems controlled with a standard min-max MPC law.

Consider a nonlinear discrete-time system described by

$$x_{k+1} = F(x_k, d_{1_k}, d_{2_k}),\ k \geq 0 \tag{3.7}$$

where $F : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$ is a nonlinear possibly discontinuous function, $x_k \in \mathbb{R}^n$ is the state, $d_{1_k} \in \mathbb{R}^p$ is the component of the uncertainty depending from the state and $d_{2_k} \in \mathcal{R}^q$ is the other component of the uncertainty. The transient of the system (3.7) with initial state $x_0 = \bar{x}$ and uncertainties d_1 and d_2 is denoted by $x(k, \bar{x}, \mathbf{d}_1, \mathbf{d}_2)$. This system is supposed to fulfill the following assumption.

Assumption 3.2

1. *The compact set $\mathcal{A} \subset \mathbb{R}^n$, containing the origin, is a zero-invariant set for the system (3.7), that is, a positively invariant set for the associated "undisturbed" system $x_{k+1} = F(x_k, 0, 0)$, that means*

$$F(x, 0, 0) \in \mathcal{A},\ \forall x \in \mathcal{A}.$$

2. *The uncertainty d_1 is such that*

$$d_1 \in \mathcal{D}_1(x) \subseteq \mathbb{R}^p$$

where, for each x, $\mathcal{D}_1(x)$ is closed and contains the origin. Moreover *there exist a $\mathcal{K}$-function η and a signal $d_{1\eta} \in \mathbb{R}^p$, limited in a compact set $\mathcal{D}_{1\eta} \subset \mathbb{R}^p$ (not necessarily including the origin as an interior point) such that*

$$d_1 = d_{1\eta}\eta(|x|)$$

for all $x \in \Xi$, where $\Xi \subset \mathbb{R}^n$ is a compact set containing the origin as an interior point.

3. *The uncertainty d_2 is such that*

$$d_2 \in \mathcal{D}_2 \subset \mathbb{R}^q$$

where $\mathcal{D}_2$ is a compact set containing the origin. □

A regional version of Input-to-State practical Stability (ISpS) [Sontag & Wang 1996, Jiang & Wang 2001] is defined in the following.

Definition 3.1 (ISpS in Ξ) *Given a compact set $\Xi \subset \mathbb{R}^n$, including the origin as an interior point, the system (3.7) with $\mathbf{d}_1 \in \mathcal{M}_{\mathcal{D}_1}$ and $\mathbf{d}_2 \in \mathcal{M}_{\mathcal{D}_2}$ is said to be ISpS (Input-to-State practical Stable) in Ξ with respect to d_2 if Ξ is a RPI set for (3.7) and if there exist a $\mathcal{KL}$-function β, a $\mathcal{K}$-function γ_2 and a constant $c \geq 0$ such that*

$$|x(k, \bar{x}, \mathbf{d}_1, \mathbf{d}_2)| \quad \leq \quad \beta(|\bar{x}|, k) + \gamma_2(||\mathbf{d}_2||) + c \tag{3.8}$$

for all $\bar{x} \in \Xi$ and $k \geq 0$. □

Note that, whenever (3.8) is satisfied with $c = 0$, and the origin is an equilibrium point for "undisturbed" system, i.e. $F(0, 0, 0) = 0$, the system (3.7) is said to be ISS (Input-to-State Stable) in Ξ with respect to d_2 (see Definition 2.3).

In many applications it is of interest to study the stability with respect to an invariant set $\mathcal{A}$, where $\mathcal{A}$ in general does not consist of a single point. The "set" version of the ISS property (also known as "compact-ISS") was originally proposed in [Sontag & Lin 1992]. A regional version of the global ISS with respect to $\mathcal{A}$, presented in [Sontag & Wang 1995a, Gao & Lin 2000], is given in the Appendix of the Thesis. In Proposition VI.3 of [Sontag & Wang 1996] and in Definition 2.4 of [Sontag & Wang 1995b], it is shown, for the continuous-time case, that concept of ISpS is equivalent to concept of ISS with respect to $\mathcal{A}$. In this case, the constant c of equation (3.8) is $\mathcal{A}^{sup}$. The same relation holds for discrete-time case. As for the ISpS property, the concepts of LpS and UpAG are equivalent to the concepts of LS and UAG with respect to $\mathcal{A}$ (see the Appendix of the Thesis for the definitions of the properties). Moreover, as discussed in Chapter 2, by examinating the proof of Lemma 2.7 in [Sontag & Wang 1995a] carefully, one can see that the equivalence between ISS in Ξ and the conjunction of UAG and LS (and consequently the equivalence between ISpS in Ξ and the conjunction of UpAG in Ξ and LpS) also applies to discontinuous systems (both continuous and discrete-time), if a RPI compact set Ξ is considered.

Summarizing, on the base of the results in [Sontag & Wang 1995a, Sontag & Wang 1995b, Sontag & Wang 1996, Gao & Lin 2000], it is possible to give the following equivalence result.

Theorem 3.1 *Let $\Xi \subset \mathbb{R}^n$ be a compact set. Consider system (3.7). Suppose that Assumption 3.2 is satisfied. The following properties are equivalent*

- *a) ISpS in Ξ*
- *b) ISS in Ξ with respect to $\mathcal{A}$*
- *c) UpAG in Ξ and LpS*
- *d) UAG in Ξ with respect to $\mathcal{A}$ and LS with respect to $\mathcal{A}$.*

Remark 3.1 *The assumption that Ξ is a RPI set could render the definitions of ISpS and UpAG in Ξ trivials. In fact ISS in Ξ with respect to Ξ and ISpS in Ξ with $\mathcal{A} \equiv \Xi$ are always satisfied. However, if in the ISpS property, c is shown to be smaller than Ξ^{sup}, and in the ISS the set $\mathcal{A}$ is shown to be smaller than Ξ, then the ISpS in Ξ (or the ISS in Ξ with respect to $\mathcal{A}$) give more information than the solely robust positively invariance of Ξ.* □

Regional ISpS will be now associated to the existence of a suitable Lyapunov function (in general, a priori non-smooth) with respect to d_2. A sufficient condition, that extends the ISS results of [Magni *et al.* 2006a], using the results of [Limon *et al.* 2006a], is introduced. In order to clarify the relation between the sets introduced in the following definition see Figure 3.1.

Definition 3.2 *(ISpS-Lyapunov function in Ξ) A function $V\colon \mathbb{R}^n \to \mathbb{R}_{\geq}0$ is called an ISpS-Lyapunov function in Ξ for system (3.7), if*

1. *Ξ is a compact RPI set including the origin as an interior point*

2. *there exist a compact set $\Omega \subseteq \Xi$ (including the origin as an interior point), a pair of suitable $\mathcal{K}_\infty$-functions α_1, α_2 and a constant $c_1 \geq 0$ such that*
$$V(x) \geq \alpha_1(|x|), \forall x \in \Xi \tag{3.9}$$
$$V(x) \leq \alpha_2(|x|) + c_1, \forall x \in \Omega \tag{3.10}$$

3. *there exist a suitable $\mathcal{K}_\infty$-function α_3, a $\mathcal{K}$-function λ_2 and a constant $c_2 \geq 0$ such that*
$$\begin{aligned} \Delta V(x) &\triangleq V(F(x, d_1, d_2)) - V(x) \\ &\leq -\alpha_3(|x|) + \lambda_2(|d_2|) + c_2 \end{aligned} \tag{3.11}$$
for all $x \in \Xi$, all $d_1 \in \mathcal{D}_1(x)$, and all $d_2 \in \mathcal{D}_2$

4. *there exist suitable $\mathcal{K}_\infty$-functions ζ and ρ (with ρ such that $(id - \rho)$ is a $\mathcal{K}_\infty$-function) and a suitable constant $c_\theta > 0$, such that, given a disturbance sequence $\mathbf{d}_2 \in \mathcal{M}_{\mathcal{D}_2}$, there exists a nonempty compact set $\Theta_{\mathbf{d}_2} \subseteq I\Omega \triangleq \{x : x \in \Omega, |x|_{\delta\Omega} > c_\theta\}$ (including the origin as an interior point) defined as follows:*
$$\Theta_{\mathbf{d}_2} \triangleq \{x : V(x) \leq b(\lambda_2(||\mathbf{d}_2||) + c_3)\} \tag{3.12}$$
where $b \triangleq \alpha_4^{-1} \circ \rho^{-1}$, with $\alpha_4 \triangleq \underline{\alpha}_3 \circ \overline{\alpha}_2^{-1}, \underline{\alpha}_3(s) \triangleq \min(\alpha_3(s/2), \zeta(s/2)), \overline{\alpha}_2 \triangleq \alpha_2(s) + s$, $c_3 \triangleq c_2 + \zeta(c_1)$. $\square$

Note that, whenever Definition 3.2 is satisfied with $c_1 = c_2 = 0$, the function V is an ISS-Lyapunov function in Ξ for system (3.7) (see Definition 2.4).

Remark 3.2 *Note that, as discussed in Chapter 2, in order to verify that $\Theta_{\mathbf{d}_2} \subseteq I\Omega$ for all $\mathbf{d}_2 \in \mathcal{M}_{\mathcal{D}_2}$, one has to verify that*
$$\Theta \triangleq \{x : V(x) \leq b(\lambda_2(\mathcal{D}_2^{sup}) + c_3)\} \subseteq I\Omega \tag{3.13}$$
$\square$

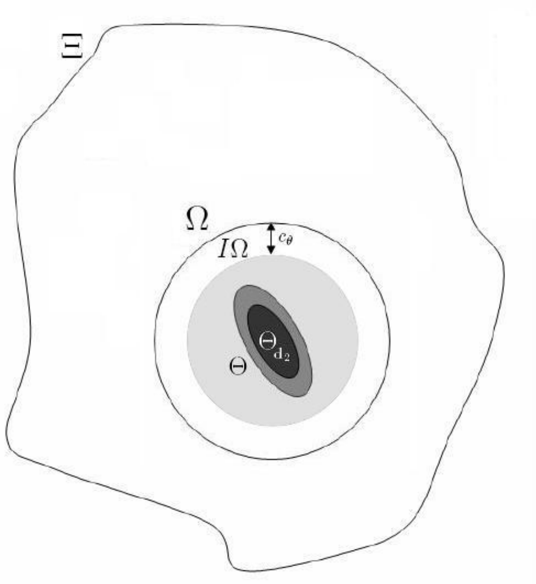


Figure 3.1: Example of sets satisfying Definition 3.2

Now, a sufficient condition for regional ISpS of system (3.7), that extends the ISS results of [Magni *et al.* 2006a] using the results of [Limon *et al.* 2006a], can be stated.

Theorem 3.2 *Suppose that Assumption 3.2 holds. If system (3.7) admits an ISpS-Lyapunov function in* Ξ *with respect to* d_2*, then it is ISpS in* Ξ *with respect to* d_2 *and, for all disturbance sequences* $\mathbf{d}_2 \in \mathcal{M}_{\mathcal{D}_2}$*,* $\lim_{k\to\infty} |x(k,\bar{x},\mathbf{d}_1,\mathbf{d}_2)|_{\Theta_{\mathbf{d}_2}} = 0$. ■

Remark 3.3 *Note that for a generic disturbance* d_1*, condition* 3 *of Definition 3.2 should be*

3. *there exist a suitable* $\mathcal{K}_\infty$*-function* α_3*, a pair of* $\mathcal{K}$*-functions* λ_1 *and* λ_2 *and a constant* $c_2 \geq 0$ *such that*

$$\begin{aligned} \Delta V(x) &\triangleq V(F(x,d_1,d_2)) - V(x) \\ &\leq -\alpha(|x|) + \lambda_1(|d_1|) + \lambda_2(|d_2|) + c_2 \end{aligned} \tag{3.14}$$

for all $x \in \Xi$*, all* $d_1 \in \mathcal{D}_1(x)$*, and all* $d_2 \in \mathcal{D}_2$*.*

However, in view of Assumption 3.2, since d_1 is a function of x, the term $\lambda_1(d_1(x))$ is incorporated in $-\alpha_3(|x|)$. In order to satisfy the condition that α_3 is a $\mathcal{K}_\infty$-function, it is necessary that the $\mathcal{K}_\infty$-function α in (3.14) compensates the effect of the disturbance d_1. This means that system (3.7) must have a stability margin: under Assumption 3.2, it is required that

$$\begin{aligned} -\alpha(|x|) + \lambda_1(|d_1|) &= -\alpha(|x|) + \lambda_1(|d_{1\eta}|\eta(|x|)) \\ &\leq -\alpha(|x|) + \lambda_1(\mathcal{D}_{1\eta}^{sup}\eta(x)) \triangleq -\alpha_3(|x|) \end{aligned}$$

with $\alpha_3 \in \mathcal{K}_\infty$-function.

Remark 3.4 *Theorem 3.2 gives an estimation of the region $\Theta_{\mathbf{d}_2}$ where the state of the system converges asymptotically. In some cases, as for example in the MPC, the function V is not known in explicit form. Hence, in order to verify that point 4 of Definition 3.2 is satisfied, considering also Remark 3.2, one has to verify that*

$$\bar{\Theta} \triangleq \{x : |x| \leq \alpha_1^{-1} \circ b(\mathcal{D}_2^{sup})\} \subseteq I\Omega \tag{3.15}$$

In fact, in view of (3.9) and Remark 3.2, $\bar{\Theta} \supseteq \Theta$. In order to clarify the relation between the sets, see Figure 3.2.

3.4 Min-max model predictive control

This section presents new results that allows the presentation of previous results in a unified framework. Firstly, the formulation of the closed-loop min-max control law is presented. Then, the stability of different approaches of this control technique is studied, deriving sufficient conditions and generalizing existing results.

As it was claimed in Section 3.2, the control law derived by means of a closed-loop min-max MPC considers a vector of feedback control policies $\kappa_{[t,t+N-1]} \triangleq [\kappa_0(x_t), \kappa_1(x_{t+1}), \cdots, \kappa_{N-1}(x_{t+N-1})]$ in the minimization of the cost in the worst disturbance case. This optimization problem can be posed as the following Finite Horizon Closed-Loop Game (FHCLG).

Definition 3.3 (FHCLG) *Consider system (3.1) with $x_t = \bar{x}$. Given the positive integer N, the stage cost l, the terminal penalty V_f and the ter-*

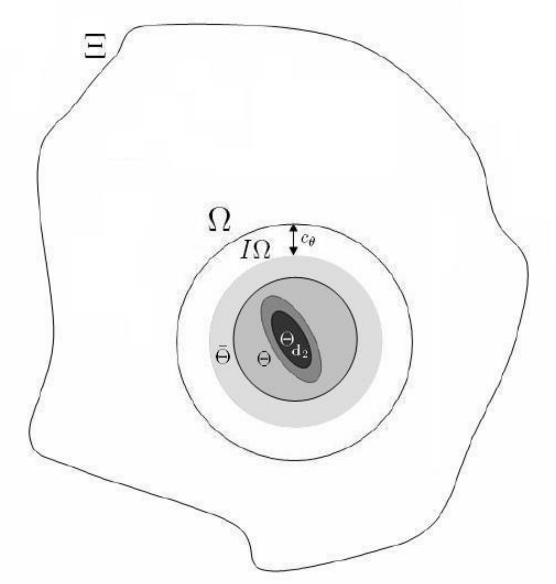


Figure 3.2: Relation between sets

minal set $\mathcal{X}_f$, the FHCLG problem consists in minimizing, with respect to $\kappa_{[t,t+N-1]}$ and maximizing with respect to $\mathbf{d}_{1[\mathbf{t},\mathbf{t}+\mathbf{N}-1]}$ and $\mathbf{d}_{2[\mathbf{t},\mathbf{t}+\mathbf{N}-1]}$ the cost function

$$\begin{aligned} J(\bar{x}, \kappa_{[t,t+N-1]}, \mathbf{d}_{1[\mathbf{t},\mathbf{t}+\mathbf{N}-1]}, \mathbf{d}_{2[\mathbf{t},\mathbf{t}+\mathbf{N}-1]}, N) \triangleq \\ \textstyle\sum_{k=t}^{t+N-1} l(x_k, u_k, d_{1_k}, d_{2_k}) + V_f(x_{t+N}) \end{aligned} \tag{3.16}$$

subject to

1. *the state dynamics (3.1)*

2. *the constraints (3.2)-(3.6), $k \in [t, t+N-1]$*

3. *the terminal constraint $x_{t+N} \in \mathcal{X}_f$.* □

In the following, let $\mathcal{X}^{MPC}(N)$ denote the set of states for which a solution of the FHCLG problem exists.

Letting $\kappa^o_{[t,t+N-1]}, \mathbf{d^o_{1[t,t+N-1]}}, \mathbf{d^o_{2[t,t+N-1]}}$ be the solution of the FH-CLG, according to the Receding Horizon (RH) paradigm, the feedback control law $u = \kappa^{MPC}(x)$ is obtained by setting

$$\kappa^{MPC}(x) = \kappa^o_0(x) \tag{3.17}$$

where $\kappa_0^o(x)$ is the first element of $\kappa_{[t,t+N-1]}^o$.

The parameters of the controller are the prediction horizon, the stage cost function, the terminal cost function and the terminal region. The stage cost defines the performance index to optimize and must satisfy the following assumption.

Assumption 3.3 *The stage cost $l(\cdot)$ is such that $l(0,0,0,0) = 0$ and $l(x,u,d_1,d_2) \geq \alpha_l(|x|) - \alpha_d(|d_2|)$ where α_l and α_d are $\mathcal{K}_\infty$-functions.* □

As it is standard in MPC [Mayne *et al.* 2000], the terminal ingredients are added to provide closed-loop stability as it can be seen in the following section.

3.4.1 Stability

In this section, tools for analyzing stability of closed-loop min-max MPC systems are provided. Firstly, it will be shown that, when persistent disturbances are present, the standard min-max approach can only guarantee ISpS. Secondly, two different solutions for overcoming this problem and guaranteeing ISS of the min-max MPC closed-loop system are given: the first one is derived using a particular design of the stage cost of the performance index, while the second one is based on a dual-mode strategy.

In order to derive the main stability and performance properties associated to the solution of FHCLG, the following assumption is introduced.

Assumption 3.4 *The design parameters V_f, $\mathcal{X}_f$ are such that, given an auxiliary control law κ_f*

1. *$X_f \subseteq \mathcal{X}$, $\mathcal{X}_f$ closed, $0 \in \mathcal{X}_f$*
2. *$\kappa_f(x) \in \mathcal{U}$, $|\kappa_f(x)| \leq \mathcal{L}_{\kappa_f}|x|$, for all $x \in \mathcal{X}_f$, where $\mathcal{L}_{\kappa f} > 0$*
3. *$f(x, \kappa_f(x), d_1, d_2) \in \mathcal{X}_f$, for all $x \in \mathcal{X}_f$, all $d_{1\eta} \in \mathcal{D}_{1\eta}$, and all $d_2 \in \mathcal{D}_2$*
4. *there exist a pair of suitable $\mathcal{K}_\infty$-functions α_{V_f} and β_{V_f} such that $\alpha_{V_f} < \beta_{V_f}$ and*
$$\alpha_{V_f}(|x|) \leq V_f(x) \leq \beta_{V_f}(|x|)$$
for all $x \in \mathcal{X}_f$

5. $V_f(f(x, \kappa_f(x), d_1, d_2) - V_f(x) \leq -l(x, \kappa_f(x), d_1, d_2) + \varrho(|d_2|)$, *for all* $x \in \mathcal{X}_f$, *all* $d_{1\eta} \in \mathcal{D}_{1\eta}$, *and all* $d_2 \in \mathcal{D}_2$, *where* ϱ *is a* $\mathcal{K}_\infty$-*function.* □

Assumption 3.4 implies that the closed-loop system formed by (3.1) and $u(k) = \kappa_f(x)$, is ISS in $\mathcal{X}_f$ (V_f is an ISS-Lyapunov function in $\mathcal{X}_f$).

Remark 3.5 *If the feedback policies* $\kappa_i(x)$, $i = 0, \ldots, N-1$ *are restricted to belong to a particular class of functions then also* κ_f *must belong to this class. This motivates the difficulty to guarantee closed-loop stability if optimization is performed with respect to open-loop strategies [Chen et al. 1997]. In fact, Assumption 3.4 should hold with* $\kappa_f(x) = 0$. *On the contrary, a natural choice, when semi-feedback controllers are used, is to include the auxiliary control law among the regressors (see the example in Section 3.6).*

□

In what follows, the optimal value of the performance index, i.e.

$$V(x, N) \triangleq J(x, \kappa^o_{[t,t+N-1]}, \mathbf{d^o_{1[t,t+N-1]}}, \mathbf{d^o_{2[t,t+N-1]}}, N) \tag{3.18}$$

is employed as an ISpS-Lyapunov function.

Assumption 3.5 *Let*

- $\Xi = \mathcal{X}^{MPC}$
- $\Omega = \mathcal{X}_f$
- $\alpha_1 = \alpha_l$
- $\alpha_2 = \beta_{V_f}$
- $\alpha_3 = \alpha_l$
- $\lambda_2 = \alpha_d$
- $c_1 = N\varrho(\mathcal{D}_2^{sup})$
- $c_2 = \varrho(\mathcal{D}_2^{sup})$

The set $\mathcal{D}_2$ is such that the set Θ (depending from $\mathcal{D}_2^{sup}$), defined in (3.13) with function V given by (3.18), is contained in $I\Omega$. □

The main result can now be stated.

Theorem 3.3 *Under Assumptions 3.1, 3.3-3.5, the closed-loop system formed by (3.1) and (3.17), subject to constraints (3.2)-(3.6), is ISpS with respect to d_2 with RPIA set $\mathcal{X}^{MPC}(N)$.* ■

Remark 3.6 *In the proof of Theorem 3.3, it is shown that, in order to prove the ISpS, the upper bound (3.10) in a local region is sufficient. However, this could be a limitation due to (3.13). In fact the uncertainty should be such that $\Theta \subseteq \Omega$. In order to enlarge the set of admissible uncertainty it could be useful to find an upper bound in a region $\Omega_1 \supseteq \Omega$ as suggested in [Limon et al. 2006a, Lazar et al. 2008]. However this idea can either enlarge or restrict the set of admissible uncertainty since $\Omega_1 \supseteq \Omega$ but the upper bound could be more conservative.*

Remark 3.7 *As discussed in Remark 2.5, it is not necessary to obtain the global optimum solution of the FHCLG in order to guarantee the ISpS (or the ISS) of the closed-loop system. In fact the vector of feedback control policies $\tilde{\kappa}_{[1,N]} \triangleq [\tilde{\kappa}_{[0,N-1]} \ \kappa_f]$, where $\tilde{\kappa}_{[0,N-1]}$ is the possible sub-optimal solution obtained at the previous step, is an available feasible solution that guarantees ISpS or ISS. Indeed this sequence is such that the value function satisfies (3.11). The only requirement on the possible sub-optimal solution is to be not worst than $\tilde{\kappa}_{[1,N]}$. On the contrary, the applicability of a sub-optimal solution of the maximization of the FHCLG is still an open issue [Alamo et al. 2005, Raimondo et al. 2007a].* □

The previous theorem formulated for the general case of standard min-max MPC states that only ISpS is guaranteed for the resulting closed-loop system, irrespective of the fact that the disturbances may vanish in reality. However, when this is the case, it would be preferable that the closed-loop system is ISS, so that nominal asymptotic stability is recovered when disturbances are no longer active.

In the following subsections some ingredients are presented, in the form of assumptions on the type of disturbances or the min-max MPC cost function, that make possible to establish ISS, instead of ISpS, of closed-loop min-max MPC systems.

3.4.1.1 Standard min-max with only state dependent uncertainty

Consider the case system (3.1) is affected only by the uncertainty d_1 satisfying (3.2) (it is known that $d_{2_k} = 0, \forall k \geq 0$). This assumption led to the result published in [Mayne 2001], which is stated in the following theorem.

Theorem 3.4 *[Mayne 2001] Consider that $d_2 = 0$. Under Assumptions 3.1, 3.3-3.5, the origin of the closed-loop system formed by (3.1) and (3.17), subject to constraints (3.2)-(3.6), is robustly asymptotically stable with RPIA set $\mathcal{X}^{MPC}(N)$.*

Remark 3.8 *Note that Assumption 3.4 states that control law $u = \kappa_f(x)$ is designed in such a way that the closed-loop system has a stability margin in $\mathcal{X}_f$. Moreover, note that the robustness of the auxiliary control law is translated to the MPC, that is, the min-max MPC controller extends to $\mathcal{X}^{MPC}(N)$ the stability margin provided by the auxiliary control law.* □

Next, the more challenging case, when both state dependent and state independent uncertainties are present, will be considered.

3.4.1.2 Standard min-max with state independent uncertainty

Consider the case system (3.1) is affected by both uncertainties of the type d_1 and of the type d_2.

A new condition on the stage cost (standard min-max stage cost) is introduced.

Assumption 3.6 *The stage cost $l(x, u)$ is disturbance independent and such that $l(0, 0) = 0$ and $l(x, u) \geq \alpha_l(|x|)$, where α_l is a $\mathcal{K}_\infty$-function.*

□

Under the above assumption, the following ISpS result (which can be recovered also from Theorem 3.3) for min-max MPC was obtained in [Limon *et al.* 2006a].

Corollary 3.1 *[Limon et al. 2006a] Under Assumptions 3.1, 3.4-3.6, the closed-loop system formed by (3.1) and (3.17), subject to constraints (3.2)-(3.6), is ISpS with respect to d_2 with RPIA set $\mathcal{X}^{MPC}(N)$.* ■

Note that, even if the auxiliary control strategy guarantees ISS, only ISpS can be established for the min-max MPC closed-loop system. As already mentioned, this is not a desirable property, as the employed control design method prevents that closed-loop asymptotic stability is attained when there are no disturbances active.

In the followings two solutions for solving this problem of standard min-max MPC will be discussed. The first solution, presented previously in Chapter 2, employs a particular design of the stage cost such that Assumption 3.4 can be satisfied also with $\varrho \equiv 0$, leading, for example, to the well-known $\mathcal{H}_\infty$ strategy. The second solution exploits the ISS property of the auxiliary control law via a dual-mode strategy to establish ISS of the dual-mode min-max MPC closed-loop system.

3.4.1.3 $\mathcal{H}_\infty$ strategy

The proof of Corollary 3.1 clearly illustrates that the difficulty of proving ISS of min-max MPC, which is attempted in Theorem 3.3, is related to the terms c_1 and c_2 that are depending on the $\mathcal{K}_\infty$-function ϱ defined in Assumption 3.4 and whose necessity is related to the stage cost considered in the standard min-max MPC optimization problem.

This observation leads to the following new condition on the stage cost, which will turn out to be sufficient for proving ISS of the corresponding min-max MPC closed-loop system.

Assumption 3.7 *The stage cost is composed by the functions $l_x : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ and $l_d : \mathbb{R}^q \to \mathbb{R}$ as follows*

$$l(x, u, d_1, d_2) = l_x(x, u, d_1) - l_d(d_2)$$

and satisfies Assumption 3.3. □

If Assumption 3.7 is satisfied, as it will be shown in Section 3.5, then Assumption 3.4 can be satisfied also with $\varrho \equiv 0$. Then, using the proof of

Theorem 3.3, ISS of the closed-loop system can be guaranteed, as established by the following result.

Corollary 3.2 *[Magni et al. 2006a] Under Assumptions 3.1, 3.4, 3.5, 3.7 with* $\varrho \equiv 0$*, the closed-loop system formed by (3.1) and (3.17), subject to constraints (3.2)-(3.6), is ISS with respect to* d_2 *with RPIA set* $\mathcal{X}^{MPC}(N)$*.*

□

The above result shows that by adding a new term to the stage cost, which depends solely on the disturbance signal, ISS of the resulting min-max MPC closed-loop system can be attained. It remains to be explored how the modified cost function affects the solvability of the min-max optimization problem and whether standard min-max MPC solvers can still be employed.

Next, a dual-mode strategy for guaranteeing ISS of min-max MPC will be presented, which relies on the same cost function as the one used in standard min-max MPC.

3.4.1.4 Dual-mode strategy

Sufficient conditions for input-to-state stability of nonlinear discrete-time systems in closed-loop with dual-mode min-max MPC controllers were recently developed in [Lazar *et al.* 2008]. Therein, only state independent uncertainties were considered. In this section the results presented in [Lazar *et al.* 2008] will be exploited and Theorem 3.2 will be used to apply the dual-mode approach to the more general class of uncertainties considered in this chapter.

First, let recall the classical dual-mode strategy. In dual-mode MPC, the receding horizon controller (3.17) is employed outside $\mathcal{X}_f$ and the auxiliary control law κ_f is used inside $\mathcal{X}_f$, i.e.

$$u^{DM} \triangleq \begin{cases} \kappa^{MPC}(x) & \text{if } x \in \mathcal{X}^{MPC}(N) \setminus \mathcal{X}_f, \\ \kappa_f(x) & \text{if } x \in \mathcal{X}_f. \end{cases} \tag{3.19}$$

Next, the ISS result for dual-mode min-max MPC with standard cost function are presented.

Theorem 3.5 *Under Assumptions 3.1, 3.3-3.5, the closed-loop system formed by (3.1) and (3.19), subject to constraints (3.2)-(3.6), is ISS with respect to d_2 with RPIA set $\mathcal{X}^{MPC}(N)$.* ■

So far, two methods for establishing ISS of min-max MPC have been presented. However, both these methods rely on a specific cost function that must satisfy certain assumptions. Therefore, the computation of suitable cost functions is equally important for synthesizing min-max MPC schemes with an a priori ISS guarantee. A solution that applies to nonlinear systems affine in control is presented in the next section.

3.5 The auxiliary control law

In this section it is shown that, if nonlinear input affine systems are considered, a nonlinear control law $u = \kappa^*(x)$ satisfying Assumption 3.4 can be derived by the solution of the $\mathcal{H}_\infty$ control problem for the linearized system. This section extends the results obtained in [Magni *et al.* 2003]) also to the presence of state independent disturbances. In this respect, consider the system

$$
\begin{aligned}
x_{k+1} &= f_1(x_k) + f_2(x_k)u_k + f_3(x_k)w_k \\
z_k &= \begin{bmatrix} h_1(x_k) \\ u_k \end{bmatrix}
\end{aligned}
\tag{3.20}
$$

where $w = [d_1^\top \ d_2^\top]^\top$, f_1, f_2, f_3 and h_1 are C^2 functions with $f_1(0) = 0$ and $h_1(0) = 0$. For convenience, let represent the corresponding discrete-time linearized system as

$$
\begin{aligned}
x_{k+1} &= F_1 x_k + F_2 u_k + F_3 w_k \\
z_k &= \begin{bmatrix} H_1 x_k \\ u_k \end{bmatrix}
\end{aligned}
$$

where $F_1 = \left.\frac{\partial f_1}{\partial x}\right|_{x=0}$, $F_2 = f_2(0)$, $F_3 = f_3(0)$, $H_1 = \left.\frac{\partial h_1}{\partial x}\right|_{x=0}$. Given a square $n \times n$ matrix P, and a positive constant γ, define also the symmetric matrix

$$
R = R(P) = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}
\tag{3.21}
$$

where

$$\begin{aligned} R_{11} &= F_2^\top P F_2 + I \\ R_{12} &= R_{21}^\top = F_2^\top P F_3 \\ R_{22} &= F_3^\top P F_3 - \gamma^2 I \end{aligned}$$

and the quadratic function

$$V_f(x) = x^\top P x.$$

Proposition 3.1 *Suppose that Assumptions 3.1 is satisfied. Suppose that*

(i) $|d_1|^2 = |d_{1\eta}\eta(|(x,u)|)|^2 \leq \mathcal{K}_{dx}|x|^2 + \mathcal{K}_{du}|u|^2$, *with* $\mathcal{K}_{dx} \geq 0$ *and* $\mathcal{K}_{du} \geq 0$

and that there exists a positive definite matrix P *such that*

(ii) $R_{22} < 0$

(iii) $-P + F_1^\top P F_1 + H_1^\top H_1 - F_1^\top P \begin{bmatrix} F_2 & F_3 \end{bmatrix} R^{-1} \begin{bmatrix} F_2 & F_3 \end{bmatrix}^\top P F_1 < 0.$

Then, there exist sets $\mathcal{D}_1(x)$ *and* $\mathcal{D}_2$ *such that for all* $w \in \bar{\mathcal{W}}_{ne} = \mathcal{D}_1(x) \times \mathcal{D}_2$ *the control law* $u = \kappa^*(x)$ *where*

$$\begin{bmatrix} \kappa^*(x) \\ \xi^*(x) \end{bmatrix} = -R(x)^{-1} \begin{bmatrix} f_2(x) & f_3(x) \end{bmatrix}^\top P f_1(x)$$

with

$$R(x) = \begin{bmatrix} f_2(x)^\top P f_2(x) + I & f_2(x)^\top P f_3(x) \\ f_3(x)^\top P f_2(x) & f_3(x)^\top P f_3(x) - \gamma^2 I \end{bmatrix} = \begin{bmatrix} r_{11}(x) & r_{12}(x) \\ r_{21}(x) & r_{22}(x) \end{bmatrix}$$

satisfies Assumption 3.4 with stage cost

(a) $l(x,u,d_1,d_2) = |z|^2 - \gamma^2|w|^2$ *and* $\varrho \equiv 0$

or

(b) $l(x,u,d_1,d_2) = |z_l|^2$ *with* $z_l = \begin{bmatrix} h_l(x) \\ u \end{bmatrix}$, *where* h_l *is such that* $h_1(x)^\top h_1(x) \geq h_l(x)^\top h_l(x) + a|x|^2 + c|x|^2$, $a \triangleq \gamma^2 \mathcal{K}_{dx}$, $b \triangleq \gamma^2 \mathcal{K}_{du}$, $c \triangleq b\mathcal{L}_{\kappa_f}^2$ *and* $\varrho(s) \triangleq \gamma^2 |s|^2$

and

$$\mathcal{X}_f \triangleq \left\{ x : x^\top P x \leq \alpha \right\} \subseteq \mathcal{X},$$

where α is a finite positive constant.

Remark 3.9 *P can be computed by solving a discrete-time $\mathcal{H}_\infty$ algebraic Riccati equation.*

It is important to underline that the proposed auxiliary control law satisfies Assumption 3.4 with both the stage cost $l(x, u, d_1, d_2) = |z|^2 - \gamma^2 |w|^2$ and $l(x, u, d_1, d_2) = |z_l|^2$.

3.6 Example

In this section, the MPC law introduced in the chapter is applied to a cart with mass M moving on a plane (the model is the same of the paper [Magni *et al.* 2003]). This carriage (see Figure 3.3) is attached to the wall via a spring with elastic constant k given by $k = k_0 e^{-x_1}$, where x_1 is the displacement of the carriage from the equilibrium position associated with the external force $u = 0$ and the external disturbance force (wind force) $d_2 = 0$. Finally a damper with damping factor h_d affects the system in a resistive way. The model of the system is given by the following continuous-

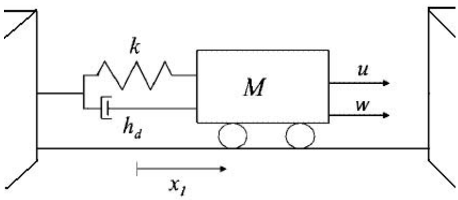


Figure 3.3: Cart and spring-damper example.

time state space nonlinear model

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{k_0}{M} e^{-x_1(t)} x_1(t) - \frac{h_d}{M} x_2(t) + \frac{u(t)}{M} + \frac{d_2(t)}{M} \end{cases}$$

where x_2 is the carriage velocity. The parameters of the system are $M = 1$ kg, $k_0 = 0.33$ $\frac{N}{m}$, while the damping factor in not well known and is given by $h_d = \bar{h}_d + d_{1\eta}$, where $\bar{h}_d = 1.1\frac{Ns}{m}$ and $|d_{1\eta}| \leq 0.1$. Wind force is limited: $-0.2 \leq d_2 \leq 0.4$. The system can be rewritten as

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{k_0}{M}e^{-x_1(t)}x_1(t) - \frac{\bar{h}_d}{M}x_2(t) + \frac{u(t)}{M} - \frac{d_{1\eta}(t)}{M}x_2(t) + \frac{d_2(t)}{M} \end{cases}$$

The state and control variables have to satisfy the following constraints $|u| \leq 4.5\ N$, $|x_1| \leq 2.65\ m$. An Euler approximation of the system with sampling time $T_c = 0.4\ s$ is given by

$$\begin{cases} x_{1_{k+1}} &= x_{1_k} + T_c x_{2_k} \\ x_{2_{k+1}} &= -T_c\frac{k_0}{M}e^{-x_{1_k}}x_{1_k} + x_{2_k} - T_c\frac{\bar{h}_d}{M}x_{2_k} \\ & \quad +T_c\frac{u_k}{M} - T_c\frac{d_{1\eta_k}}{M}x_2(k) + T_c\frac{d_{2_k}}{M} \end{cases}$$

which is a discrete-time nonlinear system. The system can be rewritten as

$$\begin{aligned} \begin{bmatrix} x_{1_{k+1}} \\ x_{2_{k+1}} \end{bmatrix} &= \underbrace{\begin{bmatrix} 1 & T_c \\ -T_c\frac{k_0}{M}e^{-x_{1_k}} & 1 - T_c\frac{\bar{h}_d}{M} \end{bmatrix}\begin{bmatrix} x_{1_k} \\ x_{2_k} \end{bmatrix}}_{f_1(x_k)} + \underbrace{\begin{bmatrix} 0 \\ \frac{T_c}{M} \end{bmatrix}}_{F_2} u_k \\ &+ \underbrace{\begin{bmatrix} 0 & 0 \\ -\frac{T_c}{M} & \frac{T_c}{M} \end{bmatrix}}_{F_3}\underbrace{\begin{bmatrix} d_{1_k} \\ d_{2_k} \end{bmatrix}}_{w_k} \end{aligned}$$

where $d_{1_k} = d_{1\eta_k}x_{2_k}$. Disturbance d_{1_k} satisfies point (i) of Proposition 3.1 with $\mathcal{K}_{dx} = 0.01$ and $\mathcal{K}_{du} = 0$. Let choose $l(x, u, d_1, d_2) = |z|^2 - \gamma^2|w|^2$ and $l(x, u) = |z_l|^2$ where

$$z_l = \begin{bmatrix} h_l(x) \\ u \end{bmatrix} = \begin{bmatrix} \overbrace{\begin{bmatrix} q_{1l} & 0 \\ 0 & q_{2l} \end{bmatrix}}^{H_L}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ u \end{bmatrix}$$

with $q_{1l} = 1$ and $q_{2l} = 1$ and $\gamma = 3$. The auxiliary control law is obtained as described in Section 3.5 with

$$z = \begin{bmatrix} h_1(x) \\ u \end{bmatrix} = \begin{bmatrix} \overbrace{\begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}}^{H_1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ u \end{bmatrix}$$

with $q_1 = 1.1$ and $q_2 = 1.1$. Note that, as required at point b) of Proposition 3.1, $h_1(x)^\top h_1(x) \geq h_l(x)^\top h_l(x) + a|x|^2 + c|x|^2$ with $a = \gamma^2 \mathcal{K}_{dx} = 0.09$, $b = \gamma^2 \mathcal{K}_{du} = 0$, $c = b\mathcal{L}^2_{\kappa_f} = 0$, where $\mathcal{L}_{\kappa_f} = 2$. In fact

$$
\begin{aligned}
q_1^2 x_1^2 + q_2^2 x_2^2 &\geq q_{1l}^2 x_1^2 + q_{2l}^2 x_2^2 + a x_1^2 + a x_2^2 \\
1.21 x_1^2 + 1.21 x_2^2 &\geq x_1^2 + x_2^2 + 0.09 x_1^2 + 0.09 x_2^2.
\end{aligned}
$$

The auxiliary control law is given by

$$
\kappa_f(x) = -[1 \;\; 0 \;\; 0] R^{-1} \begin{bmatrix} F_2 \\ F_3 \end{bmatrix} P f_1(x) = -[0.8783 \; 1.1204] f_1(x)
$$

where

$$
P = \begin{bmatrix} 7.0814 & 3.3708 \\ 3.3708 & 4.2998 \end{bmatrix}
$$

is computed solving a discrete time $\mathcal{H}_\infty$ algebraic Riccati equation

$$
P = F_1^\top P F_1 + H_2^\top H_2 - F_1^\top P \begin{bmatrix} F_2 & F_3 \end{bmatrix} R^{-1} \begin{bmatrix} F_2 & F_3 \end{bmatrix}^\top P F_1
$$

with

$$
H_2^\top H_2 = 1.2 H_1^\top H_1
$$

in order to satisfy inequality (iii) of Proposition 3.1. Matrix P satisfies inequality (ii) of Proposition 3.1. The terminal penalty is given by $V_f = x^\top P x$. The auxiliary control law satisfies Assumption 3.4, for the stages cost chosen, in the region $\mathcal{X}_f \triangleq \{x : x^\top P x \leq 4.7\}$. Region $\mathcal{X}_f$ has been obtained numerically. The length of horizon is $N = 4$. The policies $\kappa_i(x)$ are functions of the form $\kappa_i(x) = \alpha_i \kappa_f(x) + \beta_i(x_1^2 + x_2^2) + \gamma_i$. Figure 3.4 and 3.5 show the time evolution of position and velocity of the cart, starting from $x_1(0) = 0.5m$ and $x_2(0) = 0\frac{m}{s}$. Figure 3.6 shows the control sequence. Figure 3.7 show the time evolution of $d_{1\eta}$ and d_2. For $t \geq 1.6s$ the signal d_2 is equal to zero. Note that, $\mathcal{H}_\infty$ and dual mode strategies guarantee ISS: cart position goes to zero when the disturbance vanishes. On the other hand, standard min-max strategy only guarantees ISpS. In fact, when the disturbance vanishes, cart position does not tend to the origin but to $x_1 = 0.1541m$. Moreover, note that that $\mathcal{H}_\infty$ and dual mode performances are comparable since, in the neighborhood of the origin, the auxiliary control law is a good approximation of the $\mathcal{H}_\infty$ strategy.

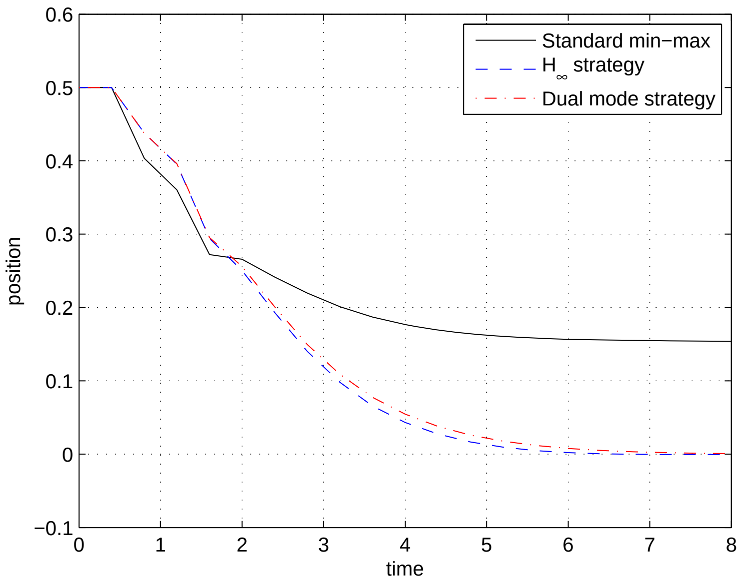


Figure 3.4: Time evolution of cart position.

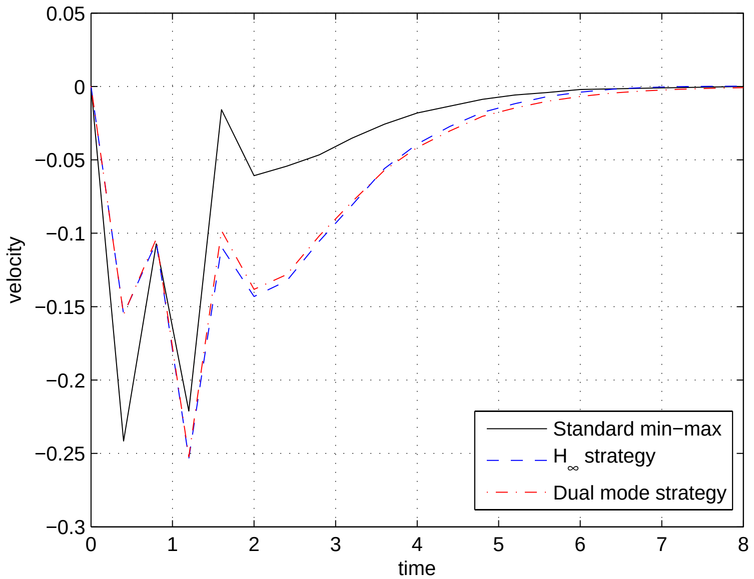


Figure 3.5: Time evolution of cart velocity.

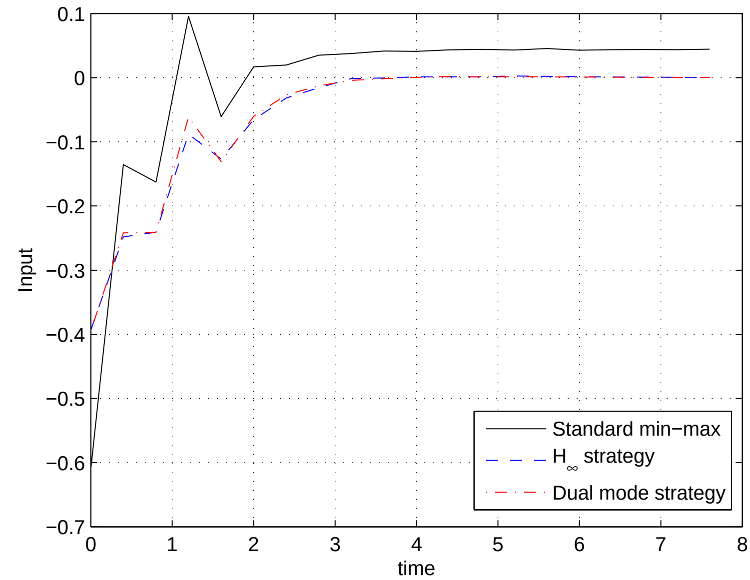


Figure 3.6: Time evolution of the control.

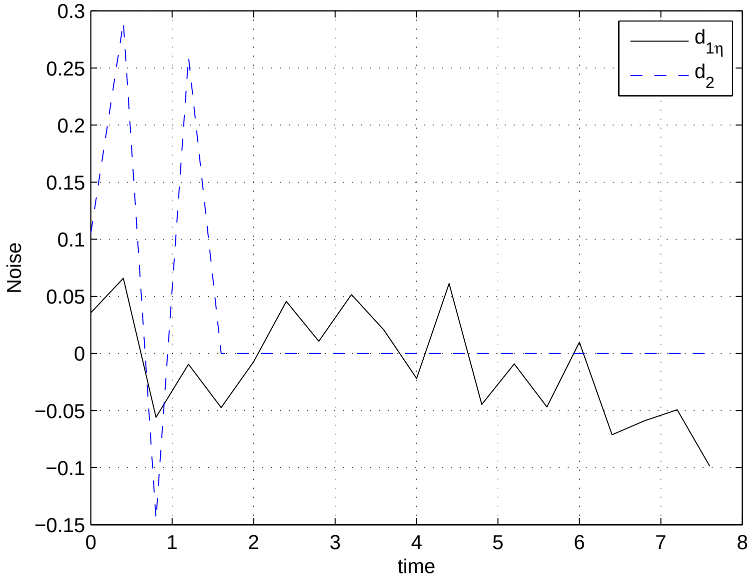


Figure 3.7: Time evolution of uncertainty $d_{1\eta}$ and disturbance d_2.

3.7 Conclusions

In this chapter a unified framework for the synthesis of min-max MPC control algorithms has been provided. The ISpS or ISS property of such algorithms is analyzed with respect to a general class of disturbances that considers both state dependent and state independent disturbances. The algorithms based on a standard stage cost, on an $\mathcal{H}_\infty$ cost and on a dual mode approach are compared. The relevance of the adopted stage cost to achieve ISS clarifies the difference between some of the results appeared in the literature.

3.8 Appendix

Proof of Theorem 3.2: The proof will be carried out in four steps.

Step 1: first, it is going to be shown that $\Theta_{\mathbf{d}_2}$ defined in (3.12) is a RPI set for system (3.7). From the definition of $\overline{\alpha}_2(s)$ it follows that $\alpha_2(|x|) + c_1 \leq \overline{\alpha}_2(|x| + c_1)$. Therefore $V(x) \leq \overline{\alpha}_2(|x| + c_1)$ and hence $|x| + c_1 \geq \overline{\alpha}_2^{-1}(V(x)), \forall x \in \Omega$. Moreover (see [Limon *et al.* 2006a]):

$$\alpha_3(|x|) + \zeta(c_1) \geq \underline{\alpha}_3(|x| + c_1) \geq \alpha_4(V(x)) \tag{3.22}$$

where $\alpha_4 \triangleq \underline{\alpha}_3 \circ \overline{\alpha}_2^{-1}$ is a $\mathcal{K}_\infty$-function. Then

$$\begin{aligned}
\Delta V(x) &\leq -\alpha_4(V(x)) + \zeta(c_1) + c_2 + \lambda_2(|d_2|) \\
&= -\alpha_4(V(x)) + \lambda_2(|d_2|) + c_3 \\
&\leq -\alpha_4(V(x)) + \lambda_2(\|\mathbf{d}_2\|) + c_3,
\end{aligned}$$

for all $x \in \Xi$, all $d_1 \in \mathcal{D}_1(x)$, and all $d_2 \in \mathcal{D}_2$, where $c_3 \triangleq c_2 + \zeta(c_1)$.

Assume now that there exists a finite time t such that $x_t \in \Theta_{\mathbf{d}_2}$. Then $V(x_t) \leq b(\lambda_2(\|\mathbf{d}_2\|) + c_3)$; this implies $\rho \circ \alpha_4(V(x_t)) \leq \lambda(\|\mathbf{d}_2\|) + c_3$. Without loss of generality, assume that $(id - \alpha_4)$ is a $\mathcal{K}_\infty$-function (see Lemma

B.1 [Jiang & Wang 2001]). Then

$$\begin{aligned}
V(F(x_t, d_{1_t}, d_{2_t})) &\leq (id - \alpha_4)(V(x_t)) + \lambda_2(||\mathbf{d}_2||) + c_3 \\
&\leq (id - \alpha_4)(b(\lambda_2(||\mathbf{d}_2||) + c_3)) + \lambda_2(||\mathbf{d}_2||) + c_3 \\
&= -(id - \rho) \circ \alpha_4(b(\lambda_2(||\mathbf{d}_2||) + c_3)) \\
&\quad + b(\lambda_2(||\mathbf{d}_2||) + c_3) - \rho \circ \alpha_4(b(\lambda_2(||\mathbf{d}_2||) + c_3)) \\
&\quad + \lambda_2(||\mathbf{d}_2||) + c_3.
\end{aligned}$$

From the definition of b, it follows that $\rho \circ \alpha_4(b(s)) = s$ and, owing to the fact that $(id - \rho)$ is a $\mathcal{K}_\infty$-function, it follows that

$$\begin{aligned}
V(F(x_{k_0}, d_{1_{k_0}}, d_{2_{k_0}})) &\leq -(id - \rho) \circ \alpha_4(b(\lambda_2(||\mathbf{d}_2||) + c_3)) \\
&\quad + b(\lambda_2(||\mathbf{d}_2||) + c_3) \\
&\leq b(\lambda_2(||\mathbf{d}_2||) + c_3).
\end{aligned}$$

By induction one can show that $V(x(t + j, \bar{x}, \mathbf{d}_1, \mathbf{d}_2)) \leq b(\lambda_2(||\mathbf{d}_2||) + c_3)$ for all $j \in \mathbf{Z}_{\geq 0}$, that is $x(t + j, \bar{x}, \mathbf{d}_1, \mathbf{d}_2) \in \Theta_{\mathbf{d}_2}$ for all $k \geq t$. Hence $\Theta_{\mathbf{d}_2}$ is a RPI set for system (3.7).

Step 2: now, it is shown that, starting from $\Xi \setminus \Theta_{\mathbf{d}_2}$, the state tends asymptotically to $\Theta_{\mathbf{d}_2}$. Firstly, if $x \in \Omega \setminus \Theta_{\mathbf{d}_2}$, then

$$\rho \circ \alpha_4(V(x)) > \lambda_2(||\mathbf{d}_2||) + c_3.$$

From the inequality (3.22), one has

$$\rho(\alpha_3(|x|) + \zeta(c_1)) > \lambda_2(||\mathbf{d}_2||) + c_3.$$

On the other hand, $(id - \rho)$ is a $\mathcal{K}_\infty$-function, hence

$$id(s) > \rho(s), \forall s > 0$$

then

$$\alpha_3(|x|) + \zeta(c_1) > \rho(\alpha_3(|x|) + \zeta(c_1)) > \lambda_2(||\mathbf{d}_2||) + \zeta(c_1) + c_2.$$

Hence

$$\alpha_3(|x|) > \lambda_2(||\mathbf{d}_2||) + c_2 \tag{3.23}$$

that means

$$\Delta V(x) < 0, \ \forall x \in \Omega \setminus \Theta_{\mathbf{d}_2}.$$

Moreover, by definition of $\Theta_{\mathbf{d}_2}$ (see point 4 of Definition 3.2), there exists $\bar{c}_\theta > 0$ such that for all $x_1 \in \Xi \setminus \Omega$, there exists $x_2 \in \Omega \setminus \Theta_{\mathbf{d}_2}$ such that $\alpha_3(|x_2|) \leq \alpha_3(|x_1|) - \bar{c}_\theta$. Then from (3.23) it follows that

$$-\alpha_3(|x_1|) + \bar{c}_\theta \leq -\alpha_3(|x_2|) < -\lambda_2(|d_2|) - c_2, \ \forall x_1 \in \Xi \setminus \Omega.$$

Then

$$\Delta V(x) < -\bar{c}_\theta, \ \forall x \in \Xi \setminus \Omega$$

so that, considering that $|x| \leq \alpha_1^{-1}(V(x))$, there exists T_1 such that

$$x(T_1, \bar{x}, \mathbf{d_1}, \mathbf{d_2}) \in \Omega.$$

Therefore, starting from Ξ, the state will reach the region Ω in a finite time. If in particular $x(T_1, \bar{x}, \mathbf{d_1}, \mathbf{d_2}) \in \Theta_{\mathbf{d}_2}$, the region $\Theta_{\mathbf{d}_2}$ is achieved in a finite time. Since $\Theta_{\mathbf{d}_2}$ is a RPI set, it is true that $\lim_{k\to\infty} |x(k, \bar{x}, \mathbf{d_1}, \mathbf{d_2})|_{\Theta_{\mathbf{d}_2}} = 0$. Otherwise, if $x(T_1, \bar{x}, \mathbf{d_1}, \mathbf{d_2}) \notin \Theta_{\mathbf{d}_2}$, $\rho \circ \alpha_4(V(x(T_1, \bar{x}, \mathbf{d_1}, \mathbf{d_2})) > \lambda_2(||\mathbf{d}_2||) + c_3$ and

$$\begin{aligned}
\Delta V(x(T_1, \bar{x}, \mathbf{d_1}, \mathbf{d_2})) &\leq -\alpha_4(V(x(T_1, \bar{x}, \mathbf{d_1}, \mathbf{d_2}))) + \lambda_2(||\mathbf{d}_2||) + c_3 \\
&= -(id - \rho) \circ \alpha_4(V(x(T_1, \bar{x}, \mathbf{d_1}, \mathbf{d_2}))) \\
&\quad -\rho \circ \alpha_4(V(x(T_1, \bar{x}, \mathbf{d_1}, \mathbf{d_2}))) + \lambda_2(||\mathbf{d}_2||) + c_3 \\
&\leq -(id - \rho) \circ \alpha_4(V(x(T_1, \bar{x}, \mathbf{d_1}, \mathbf{d_2}))) \\
&\leq -(id - \rho) \circ \alpha_4 \circ \alpha_1(|x(T_1, \bar{x}, \mathbf{d_1}, \mathbf{d_2})|)
\end{aligned}$$

where the last step is obtained using (3.9). Then, $\forall \varepsilon' > 0$, $\exists T_2(\varepsilon') \geq T_1$ such that

$$V(x(T_2, \bar{x}, \mathbf{d_1}, \mathbf{d_2})) \leq \varepsilon' + b(\lambda_2(||\mathbf{d}_2||) + c_3). \tag{3.24}$$

Therefore, starting from Ξ, the state will arrive close to $\Theta_{\mathbf{d}_2}$ in a finite time and to $\Theta_{\mathbf{d}_2}$ asymptotically. Hence $\lim_{k\to\infty} |x(k, \bar{x}, \mathbf{d_1}, \mathbf{d_2})|_{\Theta_{\mathbf{d}_2}} = 0$. Note that if $||\mathbf{d}_2|| \neq 0$ or $c_3 > 0$, there exists a $\tilde{c} > 0$ such that $\Theta_{\mathbf{d}_2}^{sup} = \tilde{c}$. Then, for $x(T_1, \bar{x}, \mathbf{d}_1, \mathbf{d}_2) \notin \Theta_{\mathbf{d}_2}$, there is $|x(T_1, \bar{x}, \mathbf{d}_1, \mathbf{d}_2)| > \tilde{c}$. Hence, there exists a $c_2 > 0$ such that, for $x(T_1, \bar{x}, \mathbf{d}_1, \mathbf{d}_2) \notin \Theta_{\mathbf{d}_2}$, $\Delta V(x(T_1, \bar{x}, \mathbf{d}_1, \mathbf{d}_2)) < -c_2$, that means, the state arrives in $\Theta_{\mathbf{d}_2}$ in a finite time.

Step 3: given $e \in \mathbb{R}_{\geq 0}$, let $\mathcal{R}(e) \triangleq \{x : V(x) \leq e\}$. Let $\Psi \triangleq \{x : V(x) \leq \bar{e} = \max_{\mathcal{R}(e) \subseteq \Omega} e\}$. It is clear that $\Psi \supseteq \Theta_{\mathbf{d}_2}$ and that Ψ is a RPI set. Since the upper bound of $V(x)$ is known in $\Psi \subseteq \Omega$ then, using the same steps of the proof of Lemma 3.5 in [Jiang & Wang 2001], that also hold for discontinuous systems, the regional ISpS in Ψ is obtained.

Step 4: finally it is shown that system (3.7) is regional ISpS in Ξ. Using (3.24) and (3.9) there is

$$\alpha_1(|x(T_2,\bar{x},\mathbf{d_1},\mathbf{d_2})|) \leq V(x(T_2,\bar{x},\mathbf{d_1},\mathbf{d_2})) \leq \varepsilon' + b(\lambda_2(||\mathbf{d}_2||)+c_3)$$

hence

$$|x(T_2,\bar{x},\mathbf{d_1},\mathbf{d_2})| \leq \alpha_1^{-1}(\varepsilon' + b(\lambda_2(||\mathbf{d}_2||)+c_3)).$$

Noting that, given a $\mathcal{K}_\infty$-function θ_1, $\theta_1(s_1+s_2) \leq \theta_1(2s_1)+\theta_1(2s_2)$, see [Limon *et al.* 2006a], it follows that

$$|x(T_2,\bar{x},\mathbf{d_1},\mathbf{d_2})| \leq \alpha_1^{-1}(2\varepsilon') + \alpha_1^{-1}(2b(2\lambda_2(||\mathbf{d}_2||))) + \alpha_1^{-1}(2b(2c_3)).$$

Now, letting $\varepsilon \triangleq \alpha_1^{-1}(2\varepsilon')$ and $\gamma(||\mathbf{d}_2||) \triangleq \alpha_1^{-1}(2b(2\lambda_2(||\mathbf{d}_2||)))$ and $c \triangleq \alpha_1^{-1}(2b(2c_3))$, the UpAG property in Ξ is proven. In view of Theorem 3.1, since the system is regional ISpS in Ψ, it is LpS and UpAG in Ψ. Finally, LpS with UpAG in Ξ imply that the system is ISpS in Ξ with respect to d_2. ■

Proof of Theorem 3.3: by Theorem 3.2, if system admits an ISpS-Lyapunov function in $\mathcal{X}^{MPC}(N)$, then it is ISpS in $\mathcal{X}^{MPC}(N)$.
In the following it will be shown that the function $V(x,N)$, defined in 3.18, is an ISpS-Lyapunov function for the closed-loop system (3.1) and (3.17) in $\mathcal{X}^{MPC}(N)$.

First, the robust invariance of $\mathcal{X}^{MPC}(N)$ is easily derived from Assumption 3.4 by taking $\bar{\kappa}_{[t+1,t+N]} \triangleq [\kappa^o_{[t+1,t+N-1|t]}\ \kappa_f(x_{t+N})]$ as admissible policy vector at time $t+1$ starting from the optimal sequence $\kappa^o_{[t,t+N-1]}$ at time t. Then, using Assumption 3.3, the lower bound is easily obtained

$$\begin{aligned} V(x,N) &= J(x,\kappa^o_{[t,t+N-1]},\mathbf{d}^o_{1_{[t,t+N-1]}},\mathbf{d}^o_{2[t,t+N-1]},N) \\ &\geq \min_{\kappa_{[t,t+N-1]}} J(x,\kappa_{[t,t+N-1]},0,0,N) \\ &\geq l(x,\kappa_0(x),0,0) \\ &\geq \alpha_l(|x|) \end{aligned} \tag{3.25}$$

for all $x \in \mathcal{X}^{MPC}(N)$.
In order to derive the upper bound, consider the following policy vector

$\tilde{\kappa}_{[t,t+N]} \triangleq [\kappa^o_{[t,t+N-1]}\ \kappa_f(x_{t+N})]$ as admissible policy vector for the FHCLG at time t with horizon $N+1$. Then

$$\begin{aligned}
&J(x, \tilde{\kappa}_{[t,t+N]}, \mathbf{d}_{1_{[t,t+N]}}, \mathbf{d}_{2_{[t,t+N]}}, N+1) = \\
&\sum_{k=t}^{t+N-1} l(x_k, u_k, d_{1_k}, d_{2_k}) + V_f(x_{t+N}) - V_f(x_{t+N}) + V_f(x_{t+N+1}) \\
&+l(x_{t+N}, u_{t+N}, d_{1_{t+N}}, d_{2_{t+N}}).
\end{aligned}$$

In view of Assumption 3.4

$$\begin{aligned}
&J(x, \tilde{\kappa}_{[t,t+N]}, \mathbf{d}_{1_{[t,t+N]}}, \mathbf{d}_{2_{[t,t+N]}}, N+1) \leq \\
&\sum_{k=t}^{t+N-1} l(x_k, u_k, d_{1_k}, d_{2_k}) + V_f(x_{t+N}) + \varrho(|d_2|)
\end{aligned}$$

which implies

$$\begin{aligned}
V(x, N+1) &\leq \max_{\mathbf{d}_{1\eta} \in \mathcal{M}_{\mathcal{D}_{1\eta}}, \mathbf{d}_2 \in \mathcal{M}_{\mathcal{D}_2}} J(x, \tilde{\kappa}_{[t,t+N]}, \mathbf{d}_{1_{[t,t+N]}}, \mathbf{d}_{2_{[t,t+N]}}, N+1) \\
&\leq \max_{\mathbf{d}_{1\eta} \in \mathcal{M}_{\mathcal{D}_{1\eta}}, \mathbf{d}_2 \in \mathcal{M}_{\mathcal{D}_2}} \sum_{k=t}^{t+N-1} l(x_k, u_k, d_{1_k}, d_{2_k}) + V_f(x_{t+N}) \\
&\quad +\varrho(|d_2|) \\
&= V(x, N) + \varrho(\mathcal{D}_2^{sup}) \qquad (3.26)
\end{aligned}$$

which holds for all $x \in \mathcal{X}^{MPC}(N)$, all $\mathbf{d}_{1\eta} \in \mathcal{M}_{\mathcal{D}_{1\eta}}$, and all $\mathbf{d}_2 \in \mathcal{M}_{\mathcal{D}_2}$.

Therefore, using Assumption 3.4, the upper bound is obtained

$$\begin{aligned}
V(x, N) &\leq V(x, N-1) + \varrho(\mathcal{D}_2^{sup}) \leq \ldots \leq V(x, 0) + N\varrho(\mathcal{D}_2^{sup}) \\
&= V_f(x) + N\varrho(\mathcal{D}_2^{sup}) \leq \beta_{V_f}(|x|) + N\varrho(\mathcal{D}_2^{sup}) \qquad (3.27)
\end{aligned}$$

for all $x \in \mathcal{X}_f$.
From the monotonicity property (3.26) and by sub-optimality there is

$$\begin{aligned}
&V(f(x, \kappa^{MPC}(x), d_1, d_2), N) - V(x, N) \\
&\leq V(f(x, \kappa^{MPC}(x), d_1, d_2), N-1) - V(x, N) \\
&\leq -l(x, \kappa^{MPC}(x), d_1, d_2) + \varrho(\mathcal{D}_2^{sup}).
\end{aligned}$$

For details, see the analogous proof of Theorem 2.3. Hence, by applying

Assumptions 3.3, the bound on the decrease of V is obtained

$$V(f(x,\kappa^{MPC}(x),d_1,d_2),N)-V(x,N)\leq -\alpha_l\left(|x|\right)+\alpha_d\left(|d_2|\right)+\varrho(\mathcal{D}_2^{sup}) \tag{3.28}$$

for all $x \in \mathcal{X}^{MPC}(N)$, all $\mathbf{d}_{1\eta} \in \mathcal{M}_{\mathcal{D}_{1\eta}}$, and all $\mathbf{d}_2 \in \mathcal{M}_{\mathcal{D}_2}$.

Therefore, by (3.25), (3.27), (3.28), $V(x)$ is an ISpS-Lyapunov function for the closed-loop system (3.1) and (3.17) in $\mathcal{X}^{MPC}(N)$ and hence, by Theorem 3.2, the closed-loop system formed by (3.1) and (3.17), subject to constraints (3.2)-(3.6), is ISpS with respect to d_2 with RPIA set $\mathcal{X}^{MPC}(N)$.

■

Proof of Corollory 3.1: Proof of Corollary 3.1 is derived by proof of Theorem 3.3. One of the key steps in the proof of Theorem 3.3 is to show that condition 3 in Definition 3.2 is satisfied. In particular, using Assumption 3.3, point 5 of Assumption 3.4 and monotonicity property (3.26), it is shown that condition 3 in Definition 3.2 is satisfied by the inequality (3.28). Only ISpS can be proven because of term $\varrho(\mathcal{D}_2^{sup})$ derived by term $\varrho(|d_2|)$ in point 5 of Assumption 3.4. The necessity of this term is related to the particular stage cost considered in the optimization problem. In Corallary 3.1, a standard stage cost $l(x,u)$ is considered. In order to guarantee the satisfaction of Assumption 3.4 for a disturbance d_2 different from zero, term $\varrho(|d_2|)$ must be different from zero. However, note that in this case, by Assumption 3.6, $\alpha_d \equiv 0$. This fact, considering Assumption 3.5, leads to a less conservative estimation of the region Θ defined in (3.13). ■

Proof of Theorem 3.5: As shown in the proof of Theorem 3.3, Assumptions 3.1, 3.3-3.5 guarantee that the closed-loop system (3.1)-(3.17) is ISpS in $X^{MPC}(N)$. Following the steps of the proof of Theorem 3.2, it can be proven that region $\mathcal{X}_f$ is achieved in a finite time. The auxiliary control law is used when the state reaches the region $\mathcal{X}_f$ or when it starts in $\mathcal{X}_f$. By Assumption 3.4, $V_f(x)$ is an ISS-Lyapunov function in $\mathcal{X}_f$. Hence the closed-loop system with the auxiliary control law is ISS in $\mathcal{X}_f$. ISS in $\mathcal{X}_f$ is equivalent to UAG in $\mathcal{X}_f$ and LS. Since $\mathcal{X}_f$ is achieved in a finite time and system satisfies UAG property in $\mathcal{X}_f$, UAG in $\mathcal{X}^{MPC}(N)$ is obtained. Finally, by Theorem 3.1, the closed-loop system (3.1)-(3.19) is ISS in $\mathcal{X}^{MPC}(N)$ since UAG in $\mathcal{X}^{MPC}(N)$ and LS are equivalent to ISS in $\mathcal{X}^{MPC}(N)$. ■

Proof of Proposition 3.1: In the following, it will be shown that the proposed auxiliary control law satisfies Assumption 3.4. First, note that there exists a positive constant r_0 such that point 2 of Assumption 3.4 is satisfied for all x belonging to

$$\Omega_0 = \{x : |x| \leq r_0\} \subset \mathcal{X}.$$

Define

$$H(x,u,w) = V_f(f_1(x) + f_2(x)u + f_3(x)w) - V_f(x) + |z|^2 - \gamma^2|w|^2. \quad (3.29)$$

Then

$$\begin{aligned}
H(x,u,w) &= (f_1(x) + f_2(x)u + f_3(x)w)^\top P(f_1(x) + f_2(x)u + f_3(x)w) \\
&\quad -x^\top Px + h_1(x)^\top h_1(x) + u^\top u - \gamma^2 w^\top w \\
&= \left(f_1(x)^\top Pf_1(x) - x^\top Px + h_1(x)^\top h_1(x)\right) \\
&\quad +u^\top \left(f_2(x)^\top Pf_2(x) + I\right) u + w^\top \left(f_3(x)^\top Pf_3(x) - \gamma^2\right) w \\
&\quad +2 \begin{bmatrix} u^\top & w^\top \end{bmatrix} \begin{bmatrix} f_2(x)^\top Pf_1(x) \\ f_3(x)^\top Pf_1(x) \end{bmatrix} + 2u^\top F_2(x)^\top PF_3(x)w \\
&= \left(f_1(x)^\top Pf_1(x) - x^\top Px + h_1(x)^\top h_1(x)\right) \\
&\quad + \begin{bmatrix} u^\top & w^\top \end{bmatrix} R(x) \begin{bmatrix} u \\ w \end{bmatrix} + 2 \begin{bmatrix} u^\top & w^\top \end{bmatrix} \begin{bmatrix} f_2(x)^\top Pf_1(x) \\ f_3(x)^\top Pf_1(x) \end{bmatrix}
\end{aligned}$$

and, computing $H(x,u,w)$ for $u = \kappa^*(x)$ and $w = \xi^*(x)$,

$$\begin{aligned}
&H(x,\kappa^*(x),\xi^*(x)) = \\
&\left(f_1(x)^\top Pf_1(x) - x^\top Px + h_1(x)^\top h_1(x)\right) + \begin{bmatrix} \kappa^*(x)^\top & \xi^*(x)^\top \end{bmatrix} R(x) \cdot \\
&\cdot \begin{bmatrix} \kappa^*(x) \\ \xi^*(x) \end{bmatrix} - 2 \begin{bmatrix} \kappa^*(x)^\top & \xi^*(x)^\top \end{bmatrix} R(x) \begin{bmatrix} \kappa^*(x) \\ \xi^*(x) \end{bmatrix} \\
&= \left(f_1(x)^\top Pf_1(x) - x^\top Px + h_1(x)^\top h_1(x)\right) \\
&- \begin{bmatrix} f_1(x)^\top Pf_2(x) & f_1(x)^\top Pf_3(x) \end{bmatrix} R(x)^{-1} \begin{bmatrix} f_2(x)^\top Pf_1(x) \\ f_3(x)^\top Pf_1(x) \end{bmatrix}
\end{aligned}$$

From (iii) it follows that there exist positive constants ε, r such that

$$H(x,\kappa^*(x),\xi^*(x)) \leq -\varepsilon|x|^2, \ \textit{for all } x \in \Omega_1 = \{x : |x| \leq r\} \subset \Omega_0. \quad (3.30)$$

By the Taylor expansion Theorem (note that the first order term is evaluated in $(u, w) = (\kappa^*(x), \xi^*(x))$ and terms of order > 2 are null)

$$H(x,u,w) = H(x,\kappa^*(x),w^*) + \frac{1}{2}\begin{bmatrix} u-\kappa^*(x) \\ w-\xi^*(x) \end{bmatrix}^\top R(x) \begin{bmatrix} u-\kappa^*(x) \\ w-\xi^*(x) \end{bmatrix}$$

Assumption (ii) implies that there exists a neighborhood $\mathcal{X}_{ne}$ of $x = 0$ such that for all $x \in \mathcal{X}_{ne}$, $r_{22}(x) < 0$. If the system is controlled by $u = \kappa^*(x)$ then

$$H(x,\kappa^*(x),w) = H(x,\kappa^*(x),\xi^*(x)) + \frac{1}{2}(w-\xi^*(x))^\top r_{22}(x)(w-\xi^*(x)).$$

Since $r_{22}(x) < 0$, it follows that, given $\mathcal{X}_{ne}$, there exist an open neighborhoods $\mathcal{W}_{ne}$ of $w = 0$ such that

$$H(x,\kappa^*(x),w) \le H(x,\kappa^*(x),\xi^*(x)),\ \forall x \in \mathcal{X}_{ne},\ \forall w \in \mathcal{W}_{ne}.$$

In view of (3.30), there exists a positive constant r_2 such that

$$H(x,\kappa^*(x),w) \le H(x,\kappa^*(x),\xi^*(x)) \le -\varepsilon|x|^2 \tag{3.31}$$

for all $x \in \Omega_2 = \{x : |x| \le r_2\} \subset \Omega_1$ and for all $w \in \mathcal{W}_{ne}$. Let choose some $\beta > 0$ such that

$$\Omega_\beta = \{x : V_f(x) \le \beta\} \subset \Omega_2. \tag{3.32}$$

From (3.29), (3.31), (3.32) follows that

$$\begin{aligned} V_f(f_1(x)+f_2(x)\kappa^*(x)+f_3(x)w) &\le V_f(x) - |z|^2 + \gamma^2|w|^2 - \varepsilon|x|^2 \\ &\le V_f(x) - |z|^2 + \gamma^2|w|^2 \end{aligned}$$

for all $x \in \Omega_\beta$ and all $w \in \mathcal{W}_{ne}$. Hence, if $\mathcal{H}_\infty$ strategy is used with $l(x,u,d_1,d_2) = |z|^2 - \gamma^2|w|^2$ and $\varrho \equiv 0$, point 5 of Assumption 3.4 is satisfied. Consider now the case of $l(x,u,d_1,d_2) = |z_l|^2$ as defined in point $b)$ of Proposition 3.1 (standard min-max stage cost)

$$\begin{aligned} V_f(f_1(x)+f_2(x)\kappa^*(x)+f_3(x)w) &\le V_f(x) - |z|^2 + \gamma^2|w|^2 \\ &= V_f(x) - h_1(x)^\top h_1(x) - u^\top u \\ &\quad +\gamma^2 d_1^\top(x)d_1(x) + \gamma^2 d_2^\top d_2. \end{aligned}$$

Using point (i) of Proposition 3.1

$$\begin{aligned} V_f(f_1(x) + f_2(x)\kappa^*(x) + f_3(x)w) &\leq V_f(x) - h_1(x)^\top h_1(x) - u^\top u \\ &\quad + \gamma^2(\mathcal{K}_{dx}|x|^2 + \mathcal{K}_{du}|u|^2) + \gamma^2 d_2^\top d_2 \\ &\leq V_f(x) - h_1(x)^\top h_1(x) - u^\top u + a|x|^2 \\ &\quad + b|u|^2 + \gamma^2 d_2^\top d_2 \end{aligned}$$

with $a \triangleq \gamma^2 \mathcal{K}_{dx}$ and $b \triangleq \gamma^2 \mathcal{K}_{du}$. Considering that point 2 of Assumption 3.4 is locally satisfied

$$\begin{aligned} V_f(f_1(x) + f_2(x)\kappa^*(x) + f_3(x)w) &\leq V_f(x) - h_1(x)^\top h_1(x) - u^\top u \\ &\quad + a|x|^2 + c|x|^2 + \gamma^2 d_2^\top d_2 \end{aligned}$$

with $c \triangleq bL^2_{\kappa_f}$. Then, using point b) of Proposition 3.1

$$\begin{aligned} V_f(f_1(x) + f_2(x)\kappa^*(x) + f_3(x)w) &\leq V_f(x) - h_l(x)^\top h_l(x) - u^\top u + \gamma^2 |d_2|^2 \\ &\leq V_f(x) - h_l(x)^\top h_l(x) - u^\top u + \varrho(|d_2|) \\ &= V_f(x) - l(x, u) + \varrho(|d_2|) \end{aligned}$$

for all $x \in \Omega_\beta$, and all $w \in \mathcal{W}_{ne}$ where $\varrho(s) \triangleq \gamma^2 |s|^2$. Hence point 5 of Assumption 3.4 is satisfied for the standard min-max stage cost too.

In order to verify that Assumption 3.4 is satisfied, it remains to prove points $1, 3, 4$. Point 1 is obtained if $\mathcal{X}_f \subseteq \Omega_\beta \subset \mathcal{X}$. Point 4 is obviously satisfied since $V_f(x) = x^\top P x$, with P positive definite matrix, is such that

$$\alpha_{V_f}(|x|) \triangleq \lambda_{min}(P)||x||^2 \leq x^\top P x \leq \lambda_{max}(P)||x||^2 \triangleq \beta_{V_f}(|x|)$$

where $\lambda_{min}(P)$ and $\lambda_{max}(P)$ are minimum and maximum eigenvalues of P.

In order to prove point 3, the robust invariance of $\mathcal{X}_f$, let consider the proof of Theorem 3.2. By step 1, $\Theta(\mathcal{D}_2^{sup})$ is a RPI set. Since $\Theta(\mathcal{D}_2^{sup}) \propto \mathcal{D}_2^{sup}$, there exists a set $\bar{\mathcal{W}}_{ne} = \mathcal{D}_1(x) \times \mathcal{D}_2 \subset \mathcal{W}_{ne}$ and a positive constant α such that, defining $\mathcal{X}_f$ as

$$\mathcal{X}_f \triangleq \{x : V_f(x) \leq \alpha\},$$

there is $\Theta \subset \mathcal{X}_f \subset \Omega_\beta$. It is clear that $\mathcal{X}_f$ is a RPI set.

The invariance of the closed-loop system (with the auxiliary control law) in $\mathcal{X}_f$ ends the proof that Assumption 3.4 is satisfied. ■

Chapter 4

Min-Max Nonlinear Model Predictive Control: a min formulation with guaranteed robust stability

Contents

4.1 Introduction

As discussed in Chapter 3, min-max model predictive control is based on the solution of a finite-horizon game, where u is the input of the minimizing player (the controller) and w is the input of the maximizing player (the nature). The controller chooses the input u_k as a function of the current state x_k so as to ensure constraint satisfaction along the predicted trajectory of the plant for any possible uncertainty, minimizing at the same time the worst case performance index of the predicted evolution of the system.

The calculation of the solution of the optimization problem results to be an NP-hard problem [Blondel & Tsitsiklis 2000], even in the case that the cost is a convex function [Scokaert & Mayne 1998] with respect to the future sequence of disturbances. This fact has limited its applications to a narrow field in spite of its benefits. This has motivated an increasing effort to find novel approaches to the min-max MPC which maintain its desirable properties with a more tractable optimization problem associated. This is typically done by using an upper bound function of the min-max cost function [Kothare *et al.* 1996, Lee & Yu 1997, Lu & Arkun 2000] or by using a relaxed maximization problem aimed to make the problem tractable [Alamo *et al.* 2005, Alamo *et al.* 2007]. The goal of this chapter is to propose a relaxed min-max predictive controller for constrained nonlinear systems. To this aim, the max stage is replaced by a simple suitable choice of an uncertain realization. This choice produces a solution that does not differ much from the original min-max problem. Moreover, the proposed predictive control is shown to inherit the convergence and the domain of attraction of the standard min-max strategy. The computational burden of this solution is much lower but at the expense of a potential minor loss of performance.

4.2 Problem statement

Assume that the plant to be controlled is described by discrete-time nonlinear dynamic model

$$x_{k+1} = f(x_k, u_k, w_k), \ k \geq 0 \tag{4.1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the current control vector, $w_k \in \mathbb{R}^p$ is the disturbance term.

The system is supposed to fulfill the following assumption.

Assumption 4.1

1. *For simplicity of notation, it is assumed that the origin is an equilibrium point, i.e.* $f(0,0,0) = 0$.

2. The disturbance w is such that

$$w \in \mathcal{W} = \{w \in \mathbb{R}^p : |w|_\infty \leq \varepsilon\} \tag{4.2}$$

where $\varepsilon > 0$.

3. The state and the control variables are restricted to fulfill the following constraints

$$x \in \mathcal{X} \tag{4.3}$$

$$u \in \mathcal{U} \tag{4.4}$$

where $\mathcal{X}$ and $\mathcal{U}$ are compact sets, both containing the origin as an interior point.

4. The state of the plant x_k can be measured at each sample time.

4.3 Min-Max Nonlinear Model Predictive Control

As discussed in Chapter 3, a practical solution, compromise between the simplicity of the open-loop min-max strategy and the performance advantages of the closed-loop one, is the so-called *semi-feedback* min-max formulation. In this case, control policies are considered as decision variables, but forcing a given structure of the control law. Thus, the decision variable of each control law is its set of defining parameters, yielding to an optimization problem similar to the open-loop case one. In the following, different control and prediction horizon, N_c and N_p respectively with $N_c \leq N_p$, will be considered. Hence, functions of x and parameters $z \in \mathbb{R}^q$, $\kappa(x_{t+j}, z_{t+j})$, $j \in [0, N_c - 1]$, will be used as feedback control strategies [Fontes & Magni 2003]. This means that, at each instant t, the controller will have only to choose $\mathbf{z}_{[t,t+N_c-1]} \triangleq [z_t, \dots, z_{t+N_c-1}]$, a sequence belonging to a finite-dimensional space. Moreover, at the end of the control horizon, i.e. in the interval $[t+N_c, t+N_p-1]$, an auxiliary state-feedback control law $u = \kappa_f(x)$ will be used. The functions κ should be such that the auxiliary control law is a particular case of the possible feedback control strategies, that is, there exists z_f such that $\kappa(x, z_f) = \kappa_f(x)$.

In the following, the optimal min-max problem can be stated.

Definition 4.1 (FHCLG) *Consider system (4.1) with $x_t = \bar{x}$. Given the positive integers N_c and N_p, the stage cost l, the terminal penalty V_f the terminal set $\mathcal{X}_f$, and the auxiliary control law κ_f, the Finite Horizon Closed-Loop Game (FHCLG) problem consists in minimizing, with respect to $\mathbf{z}_{[t,t+N_c-1]}$ and maximizing with respect to $\mathbf{w}_{[t,t+N_p-1]}$ the cost function*

$$J(\bar{x}, \mathbf{z}_{[t,t+N_c-1]}, \mathbf{w}_{[t,t+N_p-1]}) \triangleq \sum_{k=t}^{t+N_p-1} l(x_k, u_k) + V_f(x_{t+N_p}) \tag{4.5}$$

subject to:

1. *the state dynamics (4.1)*
2. *the control signal*

$$u_k = \begin{cases} \kappa(x_k, z_k), & k \in [t, t+N_c-1] \\ \kappa_f(x_k), & k \in [t+N_c, t+N_p-1] \end{cases}$$

3. *the constraints (4.2)-(4.4), $k \in [t, t+N_p-1]$*
4. *the terminal state constraints $x_{t+N_p} \in \mathcal{X}_f$.* □

Note that the parameters $\mathbf{z}_{[t,t+N_c-1]}$ have to be such that, for each possible sequence $\mathbf{w}_{[t,t+N_c-1]} \in \mathcal{M}_{\mathcal{W}}$

$$\kappa(x_k, z_k) \in \mathcal{U},\ k \in [t, t+N_c-1].$$

Moreover, the same constraint has to be satisfied between the end of the control horizon and the end of the prediction horizon, that means, for all $\mathbf{w}_{[t,t+N_p-1]} \in \mathcal{M}_{\mathcal{W}}$

$$\kappa_f(x_k) \triangleq \kappa(x_k, z_f) \in \mathcal{U},\ k \in [t+N_c, t+N_p-1].$$

Letting $\mathbf{z}^o_{[t,t+N_c-1]}$, $\mathbf{w}^o_{[t,t+N_p-1]}$ be a solution of the FHCLG, according to the RH paradigm, the feedback control law $u = \kappa^{MPC}(x)$ is obtained by setting

$$\kappa^{MPC}(x) = \kappa(x, z^o_0) \tag{4.6}$$

where z^o_0 is the first element of $\mathbf{z}^o_{[t,t+N_c-1]}$.

The stage cost defines the performance index to optimize and must satisfy the following assumption.

Assumption 4.2 *The stage cost* $l(\cdot,\cdot)$ *is such that* $l(0,0)=0$ *and* $l(x,u)\geq l(x,0)$, *for all* $x\in\mathcal{X}$, *and all* $u\in\mathcal{U}$. $\square$

In order to derive the main stability and performance properties associated to the solution of FHCLG, the following assumption is introduced.

Assumption 4.3 *The design parameters* V_f, $\mathcal{X}_f$ *are such that, given an auxiliary control law* κ_f

1. $X_f\subseteq\mathcal{X}$, $\mathcal{X}_f$ *closed,* $0\in\mathcal{X}_f$
2. $\kappa_f(x)\in\mathcal{U}$, *for all* $x\in\mathcal{X}_f$
3. $f(x,\kappa_f(x),w)\in\mathcal{X}_f$, *for all* $x\in\mathcal{X}_f$, *all* $w\in\mathcal{W}$
4. *there exist a pair of suitable* $\mathcal{K}_\infty$*-functions* α_{V_f} *and* β_{V_f} *such that* $\alpha_{V_f}<\beta_{V_f}$ *and*
$$\alpha_{V_f}(|x|)\leq V_f(x)\leq\beta_{V_f}(|x|)$$
for all $x\in\mathcal{X}_f$
5. $V_f(f(x,\kappa_f(x),w)-V_f(x)\leq -l(x,\kappa_f(x))+\varrho(|w|)$,
for all $x\in\mathcal{X}_f$ *and all* $w\in\mathcal{W}$, *where* ϱ *is a* $\mathcal{K}_\infty$*-function.*

Remark 4.1 *The computation of the auxiliary control law, the terminal penalty and the terminal inequality constraint satisfying Assumption 4.3, could be obtained by, for example, LDIs (Linear Differential Inclusions) around the origin and by using standard linear robust strategy [Alamo et al. 2005]. A solution for affine systems has been proposed in Chapter 3 and [Magni et al. 2003], where it is shown how to compute a nonlinear auxiliary control law based on the solution of a suitable* H_∞ *problem for the linearized system under control.* $\square$

In order to simplify the notation, let denote $\mathbf{z}\triangleq\mathbf{z}_{[t,t+N_c-1]}$, $\mathbf{w}\triangleq\mathbf{w}_{[t,t+N_p-1]}$ and $\mathbf{z}^o\triangleq\mathbf{z}^o_{[t,t+N_c-1]}$. In the following let denote

$$\bar{J}(x,\mathbf{z}) \quad \triangleq \quad \max\nolimits_{\mathbf{w}\in\mathcal{M}_\mathcal{W}} J(x,\mathbf{z},\mathbf{w}).$$

Moreover, let denote with $V(x)$ the cost associated with the solution of the FHCLG problem $V(x)\triangleq\min_{\mathbf{z}}\bar{J}(x,\mathbf{z})$, where the minimization with respect to $\mathbf{z}$ is carried out under the robust constraints of the FHCLG problem.

4.4 Guaranteed bound of the max function

In this section a new cost function that avoids to evaluate the maximum of FHCLG, will be proposed. For this aim, the following assumption has to be introduced.

Assumption 4.4 *Let assume $J(x, \mathbf{z}, \mathbf{w})$ be C^2 with respect to $\mathbf{w}$.*

Remark 4.2 *Note that Assumption 4.4 can be ensured, for example, if the functions $f(\cdot,\cdot,\cdot)$, $l(\cdot,\cdot)$, $V_f(\cdot)$ and $\kappa(\cdot,\cdot)$ are C^2 with respect to the variables x, u, $\mathbf{z}$ and $\mathbf{w}$. It is worth to note that Assumption 4.4 is different from requiring $V(x)$ is C^2, that, a priori is not guaranteed. Moreover, note that the requirement $\kappa(\cdot,\cdot)$ is C^2 is very different from requiring $\kappa^{MPC}(x)$ is C^2, control law that, as discussed in Chapter 1, could be discontinuous.* □

Since sets $\mathcal{X}, \mathcal{U}$, and $\mathcal{W}$ are compact, under Assumption 4.4, there exists a constant $\lambda \in [0, \infty)$ such that

$$\bar{\sigma}(\nabla^2_{\mathbf{w}} J(x, \mathbf{z}, \mathbf{w})) \leq \lambda$$

for all $x \in \mathcal{X}$, all $\mathbf{u} \in \mathcal{M}_{\mathcal{U}}$, and all $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$.

Note that the results of this chapter rely *only* on the existence of the constant λ, which means that the proposed MPC formulation can be implemented without requiring an estimation of its value. Let introduce now the following lemma.

Lemma 4.1 *Suppose that Assumptions 4.1 and 4.4 hold. Let define*

$$E_L(x, \mathbf{z}, \mathbf{w}) \triangleq J(x, \mathbf{z}, \mathbf{w}) - J(x, \mathbf{z}, 0) - [\nabla_{\mathbf{w}} J(x, \mathbf{z}, 0)]^T \mathbf{w}$$

as the error between the cost function $J(x, \mathbf{z}, \mathbf{w})$ and its linearization at $\mathbf{w} = 0$. Then

$$|E_L(x, \mathbf{z}, \mathbf{w})| \leq \frac{\sigma}{2}\varepsilon^2, \tag{4.7}$$

for all $x \in \mathcal{X}$, all $\mathbf{u} \in \mathcal{M}_U$, and all $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$, where $\sigma = pN_p\lambda$. ■

Let denote

$$\begin{aligned} \mathbf{w}_g(x, \mathbf{z}) &\triangleq arg \max\, [\nabla_{\mathbf{w}} J(x, \mathbf{z}, 0)]^T \mathbf{w} \\ &= sign\, (\nabla_{\mathbf{w}} J(x, \mathbf{z}, 0))\, \varepsilon \end{aligned} \tag{4.8}$$

the uncertainty sequence maximizing the first order approximation of the cost function $J(x, \mathbf{z}, 0)$ (remember that by constraint (4.2), $|w|_\infty \leq \varepsilon$). Define $\tilde{J}(x, \mathbf{z}) \triangleq J(x, \mathbf{z}, \mathbf{w}_g(x, \mathbf{z}))$.
Now it is possible to state the following result.

Lemma 4.2 *Consider the FHCLG given in Definition 4.5. Under Assumptions 4.1 and 4.4*

$$\bar{J}(x, \mathbf{z}) - \sigma\varepsilon^2 \leq \tilde{J}(x, \mathbf{z}) \leq \bar{J}(x, \mathbf{z}) \tag{4.9}$$

where $\sigma = pN_p\lambda$. ■

4.5 Proposed Min formulation: stability

A new formulation of the min-max model predictive control is presented in this section. The objective is to circumvent the computation of the worst-case disturbance realization, which requires the solution of a maximization problem. The idea is to evaluate the cost function only for an appropriately calculated sequence of disturbances so reducing dramatically the computational burden of the maximization problem.

Consider the proposed cost function $\tilde{J}(x, \mathbf{z})$. The new min-max problem is stated in the following definition.

Definition 4.2 (FHACLG) *Consider system (4.1) with* $x_t = \bar{x}$. *Given the positive integers* N_c *and* N_p, *the stage cost* l, *the terminal penalty* V_f *the terminal set* $\mathcal{X}_f$, *and the auxiliary control law* κ_f, *the uncertainty sequence* $\mathbf{w}_g(x, \mathbf{z})$ *as defined in (4.8), the Finite Horizon Approximated Closed-Loop Game (FHACLG) problem consists in minimizing, with respect to* $\mathbf{z}_{[t,t+N_c-1]}$, *the cost function* $\tilde{J}(x, \mathbf{z})$ *subject to:*

1. *the state dynamics (4.1)*

2. *the control signal*

$$u_k = \begin{cases} \kappa(x_k, z_k), & k \in [t, t + N_c - 1] \\ \kappa_f(x_k), & k \in [t + N_c, t + N_p - 1] \end{cases}$$

3. *the constraints (4.3)-(4.4), for all* $\mathbf{w}_{[t,t+N_p-1]} \in \mathcal{M}_\mathcal{W}$ *and all* $k \in [t, t + N_p - 1]$

4. *the terminal state constraints* $x_{t+N_p} \in \mathcal{X}_f$, *for all* $\mathbf{w}_{[t,t+N_p-1]} \in \mathcal{M}_{\mathcal{W}}$.

□

Letting $\tilde{\mathbf{z}}^o \triangleq \tilde{\mathbf{z}}^o_{[t,t+N_c-1]}$, be a solution of the FHACG, according to the RH paradigm, the feedback control law $u = \tilde{\kappa}^{MPC}(x)$ is obtained by setting

$$u = \tilde{\kappa}^{MPC}(x) = \kappa(x, \tilde{z}^o_0) \tag{4.10}$$

where $\tilde{z}^o_0$ is the first element of $\tilde{\mathbf{z}}^o_{[t,t+N_c-1]}$.

Remark 4.3 *Even if, by solving the FHACLG instead of the FHCLG problem, the computational burden of the maximization problem is dramatically reduced, the constraints satisfaction must be checked for all possible disturbances. In order to guarantee this, a tube-based formulation for the robust constraint satisfaction could be used (see for example [Limon et al. 2008]).*

□

In the following, let $\tilde{\mathcal{X}}^{MPC}$ denote the set of states for which a solution of the FHACLG problem exists.

It is clear that, since the constraints of the FHCLG and the FHACLG problems are the same, the optimal solution $\tilde{\mathbf{z}}^o$ of the FHACLG is a sub-optimal feasible solution for the FHCLG. As it is claimed in the following lemma, the difference between the optimal value of the original objective function and the value obtained with $\tilde{\mathbf{z}}^o$ is bounded by $\sigma\varepsilon^2$.

Lemma 4.3 *Denote with* $V(x)$ *and* $\tilde{V}(x) \triangleq \tilde{J}(x, \tilde{\mathbf{z}}^o)$ *the optimal solution of the FHCLG and the FHACLG problem respectively. Under Assumption 4.1 and 4.4 it results that*

1. $\bar{J}(x, \tilde{\mathbf{z}}^o) - \sigma\varepsilon^2 \le V(x) \le \bar{J}(x, \tilde{\mathbf{z}}^o)$
2. $\tilde{V}(x) \le V(x) \le \tilde{V}(x) + \sigma\varepsilon^2$. ■

In what follows, the optimal value of the performance index, i.e. $V(x)$ is employed as an ISpS-Lyapunov function for the closed-loop system (4.1), (4.10).

Assumption 4.5 *Let*

- $\Xi = \tilde{\mathcal{X}}^{MPC}$
- $\Omega = \mathcal{X}_f$
- $\alpha_1 = \alpha_l$
- $\alpha_2 = \beta_{V_f}$
- $\alpha_3 = \alpha_l$
- $\lambda_2 = \varrho$
- $c_1 = N_p \varrho(\varepsilon)$
- $c_2 = \sigma \varepsilon^2$

The set $\mathcal{W}$ is such that the set Θ (depending from ε), defined in (3.13) (in the definition replace d_2 with w) with function $V(x) = \bar{J}(x, \mathbf{z}^o)$, is contained in $I\Omega$. □

The main result can now be stated.

Theorem 4.1 *Under Assumptions 4.1-4.5, the closed-loop system formed by (4.1), (4.10), subject to constraints (4.2)-(4.4), is ISpS with respect to w with RPIA set $\tilde{\mathcal{X}}^{MPC}$.* ■

Remark 4.4 *The control applied at time t is $\tilde{\kappa}^{MPC}(x)$ and not $\kappa^{MPC}(x)$. In spite of this, the proposed controller guarantees that the optimal original cost function is an ISpS Lyapunov function for the resulting closed-loop system.*

4.6 Example

In this section, the MPC law introduced in the chapter is applied to a cart with a mass M moving on a plane (the model is the same of the paper [Magni *et al.* 2003]). This carriage (see Figure 4.1) is attached to the wall via a spring with elastic k given by $k = k_0 e^{-x_1}$, where x_1 is the displacement of the carriage from the equilibrium position associated with the external force $u = 0$ and the external disturbance force (wind force) $w = 0$. Finally a damper with damping factor h_d affects the system in a resistive way. The model of the system is given by the following continuous-time state space nonlinear model

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{k_0}{M}e^{-x_1(t)}x_1(t) - \frac{h_d}{M}x_2(t) + \frac{u(t)}{M} + \frac{w(t)}{M} \end{cases}$$

where x_2 is the carriage velocity. The parameters of the system are $M = 1$ kg, $k_0 = 0.33$ $\frac{N}{m}$, $h_d = 1.1$ $\frac{Ns}{m}$. The wind force is assumed to be bounded, $|w| \leq 0.4$ N. The state and control variables have to satisfy the following constraints $|u| \leq 4.5$ N, $|x_1| \leq 2.65$ m. An Euler approximation of the system with sampling time $T_c = 0.4$ s is given by

$$\begin{cases} x_{1_{k+1}} &= x_{1_k} + T_c x_{2_k} \\ x_{2_{k+1}} &= -T_c\frac{k_0}{M}e^{-x_{1_k}}x_{1_k} + x_{2_k} - T_c\frac{h_d}{M}x_{2_k} + T_c\frac{u_k}{M} + T_c\frac{w_k}{M} \end{cases}$$

which is a discrete-time nonlinear system. For this system the MPC control law is computed according to the algorithm presented in the paper. The auxiliary control law $\kappa_f = -Kx$, with $K = [0.5332 \quad 0.7159]$, is the LQR obtained for the system linearized in $x_{eq} = 0$, $u_{eq} = 0$, using $Q = I^{2\times 2}$, identity matrix of dimension 2×2 and $R = 1$ Let $\tilde{Q} \triangleq \beta(Q + K^\top RK)$, with $\beta = 1.2$, and denote with P the unique symmetric positive definite solution of the Lyapunov equation $(A - BK)^\top P(A - BK) - P + \tilde{Q} = 0$

$$P = \begin{bmatrix} 5.9091 & 2.8639 \\ 2.8639 & 3.6616 \end{bmatrix}$$

Using as terminal penalty $V_f(x) = x^\top Px$, the Assumption 4.3 is satisfied in the terminal set $X_f = \{x : x^\top Px \leq 6.1\}$ with $\varrho(s) = 10s$. The stage cost adopted in the cost function is quadratic, $x^\top Qx + u^\top Ru$, with Q and R the same used for the LQR. The lengths of the control and prediction horizon in the MPC implementation are respectively $N_c = 6$ and $N_p = 10$. The feedback control strategies are assumed to have the following structure $k(x, z) = -z_1\kappa_f(x) + z_2$. This means that the minimization problem, at each time, has to choose the value of the parameters z_1 and z_2. The proposed strategy, as described in Section 4.4, at each instant time, finds the maximum of the linear approximation of the cost function. The problem of solving the maximization of the FHCLG in exact manner is unrealistic, so that only an approximated solution is possible in real-time. However, considering the case of a cost function convex in $\mathbf{w}$, the exact solution of the max requires the evaluation, at each time instant, of all the vertexes of the disturbance, which in this case are $2^{N_p} = 2^{10} = 1024$. Instead, the solution presented in the paper requires only one evaluation. A tube-based formulation for the robust constraint satisfaction could be used to check that, for all possible sequence of disturbances, constraints are never violated. The following Figure 4.2 shows the evolution of the system for the initial state $x_1 = -2.61m$, $x_2 = 0\frac{m}{s}$, with a particular realization of the disturbance. The dashed lines show the minimum and maximum values corresponding to all possible disturbances along the evolution of the system.

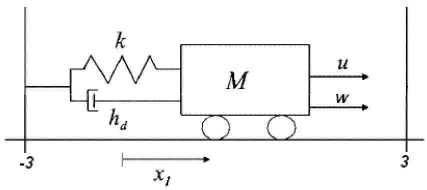
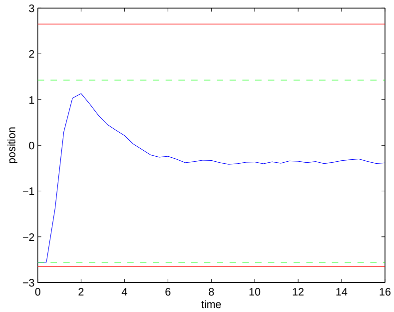


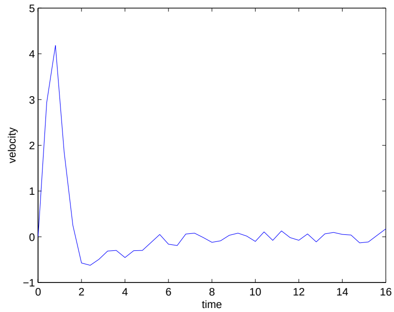
Figure 4.1: Cart and spring example.

4.7 Conclusions

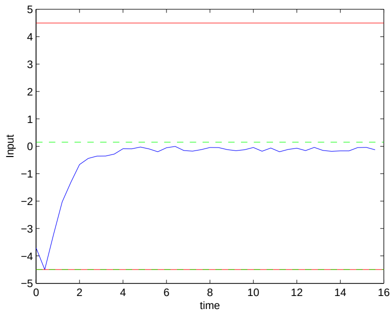
A new formulation of the min-max model predictive control has been presented in this chapter. Its objective is to make affordable the computation of the worst-case disturbance realization, which requires the solution of a maximization problem. The proposed controller is based on the simple evaluation of the cost function for an appropriately calculated sequence of future disturbances. The controller derived from the minimization of the relaxed maximization problem ensures the same domain of attraction of the min-max one and guarantees ISpS under the same conditions. In order to illustrate the proposed controller, this has been tested on a model. The implementation of the MPC for the illustrative example demonstrates the difficulty to implement a real min-max MPC compared with the proposed one. The obtained results show the stability and admissibility of the proposed controller.



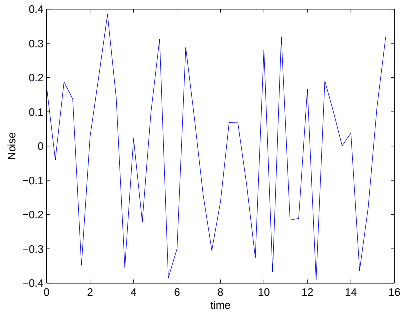
(a) Position of the cart



(b) Velocity of the cart



(c) Control applied to the cart



(d) Wind force

Figure 4.2: Evolution of the system for initial state $x_1 = -2.61m$, $x_2 = 0\frac{m}{s}$

4.8 Appendix

Proof of Lemma 4.1: using the Maclaurin series of the function $J(x, \mathbf{z}, \mathbf{w})$, for every $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$ there exists a $\hat{\mathbf{w}} \in \mathcal{M}_{\mathcal{W}}$ such that

$$J(x, \mathbf{z}, \mathbf{w}) \quad = \quad J(x, \mathbf{z}, 0) + [\nabla_{\mathbf{w}} J(x, \mathbf{z}, 0)]^T \mathbf{w} + \tfrac{1}{2}\mathbf{w}^T [\nabla^2_{\mathbf{w}} J(x, \mathbf{z}, \hat{\mathbf{w}})]\mathbf{w}.$$

This means that

$$E_L(x, \mathbf{z}, \mathbf{w}) = \frac{1}{2}\mathbf{w}^T [\nabla^2_{\mathbf{w}} J(x, \mathbf{z}, \hat{\mathbf{w}})]\mathbf{w}$$

$$|E_L(x, \mathbf{z}, \mathbf{w})| \leq \frac{1}{2}\bar{\sigma}(\nabla^2_{\mathbf{w}} J(x, \mathbf{z}, \hat{\mathbf{w}}))\mathbf{w}^T \mathbf{w}.$$

Assumptions 4.1 and 4.4 guarantee that $\bar{\sigma}(\nabla^2_{\mathbf{w}} V(x, \mathbf{z}, \hat{\mathbf{w}})) \leq \lambda$, for all $x \in \mathcal{X}$, all $\mathbf{u} \in \mathcal{M}_{\mathcal{U}}$ and all $\hat{\mathbf{w}} \in \mathcal{M}_{\mathcal{W}}$.

By this, remembering that $w \in \mathbb{R}^p$ and $\mathbf{w} \in \mathbb{R}^{pN_p}$, one has

$$|E_L(x, \mathbf{z}, \mathbf{w})| \leq \frac{\lambda}{2}\mathbf{w}^T\mathbf{w} \leq \frac{\lambda p N_p}{2}|\mathbf{w}|^2_\infty \leq \frac{\lambda p N_p}{2}\varepsilon^2 = \frac{\sigma}{2}\varepsilon^2.$$

■

Proof of Lemma 4.2: by definition,

$$J(x, \mathbf{z}, \mathbf{w}) = E_L(x, \mathbf{z}, \mathbf{w}) + J(x, \mathbf{z}, 0) + [\nabla_{\mathbf{w}} J(x, \mathbf{z}, 0)]^T \mathbf{w}.$$

Adding and subtracting the term $E_L(x, \mathbf{z}, \mathbf{w}_g(x, \mathbf{z}))$ one obtains

$$\begin{aligned} J(x, \mathbf{z}, \mathbf{w}) \quad = \quad & E_L(x, \mathbf{z}, \mathbf{w}) + J(x, \mathbf{z}, 0) + [\nabla_{\mathbf{w}} J(x, \mathbf{z}, 0)]^T \mathbf{w} \\ & + E_L(x, \mathbf{z}, \mathbf{w}_g(x, \mathbf{z})) - E_L(x, \mathbf{z}, \mathbf{w}_g(x, \mathbf{z})). \end{aligned}$$

Using Lemma 4.1, equation (4.7)

$$\begin{aligned} E_L(x, \mathbf{z}, \mathbf{w}) - E_L(x, \mathbf{z}, \mathbf{w}_g(x, \mathbf{z})) \quad & \leq \quad |E_L(x, \mathbf{z}, \mathbf{w})| + |E_L(x, \mathbf{z}, \mathbf{w}_g(x, \mathbf{z}))| \\ & \leq \quad \tfrac{\sigma}{2}\varepsilon^2 + \tfrac{\sigma}{2}\varepsilon^2 = \sigma\varepsilon^2. \end{aligned}$$

Then

$$J(x, \mathbf{z}, \mathbf{w}) \leq J(x, \mathbf{z}, 0) + [\nabla_{\mathbf{w}} J(x, \mathbf{z}, 0)]^T \mathbf{w} + E_L(x, \mathbf{z}, \mathbf{w}_g(x, \mathbf{z})) + \sigma\varepsilon^2.$$

Now, maximizing the cost with respect to $\mathbf{w}$, and considering (4.8), one has

$$\begin{aligned} & \bar{J}(x, \mathbf{z}) = \max_{\mathbf{w} \in \mathcal{M}_{\mathcal{W}}} J(x, \mathbf{z}, \mathbf{w}) \\ & \leq \max_{\mathbf{w} \in \mathcal{M}_{\mathcal{W}}} J(x, \mathbf{z}, 0) + [\nabla_{\mathbf{w}} J(x, \mathbf{z}, 0)]^T \mathbf{w} + E_L(x, \mathbf{z}, \mathbf{w}_g(x, \mathbf{z})) + \sigma\varepsilon^2 \\ & = J(x, \mathbf{z}, 0) + [\nabla_{\mathbf{w}} J(x, \mathbf{z}, 0)]^T \mathbf{w}_g(x, \mathbf{z}) + E_L(x, \mathbf{z}, \mathbf{w}_g(x, \mathbf{z})) + \sigma\varepsilon^2 \\ & = \tilde{J}(x, \mathbf{z}) + \sigma\varepsilon^2 \end{aligned}$$

and hence

$$\bar{J}(x, \mathbf{z}) - \sigma\varepsilon^2 \leq \tilde{J}(x, \mathbf{z}). \tag{4.11}$$

In order to conclude the proof, note that $\tilde{J}(x, \mathbf{z}) \leq \bar{J}(x, \mathbf{z})$, since

$$\bar{J}(x, \mathbf{z}) = \max_{\mathbf{w} \in \mathcal{M}_{\mathcal{W}}} J(x, \mathbf{z}, \mathbf{w}) \geq J(x, \mathbf{z}, \mathbf{w}_g(x, \mathbf{z})) = \tilde{J}(x, \mathbf{z})$$

■

Proof of Lemma 4.3

First claim: denoting $\mathbf{z}^o$ the optimal solution of the FHCLG and $\tilde{\mathbf{z}}^o$ the optimal solution of the FHACLG, the first inequality is obtained by Lemma 4.2, equation (4.9)

$$V(x) = \bar{J}(x, \mathbf{z}^o) \geq \tilde{J}(x, \mathbf{z}^o) \geq \tilde{J}(x, \tilde{\mathbf{z}}^o) \geq \bar{J}(x, \tilde{\mathbf{z}}^o) - \sigma\varepsilon^2.$$

The second inequality stems directly from the suboptimality of $\tilde{\mathbf{z}}^o$: $V(x) \leq \bar{J}(x, \tilde{\mathbf{z}}^o)$.

Second claim: again, from (4.9) the first inequality

$$V(x) \quad = \quad \bar{J}(x, \mathbf{z}^o) \geq \tilde{J}(x, \mathbf{z}^o) \geq \tilde{J}(x, \tilde{\mathbf{z}}^o) = \tilde{V}(x)$$

and the second one

$$\tilde{V}(x) \quad = \quad \tilde{J}(x, \tilde{\mathbf{z}}^o) \geq \bar{J}(x, \tilde{\mathbf{z}}^o) - \sigma\varepsilon^2 \geq \bar{J}(x, \mathbf{z}^o) - \sigma\varepsilon^2 = V(x) - \sigma\varepsilon^2$$

and hence $V(x) \leq \tilde{V}(x) + \sigma\varepsilon^2$. ■

Proof of Theorem 4.1: by Theorem 3.2, if system admits an ISpS-Lyapunov function in $\tilde{\mathcal{X}}^{MPC}$, then it is ISpS in $\tilde{\mathcal{X}}^{MPC}$.
In the following it will be shown that the function $V(x, N_c, N_p) = \bar{J}(x, \mathbf{z}^o, N_c, N_p)$, is an ISpS-Lyapunov function for the closed-loop system (4.1), (4.10) in $\tilde{\mathcal{X}}^{MPC}(N_c, N_p)$. For brevity, when not strictly necessary, the arguments N_c and N_p will be omitted.

The robust invariance of $\tilde{X}^{MPC}(N_c, N_p)$ is easily derived from Assumption 4.3 by taking

$$\mathbf{z}_s = \begin{cases} z_k, & k \in [t+1, t+N_c-1] \\ z_f, & k \in [t+N_c, t+N_p] \end{cases}$$

as policy parameter vector at time $t+1$ starting from the sequence $\tilde{\mathbf{z}}^o_{[t,t+N_c-1]} = \arg\min_{\mathbf{z}} \tilde{J}(x, \mathbf{z}, N_c, N_p)$ at time t. By the optimal solution of the FHACLG at time t, there is $x_{t+N_p} \in \mathcal{X}_f$, for all $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$. By Assumption 4.3, applying $\kappa_f(x)$ at time $t+N_p$, $x_{t+N_p+1} \in \mathcal{X}_f$, for all $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$ and hence, the policy parameters $\mathbf{z}_s$ is admissible; therefore $\tilde{\mathcal{X}}^{MPC}(N_c, N_p)$ is a RPIA set for the closed-loop system.

Then, using Assumption 4.2, the lower bound is easily obtained

$$\begin{aligned} V(x) &= J(x, \mathbf{z}^o, \mathbf{w}^o) \geq \min_{\mathbf{z}} J(x, \mathbf{z}, 0) \\ &\geq l(x, \kappa(x, z_0)) \geq \alpha_l(|x|) \end{aligned} \tag{4.12}$$

for all $x \in \tilde{\mathcal{X}}^{MPC}$.

In order to derive the upper bound, consider the following policy vector $\hat{\mathbf{z}}_{[t,t+N_c]} \triangleq [\mathbf{z}^o_{[t,t+N_c-1]} \; z_f]$ as admissible policy vector for the FHCLG at time t with horizons $N_c + 1$ and $N_p + 1$. Then

$$\begin{aligned} &J(x, \hat{\mathbf{z}}_{[t,t+N_c]}, \mathbf{w}_{[t,t+N_p]}, N_c + 1, N_p + 1) = \\ &\sum_{k=t}^{t+N_p-1} l(x_k, u_k) + V_f(x_{t+N_p}) - V_f(x_{t+N_p}) + V_f(x_{t+N_p+1}) \\ &+ l(x_{t+N_p}, \kappa_f(x_{t+N_p})). \end{aligned}$$

In view of Assumption 4.3

$$\begin{aligned} &J(x, \hat{\mathbf{z}}_{[t,t+N_c]}, \mathbf{w}_{[t,t+N_p]}, N_c + 1, N_p + 1) \leq \\ &\sum_{k=t}^{t+N_p-1} l(x_k, u_k) + V_f(x_{t+N_p}) + \varrho(|w|) \end{aligned}$$

which implies

$$\begin{aligned} V(x, N_c + 1, N_p + 1) &\leq \max_{\mathbf{w} \in \mathcal{M}_{\mathcal{W}}} J(x, \hat{\mathbf{z}}_{[t,t+N_c]}, \mathbf{w}_{[t,t+N_p]}, N_c + 1, N_p + 1) \\ &\leq \max_{\mathbf{w} \in \mathcal{M}_{\mathcal{W}}} \sum_{k=t}^{t+N_p-1} l(x_k, u_k) + V_f(x_{t+N_p}) + \varrho(|w|) \\ &= V(x, N_c, N_p) + \varrho(\varepsilon) \end{aligned}$$

which holds for all $x \in \tilde{\mathcal{X}}^{MPC}(N_c, N_p)$, and all $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$.

Therefore, using Assumption 4.3, the upper bound is obtained

$$\begin{aligned} V(x, N_c, N_p) &\leq V(x, N_c - 1, N_p - 1) + \varrho(\varepsilon) \leq \ldots \leq V(x, 0) + N_p \varrho(\varepsilon) \\ &= V_f(x) + N_p \varrho(\varepsilon) \leq \beta_{V_f}(|x|) + N_p \varrho(\varepsilon) \end{aligned} \tag{4.13}$$

for all $x \in \mathcal{X}_f$.

Now, in order to find the bound on ΔV, define

$$\tilde{\mathbf{w}} = \arg\max_{\mathbf{w}} J(x_{t+1}, \mathbf{z}_s, \mathbf{w}, N_c, N_p)$$

with $\mathbf{z}_s$ defined as previously, and take $\bar{\mathbf{w}} = (w_t, \tilde{w}_{t+1}, \cdots, \tilde{w}_{t+N_p-2})$. From the feasibility of $\tilde{\mathbf{z}}^o$, it is easy to derive that

$$\begin{array}{rcl} J(x_{t+1}, \mathbf{z}_s, \tilde{\mathbf{w}}) - J(x_t, \tilde{\mathbf{z}}^o, \bar{\mathbf{w}}) & \leq & V_f(x_{t+N_p+1}) - V_f(x_{t+N_p}) \\ & & +l(x_{t+N_p}, \kappa_f(x_{t+N_p})) \\ & & -l(x_t, \tilde{\kappa}^{MPC}(x_t)). \end{array}$$

Then, by Assumption 4.3

$$J(x_{t+1}, \mathbf{z}_s, \tilde{\mathbf{w}}) - J(x_t, \tilde{\mathbf{z}}^o, \bar{\mathbf{w}}) \leq -l(x_t, \tilde{\kappa}^{MPC}(x_t)) + \varrho(|w_t|).$$

By optimality of $\tilde{\mathbf{w}}$

$$J(x_{t+1}, \mathbf{z}_s, \tilde{\mathbf{w}}) = \bar{J}(x_{t+1}, \mathbf{z}_s) \geq V(x_{t+1}).$$

In virtue of Lemma 4.3 and by the optimality of $\tilde{\mathbf{z}}^o$

$$\begin{array}{rcl} J(x_t, \tilde{\mathbf{z}}^o, \bar{\mathbf{w}}) & \leq & \bar{J}(x_t, \tilde{\mathbf{z}}^o) \leq \tilde{J}(x_t, \tilde{\mathbf{z}}^o) + \sigma\varepsilon^2 \\ & = & \tilde{V}(x_t) + \sigma\varepsilon^2 \leq V(x_t) + \sigma\varepsilon^2. \end{array}$$

Therefore, it is possible to derive that

$$V(x_{t+1}) - V(x_t) \leq -l(x_t, \tilde{\kappa}^{MPC}(x_t)) + \varrho(|w_t|) + \sigma\varepsilon^2. \tag{4.14}$$

for all $x \in \tilde{\mathcal{X}}^{MPC}$, and all $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$.

Therefore, by (4.12), (4.13), (4.14), in view also of Assumption 4.5, $V(x)$ is an ISpS-Lyapunov function for the closed-loop system (4.1), (4.10) in $\tilde{\mathcal{X}}^{MPC}(N_c, N_p)$ and hence, by Theorem 3.2, the closed-loop system formed by (4.1) and (4.10), subject to constraints (4.2)-(4.4), is ISpS with respect to w with RPIA set $\tilde{\mathcal{X}}^{MPC}(N_c, N_p)$. ■

CHAPTER 5

Robust MPC of Nonlinear Systems with Bounded and State-Dependent Uncertainties

Contents

5.1 Introduction

Nominal MPC, even if it has important inherent robustness properties in the unconstrained case, cannot guarantee the robust satisfaction of state constraints. In order to achieve this goal, it is necessary to introduce some knowledge on the uncertainty in the optimization problem. Min-max model predictive control, is very computational demanding and the range of processes to which it can be applied is limited to those that are small-size and with very slow dynamics (see for example [Lee & Yu 1997, Scokaert & Mayne 1998, Bemporad *et al.* 2003, Magni *et al.* 2003]). A more efficient technique is based on solving the nominal open-loop optimization problem using tightened state constraints in order to guarantee that the original constraints are fulfilled from the real system for any possible realization of the uncertainties. The idea was introduced in [Chisci *et al.* 2001] for linear systems and applied to nonlinear systems in [Limon *et al.* 2002a] and [Raimondo & Magni 2006]. The main drawback of this open-loop strategy is the large spread of trajectories along the optimization horizon due to the effect of the disturbances. In order to reduce this, a closed-loop term was considered for nonlinear systems in [Raković *et al.* 2006b], where the concept of tube is explored. All this works consider additive disturbance. If the system is affected by state dependent disturbances, and the state is limited in a compact set, it is always possible to find the maximum value of the disturbance and to apply the algorithms described in [Limon *et al.* 2002a, Raimondo & Magni 2006, Raković *et al.* 2006b]. However, if the particular structure of the disturbance is considered, significative advantages can be clearly obtained.

The goal of this chapter is to modify the algorithm presented in [Limon *et al.* 2002a] in order to efficiently consider state dependent disturbances. The restricted sets are computed on-line iteratively by exploiting the state sequence obtained by the open-loop optimization, thus accounting for a possible reduction of the state dependent component of the uncertainty due to the control action. In this regard, it is possible to show that the devised technique yields to an enlarged feasible region compared to the one obtainable if just an additive disturbance approximation is considered. Moreover, in the proposed algorithm, in order to limit the spread of the trajectories, 1) the horizon, along which the propagation of the uncertainty must be taken into account, is reduced by using a control horizon shorter than the prediction one, 2) the terminal constraint is imposed at the end of the control horizon. On the contrary, a long prediction horizon, is useful to better approximate the performance of the Infinite Horizon control law (see e.g. [Magni *et al.* 2001a]). In order to analyze the stability properties in the presence of bounded persistent disturbances, the concept of regional ISS is used in order to show that the obtained closed-loop system is regional ISS with respect to the bounded persistent disturbance. The robustness with respect to state dependent disturbance is analyzed using the stability margin concept.

5.2 Problem formulation

Assume that the plant to be controlled is described by discrete-time nonlinear dynamic model

$$x_{k+1} = f(x_k, u_k, \upsilon_k), \ k \geq 0\,, \tag{5.1}$$

where $x_k \in \mathbb{R}^n$ denotes the system state, $u_k \in \mathbb{R}^m$ the control vector and $\upsilon_k \in \Upsilon \subseteq \mathbb{R}^r$ an exogenous input which models the disturbance, with Υ compact set containing the origin. Given the system (5.1), let $\hat{f}(x_k, u_k)$, with $\hat{f}(0,0) = 0$, denote the *nominal* model used for control design purposes, such that

$$x_{k+1} = \hat{f}(x_k, u_k) + d_k, \ k \geq 0\,, \tag{5.2}$$

where $d_k = d_k(x_k, u_k, \upsilon_k) \triangleq f(x_k, u_k, \upsilon_k) - \hat{f}(x_k, u_k) \in \mathbb{R}^n$ denotes the discrete-time state transition uncertainty. In the sequel, for the sake of brevity, we will not point out the functional dependence of $d_k(x_k, u_k, \upsilon_k)$ on its arguments except where strictly needed.

The system is supposed to fulfill the following assumptions.

Assumption 5.1

1. *For simplicity of notation, it is assumed that the origin is an equilibrium point, i.e.* $\hat{f}(0,0) = 0$.

2. *The state and control variables are restricted to fulfill the following constraints*

$$x \in \mathcal{X} \tag{5.3}$$

$$u \in \mathcal{U} \tag{5.4}$$

where $\mathcal{X}$ *and* $\mathcal{U}$ *are compact subsets of* $\mathbb{R}^n$ *and* $\mathbb{R}^m$*, respectively, containing the origin as an interior point.*

3. *The map* $\hat{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *is Lipschitz in* x *in the domain* $\mathcal{X} \times \mathcal{U}$*, i.e. there exists a positive constant* $\mathcal{L}_{f_x}$ *such that*

$$|\hat{f}(a,u) - \hat{f}(b,u)| \leq \mathcal{L}_{f_x} |a - b| \tag{5.5}$$

for all $a, b \in \mathcal{X}$ *and all* $u \in \mathcal{U}$.

4. The state of the plant x_k can be measured at each sample time.

Assumption 5.2 *The additive transition uncertainty d_k is limited in a time varying compact ball $\mathcal{D}_k$, that is*

$$d_k(x_k, u_k, \upsilon_k) \in \mathcal{D}_k \triangleq \mathcal{B}(\delta(|x_k|) + \mu(\Upsilon^{sup}))$$

for all $x_k \in \mathcal{X}$, all $u_k \in \mathcal{U}$, and all $\upsilon_k \in \Upsilon$, where δ and μ are two $\mathcal{K}$-functions. The $\mathcal{K}$-function δ is such that $L_\delta \triangleq \min\{L \in \mathbb{R}_{>0} : \delta(|x|) \leq L|x|, \forall x \in \mathcal{X}\}$ exists finite. It follows that d_k is bounded by the sum of two contributions: a state-dependent component and a non-state dependent one. □

The control objective consists in designing a state-feedback control law capable to achieve ISS closed-loop stability and to satisfy state and control constraints in presence of state dependent uncertainties and persistent disturbances.

In order to introduce the MPC algorithm formulated according to an open-loop strategy, first let $\mathbf{u}_{[t_2,t_3|t_1]} \triangleq [u_{t_2|t_1} \; u_{t_2+1|t_1} \dots u_{t_3|t_1}]$, with $t_1 \leq t_2 \leq t_3$, be a control sequence. Moreover, given $k \geq 0$, $j \geq 1$, let $\hat{x}_{k+j|k}$ be the predicted state at $k+j$ obtained with the nominal model $\hat{f}(x_k, u_k)$, with initial condition x_k and input $\mathbf{u}_{[k,k+j-1|k]}$. Then, the following Finite-Horizon Optimal Control Problem (FHOCP) can be stated.

Definition 5.1 (FHOCP) *Consider system (5.1) with $x_t = \bar{x}$. Given the positive integer N_c, N_p, with $N_c \leq N_p$, the stage cost l, the terminal penalty V_f, an auxiliary control law κ_f, the terminal set $\mathcal{X}_{N_c}$ and a sequence of constraint sets $\hat{\mathcal{X}}_{k|t} \subseteq \mathcal{X}$, $k \in [t, \dots, t+N_c-1]$ (to be described later on) the Finite Horizon Optimal Control Problem (FHOCP) consists in minimizing, with respect to $\mathbf{u}_{[t,t+N_c-1|t]}$, the performance index*

$$\begin{aligned} &J(\bar{x}, \mathbf{u}_{[t,t+N_c-1|t]}, N_c, N_p) \triangleq \\ &\sum_{k=t}^{t+N_c-1} l(\hat{x}_{k|t}, u_{k|t}) + \sum_{k=t+N_c}^{t+N_p-1} l(\hat{x}_{k|t}, \kappa_f(x_{k|t})) + V_f(\hat{x}_{t+N_p|t}) \end{aligned} \tag{5.6}$$

subject to

1. *the nominal state dynamics $\hat{x}_{k+1} = \hat{f}(\hat{x}_k, u_k)$, with $\hat{x}_t = \bar{x}$*

2. *the control constraint (5.4) for all* $k \in [0, \ldots, N_p - 1]$

3. *the state constraints* $\hat{x}_{t+j|t} \in \hat{\mathcal{X}}_{k|t}$*, for all* $k \in [t, \ldots, t + N_c - 1]$

4. *the terminal state constraints* $\hat{x}_{t+N_c|t} \in \mathcal{X}_{N_c}$

5. *the auxiliary control law* $u_{k|t} = \kappa_f(\hat{x}_{k|t})$*, for all* $k \in [t + N_c, \ldots, t + N_p - 1]$. □

The stage cost defines the performance index to optimize and satisfies the following assumption.

Assumption 5.3 *The stage cost function* $l(x, u)$ *is such that* $l(0, 0) = 0$*, and* $\underline{l}(|x|) \leq l(x, u)$*, for all* $x \in \mathcal{X}$*, all* $u \in \mathcal{U}$*, where* $\underline{l}$ *is a* $\mathcal{K}_\infty$*-function. Moreover,* $l(x, u)$ *is Lipschitz with respect to* x *and* u *in* $\mathcal{X} \times \mathcal{U}$*, with Lipschitz constants* $\mathcal{L}_l \in \mathbb{R}_{\geq 0}$ *and* $\mathcal{L}_{lu} \in \mathbb{R}_{\geq 0}$ *respectively.* □

The usual RH control technique can now be stated as follows: at every time instants t, given $x_t = \bar{x}$, find the optimal control sequence $\mathbf{u}^o_{[t,t+N_c-1|t]}$ by solving the FHOCP. Then, according to the Receding Horizon (RH) strategy, define

$$\kappa^{MPC}(\bar{x}) \triangleq u^o_{t|t}(\bar{x})$$

where $u^o_{t|t}(\bar{x})$ is the first column of $\mathbf{u}^o_{[t,t+N_c-1|t]}$, and apply the control law

$$u = \kappa^{MPC}(x). \tag{5.7}$$

With particular reference to the underlined definition of the FHOCP, note that, with respect to the usual formulation, in this case the constraint sets are defined only within the control horizon and the terminal constraint is stated at the end of the control horizon. Another peculiarity is the use of a state constraint that changes along the horizon. In the following, it will be shown how to choose accurately the stage cost l, the terminal cost function V_f, the control and prediction horizon N_c and N_p, the constraint sets $\hat{\mathcal{X}}_{k|t}$, $k \in [t, \ldots, t + N_c - 1]$, the terminal constraint $\mathcal{X}_{N_c}$ and the auxiliary control law κ_f in order to guarantee closed-loop ISS. In particular the set $\mathcal{X}_{N_c}$ will be chosen such that, starting from any $x \in \mathcal{X}_{N_c}$ in $N_p - N_c$ steps the auxiliary control law can steer the state of the nominal system into a set $\mathcal{X}_f$ which satisfies the assumption asked for the terminal set of standard stabilizing MPC control algorithm [Mayne *et al.* 2000]. In the following, $\mathcal{X}^{MPC}$ will denote the set containing all the state vectors for which a feasible control sequence exists, i.e. a control sequence $\mathbf{u}_{[t,t+N_c-1|t]}$ satisfying all the constraints of the FHOCP.

5.3 Robust MPC strategy

In order to formulate the robust MPC algorithm, let introduce the following further assumptions.

Assumption 5.4 *Given an auxiliary control law κ_f, and a set $\mathcal{X}_f$, the design parameters V_f is such that*

1. $\mathcal{X}_f \subseteq \mathcal{X}$, $\mathcal{X}_f$ *closed,* $0 \in \mathcal{X}_f$
2. $\kappa_f(x) \in \mathcal{U}$, *for all* $x \in \mathcal{X}_f$*;* $\kappa_f(x)$ *is Lipschitz in* $\mathcal{X}_f$*, with constant* $\mathcal{L}_{\kappa_f} \in \mathbb{R}_{>0}$
3. *the closed loop map* $\hat{f}(x, \kappa_f(x))$*, is Lipschitz in* $\mathcal{X}_f$ *with constant* $\mathcal{L}_{f_c} \in \mathbb{R}_{>0}$
4. $\hat{f}(x, \kappa_f(x)) \in \mathcal{X}_f$*, for all* $x \in \mathcal{X}_f$
5. $V_f(x)$ *is Lipschitz in* $\mathcal{X}_f$*, with constant* $\mathcal{L}_{V_f} \in \mathbb{R}_{>0}$
6. $V_f(\hat{f}(x, \kappa_f(x))) - V_f(x) \leq -l(x, \kappa_f(x))$*, for all* $x \in \mathcal{X}_f$
7. $\tilde{\mathbf{u}}_{[t,t+N_p-1|t]} \triangleq [\kappa_f(\hat{x}_{t|t}),\ \kappa_f(\hat{x}_{t+1|t}), \ldots,\ \kappa_f(\hat{x}_{t+N_p-1|t})]$*, with* $\hat{x}_{t|t} = x_t$*, is a feasible control sequence for the FHOCP, for all* $x_t \in \mathcal{X}_f$*.*

Assumption 5.5 *The robust terminal constraint set of the FHOCP,* $\mathcal{X}_{N_c}$*, is chosen such that*

1. $\mathcal{X}_{N_c} \supseteq \mathcal{X}_f$
2. *for all* $x \in \mathcal{X}_{N_c}$ *the state can be steered to* $\mathcal{X}_f$ *in* $N_p - N_c$ *steps under the nominal dynamics in closed-loop with the auxiliary control law* κ_f
3. *there exists a positive scalar* $\varepsilon \in \mathbb{R}_{>0}$ *such that* $\hat{f}(x, \kappa_f(x)) \in \mathcal{X}_{N_c} \backsim \mathcal{B}(\varepsilon)$*, for all* $x \in \mathcal{X}_{N_c}$

4. $\tilde{\mathbf{u}}_{[t,t+N_p-1|t]} \triangleq [\kappa_f(\hat{x}_{t|t}),\ \kappa_f(\hat{x}_{t+1|t}), \ldots,\ \kappa_f(\hat{x}_{t+N_p-1|t})]$, *with* $\hat{x}_{t|t}{=}x_t$, *is a feasible control sequence for the FHOCP, for all* $x_t \in \mathcal{X}_{N_c}$. □

5.3.1 Shrunk State Constraints

In the following, under Assumption 5.2, given x_t, a norm-bound on the state prediction error will be derived. Subsequently, it is shown that the satisfaction of the original state constraints is ensured, for any admissible disturbance sequence, by imposing restricted constraints to the predicted open-loop trajectories.

Throughout this section, the following notation will be used: given an optimal sequence $\mathbf{u}^{\circ}_{[t,t+N_c-1|t]}$ of control actions obtained by solving the FHOCP at time t, let define the sequence $\bar{\mathbf{u}}_{[t+1,t+N_c|t+1]} \triangleq [u^{\circ}_{t+1|t}, \ldots, u^{\circ}_{t+N_c-1|t}, \bar{u}]$, where $\bar{u} \in \mathcal{U}$ is a suitably defined feasible control action implicitly depending on $\hat{x}_{t+N_c|t+1}$. The following result will be instrumental for the subsequent analysis.

Lemma 5.1 *Under Assumptions 5.1 and 5.2, given the state vector* x_t *at time* t*, let a control sequence,* $\bar{\mathbf{u}}_{[t,t+N_c-1|t]}$*, be feasible with respect to the restricted state constraints of the FHOCP,* $\hat{\mathcal{X}}_{k|t}$*, computed as follows*

$$\hat{\mathcal{X}}_{k|t} \triangleq \mathcal{X} \backsim \mathcal{B}(\hat{\rho}_{k|t}), \tag{5.8}$$

where

$$\begin{cases} \hat{\rho}_{t+1|t} & \triangleq \bar{\mu} + L_\delta |x_t|, \\ \hat{\rho}_{k|t} & = (L_\delta + \mathcal{L}_{f_x})\hat{\rho}_{k-1|t} + \bar{\mu} + L_\delta |\hat{x}_{k-1|t}|, \ \ j \in [t+2, \ldots, t+N_c-1] \end{cases} \tag{5.9}$$

with $\bar{\mu} \triangleq \mu(\Upsilon^{\sup})$*. Then, the sequence* $\bar{\mathbf{u}}_{[t,t+N_c-1|t]}$*, applied to the perturbed system (5.1), guarantees* $x_k \in \mathcal{X}$*, for all* $k \in [t+1, \ldots, t+N_c-1]$*, all* $x_t \in \mathcal{X}^{MPC}$*, and all* $\upsilon \in \mathcal{M}_\Upsilon$*.* □

Remark 5.1 *The constraint tightening (5.8), compared to previous approaches [Limon et al. 2002a, Raimondo & Magni 2006], may lead to less conservative computations. In fact, rather then using only the state information* x_t *at time* t*, it relies on the whole predicted state sequence* $\hat{x}_{k|t}, k \in [t+1, \ldots, t+N_c-1]$*, thus accounting for a possible reduction of the state-dependent component of the uncertainty along the horizon. The* *effectiveness of the proposed approach in enlarging the feasible region of the FHOCP will be shown in Section 5.4 by a simulation example.*

5.3.2 Feasibility

In order to show that $\mathcal{X}^{MPC}$ is a RPIA set, under the closed loop dynamics given by (5.1) and (5.7), an upper norm bound for the maximal admissible uncertainty will be stated in Assumption 5.6, motivated by the following Definition 5.2 and Lemma 5.2.

Definition 5.2 ($\mathcal{P}(\Xi)$) *Given a set $\Xi \subset \mathcal{X}$, the (one-step) predecessor set, $\mathcal{P}(\Xi)$, is defined as $\mathcal{P}(\Xi) \triangleq \left\{ x \in \mathbb{R}^n \middle| \exists u \in \mathcal{U} : \hat{f}(x,u) \in \Xi \right\}$, i.e., $\mathcal{P}(\Xi)$ is the set of states which can be steered to Ξ by a control action under $\hat{f}(x_t, u_t)$, subject to (5.4).*

Lemma 5.2 *Consider that Assumptions 5.1 and 5.5 hold. Given a set $\mathcal{X}_{N_c}$ and a positive constant ε as in Assumption 5.5, let define $\bar{d}_{\kappa_f} \triangleq \varepsilon / \mathcal{L}_{f_x}$ and $\bar{d} \triangleq \mathrm{dist}(\mathbb{R}^n \backslash \mathcal{P}(\mathcal{X}_{N_c}), \mathcal{X}_{N_c})$. Then, it holds that*

1. *$\mathcal{X}_{N_c} \subset \mathcal{X}_{N_c} \oplus \mathcal{B}(\bar{d}_{\kappa_f}) \subseteq \mathcal{P}(\mathcal{X}_{N_c})$;*
2. *$\bar{d} \geq \bar{d}_{\kappa_f}$.* ■

It must be remarked that, for nonlinear systems, the numerical computation of $\mathcal{P}(\mathcal{X}_{N_c})$ is a very difficult task, although the underlying theory is well established and many different methods have been proposed since the seminal paper [Bertsekas & Rhodes 1971]. In this regard, for some classes of nonlinear systems, there exist efficient numerical procedures for the computation of pre-images and predecessor sets (see [Bravo *et al.* 2005, Kerrigan *et al.* 2002, Raković *et al.* 2006a]).

Assumption 5.6 *The $\mathcal{K}$-functions δ and μ are such that $\delta(|x_t|)+\mu(\Upsilon^{\sup}) \leq \mathcal{L}_{f_x}^{1-N_c}\bar{d}$, for all $x_t \in \mathcal{X}$.* □

Now it will be stated and proved that $\mathcal{X}^{MPC}$, is a RPIA set under the closed-loop dynamics.

Theorem 5.1 *Let a system be described by equation (5.1) and subject to (5.3) and (5.4). Under Assumptions 5.1-5.6, the set in which the FHOCP is feasible, $\mathcal{X}^{MPC}$, is also RPI for the closed-loop system under the action of the control law given by (5.7).* ■

Remark 5.2 *With respect to previous literature [Limon et al. 2002a, Raimondo & Magni 2006], the possibility to compute $\bar{d}$ relying on $\mathcal{P}(\mathcal{X}_{N_c})$ and the use of $\mathcal{X}_{N_c}$ instead of $\mathcal{X}_f$ as stabilizing constraint set, allow to enlarge the bound on admissible uncertainties which the controller can cope with. In fact, considering that the restricted constraints are based on Lipschitz constants that are conservative since globals (or just regionals due to the presence of constraints), the use of them along a longer horizon (the entire prediction horizon) could reduce the feasible set of the FHOCP.*

5.3.3 Regional Input-to-State Stability

In the following, the stability properties of system (5.1) in closed-loop with (5.7) are analyzed.

Assumption 5.7 *The solution of closed-loop system formed by (5.2), (5.7) is continuous at $\bar{x} = 0$ and $\mathbf{w} = 0$ with respect to disturbances and initial conditions.*

In what follows, the optimal value of the performance index, i.e.

$$V(x) \triangleq J(x, \mathbf{u}^o_{[t,t+N_c-1|t]}, N_c, N_p) \tag{5.10}$$

is employed as an ISS-Lyapunov function for the closed-loop system formed by (5.1) and (5.7).

Assumption 5.8 *The stage transition cost $l(x,u)$, is such that $\alpha_3(|x_t|) \triangleq \underline{l}(|x_t|) - \varphi_x(|x_t|)$ is a $\mathcal{K}_\infty$-function for all $x_t \in \mathcal{X}^{MPC}$, with $\varphi_x(|x_t|) \triangleq \left[\mathcal{L}_l \frac{\mathcal{L}_{f_x}^{N_c}-1}{\mathcal{L}_{f_x}-1} + \mathcal{L}_{V_f}\mathcal{L}_{f_c}^{N_p-(N_c+1)}\mathcal{L}_{f_x}^{N_c} + (\mathcal{L}_l + \mathcal{L}_{lu}\mathcal{L}_{\kappa_f})\frac{\mathcal{L}_{f_c}^{N_p-(N_c+1)}-1}{\mathcal{L}_{f_c}-1}\mathcal{L}_{f_x}^{N_c}\right]\delta(|x_t|)$.*

Assumption 5.9 *The $\mathcal{K}$-functions δ and μ are such that $\delta(|x_t|)+\mu(|\upsilon_t|) \leq \mathcal{L}_{f_x}^{1-N_c}\bar{d}_{\kappa_f}$, for all $x_t \in \mathcal{X}$ and all $\upsilon_t \in \Upsilon$.*

Assumption 5.10 *Let*

- $\Xi = \mathcal{X}^{MPC}$
- $\Omega = \mathcal{X}_f$
- $\alpha_1(s) = \underline{l}(s)$
- $\alpha_2(s) = \mathcal{L}_{V_f} s$
- $\alpha_3(s) = \underline{l}(s) - \varphi_x(s)$, *with* $\varphi_x(s)$ *as in Assumption 5.8*
- $\sigma(s) = \left[\mathcal{L}_l \frac{\mathcal{L}_{f_x}^{N_c}-1}{\mathcal{L}_{f_x}-1} + (\mathcal{L}_l + \mathcal{L}_{lu}\mathcal{L}_{\kappa_f})\frac{\mathcal{L}_{f_c}^{N_p-N_c-1}}{\mathcal{L}_{f_c}-1}\mathcal{L}_{f_x}^{N_c} + \mathcal{L}_{V_f}\mathcal{L}_{f_c}^{N_p-(N_c+1)}\mathcal{L}_{f_x}^{N_c}\right]\mu(s)$

The set Υ *is such that the set* Θ *(depending from* Υ^{sup}*), defined in (2.17), with function* V *given by (5.10), is contained in* $I\Omega$.

Theorem 5.2 *Under Assumptions 5.1-5.10, the system (5.1) under the action of the MPC control law (5.7) is regional ISS in* $\mathcal{X}^{MPC}$ *with respect to* $v \in \Upsilon$. ■

5.4 Simulation Results

Consider the following discrete-time model of an undamped nonlinear oscillator

$$\begin{cases} x_{1k+1} = x_{1k} + 0.05\,[-x_{2k} + 0.5\,(1 + x_{1k})\,u_k] + d_{1k} \\ x_{2k+1} = x_{2k} + 0.05\,[x_{1k} + 0.5\,(1 - 4x_{2k})\,u_k] + d_{2k} \end{cases} \tag{5.11}$$

where the subscript (i) denotes the i-th component of a vector. The uncertainty vector is given by $d_k = 1\cdot 10^{-3}x_k + v_k$, with $|v| \leq 1\cdot 10^{-4}$. System (5.11) is subject to state and input constraints (5.3) and (5.4), where the set $\mathcal{X}$ is depicted in Figure 5.1, while $\mathcal{U} \triangleq \{u \in \mathbb{R} : |u| \leq 2\}$. The Lipschitz constant of the system is $\mathcal{L}_{f_x}$=1.1390. Since affordable algorithms exist for the numerical computation of the Pontryagin difference set of polytopes, for implementation purposes the balls to be "subtracted" (in the Potryagin sense) from the constraint set $\mathcal{X}$ to obtain $\hat{\mathcal{X}}_{k|t}$, for all $k \in [t+1, \ldots, t+N_c-1]$ are outer approximated by convex parallelotopes.

A linear state feedback control law $u = \kappa_f(x) = k^\top x$, with $k \in \mathbb{R}^2$, stabilizing (5.11) in a neighborhood of the origin, can be designed as described in [Parisini *et al.* 1998]. Choosing $k = [0.5955\ \ 0.9764]^\top$ and N_c=8, the following ellipsoidal sets, $\mathcal{X}_f$ and $\mathcal{X}_{N_c}$, satisfy Assumption 5.4 and 5.5 respectively

$$\mathcal{X}_f \triangleq \left\{ x \in \mathbb{R}^n \colon x^\top \begin{bmatrix} 167.21 & -43.12 \\ -43.12 & 305.50 \end{bmatrix} x \leq 1 \right\}$$

$$\mathcal{X}_{N_c} \triangleq \left\{ x \in \mathbb{R}^n \colon x^\top \begin{bmatrix} 114.21 & -29.45 \\ -29.45 & 208.67 \end{bmatrix} x \leq 1 \right\}$$

with $\mathcal{L}_{\kappa_f}$=1.1437, $\mathcal{L}_{f_c}$=1.0504 and N_p=50. Let the stage cost l be given by $l(x,u) \triangleq x^\top \mathrm{Q} x + u^\top \mathrm{R} u$, and the final cost V_f by $V_f(x) \triangleq x^\top P x$, with

$$\mathrm{Q} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},\ \mathrm{R} = 1,\ \mathrm{P} = \begin{bmatrix} 91.56 & -23.61 \\ -23.61 & 167.28 \end{bmatrix},$$

then Assumption 5.4 is satisfied. The following values for $\bar{d}_{\kappa_f}$ and $\bar{d}$ can be computed: $\bar{d}_{\kappa_f}$=$2.3311 \cdot 10^{-4}$ and $\bar{d}$=$1.2554 \cdot 10^{-3}$. It follows that the admissible uncertainties, for which the feasibility set $\mathcal{X}^{MPC}$ is RPI under the closed-loop dynamics, are bounded by

$$\delta(|x_t|) + \mu(\Upsilon^{\sup}) \leq 5.0479 \cdot 10^{-4},$$

for all $x_t \in \mathcal{X}$. Figure 5.2 shows the closed-loop trajectories of the system under nominal conditions (dashed) and with uncertainties (solid).

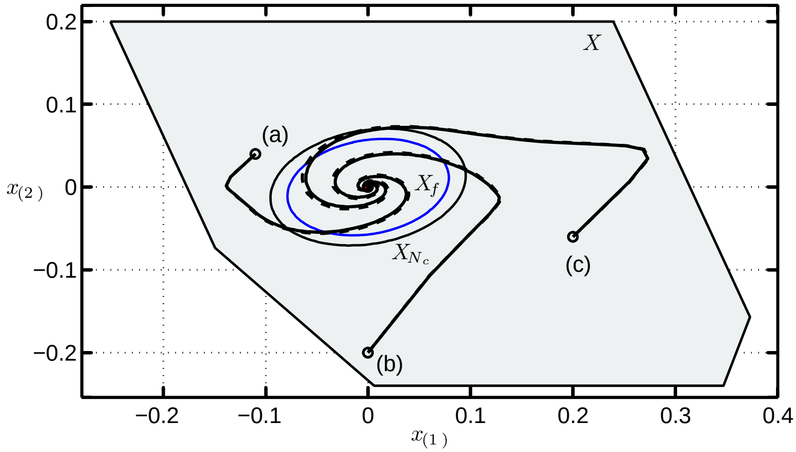


Figure 5.1: Three examples of closed-loop trajectories with initial points: (a)=$(-0.11, 0.04)^T$, (b)=$(0, -0.2)^T$, (c)=$(0.2, -0.06)^T$. The state constraint set $\mathcal{X}$, the robust constraint set $\mathcal{X}_{N_c}$ and the set $\mathcal{X}_f$ are emphasized

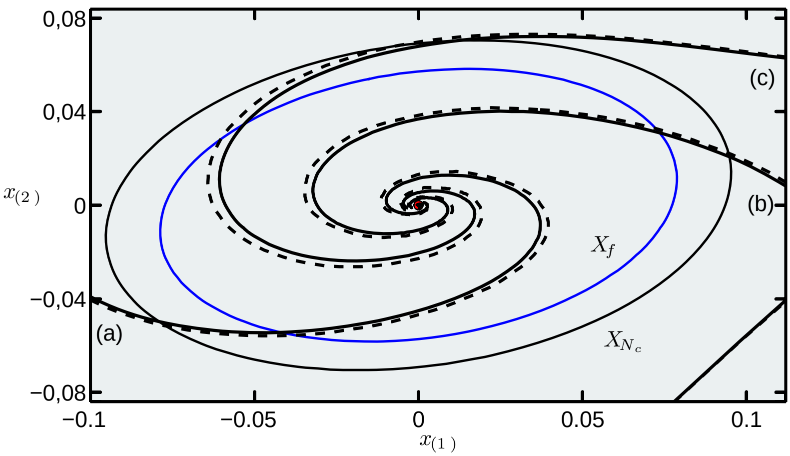


Figure 5.2: Confrontation of closed-loop trajectories without model uncertainty (solid) and with state-dependent uncertainty (dashed)

5.5 Conclusions

In this chapter, a robust MPC controller for constrained discrete-time nonlinear systems with state-dependent uncertainty and persistent disturbance is presented. In particular, under suitable assumptions, the robust constraints satisfaction is guaranteed for the considered class of uncertainties, employing a constraint tightening technique. Furthermore, the closed-loop system under the action of the MPC control law is shown to be ISS under the considered class of uncertainties. Finally, a nonlinear stability margin with respect to state dependent uncertainties is given. Future research efforts will be devoted to further increase the degree of robustness of the MPC control law, to enlarge the class of uncertainties, to allow for less conservative results and finally to address the unavoidable approximation errors involved in the computation of the optimal control actions.

5.6 Appendix

Proof of Lemma 5.1: Given x_t, consider the state x_k obtained applying the first $k-t-1$ elements of a feasible control sequence $\bar{\mathbf{u}}_{[t,t+N_c-1|t]}$ to the uncertain system (5.1). Then, by Assumptions 5.1 and 5.2, the prediction error $\hat{e}_{k|t} \triangleq x_k - \hat{x}_{k|t}$, with $k \in [t+1, \ldots, t+N_c-1]$, is upper bounded by

$$\begin{aligned} |\hat{e}_{k|t}| &= |\hat{f}(x_{k-1}, u_{k-1|t}) + d_{k-1} - \hat{f}(\hat{x}_{k-1|t}, u_{k-1|t})| \\ &\leq \mathcal{L}_{f_x}|\hat{e}_{k-1|t}| + |d_{k-1}| \leq \mathcal{L}_{f_x}|\hat{e}_{k-1|t}| + \bar{\mu} + L_\delta |x_{t+j-1}| \\ &\leq (\mathcal{L}_{f_x} + L_\delta)|\hat{e}_{k-1|t}| + \bar{\mu} + L_\delta |\hat{x}_{k-1|t}|. \end{aligned} \tag{5.12}$$

Finally, comparing (5.12) with (5.9), it follows that $|\hat{e}_{k|t}| \leq \hat{\rho}_{k|t}$, which in turn proves the statement. ■

Proof of Lemma 5.2: Notice that, given a vector $x \in \mathcal{X}_{N_c} \oplus \mathcal{B}(\bar{d}_{\kappa_f})$, there exists at least one vector $x' \in \mathcal{X}_{N_c}$ such that $|x - x'| \leq \varepsilon/\mathcal{L}_{f_x}$. Since $\hat{f}(x', \kappa_f(x')) \in \mathcal{X}_{N_c} \backsim \mathcal{B}(\varepsilon)$, with $\kappa_f(x') \in \mathcal{U}$, then, by Assumption 5.1, it follows that $\hat{f}(x, \kappa_f(x')) \in \mathcal{B}(\hat{f}(x', \kappa_f(x')), \varepsilon) \subseteq \mathcal{X}_{N_c}$, and hence $x \in \mathcal{P}(\mathcal{X}_{N_c})$, for all $x \in \mathcal{X}_{N_c} \oplus \mathcal{B}(\bar{d}_{\kappa_f})$, thus proving the statement. ■

Let introduce the following lemma that will be used in the proof of Theorem 5.1.

Lemma 5.3 *Suppose that Assumptions 5.1,5.2,5.4-5.6 hold. Given x_t and $x_{t+1} = \hat{f}(x_t, \kappa^{MPC}(x_t)) + d_t$, with $d_t \in \mathcal{D}_t$, consider the predictions $\hat{x}_{t+N_c|t}$ and $\hat{x}_{t+N_c+1|t+1}$, obtained respectively using the input sequences $\mathbf{u}^\circ_{[t,t+N_c-1|t]}$ and $\bar{\mathbf{u}}_{[t+1,t+N_c-1|t+1]} = [u^\circ_{t+1|t}, \ldots, u^\circ_{t+N_c-1|t}]$, and initialized with $\hat{x}_{t|t} = x_t$ and $\hat{x}_{t+1|t+1} = x_{t+1}$. Then $\hat{x}_{t+N_c|t+1} \in \mathcal{P}(\mathcal{X}_{N_c})$. Moreover, if $\delta(|x_t|) + \mu(\Upsilon^{sup}) \leq \mathcal{L}_{f_x}^{1-N_c} \bar{d}_{\kappa_f}$, $\hat{x}_{t+N_c|t+1} \in \mathcal{X}_{N_c} \oplus \mathcal{B}(\bar{d}_{\kappa_f})$.*

Proof: Given $x_t \in \mathcal{X}^{MPC}$, let $\xi \triangleq \hat{x}_{t+N_c|t+1} - \hat{x}_{t+N_c|t}$; then, by Assumptions 5.1 and 5.2, $|\xi| \leq |\hat{x}_{t+N_c|t+1} - \hat{x}_{t+N_c|t}| \leq \mathcal{L}_{f_x}^{N_c-1}(\delta(|x_t|) + \bar{\mu})$. Hence, $\xi \in \mathcal{B}\left(\mathcal{L}_{f_x}^{N_c-1}(\delta(|x_t|) + \bar{\mu})\right)$. Since $\mathbf{u}^\circ_{[t,t+N_c-1|t]}$ is the solution of the FHOCP, $\hat{x}_{t+N_c|t} \in \mathcal{X}_{N_c}$. Using also Lemma 5.2 and Assumption 5.6, it follows that $\hat{x}_{t+N_c|t} + \xi = \hat{x}_{t+N_c|t+1} \in \mathcal{P}(\mathcal{X}_{N_c})$. Moreover, by Lemma 5.2, if $\delta(|x_t|) + \mu(\Upsilon^{sup}) \leq \mathcal{L}_{f_x}^{1-N_c} \bar{d}_{\kappa_f}$, $\hat{x}_{t+N_c|t+1} \in \mathcal{X}_{N_c} \oplus \mathcal{B}(\bar{d}_{\kappa_f})$. ■

Proof of Theorem 5.1: It will be shown that the region $\mathcal{X}^{MPC}$ is a RPI set for the closed-loop system, proving that, for all $x_t \in \mathcal{X}^{MPC}$, there exists a feasible solution of the FHOCP at time instant $t+1$, based on the optimal solution in t, $\mathbf{u}^\circ_{[t,t+N_c-1|t]}$. In particular, a possible feasible control sequence is given by $\bar{\mathbf{u}}_{[t+1,t+N_c|t+1]} = \left[u^\circ_{t+1|t}, \ldots, u^\circ_{t+N_c-1|t}, \bar{u}\right]$, where $\bar{u} = \bar{u}(\hat{x}_{t+N_c|t+1}) \in \mathcal{U}$ is a feasible control action, suitably chosen to satisfy the robust constraint $\hat{x}_{t+N_c+1|t+1} \in \mathcal{X}_{N_c}$. Now, the proof will be divided in two steps.

Step 1: $\hat{x}_{t+j|t+1} \in \hat{\mathcal{X}}_{t+j|t+1}$: First, in view of Assumptions 5.1, 5.2 and (5.9), it follows that

$$\hat{\rho}_{t+1|t} - \hat{\rho}_{t+1|t+1} = L_\delta |x_t| + \bar{\mu},$$

and

$$\begin{aligned}\hat{\rho}_{k|t} - \hat{\rho}_{k|t+1} &= (\mathcal{L}_{f_x} + L_\delta)\left(\hat{\rho}_{k-1|t} - \hat{\rho}_{k-1|t+1}\right) + L_\delta(|\hat{x}_{k|t}| - |\hat{x}_{k|t+1}|) \\ &\geq (\mathcal{L}_{f_x} + L_\delta)\left(\hat{\rho}_{k-1|t} - \hat{\rho}_{k-1|t+1}\right) - L_\delta \mathcal{L}_{f_x}^{k-t-2}(L_\delta |x_t| + \bar{\mu}),\end{aligned}$$

for all $k \in [t+2, \ldots, t+N_c-1]$. Proceeding by induction, it follows that, for all $k \in [t+2, \ldots, t+N_c-1]$

$$\begin{aligned}&\hat{\rho}_{k|t} - \hat{\rho}_{k|t+1} \\ &\geq \left[(\mathcal{L}_{f_x} + L_\delta)^{k-t-1} - L_\delta\left(\mathcal{L}_{f_x} + L_\delta\right)^{k-t-2} \sum_{j=0}^{k-t-2} \left(\frac{\mathcal{L}_{f_x}}{\mathcal{L}_{f_x} + L_\delta}\right)^j\right](L_\delta |x_t| + \bar{\mu})\end{aligned}$$

which yields

$$\hat{\rho}_{k|t} - \hat{\rho}_{k|t+1} \geq \mathcal{L}_{f_x}^{k-t-1}(L_\delta |x_t| + \bar{\mu}), \tag{5.13}$$

for all $k \in [t+2, \ldots, t+N_c-1]$. Now, consider the predictions $\hat{x}_{k|t}$ and $\hat{x}_{k|t+1}$, with $k \in [t+2, \ldots, t+N_c-1]$, made respectively using the input sequences $\mathbf{u}^\circ_{[t,t+N_c-1|t]}$ and $\bar{\mathbf{u}}_{[t+1,t+N_c-1|t+1]}$, and initialized with $\hat{x}_{t|t} = x_t$ and $\hat{x}_{t+1|t+1} = \hat{f}(x_t, \kappa^{MPC}(x_t))$. Assuming that $\hat{x}_{k|t} \in \hat{\mathcal{X}}_{k|t} \triangleq \mathcal{X} \backsim \mathcal{B}(\hat{\rho}_{k|t})$, with $\hat{\rho}_{k|t}$ given by (5.9), let introduce $\eta \in \mathcal{B}(\hat{\rho}_{k|t+1})$. Furthermore, let $\xi \triangleq \hat{x}_{k|t+1} - \hat{x}_{k|t} + \eta$. Then, under Assumption 5.1, it follows that $|\xi| \leq |\hat{x}_{k|t+1} - \hat{x}_{k|t}| + \hat{\rho}_{k|t+1} \leq \mathcal{L}_{f_x}^{j-1}(L_\delta |x_t| + \bar{\mu}) + \hat{\rho}_{k|t+1}$. In view of (5.13), it turns out that $|\xi| \leq \hat{\rho}_{k|t}$, and hence, $\xi \in \mathcal{B}(\hat{\rho}_{k|t})$. Since $\hat{x}_{k|t} \in \hat{\mathcal{X}}_{k|t}$, it follows that $\hat{x}_{k|t} + \xi = \hat{x}_{k|t+1} + \eta \in \mathcal{X}$, $\forall \eta \in \mathcal{B}(\hat{\rho}_{k|t+1})$, which finally yields $\hat{x}_{k|t+1} \in \hat{\mathcal{X}}_{k|t+1}$.

Step 2: $\hat{x}_{t+N_c+1|t+1} \in \mathcal{X}_{N_c}$: if $\mathcal{L}_{f_x}^{N_c-1}(\delta(|x_t|) + \bar{\mu}) \leq \bar{d}_{\kappa_f}$, in view of

Lemma 5.2 there exists a feasible control action such that the statement holds. If $\bar{d}_{\kappa_f} < \mathcal{L}_{f_x}^{N_c-1}(\delta(|x_t|)+\bar{\mu}) \leq \bar{d}$, thanks to Lemma 5.3, it follows that $\hat{x}_{t+N_c|t+1} \in \mathcal{P}(\mathcal{X}_{N_c})$. Hence, there exists a feasible control action, namely $\bar{u} \in \mathcal{U}$, such that $\hat{x}_{t+N_c+1|t+1} = \hat{f}(\hat{x}_{t+N_c|t+1}, \bar{u}) \in \mathcal{X}_{N_c}$, thus ending the proof. ■

Proof of Theorem 5.2: In view of Assumptions 5.1-5.6 it follows from Theorem 5.1 that $\mathcal{X}^{MPC}$ is a RPI set for system (5.1) under the action of the control law (5.7). So, the proof consists in showing that function V defined in 5.10 is an ISS-Lyapunov function in $\mathcal{X}^{MPC}$. First, by Assumption 5.5, the set $\mathcal{X}^{MPC}$ is not empty. In fact, for any $x_t \in \mathcal{X}_{N_c}$, a feasible control sequence for the FHOCP is given by $\tilde{\mathbf{u}}_{[t,t+N_c-1|t]} \triangleq \left[\kappa_f(\hat{x}_{t|t}), \kappa_f(\hat{x}_{t+1|t}), \ldots, \kappa_f(\hat{x}_{t+N_c-1|t})\right]$. Then $\mathcal{X}^{MPC} \supseteq \mathcal{X}_{N_c} \supseteq \mathcal{X}_f$. Then, in view of Assumption 5.4 it holds that

$$
\begin{aligned}
V(x_t) &\leq J(x_t, \tilde{\mathbf{u}}_{[t,t+N_c-1|t]}, N_c, N_p) = \sum_{k=t}^{t+N_p-1} l(\hat{x}_{k|t}, \kappa_f(\hat{x}_{k|t})) + V_f(\hat{x}_{t+N_p|t}) \\
&\leq \sum_{k=t}^{t+N_p-1} \left[V_f(\hat{x}_{k|t}) - V_f(\hat{x}_{k+1|t})\right] + V_f(\hat{x}_{t+N_p|t}) \\
&\leq \ V_f(\hat{x}_{t|t}) \leq \mathcal{L}_{V_f}|x_t|
\end{aligned}
\tag{5.14}
$$

for all $x_t \in \mathcal{X}_f$. The lower bound on $V(x_t)$ can be easily obtained using Assumption 5.3:

$$
V(x_t) \geq \underline{l}(|x_t|) \tag{5.15}
$$

for all $x_t \in \mathcal{X}^{MPC}$. Suppose[1] that $\mathcal{L}_{f_c}$ and $\mathcal{L}_{f_x} \neq 1$. Now, in view of Theorem 5.1, given the optimal control sequence at time t, $\mathbf{u}^{\circ}_{[t,t+N_c-1|t]}$, the

[1] The very special case $\mathcal{L}_{f_c}, \mathcal{L}_{f_x} = 1$ can be trivially addressed by a few suitable modifications to the proof of Theorem 5.2.

sequence $\bar{\mathbf{u}}_{[t+1,t+N_c|t]} \triangleq \left[u^\circ_{t+1|t}, \dots, u^\circ_{t+N_c-1|t}, \bar{u}\right]$ with

$$\bar{u}=\begin{cases} \kappa_f(\hat{x}_{t+N_c|t}), & \text{if } \delta(|x_t|)+\mu(|\upsilon_t|) \leq \mathcal{L}_{f_x}^{1-N_c}\bar{d}_{\kappa_f} \\ \bar{u}\in\mathcal{U}:\hat{f}(\hat{x}_{t+N_c|t+1},\bar{u})\in\mathcal{X}_{N_c}, & \text{if } \mathcal{L}_{f_x}^{1-N_c}\bar{d}_{\kappa_f} < \delta(|x_t|)+\mu(|\upsilon_t|) \leq \mathcal{L}_{f_x}^{1-N_c}\bar{d} \end{cases}$$

is a feasible (in general, suboptimal) control sequence for the FHOCP at time $t+1$, with cost

$$\begin{aligned}
&J(x_{t+1}, \bar{\mathbf{u}}_{[t+1,t+N_c|t+1]}, N_c, N_p) = \\
&V(x_t) - l(x_t, u^\circ_{t,t}) + \sum_{k=t+1}^{t+N_c-1}\left[l(\hat{x}_{k|t+1}, u^\circ_{k|t}) - l(\hat{x}_{k|t}, u^\circ_{k|t})\right] + l(\hat{x}_{t+N_c|t+1}, \bar{u}) \\
&-l(\hat{x}_{t+N_c|t}, \kappa_f(\hat{x}_{t+N_c|t})) + \sum_{k=t+(N_c+1)}^{t+N_p-1}\left[l(\hat{x}_{k|t+1}, \kappa_f(\hat{x}_{k|t+1})) - l(\hat{x}_{k|t}, \kappa_f(\hat{x}_{k|t}))\right] \\
&+l(\hat{x}_{t+N_p|t+1}, \kappa_f(\hat{x}_{t+N_p|t+1})) + V_f(\hat{f}(\hat{x}_{t+N_p|t+1}, \kappa_f(\hat{x}_{t+N_p|t+1}))) \\
&-V_f(\hat{x}_{t+N_p|t}).
\end{aligned}$$

Using Assumptions 5.1 and 5.3, it follows that

$$\left|l(\hat{x}_{k|t+1}, u^\circ_{k|t}) - l(\hat{x}_{k|t}, u^\circ_{k|t})\right| \leq \mathcal{L}_l\mathcal{L}_{f_x}^{k-t-1}(\delta(|x_t|)+\mu(|\upsilon_t|)), \tag{5.16}$$

for all $k \in [t+1, \dots, t+N_c-1]$. Moreover, for $k = t+N_c$,

$$\begin{aligned}\left|l(\hat{x}_{t+N_c|t+1}, \bar{u}) - l(\hat{x}_{t+N_c|t}, \kappa_f(\hat{x}_{t+N_c|t}))\right| \;\leq\; & \mathcal{L}_l\mathcal{L}_{f_x}^{N_c-1}(\delta(|x_t|)+\mu(|\upsilon_t|)) \\ &+\mathcal{L}_{lu}\Delta_u(\delta(|x_t|)+\mu(|\upsilon_t|))\end{aligned} \tag{5.17}$$

where

$$\Delta_u(s)\triangleq\begin{cases} 0, & \text{if } s \leq \mathcal{L}_{f_x}^{1-N_c}\bar{d}_{\kappa_f} \\ \max\{|u-w|, (u,w)\in\mathcal{U}\times\mathcal{U}\}, & \text{if } \mathcal{L}_{f_x}^{1-N_c}\bar{d}_{\kappa_f} < s \leq \mathcal{L}_{f_x}^{1-N_c}\bar{d}. \end{cases} \tag{5.18}$$

Finally, under Assumptions 5.1, 5.3 and 5.4, for all $k \in [t+N_c+1, \dots, t+N_p-1]$, the following intermediate results hold

$$\begin{aligned}&\left|l(\hat{x}_{k|t+1},\kappa_f(\hat{x}_{k|t+1}))-l(\hat{x}_{k|t},\kappa_f(\hat{x}_{k|t}))\right|\leq\\&(\mathcal{L}_l+\mathcal{L}_{lu}\mathcal{L}_{\kappa_f})\mathcal{L}_{f_c}^{k-t-(N_c+1)}\Big[\mathit{\Delta}_x(\delta(|x_t|)+\mu(|\upsilon_t|))+\mathcal{L}_{f_x}^{N_c}(\delta(|x_t|)+\mu(|\upsilon_t|))\Big]\end{aligned}\tag{5.19}$$

and

$$\begin{aligned}&\left|V_f(\hat{x}_{t+N_p|t+1})-V_f(\hat{x}_{t+N_p|t})\right|\\&\leq\mathcal{L}_{V_f}\mathcal{L}_{f_c}^{N_p-(N_c+1)}\Big[\mathit{\Delta}_x(\delta(|x_t|)+\mu(|\upsilon_t|))+\mathcal{L}_{f_x}^{N_c}(\delta(|x_t|)+\mu(|\upsilon_t|))\Big]\end{aligned}\tag{5.20}$$

where $\mathit{\Delta}_x(s)=0$ if $s\leq\mathcal{L}_{f_x}^{1-N_c}\bar{d}_{\kappa_f}$ and

$$\mathit{\Delta}_x(s)=\max\{|x-\xi|,(x,\xi)\in X_{N_c}\times(X_{N_c}\oplus\mathcal{B}(\bar{d}))\}-\mathcal{L}_{f_x}^{N_c}s\tag{5.21}$$

if $\mathcal{L}_{f_x}^{1-N_c}\bar{d}_{\kappa_f}<s\leq\mathcal{L}_{f_x}^{1-N_c}\bar{d}$. Consider now the case $\delta(|x_t|)+\mu(|\upsilon_t|)\leq\bar{d}_{\kappa_f}$, for all $x_t\in\mathcal{X}$, and all $\upsilon\in\mathcal{M}_\Upsilon$. Then, using (5.16)-(5.21), Assumptions 5.4, 5.5 and 5.9, if $\delta(|x_t|)+\mu(|\upsilon_t|)\leq\mathcal{L}_{f_x}^{1-N_c}\bar{d}_{\kappa_f}$, the following inequalities hold

$$\begin{aligned}&J(x_{t+1},\bar{\mathbf{u}}_{[t+1,t+N_c|t]},N_c,N_p)\\&\leq V(x_t)-l(x_t,u^\circ_{t,t})+\sum_{j=1}^{N_c}\mathcal{L}_l\mathcal{L}_{f_x}^{j-1}(\delta(|x_t|)+\mu(|\upsilon_t|))\\&\quad+\sum_{j=N_c+1}^{N_p-1}(\mathcal{L}_l+\mathcal{L}_{lu}\mathcal{L}_{\kappa_f})\mathcal{L}_{f_c}^{j-(N_c+1)}\mathcal{L}_{f_x}^{N_c}(\delta(|x_t|)+\mu(|\upsilon_t|))\\&\quad+l(\hat{x}_{t+Np|t+1},\kappa_f(\hat{x}_{t+Np|t+1}))+V_f(\hat{x}_{t+N_p+1|t+1})-V_f(\hat{x}_{t+N_p|t+1})\\&\quad+\mathcal{L}_{V_f}\mathcal{L}_{f_c}^{N_p-(N_c+1)}L_{f_x}^{N_c}(\delta(|x_t|)+\mu(|\upsilon_t|))\\&\leq V(x_t)-l(x_t,u^\circ_{t,t})+\varphi_x(|x_t|)+\varphi_\upsilon(|\upsilon_t|)\,,\end{aligned}$$

for all $x_t\in\mathcal{X}^{MPC}$, and all $\upsilon\in\mathcal{M}_\Upsilon$, where $\varphi_\upsilon(|\upsilon_t|)\triangleq\Big[\mathcal{L}_l\frac{\mathcal{L}_{f_x}^{N_c}-1}{\mathcal{L}_{f_x}-1}+(\mathcal{L}_l+\mathcal{L}_{lu}\mathcal{L}_{\kappa_f})\frac{\mathcal{L}_{f_c}^{N_p-N_c-1}-1}{\mathcal{L}_{f_c}-1}\mathcal{L}_{f_x}^{N_c}+\mathcal{L}_{V_f}\mathcal{L}_{f_c}^{N_p-(N_c+1)}\mathcal{L}_{f_x}^{N_c}\Big]\mu(|\upsilon_t|)$ is a $\mathcal{K}$-function. Now, from inequality $V(x_{t+1})\leq J(x_{t+1},\bar{\mathbf{u}}_{[t+1,t+N_c|t]},N_c,N_p)$ it follows that

$$V(x_{t+1})-V(x_t)\leq-\alpha_3(|x_t|)+\sigma(|\upsilon_t|),\tag{5.22}$$

where $\alpha_3(|x_t|) \triangleq \underline{l}(|x_t|) - \varphi_x(|x_t|)$ and $\sigma(|\upsilon_t|) \triangleq \varphi_\upsilon(|\upsilon_t|)$, for all $x_t \in \mathcal{X}^{MPC}$, with $\upsilon \in \mathcal{M}_\Upsilon$, N_c and N_p fixed. By (5.14), (5.15) and (5.22), if $\delta(|x_t|)$ is such that $\alpha_3(|x_t|)$ is a $\mathcal{K}_\infty$-function, then it is a stability margin for the closed-loop system [Jiang & Wang 2001]. Therefore, if Assumption 5.9 hold, the optimal cost $J(x_t, \mathbf{u}^\circ_{t,t+N_c-1|t}, N_c, N_p)$ is an ISS-Lyapunov function for the closed-loop system in $\mathcal{X}^{MPC}$. Since Assumption 5.7 hold, in view of Theorem 2.1, it is possible to conclude that the closed-loop system is regional ISS in $\mathcal{X}^{MPC}$ with respect to $\upsilon \in \Upsilon$.

Conversely, if $\mathcal{L}_{f_x}^{1-N_c}\bar{d}_{\kappa_f} < \delta(|x_t|) + \mu(|\upsilon_t|) \leq \mathcal{L}_{f_x}^{1-N_c}\bar{d}$, the following inequality can be straightforwardly obtained

$$V(x_{t+1}) - V(x_t) \leq -\tilde{\alpha}_3(|x_t|) + \tilde{\sigma}(|\upsilon_t|) + \bar{\sigma}, \tag{5.23}$$

where

$$\tilde{\alpha}_3(|x_t|) \triangleq \alpha_3(|x_t|) - \left[(\mathcal{L}_l + \mathcal{L}_{lu}\mathcal{L}_{\kappa_f})\frac{\mathcal{L}_{f_c}^{N_p-(N_c+1)} - 1}{\mathcal{L}_{f_c} - 1} + \mathcal{L}_{V_f}\mathcal{L}_{f_c}^{N_p-(N_c+1)}\right]\mathcal{L}_{f_x}^{N_c}\delta(|x_t|)$$

$$\tilde{\sigma}(|\upsilon_t|) \triangleq \sigma(|\upsilon_t|) - \left[(\mathcal{L}_l + L_{lu}L_{\kappa_f})\frac{\mathcal{L}_{f_c}^{N_p-(N_c+1)} - 1}{\mathcal{L}_{f_c} - 1} + \mathcal{L}_{V_f}\mathcal{L}_{f_c}^{N_p-(N_c+1)}\right]\mathcal{L}_{f_x}^{N_c}\mu(|\upsilon_t|)$$

and $\bar{\sigma} \triangleq \left[(\mathcal{L}_l + \mathcal{L}_{lu}\mathcal{L}_{\kappa_f})\frac{\mathcal{L}_{f_c}^{N_p-(N_c+1)} - 1}{\mathcal{L}_{f_c} - 1} + \mathcal{L}_{h_f}\mathcal{L}_{f_c}^{N_p-(N_c+1)}\right]) \max\{|x - \xi|, (x, \xi) \in \mathcal{X}_{N_c} \times (\mathcal{X}_{N_c} \oplus \mathcal{B}(\bar{d}))\} + \mathcal{L}_{lu} \max\{|u - w|, (u, w) \in \mathcal{U} \times \mathcal{U}\}$. Hence, in the latter case, only ISpS can be guaranteed, although the invariance of $\mathcal{X}^{MPC}$ and the fulfillment of constraints are preserved thanks to Theorem 5.1. ■

Chapter 6

Decentralized NMPC: an ISS approach

Contents

6.1 Introduction

The design and analysis of decentralized control systems have been under investigation for more than thirty years. Many problems falling into this category have been addressed with various mathematical tools, while new theoretical and application issues are arising as a result of current trends in distributed systems, such as the increasing size and complexity of feedback control systems, the availability of spatially distributed sensors and actuators, and the need to come up with more autonomous systems.

When dealing with large scale systems, one of the key objectives is to find conditions guaranteeing closed-loop stability, while reducing the computational load stemming from a centralized approach. Starting with the notion of "fixed modes" introduced in the 1970s for linear large scale systems [Wang & Davidson 1973], other investigations focused on the structure and size of interconnections [Siljak 1978]. Specific emphasis on the structural properties of decentralized controlled large-scale systems is given in the research work of D'Andrea and co-workers (see, for instance, [D'Andrea & Dullerud 2003]), which is used in several applications, such as flight formation and distributed sensors. Studies on topology independent control have also been recently reported [Cogill & Lall 2004]. Works on the size of interconnections have been proposed in adaptive control [Ioannou 1986], and more recently in model predictive control.

In this chapter, decentralized MPC techniques are considered. Decentralized MPC is of paramount interest in the process industry; in fact a decentralized control structure is often the most appropriate one due to topological constraints and limited exchange of information between subsystems, while the MPC approach allows one to include in the problem formulation both performance requirements and state and control constraints. For these reasons, decentralized MPC has already been studied for discrete time linear systems in e.g. [Dunbar & Murray 2006], [Camponogara *et al.* 2002] and in a number of papers quoted there. In [Magni & Scattolini 2006] a decentralized MPC algorithm for nonlinear systems has been proposed, where closed loop stability of the origin is achieved through the inclusion of a contractive constraint (see also [de Oliveira Kothare & Morari 2000]).

Distributed MPC algorithms can be developed either by assuming that there is a partial exchange of information between the subsystems, as in [Dunbar & Murray 2006], [Venkat *et al.* 2005], or by considering a fully decentralized control structure, as in [Magni & Scattolini 2006]. This second possibility is obviously more critical than the previous and requires a more conservative solution, since the amount of information available to any local controller is less. However this setting more closely resembles most of real world cases, where complex control structures are built according to fully decentralized schemes. In this chapter, stabilizing fully decentralized MPC algorithms for nonlinear, discrete-time systems are derived under the assumption that no information is exchanged between subsystems. Relying on the concept of regional ISS, the approach taken in the following to derive decentralized MPC implementations consists in considering the overall system as composed by a number of interconnected subsystems, each one of them controlled by a robust (open-loop or closed-loop) MPC algorithm guaranteeing ISS, and by considering the effect of interconnections as a perturbation term. A similar approach has also been taken in [Dashkovskiy *et al.* 2007], [Dashkovskiy *et al.* 2005] where global results are given for interconnected systems. Then, by suitably combining and extending the results reported in [Magni *et al.* 2006a], [Dashkovskiy *et al.* 2007] and [Dashkovskiy *et al.* 2005], it is shown that under suitable assumptions the ISS property of the controlled subsystems guarantees the ISS of the overall (controlled) system.

6.2 Problem statement

Assume that the plant to be controlled is composed by the interconnection of S local subsystems described by the following nonlinear, discrete-time models

$$x_{i_{k+1}} = f_i(x_{i_k}, u_{i_k}) + g_i(x_k) + d_{i_k},\ k \geq 0 \tag{6.1}$$

where $x_{i_k} \in \mathbb{R}^{n_i}$ is the state of the i-th subsystem, $u_{i_k} \in \mathbb{R}^{m_i}$ is the current control vector, $d_{i_k} \in \mathbb{R}^{n_i}$ is an unknown disturbance, and $g_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$, which depends on the overall state

$$x_k \triangleq [x_{1_k}^\top\ x_{2_k}^\top\ \ldots\ x_{S_k}^\top]^\top \in \mathbb{R}^n,\ n \triangleq \sum_{i=1}^{S} n_i$$

describes the influence of the S subsystems on the i-th subsystem. Defining $f(x,u) \triangleq [f_1^\top(x_1,u_1), \ldots, f_S^\top(x_S,u_S)]^\top$, $g(x) \triangleq [g_1^\top(x), \ldots, g_S^\top(x)]^\top$ and $d \triangleq [d_1^\top, \ldots, d_S^\top]^\top$, the whole system can be written as

$$x_{k+1} = f(x_k, u_k) + g(x_k) + d_k,\ k \geq 0. \tag{6.2}$$

Each subsystem is supposed to fulfill the following assumptions.

Assumption 6.1

1. *For simplicity of notation, it is assumed that the origin is an equilibrium point, i.e.* $f_i(0,0) = 0$.

2. *The disturbance* d_i *is such that*

$$d_i \in \mathcal{D}_i \tag{6.3}$$

where $\mathcal{D}_i$ *is a compact set containing the origin as an interior point, with* $\mathcal{D}_i^{sup}$ *known.*

3. *The state and the control variables are restricted to fulfill the following*

constraints

$$x_i \in \mathcal{X}_i \tag{6.4}$$

$$u_i \in \mathcal{U}_i \tag{6.5}$$

where $\mathcal{X}_i$ and $\mathcal{U}_i$ are compact sets, both containing the origin as an interior point. Let denote $\mathcal{X} \triangleq [\mathcal{X}_1^\top \; \mathcal{X}_2^\top \; \dots \; \mathcal{X}_S^\top]^\top$ the overall constraint set on the state.

4. *The map $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{n_i}$ is Lipschitz in x_i in the domain $\mathcal{X}_i \times \mathcal{U}_i$, i.e. there exists a positive constant $\mathcal{L}_{if}$ such that*

$$|f_i(a,u) - f_i(b,u)| \leq \mathcal{L}_{if}|a-b| \tag{6.6}$$

for all $a, b \in \mathcal{X}_i$ and all $u \in \mathcal{U}_i$.

5. *The map $g_i : \mathbb{R}^n \rightarrow \mathbb{R}^{n_i}$ is such that $g_i(0) = 0$.*

6. *There exist positive constants L_{ij}, $i, j \in [1, 2, \dots, S]$ such that*

$$|g_i(x)| \leq \sum_{j=1}^{S} L_{ij}|x_j|$$

for all $x_i \in \mathcal{X}_i$.

7. *The state of the plant x_{i_k} can be measured at each sample time.*

□

In the following, let denote a generic interaction sequence as $\mathbf{g}_i \triangleq \{g_i(x_1), g_i(x_2), \dots\}$. Note that, by point 6 of Assumption 6.1, the interaction sequence $\mathbf{g}_i$ is bounded, i.e. $\mathbf{g}_i \in \mathcal{M}_{\mathcal{G}_i}$ with $\mathcal{G}_i \triangleq g_i(\mathcal{X}^{sup})$.

The control objective consists in designing, for each subsystem, a control law $u_i = \kappa(x_i)$, without taking into account explicitly the interaction with the other subsystems, such that the overall system is steered to (a neighborhood of) the origin fulfilling the constraints on the input and the state along the system evolution for any possible disturbance and yielding, for each local subsystem, an optimal closed-loop performance according to certain performance index.

In the following section it is presented a suitable framework for the analysis of stability of such class of closed loop systems: the regional ISS for nonlinear interconnected subsystems.

6.3 Regional Input-to-State Stability for nonlinear interconnected subsystems

Consider a system composed by the interconnection of S local subsystems described by the following nonlinear, discrete-time models

$$x_{i_{k+1}} = F_i(x_{i_k}) + g_i(x_k) + d_{i_k},\ k \geq 0 \tag{6.7}$$

where the map $F_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ is nonlinear possibly discontinuous, $x_{i_k} \in \mathbb{R}^{n_i}$ is the state of the i-th subsystem, $d_{i_k} \in \mathbb{R}^{n_i}$ is an unknown disturbance, and $g_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$, which depends on the overall state

$$x_k \triangleq [x_{1_k}^\top\ x_{2_k}^\top\ \dots\ x_{S_k}^\top]^\top \in \mathbb{R}^n,\ n \triangleq \sum_{i=1}^{S} n_i$$

describes the influence of the S subsystems on the i-th subsystem. Defining $F(x) \triangleq [F_1^\top(x_1), \dots, F_S^\top(x_S)]^\top$, $g(x) \triangleq [g_1^\top(x), \dots, g_S^\top(x)]^\top$ and $d \triangleq$

$[d_1^\top, \ldots, d_S^\top]^\top$, the whole system can be written as

$$x_{k+1} = F(x_k) + g(x_k) + d_k, \;\; k \geq 0. \tag{6.8}$$

The transient of the system (6.7) with initial state $x_{i_0} = \bar{x}_i$, disturbance sequence $\mathbf{d}_i$ and interaction sequence $\mathbf{g}_i \triangleq \{g_i(x_1), g_i(x_2), \ldots\}$ is denoted by $x_i(k, \bar{x}_i, \mathbf{d}_i, \mathbf{g}_i)$. Each subsystem is supposed to fulfill the following assumptions.

Assumption 6.2

1. *The origin of the system is an equilibrium point, i.e.* $F_i(0) = 0$.

2. *The map* $g_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ *is such that* $g_i(0) = 0$.

3. *The state variables* x_i *fulfill the constraint*

$$x_i \in \mathcal{X}_i \tag{6.9}$$

   where $\mathcal{X}_i$ *is a compact set containing the origin as an interior point. Let denote* $\mathcal{X} \triangleq [\mathcal{X}_1^\top \; \mathcal{X}_2^\top \; \ldots \; \mathcal{X}_S^\top]^\top$ *the overall constraint set on the state.*

4. *The disturbance* d_i *is such that*

$$d_i \in \mathcal{D}_i \tag{6.10}$$

   for all $k \geq 0$, *where* $\mathcal{D}_i$ *is a compact set containing the origin as an interior point, with* $\mathcal{D}_i^{sup}$ *known.*

5. *There exist positive constants* L_{ij}, $i, j \in [1, 2, \ldots, S]$ *such that*

$$|g_i(x)| \leq \sum_{j=1}^{S} L_{ij}|x_j|$$

for all $x_i \in \mathcal{X}_i$.

6. *The solution of (6.7) is continuous at* $\bar{x}_i = 0$, $\mathbf{g}_i = 0$ *and* $\mathbf{d}_i = 0$ *with respect to initial conditions, interactions and disturbances.*

Assumption 6.3 *Each subsystem (6.7), satisfying Assumption 6.2, admits an ISS-Lyapunov function* V_i *in* $\Xi_i \subseteq \mathcal{X}_i$, *with* $g_i(x) + d_i$ *as disturbance terms, that means*

1. Ξ_i *is a compact RPI set including the origin as an interior point*

2. *there exist a compact set* $\Omega_i \subseteq \Xi_i$ *(including the origin as an interior point), and a pair of suitable* $\mathcal{K}_\infty$*-functions* α_{i1}, α_{i2} *such that*

$$V_i(x_i) \geq \alpha_{i1}(|x_i|), \ \forall x_i \in \Xi_i \tag{6.11}$$

$$V_i(x_i) \leq \alpha_{i2}(|x_i|), \ \forall x_i \in \Omega_i \tag{6.12}$$

3. *there exist a suitable* $\mathcal{K}_\infty$*-function* α_{i3} *and some suitable* $\mathcal{K}$*-functions* σ^x_{ij}, σ^d_i *such that*

$$\Delta V_i(x_i) \leq -\alpha_{i3}(|x_i|) + \textstyle\sum_{j=1}^{S} \sigma^x_{ij}(|x_j|) + \sigma^d_i(|d_i|), \tag{6.13}$$

for all $x_i \in \Xi_i$, *all* $x_j \in \mathcal{X}_j$, *and all* $d_i \in \mathcal{D}_i$

4. *there exist a suitable* $\mathcal{K}_\infty$*-function* ρ_i *(with* ρ_i *such that* $(id - \rho_i)$ *is a* $\mathcal{K}_\infty$*-function) and a suitable constant* $c_{\theta_i} > 0$, *such that, given a disturbance sequence* $\mathbf{d}_i \in \mathcal{M}_{\mathcal{D}_i}$ *and an interaction sequence* $\mathbf{g}_i \in$

$\mathcal{M}_{\mathcal{G}_i}$, there exists a nonempty compact set $\Theta_i(\mathbf{d}_i, \mathbf{g}_i) \subseteq I\Omega_i\{x_i : x_i \in \Omega_i, |x_i|_{\delta\Omega_i} > c_{\theta_i}\}$ (including the origin as an interior point) defined as follows

$$\Theta_i(\mathbf{d}_i, \mathbf{g}_i) \triangleq \left\{x_i : V(x_i) \leq b_i\left(\sum_{j=1}^{S} \sigma_{ij}^x(||\mathbf{x}_j||) + \sigma_i^d(||\mathbf{d}_i||)\right)\right\} \tag{6.14}$$

where $b_i \triangleq \alpha_{i4}^{-1} \circ \rho_i^{-1}$, with $\alpha_{i4} \triangleq \alpha_{i3} \circ \alpha_{i2}^{-1}$. □

Note that, by (6.11) and (6.12), function V_i is continuous at the origin.

Remark 6.1 *Note that, in order to verify that $\Theta_i(\mathbf{d}_i, \mathbf{g}_i) \subseteq I\Omega$ for all sequences $\mathbf{d}_i \in \mathcal{M}_{\mathcal{D}_i}$ and all $\mathbf{g}_i \in \mathcal{M}_{\mathcal{G}_i}$, one has to verify that*

$$\Theta_i \;\; \triangleq \;\; \{x : V_i(x) \leq b_i\left(\sum_{j=1}^{S} \sigma_{ij}^x(\mathcal{X}_j^{sup}) + \sigma_i^d(\mathcal{D}_i^{sup}))\right)\} \subseteq I\Omega \tag{6.15}$$

□

Let introduce now an intermediate result useful in the following.

Lemma 6.1 *Given any $\mathcal{K}_\infty$-function ρ_i such that $(id - \rho_i)$ is a $\mathcal{K}_\infty$-function too, there exist some $\mathcal{K}_\infty$-functions a_{i1}, a_{i2}, ..., a_{id} such that*

$$\begin{aligned}\rho_i^{-1}(\theta_1 + \ldots + \theta_S + \theta_d) \leq & \max\{(id + a_{i1})(\theta_1), (id + a_{i2})(\theta_2), \ldots \\ & \ldots, (id + a_{iS})(\theta_S), (id + a_{id})(\theta_d)\}.\end{aligned}$$

■

In order to introduce the next theorem, first, let define $\Xi \triangleq \Xi_1 \times \ldots \times \Xi_S$ the composition of RPI sets of all the subsystems. Moreover, let define the maps $\Delta : \mathbb{R}_{\geq 0}^S \to \mathbb{R}_{\geq 0}^S$ as

$$\Delta(s_1, \ldots, s_S)^\top \triangleq ((id + \alpha_1)(s_1), \ldots, (id + \alpha_S)(s_S))^\top \tag{6.16}$$

with $\alpha_i \in \mathcal{K}_\infty$, $i = 1, \dots, S$ and $\Gamma : \mathbb{R}^S_{\geq 0} \to \mathbb{R}^S_{\geq 0}$ as

$$\Gamma(s_1, \dots, s_S)^\top \triangleq \\ \left(\sum_{j=1}^S \alpha_{14}^{-1} \circ (id + a_{1j}) \circ \eta_{1j}^x(s_j), \dots, \sum_{j=1}^S \alpha_{S4}^{-1} \circ (id + a_{Sj}) \circ \eta_{Sj}^x(s_j)\right)^\top \tag{6.17}$$

where $\alpha_{i4} = \alpha_{i3} \circ \alpha_{i2}^{-1}$, $\eta_{ij}^x = \sigma_{ij}^x \circ \alpha_{i1}^{-1}$, while $(id + a_{ij})$ are obtained starting from ρ_i in (6.14) using Lemma 6.1.

Theorem 6.1 *Consider systems (6.7) and suppose that Assumptions 6.2 and 6.3 are satisfied. Let Γ be given by (6.17). If there exists a mapping Δ as in (6.16), such that*

$$(\Gamma \circ \Delta)(s) \not\geq s, \ \forall s \in \mathbb{R}^S_{\geq 0} \setminus \{0\} \tag{6.18}$$

then the overall system (6.8) is ISS in Ξ from d to x. ■

Remark 6.2 *As discussed in [Dashkovskiy et al. 2005], condition (6.18) is the generalization to nonlinear interconnected systems of the well known small gain theorem, which, in the case of only two interconnected systems was previously given in [Jiang et al. 1994]. Many interesting interpretations of this conditions are given in [Dashkovskiy et al. 2005]. Note also that condition (6.18) together with point 6 of Assumption 6.2 is necessary to guarantee that the interconnections between the subsystems do not cause instability.*

6.4 Nonlinear Model Predictive Control

In this section, the results derived in Theorems 2.1 and 6.1 are used to analyze the ISS property of open-loop and closed-loop min-max formulations of stabilizing MPC for nonlinear systems. Notably, in the following it is not necessary to assume the regularity of the value function and of the resulting control law.

6.4.1 Open-loop formulation

In order to introduce the MPC algorithm formulated according to an open-loop strategy, first let $\mathbf{u}_{i[t_2,t_3|t_1]} \triangleq [u_{i_{t_2|t_1}}\ u_{i_{t_2+1|t_1}} \dots u_{i_{t_3|t_1}}]$, with $t_1 \le t_2 \le t_3$ a control sequence. Moreover, given $k \ge 0$, $j \ge 1$, let $\hat{x}_{i_{k+j|k}}$ be the predicted state at $k+j$ obtained with the nominal model $f_i(x_{i_k}, u_{i_k})$ with initial condition x_k and input $\mathbf{u}_{i[k,k+j-1|k]}$.

Then, the following finite-horizon optimization problem can be stated.

Definition 6.1 (FHOCP) *Consider system (6.1) with $x_{i_t} = \bar{x}_i$. Given the positive integer N_i, the stage cost l_i, the terminal penalty V_{if} and the terminal set $\mathcal{X}_{if}$, the Finite Horizon Optimal Control Problem (FHOCP) consists in minimizing, with respect to $\mathbf{u}_{i[t,t+N-1|t]}$, the performance index*

$$J_i(\bar{x}_i, \mathbf{u}_{i[t,t+N_i-1|t]}, N_i) \triangleq \sum_{k=t}^{t+N_i-1} l_i(\hat{x}_{i_{k|t}}, u_{i_{k|t}}) + V_{if}(\hat{x}_{i_{t+N_i|t}})$$

subject to

1. *the nominal state dynamics $\hat{x}_{i_{k+1}} = f_i(\hat{x}_{i_k}, u_{i_k})$, with $\hat{x}_{i_t} = \bar{x}_i$*
2. *the constraints (6.4), (6.5), $k \in [t, t+N_i-1]$*
3. *the terminal state constraints $\hat{x}_{t+N_i|t} \in \mathcal{X}_{if}$.* $\square$

The stage cost defines the performance index to optimize and satisfies the following assumption.

Assumption 6.4 *The stage cost $l_i(x,u)$ is such that $l_i(0,0) = 0$ and $l_i(x_i,u_i) \geq \alpha_{il}(|x_i|)$ where α_{il} is a $\mathcal{K}_\infty$-function. Moreover, $l_i(x_i,u_i)$ is Lipschitz in x_i, in the domain $\mathcal{X}_i \times \mathcal{U}_i$, i.e. there exists a positive constant $\mathcal{L}_{il}$ such that*

$$|l_i(a,u) - l_i(b,u)| \leq \mathcal{L}_{il}|a-b|$$

for all $a,b \in \mathcal{X}_i$ and all $u \in \mathcal{U}_i$. □

It is now possible to define a "prototype" of a nonlinear MPC algorithm: at every time instants t, given $x_{i_t} = \bar{x}_i$, find the optimal control sequence $\mathbf{u}^o_{i[t,t+N_i-1|t]}$ by solving the FHOCP. Then, according to the Receding Horizon (RH) strategy, define

$$\kappa_i^{MPC}(\bar{x}_i) \triangleq u^o_{i_{t|t}}(\bar{x})$$

where $u^o_{i_{t|t}}(\bar{x})$ is the first column of $\mathbf{u}^o_{i[t,t+N-1|t]}$, and apply the control law

$$u_i = \kappa_i^{MPC}(x). \tag{6.19}$$

Define the overall control law

$$u \triangleq \left[\kappa_1^{MPC}(x_1)^\top, \kappa_2^{MPC}(x_2)^\top, \ldots, \kappa_S^{MPC}(x_S)^\top\right]^\top \tag{6.20}$$

Although the FHOCP has been stated for nominal conditions, under suitable assumptions and by choosing accurately the terminal cost function V_{if} and the terminal constraint $\mathcal{X}_{if}$, it is possible to guarantee the ISS property of the closed-loop system formed by (6.1) and (6.19), subject to constraints (6.3)-(6.5).

Assumption 6.5 *The solution of closed-loop system formed by (6.1),* *(6.19) is continuous at* $\bar{x}_i = 0$, $\mathbf{g}_i = 0$ *and* $\mathbf{d}_i = 0$ *with respect to initial conditions, interactions and disturbances.*

Assumption 6.6 *The design parameters* V_{if} *and* $\mathcal{X}_{if}$ *are such that, given an auxiliary control law* κ_{if},

1. $\mathcal{X}_{if} \subseteq \mathcal{X}_i$, $\mathcal{X}_{if}$ *closed,* $0 \in \mathcal{X}_{if}$
2. $\kappa_{if}(x_i) \in \mathcal{U}_i$, *for all* $x_i \in \mathcal{X}_{if}$
3. $f_i(x_i, \kappa_{if}(x_i)) \in \mathcal{X}_{if}$, *for all* $x_i \in \mathcal{X}_{if}$
4. *there exist a pair of suitable* $\mathcal{K}_\infty$*-functions* $\alpha_{V_{if}}$ *and* $\beta_{V_{if}}$ *such that* $\alpha_{V_{if}} < \beta_{V_{if}}$ *and*

$$\alpha_{V_{if}}(|x_i|) \leq V_{if}(x_i) \leq \beta_{V_{if}}(|x_i|)$$

5. $V_{if}(f_i(x_i, \kappa_{if}(x_i))) - V_{if}(x_i) \leq -l_i(x_i, \kappa_{if}(x_i))$, *for all* $x_i \in \mathcal{X}_{if}$
6. V_{if} *is Lipschitz in* $\mathcal{X}_{if}$ *with a Lipschitz constant* $\mathcal{L}_{V_{if}}$. □

Assumption 6.6 implies that the closed-loop system formed by the nominal system $f_i(x_{i_k}, \kappa_{if}(x_{i_k}))$ is asymptotically stable $\mathcal{X}_{if}$ (V_{if} is a Lyapunov function in $\mathcal{X}_{if}$ for the nominal system).

In the following, let $\mathcal{X}_i^{MPC}(N_i)$ denote the set of states for which a solution of the FHOCP problem exists.

Assumption 6.7 *Consider closed-loop system (6.1) and (6.19). For each* $x_{i_t} \in \mathcal{X}_i^{MPC}(N_i)$, $\tilde{\mathbf{u}}_{i[t+1,t+N_i|t+1]} \triangleq [\mathbf{u}^o_{i[t+1,t+N_i-1|t]}\ \kappa_{if}(\hat{x}_{i_{t+N_i|t+1}})]$ *is an admissible, possible suboptimal, control sequence for the FHOCP at time* $t+1$, *for all possible* $\mathbf{d}_i \in \mathcal{D}_i$ *and all possible* $\mathbf{g}_i \in \mathcal{G}_i$. □

Note that Assumption 6.7 implies that $\mathcal{X}_i^{MPC}(N_i)$ is a RPIA set for the closed-loop system (6.1) and (6.19).

In what follows, the optimal value of the performance index, i.e.

$$V_i(x) \triangleq J_i(\bar{x}_i, \mathbf{u}^o_{i[t,t+N_i-1|t]}, N_i) \tag{6.21}$$

is employed as an ISS-Lyapunov function for the closed-loop system formed by (6.1) and (6.19).

Assumption 6.8 *Let*

- $\Xi_i = \mathcal{X}_i^{MPC}$
- $\Omega_i = \mathcal{X}_{if}$
- $\alpha_{i1} = \alpha_{il}$
- $\alpha_{i2} = \beta_{V_{if}}$
- $\alpha_{i3} = \alpha_{il}$
- $\sigma^x_{ij} = \mathcal{L}_{iJ} L_{ij}$
- $\sigma^d_i = \mathcal{L}_{iJ}$, *where* $\mathcal{L}_{iJ} \triangleq \mathcal{L}_{V_{if}} \mathcal{L}_{if}^{N_i-1} + \mathcal{L}_{il} \frac{\mathcal{L}_{if}^{N_i-1}-1}{\mathcal{L}_{if}-1}$.

The sets $\mathcal{D}_i$ and $\mathcal{X}_j$ are such that the set Θ_i (depending from $\mathcal{D}_i^{sup}$ and $\mathcal{X}_j^{sup}$), defined in (6.15), with function V_i given by (6.21), is contained in $I\Omega_i$.

Remark 6.3 *The assumptions above can appear quite difficult to be satisfied, but they are standard in the development of nonlinear stabilizing MPC algorithms. Moreover, many methods have been proposed in the literature to compute V_{if}, $\mathcal{X}_{if}$ satisfying Assumption 6.6* *(see for example [Keerthi & Gilbert 1988, Mayne & Michalska 1990, Chen & Allgöwer 1998, De Nicolao et al. 1998c, Magni et al. 2001a]). However, with the MPC based on the FHOCP defined above, Assumption 6.7 is not a-priori satisfied. A way to fulfill it is shown in [Limon et al. 2002a] by properly restricting the state constraints 2 and 3 in the formulation of the FHOCP.* □

Theorem 6.2 *Under Assumptions 6.1, 6.4-6.8, V_i is an ISS-Lyapunov function in Ξ_i for the closed-loop system formed by (6.1) and (6.19) subject to constraints (6.3)-(6.5), that means, system (6.1), (6.19) is ISS from $g_i(x) + d_i$ to x_i with RPIA set $\mathcal{X}_i^{MPC}(N_i)$.* ■

Assumption 6.9 *Given the systems (6.1), (6.19), $i = 1, \ldots, S$, there exists a mapping Δ as in (6.16), such that condition (6.18) is satisfied.*

□

Define $\mathcal{X}^{MPC}(N) \triangleq \mathcal{X}_1^{MPC}(N_1) \times \ldots \times \mathcal{X}_S^{MPC}(N_S)$ as the vector of RPI sets of all the subsystems.

The main result can now be stated.

Theorem 6.3 *Under Assumptions 6.1, 6.4-6.9 the overall system (6.2), (6.20) is ISS in $\mathcal{X}^{MPC}(N)$ from d to x.* ■

6.4.2 Closed-loop formulation

In the following, it is shown that the ISS result of the previous section is also useful to derive the ISS property of min-max MPC. In this framework, at any time instant the controller for the i-th subsystem chooses the input u_i as a function of the current state x_i, so as to guarantee that the influence of the disturbance of the S subsystems are compensated. Hence, instead of optimizing with respect to a control sequence, at any time t the controller has to choose a vector of feedback control policies $\kappa_{i[t,t+N_i-1]} = [\kappa_{i0}(x_{i_t})\ \kappa_{i1}(x_{i_{t+1}})\ \dots\ \kappa_{iN_i-1}(x_{i_{t+N_i-1}})]$ minimizing the cost in the worst case.

Assumption 6.10 *For each subsystem, the sum of the interaction with the other subsystems and the disturbance is restricted to fulfill the following constraint*

$$w_i \triangleq \{g_i(x) + d_i\} \in \mathcal{W}_i,\ \forall x \in \mathcal{X},\ \forall d_i \in \mathcal{D}_i \tag{6.22}$$

where $\mathcal{W}_i$ is a compact set of $\mathbb{R}^{n_i}$, containing the origin as an interior point.

□

Note that, in view of Assumption 6.10, the sets $\mathcal{W}_i$ can be derived in view of the knowledge of $\mathcal{X}$ and $\mathcal{D}_i$.

The following optimal min-max problem can be stated for the i-th subsystem.

Definition 6.2 (FHCLG) *Consider system (6.1) with $x_{i_t} = \bar{x}_i$. Given the positive integer N_i, the stage cost $l_i - l_{iw}$, the terminal penalty V_{if} and the terminal set $\mathcal{X}_{if}$, the Finite Horizon Closed-loop Game (FHCLG) problem consists in minimizing, with respect to $\kappa_{i[t,t+N_i-1]}$ and maximizing with respect to $w_{i[t,t+N_i-1]}$ the cost function*

$$\begin{aligned} &J_i(\bar{x}_i, \kappa_{i[t,t+N_i-1]}, w_{i[t,t+N_i-1]}, N_i) \triangleq \\ &\sum_{k=t}^{t+N_i-1} \{l_i(x_{i_k}, u_{i_k}) - l_{iw}(w_{i_k})\} + V_{if}(x_{i_{t+N_i}}) \end{aligned} \tag{6.23}$$

subject to:

1. *the state dynamics (6.1)*

2. *the constraints (6.3)-(6.5)* $k \in [t, t+N_i-1]$;

3. *the terminal state constraint* $x_{i_{t+N_i}} \in X_{if}$. $\square$

Letting $\kappa^o_{i[t,t+N_i-1]}$, $\mathbf{w}^o_{i[t,t+N_i-1]}$ be the solution of the $FHCLG$, according to the RH paradigm, the feedback control law

$$u_i = \kappa_i^{MPC}(x_i) \tag{6.24}$$

is obtained by setting $\kappa_i^{MPC}(x_i) = \kappa^o_{i0}(x_i)$ where $\kappa^o_{i0}(x_i)$ is the first element of $\kappa^o_{i[t,t+N_i-1]}$.
Define the overall control law

$$u \triangleq [\kappa_1^{MPC}(x_1)^\top, \kappa_2^{MPC}(x_2)^\top, \ldots, \kappa_S^{MPC}(x_S)^\top]^\top. \tag{6.25}$$

In order to derive the main stability and performance properties associated to the solution of FHCLG, the following assumptions are introduced.

Assumption 6.11 *The solution of each closed-loop subsystem (6.1), (6.24) is continuous at* $\bar{x}_i$, $\mathbf{g}_i = 0$ *and* $\mathbf{d}_i = 0$ *with respect to initial conditions, interactions and disturbances.*

Assumption 6.12 *The function* $l_{iw}(w_i)$ *is such that* $\alpha_{iw}(|w_i|) \leq l_{iw}(w_i) \leq \beta_{iw}(|w_i|)$, *where* α_{iw} *and* β_{iw} *are* $\mathcal{K}_\infty$*-functions.* $\square$

Observe that, in view of Assumptions 6.1, 6.10 and 6.12,

$$\beta_{iw}(|w_i|) = \beta_{iw}(|g_i(x) + d_i|) \leq \beta_{iw}\left(\sum_{j=1}^{S} L_{ij}|x_j| + |d_i|\right)$$

and, in view of Lemma 6.1 there exist some $\mathcal{K}_\infty$-functions τ_{i1}, τ_{i2}, ..., τ_{id} such that

$$\begin{aligned}\beta_{iw}(|w_i|) &\leq \beta_{iw}\left(\sum_{j=1}^{S} L_{ij}|x_j| + |d_i|\right) \qquad (6.26)\\ &\leq \beta_{iw}\circ(id+\tau_{i1})\circ L_{i1}(|x_1|) + \beta_{iw}\circ(id+\tau_{i2})\circ L_{i2}(|x_2|) + \ldots\\ &\quad \ldots + \beta_{iw}\circ(id+\tau_{iN})\circ L_{iS}(|x_S|) + \beta_{iw}\circ(id+\tau_{id})(|d_i|).\end{aligned}$$

Assumption 6.13 *The design parameters V_{if} and $\mathcal{X}_{if}$ are such that, given an auxiliary law κ_{if},*

1. $\mathcal{X}_{if} \subseteq \mathcal{X}_i$, $\mathcal{X}_{if}$ *closed,* $0 \in \mathcal{X}_{if}$
2. $\kappa_{if}(x_i) \in U_i$, *for all* $x_i \in \mathcal{X}_{if}$
3. $f_i(x_i, \kappa_{if}(x_i)) + w_i \in \mathcal{X}_{if}$, *for all* $x_i \in \mathcal{X}_{if}$, *and all* $w_i \in \mathcal{W}_i$
4. *there exist a pair of suitable $\mathcal{K}_\infty$-functions $\alpha_{V_{if}}$ and $\beta_{V_{if}}$ such that* $\alpha_{V_{if}} < \beta_{V_{if}}$ *and*

$$\alpha_{V_{if}}(|x_i|) \leq V_{if}(x_i) \leq \beta_{V_{if}}(|x_i|)$$

for all $x_i \in \mathcal{X}_{if}$

5. $V_{if}(f_i(x_i, \kappa_{if}(x_i)) + w_i) - V_{if}(x_i) \leq -l_i(x_i, \kappa_{if}(x_i)) + l_{iw}(w_i)$, *for all* $x_i \in \mathcal{X}_{if}$ *and all* $w_i \in \mathcal{W}_i$. □

Assumption 6.13 implies that the closed-loop system formed by the system (6.1) and $u_i = \kappa_{if}(x_{i_k})$ is ISS in $\mathcal{X}_f$ (V_f is an ISS-Lyapunov function in $\mathcal{X}_f$).

Remark 6.4 *The computation of the auxiliary control law, of the terminal penalty and of the terminal inequality constraint satisfying Assumption 6.13,* *is not trivial at all. In this regard, a solution for affine system is discuss in Chapter 3, where it is shown how to compute a nonlinear auxiliary control law based on the solution of a suitable H_∞ problem for the linearized system under control.* □

In what follows, the optimal value of the performance index, i.e.

$$V(x) \triangleq J_i(\bar{x}_i, \kappa^o_{i[t,t+N_i-1]}, \mathbf{w}^o_{i[t,t+N_i-1]}, N_i) \tag{6.27}$$

is employed as an ISS-Lyapunov function for the closed-loop system formed by (6.1) and (6.24).

Assumption 6.14 *Let*

- $\Xi_i = \mathcal{X}_i^{MPC}$
- $\Omega_i = \mathcal{X}_{if}$
- $\alpha_1 = \alpha_l$
- $\alpha_2 = \beta_{V_f}$
- $\alpha_3 = \alpha_l$
- $\sigma^x_{ij} = \beta_{iw} \circ (id + \tau_{ij}) \circ L_{ij}$
- $\sigma^d_i = \beta_{iw} \circ (id + \tau_{id})$.

The sets $\mathcal{D}_i$ and $\mathcal{X}_j$ are such that the set Θ_i (depending from $\mathcal{D}_i^{sup}$ and $\mathcal{X}_j^{sup}$), defined in (6.15), with function V_i given by (6.27), is contained in $I\Omega_i$.

Theorem 6.4 *Under Assumptions 6.1, 6.4, 6.10-6.14, V_i is an ISS-Lyapunov function in $\mathcal{X}_i^{MPC}(N_i)$ for the closed-loop system formed by (6.1)* *and (6.24) subject to constraints (6.3)-(6.5), that means, system (6.1), (6.24) is ISS from* $g_i(x) + d_i$ *to* x_i *with RPIA set* $\mathcal{X}_i^{MPC}(N_i)$. ■

Assumption 6.15 *Given the systems (6.1), (6.24),* $i = 1, \ldots, S$, *there exists a mapping* Δ *as in (6.16), such that condition (6.18) is satisfied.* □

The main result can now be stated.

Theorem 6.5 *Under Assumptions 6.1, 6.4, 6.10-6.15 the overall system (6.2), (6.25) is ISS in* $\mathcal{X}^{MPC}(N)$ *from* d *to* x. ■

Remark 6.5 *Note that the term* $l_{iw}(w_i(k))$ *is included in the performance index (6.23) in order to obtain the ISS. In fact, without this term only ISpS can be proven, see Chapter 3.*

6.5 Conclusions

Regional Input-to-State Stability have been used in this paper to study the properties of two classes of decentralized MPC algorithms applied for control of nonlinear discrete time systems. Specifically, the stability analysis has been performed by considering the interconnections between the subsystems composing the overall system under control like perturbation terms and by using local MPC control laws with robustness properties. Both open-loop and closed-loop MPC formulations have been studied. Further research is required to establish the effect of partial exchange of information between subsystems on the stability conditions to be fulfilled.

6.6 Appendix

Proof of Lemma 6.1: As it is proven in [Jiang & Wang 2001], observe that given ρ_i, there exists a $\mathcal{K}_\infty$-function ζ_i such that $\rho_i^{-1} = (id + \zeta_i)$ and for any $\mathcal{K}$-function γ

$$\gamma\left(\sum_{i=1}^{S} r_i\right) \leq \max\{\gamma(Sr_1), \gamma(Sr_2), \ldots, \gamma(Sr_S)\}. \tag{6.28}$$

Using these observations, it is obtained that

$$\begin{aligned}\rho_i^{-1}(\theta_1 + \ldots + \theta_S + \theta_d) \quad &= \quad (id + \zeta_i) \circ (\theta_1 + \ldots + \theta_S + \theta_d) \\ &\leq \max\{(id + \zeta_i)((S+1)\theta_1), (id + \zeta_i)((S+1)\theta_2), \\ &\quad (id + \zeta_i)((S+1)\theta_3), \ldots, (id + \zeta_i)((S+1)\theta_d)\}.\end{aligned}$$

Finally, it is obtained that there exist some $\mathcal{K}_\infty$-functions a_{i1}, a_{i2}, $\ldots$, a_{id} such that

$$\begin{aligned}\rho_i^{-1}(\theta_1 + \ldots + \theta_S + \theta_d) \leq \quad \max \quad &\{(id + a_{i1})(\theta_1), (id + a_{i2})(\theta_2), \ldots \\ &\ldots, (id + a_{iS})(\theta_S), (id + a_{id})(\theta_d)\}.\end{aligned}$$

■

Proof of Theorem 6.1: From equation (6.13) it follows

$$\Delta V_i(x_i) \leq -\alpha_{i4}(V_i(x_i)) + \textstyle\sum_{j=1}^{N} \sigma_{ij}^x(|x_j|) + \sigma_i^d(|d_i|), \tag{6.29}$$

for all $x_i \in \Omega_i$, all $x_j \in \Xi_j$, and all $d_i \in \mathcal{D}_i$, where $\alpha_{i4} = \alpha_{i3} \circ \alpha_{i2}^{-1}$. Without loss of generality, assume that $(id - \alpha_{i4})$ is a $\mathcal{K}$-function. Using (6.11), equation (6.29) implies

$$\Delta V_i(x_i) \leq -\alpha_{i4}(V_i(x_i)) + \textstyle\sum_{j=1}^{N} \eta_{ij}^x(V_j(x_j)) + \sigma_i^d(|d_i|), \tag{6.30}$$

for all $x_i \in \Omega_i$, all $x_j \in \Xi_j$, and all $d_i \in \mathcal{D}_i$, where $\eta^x_{ij} = \sigma^x_{ij} \circ \alpha^{-1}_{i1}$. Given $e_i \in \mathbb{R}_{\geq 0}$, let $\mathcal{R}_i(e_i) \triangleq \{x_i : V_i(x_i) \leq e_i\}$. Define $\Psi_i \triangleq \{x_i : V_i(x_i) \leq \bar{e}_i = \max_{R_i \subseteq \Omega_i} e_i\}$. Note that $\bar{e}_i > b_i$ and $\Theta_i \subset \Psi_i$. By Theorem 2.1, the region Θ_i is reached asymptotically. This means that the state will arrive in Ψ_i in a finite time, that is there exists T_{ψ_i} such that $V_i(x_i(k)) \leq \bar{e}_i$, $\forall k \geq T_{\psi_i}$. Hence, the region Ψ_i is a RPI set for the subsystem (6.7). By Remark 3.6 [Jiang & Wang 2001], there exist $\beta_i \in \mathcal{KL}$ such that

$$V_i(x_{i_k}) \leq \max\{\beta_i(V_i(x_{i_t}), k), \alpha^{-1}_{i4} \circ \rho^{-1}_i(\sum_{j=1}^{S} \eta^x_{ij}(||\mathbf{V}_j(\mathbf{x}_j)||) + \sigma^d_i(||\mathbf{d}_i||))\},$$

for all $x_i \in \Psi_i$, all $x_j \in \Xi_j$, and all $d_i \in \mathcal{D}_i$.

Now, using Lemma 6.1, one has

$$\begin{aligned} V_i(x_{i_k}) \leq\ & \max\{\beta_i(V_i(x_{i_t}), k), \alpha^{-1}_{i4} \circ (id + a_{i1}) \circ \eta^x_{i1}(||\mathbf{V}_1(\mathbf{x}_1)||), \ldots, \\ & \alpha^{-1}_{i4} \circ (id + a_{iS}) \circ \eta^x_{iS}(||\mathbf{V}_S(\mathbf{x}_S)||), \alpha^{-1}_{i4} \circ (id + a_{id}) \circ \sigma^d_i(||\mathbf{d}_i||)\} \end{aligned} \tag{6.31}$$

where $a_{i1}, a_{i2}, \ldots, a_{iS}$ and a_{id} are $\mathcal{K}_\infty$-functions. Then

$$\begin{aligned} V_i(x_{i_k}) \leq\ & \beta_i(V_i(x_{i_t}), k) + \textstyle\sum_{j=1}^{S} \alpha^{-1}_{i4} \circ (id + a_{ij}) \circ \eta^x_{ij}(||\mathbf{V}_j(\mathbf{x}_j)||) \\ & + \alpha^{-1}_{i4} \circ (id + a_{id}) \circ \sigma^d_i(||\mathbf{d}_i||), \end{aligned} \tag{6.32}$$

for all $x_i \in \Psi_i$, all $x_j \in \Xi_j$, and all $d_i \in \mathcal{D}_i$. Moreover, using (6.16) and (6.17), and following the same steps of the Proof of Theorem 4 in [Dashkovskiy *et al.* 2007], using (6.32) instead of (2.4) in [Dashkovskiy *et al.* 2007], it can be shown that (6.32) satisfies AG property and is 0-AS in Ψ_i and hence, by using point 7 of Assumption 6.2 and the compactness of Ψ_i, there exist a $\mathcal{KL}$-function $\widetilde{\beta}_i$ and a $\mathcal{K}$-function $\widetilde{\gamma}_i$ such that

$$V_i(x_{i_k}) \leq \widetilde{\beta}_i(V_i(x_{i_t}), k) + \widetilde{\gamma}_i(||\mathbf{d}_i||), \tag{6.33}$$

for all $x_i \in \Psi_i$, and all $d_i \in \mathcal{D}_i$. In fact, by using point 7 of Assumption 6.2

and the compactness of Ψ_i, AG in Ψ_i + 0-AS in Ψ_i is equivalent to ISS in Ψ_i (see [Sontag & Wang 1996] for the continuous time and [Gao & Lin 2000] for the discrete time case). Now, using the properties (6.11) and (6.12), considering that, from (6.28), for any $\mathcal{K}_\infty$-function γ, $\gamma(r+s) \leq \gamma(2r) + \gamma(2s)$, one has

$$|x_{i_k}| \quad \leq \quad \bar{\beta}_i(|x_{i_t}|, k) + \bar{\gamma}_i(||\mathbf{d}_i||), \tag{6.34}$$

for all $x_i \in \Psi_i$, and all $d_i \in \mathcal{D}_i$, where $\bar{\beta}_i(|x_{i_t}|, k) = \alpha_{i1}^{-1} \circ 2\widetilde{\beta}_i(\alpha_{i2}(|x_{i_t}|), k)$ and $\bar{\gamma}_i(||\mathbf{d}_i||) = \alpha_{i1}^{-1} \circ 2\widetilde{\gamma}_i(||\mathbf{d}_i||)$. Hence the system (6.7) is ISS in Ψ_i from d_i to x_i. Then, by Lemma 2.1, ISS in Ψ_i implies *UAG* in Ψ_i and LS (see Chapter... for the definitions). Considering that starting from Ξ_i, the state will reach the region Ψ_i in a finite time, UAG in Ψ_i implies UAG in Ξ_i. Hence, using again Lemma 2.1, the system (6.7) is ISS in Ξ_i from d_i to x_i and hence the overall system (6.8) is ISS in Ξ from d to x. ■

Proof of Theorem 6.2: the proof of Theorem 6.2 can be derived from the proof of Theorem 2.1 by substituting at $|w|$, the term $\sum_{j=1}^{S} L_{ij}|x_j| + |d_i|$.

■

Proof of Theorem 6.3: by Theorem 6.2, each subsystem (6.1), (6.19) admits an ISS-Lyapunov function satisfying Assumption 6.3. Together with Assumption 6.9, and using Theorem 6.1, this proves that Theorem 6.3 holds. ■

Proof of Theorem 6.4: the proof of Theorem 6.4 can be derived from the proof of Theorem 2.3 by substituting at $|w|$, the term $\sum_{j=1}^{S} L_{ij}|x_j| + |d_i|$ and by using (6.26). ■

Proof of Theorem 6.5: by Theorem 6.4, each subsystem (6.1), (6.24) admits an ISS-Lyapunov function satisfying Assumption 6.3. Together with Assumption 6.15, and using Theorem 6.1, this proves that Theorem 6.5 holds. ■

Chapter 7

Cooperative NMPC for Distributed Agents

Contents

7.1 Introduction

Another research direction in decentralized control considers the problem of controlling a team of dynamically decoupled cooperating systems. For instance, there have been some important theoretical results on the stability of swarms [Liu *et al.* 2003], but a considerable number of publications in this area focus on specific issues related to Uninhabited Autonomous/Air Vehicles (UAVs) applications (see, for instance, [Chandler *et al.* 2000, Polycarpou *et al.* 2001, Y. Jin *et al.* 2006]).

One of the approaches used in this area is based on the selection of a suitable cost function and its optimization in a model-predictive control (MPC) framework. The cost function used for MPC framework can take into account several issues, such as collision avoidance and formation constraints, and may reward the tracking of a certain path. In [Li & Cassandras 2002], [Li & Cassandras 2003] and [Li & Cassandras 2004], the authors consider a two-degrees of freedom team of UAVs assigned to visit a certain number of points. The team of UAVs is controlled in a centralized receding–horizon (RH) framework and by exploiting global potential functions, the authors prove certain stationarity properties of the generated trajectories in the case of two agents searching for multiple targets. A RH control scheme has also been proposed in [Kevicky *et al.* 2004a], [Kevicky *et al.* 2004b], where a centralized problem is decomposed to allow local computations and feasibility issues are thoroughly examined; stability is obtained in [Kevicky *et al.* 2004a] exploiting a hierarchical decomposition of the team in suitable subgraphs with assigned priorities.

Coordination of a large group of cooperating nonlinear vehicles is considered in [Dunbar & Murray 2004] and related works, where a centralized RH problem is decomposed and solved locally. Convergence to the formation equilibrium point is assured by guaranteeing frequent updates and a bounded error between the assumed and the predicted trajectories, which every agent computes for itself and its neighbors in the model predictive control process.

Towards a broad analysis of the structural properties of cooperative systems, an ISS analysis has recently been proposed by several authors. In [Tanner *et al.* 2002], [Tanner *et al.* 2004] the concept of Leader to Formation Stability is developed. A discussion of some of the issues arising in the study of non-holonomic vehicles using ISS can be found in [Chen & Serrani 2004]. ISS tools have been successfully applied to the specific case of networked systems with serial communication, where Nesic and Teel propose a new unified framework for modelling and analyzing networked control systems [Nesic & Teel 2004b], [Nesic & Teel 2004c].

In this chapter, a cooperative control problem for a team of distributed agents with nonlinear discrete-time dynamics is considered. The problem formulation is based on a completely decentralized MPC control algorithm, which is analyzed using an ISS approach. The proposed scheme generalizes the approach presented in [Franco *et al.* 2004] to a nonlinear framework. Each agent is assumed to evolve in discrete-time by means of locally computed control laws, which takes into consideration delayed state information from a subset of neighboring cooperating agents. The cooperative control problem is first formulated in a MPC framework, where the control laws depend on the local state variables (feedback action) and on delayed information gathered from cooperating neighboring agents (feedforward action). A rigorous stability analysis is carried out, exploiting the regional ISS properties of the MPC local control laws. The asymptotic stability of the team of agents is then proved by utilizing small–gain theorem results. The information flow among the agents is considered as a set of interconnections whose size is measured by the weight this information has in the computation of the control action. Hence, the derived result confirms that, in this framework, a suitable “interconnection” boundedness is necessary to guarantee stability.

7.2 Problem statement

In this section, the cooperative control problem addressed in the chapter will be formulated in general terms, whereas, in Section 7.3, the stability properties of the distributed controlled system will be analyzed and the main results will be proved.

A *distributed dynamic system* made of a set of M agents denoted as $\mathcal{A} \triangleq \{\mathcal{A}_i \ : \ i = 1, ..., M\}$ is considered. Each agent $\mathcal{A}_i$ is described by the nonlinear time-invariant state equation:

$$x_{i_{k+1}} = f_i(x_{i_k}, u_{i_k}), \ k \geq 0, \ x_{i_0} = \bar{x}_i \tag{7.1}$$

where, for each $i = 1, ..., M$, $x_{i_k} \in \mathrm{I\!R}^{n_i}$ denotes the local state vector and $u_{i_k} \in \mathrm{I\!R}^{m_i}$ denotes the local control vector of agent $\mathcal{A}_i$ at time k, and where it is assumed that $f_i(0,0) = 0$, $i = 1, ..., M$. Let also suppose that the dynamics of all M agents evolve on the same discrete-time space (that is, the agents are synchronized).

The state vector x_i of each agent $\mathcal{A}_i \ : \ i = 1, ..., M$ is constrained to belong to a compact set $\mathcal{X}_i$, that is,

$$x_i \in \mathcal{X}_i \subset \mathrm{I\!R}^{n_i}. \tag{7.2}$$

Analogously, the control vector u_i is constrained to take values in a compact set $\mathcal{U}_i$, that is,

$$u_i \in \mathcal{U}_i \subset \mathrm{I\!R}^{m_i}. \tag{7.3}$$

In open–loop mode, each agent is dynamically decoupled from the remaining agents and the dynamics of the other agents are not assumed to be known. The coupling between agents arises due to the fact that they operate in the same environment and due to the "cooperative" objective imposed on each agent by a cost function defined later on.

To achieve some degree of cooperation, each agent $\mathcal{A}_i$ exchanges an information vector w_i with a given set of neighboring agents $\mathcal{G}_i \triangleq \{\mathcal{A}_j \,:\, j \in G_i\}$, where G_i denotes the set of indexes identifying the agents belonging to the set $\mathcal{G}_i$. More precisely, the information exchange pattern is defined as follows. Let us consider a generic time–instant t; then for each $i = 1, ..., M$, the agent $\mathcal{A}_i$ receives from each neighboring cooperating agent $\mathcal{A}_j \in \mathcal{G}_i$ the value of its local state vector with a delay of Δ_{ij} time steps, that is, agent $\mathcal{A}_i$ receives the vector $x_{j_{t-\Delta_{ij}}}$ from agent $\mathcal{A}_j \in \mathcal{G}_i$. To gain some more insight into the information exchange pattern, refer to Fig. 7.1, where a simple three–agent example is shown pictorially. In this specific example, each agent receives information from all remaining agents. At each time–instant t, let group all inputs to agent $\mathcal{A}_i$ into a vector $\bar{w}_{i_t}$ defined as $\bar{w}_{i_t} \triangleq \mathrm{col}\,(x_{i_{t-\Delta_{ij}}},\, j \in G_i)$. The size of vector $\bar{w}_i$ is equal to $n_{w_i} = \sum\limits_{j \in G_i} n_j$ and clearly

$$\bar{w}_i \in \mathcal{W}_i \tag{7.4}$$

where $\mathcal{W}_i$ denotes the cartesian product of all sets $\mathcal{X}_j$, $j \in G_i$, that is, $\mathcal{W}_i \triangleq \prod\limits_{j \in G_i} \mathcal{X}_j$.

It is worth noting that the above setting allows the investigation of quite a large class of distributed cooperating dynamic systems like teams of mobile vehicles, cooperating robotic arms, routing nodes in communications and/or transportation networks where agents cooperate to minimize the total traffic delay, networks of reservoirs in water-distribution networks, etc..

For each $i = 1, \ldots, M$ and for given values of the state vector $x_{i_t} \in \mathcal{X}_i$ and of the information vector $\bar{w}_{i_t} \in \mathcal{W}_i$ at time–instant t, let now introduce

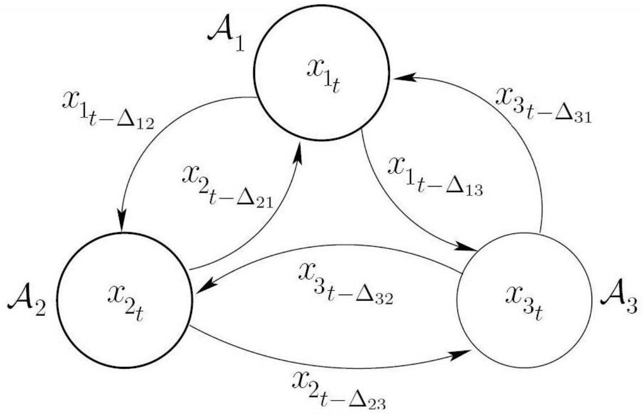


Figure 7.1: Three agents exchanging delayed state information.

the following finite–horizon (FH) cost function (in general, nonquadratic):

$$J_i(x_{i_t}, w_{i_t}, \mathbf{d}_{l_i[t,t+N_{ip}]}, \mathbf{d}_{q_i[t,t+N_{ip}-1]}, \mathbf{u}_{i[t,t+N_{ic}-1]}, N_{ic}, N_{ip}) =$$
$$\sum_{k=t}^{t+N_{ip}-1} \left[l_i\left(x_{i_k}, u_{i_k}, d_{l_{i_k}}\right) + q_i\left(x_{i_k}, w_{i_k}, d_{q_{i_k}}\right)\right] + V_{if}\left(x_{i_{t+N_{ip}}}, d_{l_{i_{t+N_{ip}}}}\right)$$

The positive integers N_{ic} and N_{ip}, $i = 1, \ldots, M$ denote the lengths of the so-called control and prediction horizons, respectively, according to the framework proposed in [Magni *et al.* 2001a]. The local cost function is composed of two terms: a partial cost term given by

$$\sum_{k=t}^{t+N_{ip}-1} l_i\left(x_{i_k}, u_{i_k}, d_{l_{i_k}}\right) + V_{if}\left(x_{i_{t+N_{ip}}}, d_{l_{i_{t+N_{ip}}}}\right)$$

where l_i is a transition cost function and V_{if} is a terminal cost function, and a "cooperation" cost term given by

$$\sum_{k=t}^{t+N_{ip}-1} q_i\left(x_{i_k}, w_{i_k}, d_{q_{i_k}}\right)$$

The quantities $\mathbf{d}_{l_i}$, $\mathbf{d}_{q_i}$, $i = 1, \ldots, M$, denote some given vectors of appropriate dimensions. In general, the vectors $\mathbf{d}_{l_i}$ are useful to specify some reference value for some or all components of the local state variables, whereas the vectors $\mathbf{d}_{q_i}$ can be used to parametrize the cooperation between the agents. For example (see also Section 7.4), if the agents represent UAV vehicles, then vectors $\mathbf{d}_{l_i}$, $\mathbf{d}_{q_i}$ could be defined so as to specify given trajectories to be followed by each agent and also given "formation structures" for the agents. As will be subsequently clarified, the control variables u_{i_k}, $k = t, \ldots, t + N_{ic} - 1$ will be the argument of a suitable optimization problem, whereas the control variables u_{i_k}, $k = t + N_{ic}, \ldots, t + N_{ip} - 1$ will be obtained through some auxiliary control law $u_{i_k} = \kappa_{if}(x_{i_k})$. The vector w_i denotes the state of the dynamic system

$$w_{i_{k+1}} = A_{w_i} w_{i_k}, \, k = t, \ldots, t + N_{ip} - 2 \, ; \quad w_{i_t} \triangleq \bar{w}_i \tag{7.5}$$

where $A_{w_i} \triangleq \alpha_{w_i} I_{n_{w_i}}$ with $\alpha_{w_i} < 1$ and with $I_{n_{w_i}}$ denoting the identity matrix of dimension n_{w_i}. The dynamic system (7.5) is introduced in order to decrease the "importance" of the information vector in the FH cost function along the prediction horizon (e.g., a "forward-forgetting-factor" is introduced in the cost function as regards the information vector exchanged at time-instant t). It is worth noting that, at time–instant t, vectors w_{i_t} can be considered as known external inputs in the cost function.

In the following, for the sake of simplicity, let suppose that, by a suitable change of state coordinates, it is possible to consider an equivalent formulation where the cost function (with straightforward re-definitions of the symbols) can be re-written in the simpler form

$$J_i(x_{i_t}, w_{i_t}, \mathbf{u}_{i[t,t+N_{ic}-1]}, N_{ic}, N_{ip}) = \\ \sum_{k=t}^{t+N_{ip}-1} \left[l_i\left(x_{i_k}, u_{i_k}\right) + q_i\left(x_{i_k}, w_{i_k}\right) \right] + V_{if}\left(x_{i_{t+N_{ip}}}\right) \tag{7.6}$$

where $l_i(0,0) = 0$, $q_i(0,0) = 0$, and $V_{if}(0) = 0$. Moreover, the origin is an interior point of the sets $\mathcal{X}_i$ and $\mathcal{U}_i$.

Remark 7.1 *Beyond allowing for a simpler problem formulation, the reduction of the original FH cost function to the form (7.6) will allow for the design of time-invariant control laws; after a change of coordinates it will also be possible to carry on the stability analysis with reference to the origin as equilibrium state of the time-invariant system (see Section 7.4 for some details about the above change of coordinates in a practical simple case). However, considering the general case would not involve major conceptual difficulties.* □

The local control law is designed according to a RH strategy. In the literature several different problem formulations can be found depending on the particular setting. In this chapter, the MPC control problem has been stated according to [Magni *et al.* 2001a] (see also the well–known survey paper [Mayne *et al.* 2000]).

Problem 7.2.1 (FHOCP) *Consider, for every agent $\mathcal{A}_i$, with $i = 1, \ldots, M$, systems (7.1) and (7.5) with $x_{i_t} \triangleq \bar{x}_i \in \mathcal{X}_i$ and $w_{i_t} \triangleq \bar{w}_i \in \mathcal{W}_i$ as initial conditions. Given the positive integers N_{ic}, N_{ip}, the transition, cooperation and terminal cost functions l_i, q_i, V_{if}, the terminal set $\mathcal{X}_{if}$, and the auxiliary control law κ_{if}, the Finite Horizon Optimal Control Problem*

(FHOCP) consists in minimizing with respect to $\mathbf{u}_{i[t,t+N_{ic}-1]}$ *the performance index (7.6) subject to:*

1. *the agent's dynamics (7.1)*
2. *the system's dynamics (7.5)*
3. *the auxiliary control law*

$$u_{i_k} = \kappa_{if}(x_{i_k}),\ k = t + N_{ic}, \ldots, t + N_{ip} - 1$$

4. *the constraints (7.2) and (7.3)*
5. *the terminal state constraint* $x_{i_{t+N_{ip}}} \in \mathcal{X}_{if}$ *.* □

Clearly, by definition, the optimal FH control sequence $\mathbf{u}^o_{i[t,t+N_{ic}-1]}$ solving Problem 7.2.1 is such that, when applied to (7.1), the constraints (7.2), (7.3), and the terminal constraint $x_{i_{t+N_{ip}}} \in \mathcal{X}_{if}$ are simultaneously satisfied. Indeed, the following definition regarding a generic control sequence $\mathbf{u}_{i[t,t+N_{ic}-1]}$ will be useful in the analysis reported in Section 7.3.

Definition 7.1 (Admissible control sequence) *Given an initial state* x_{i_t}*, the sequence* $\mathbf{u}_{i[t,t+N_{ic}-1]}$ *is said to be an admissible control sequence for the FHOCP if its application to (7.1) under the action of the auxiliary control law* $u_{i_k} = \kappa_{if}(x_{i_k}),\ k = t + N_{ic}, \ldots, t + N_{ip} - 1$ *allows simultaneous satisfaction of (7.2), (7.3) and of the terminal constraint* $x_{i_{t+N_{ip}}} \in \mathcal{X}_{if}$ *.* □

Now, the RH procedure can be described in the usual way as follows. When the controlled agent $\mathcal{A}_i$ is in the state x_{i_t} at stage t, the FHOCP is solved, thus obtaining the sequence of optimal control vectors, $\mathbf{u}^o_{i[t,t+N_{ic}-1]}$. Then, according to the Receding Horizon (RH) strategy, define $\kappa_i^{MPC}(\bar{x}_i, \bar{w}_i) \triangleq u^o_{i_t}(\bar{x}_i, \bar{w}_i)$ where $u^o_{i_t}(\bar{x}_i, \bar{w}_i)$ is the first element of

$\mathbf{u}^o_{i[t,t+N_{ic}-1]}$. This procedure is repeated stage after stage and a *feedback–feedforward* control law

$$u_{i_t} = \kappa_i^{MPC}(x_{i_t}, w_{i_t}) \tag{7.7}$$

is obtained, that depends on the local current state x_{i_t} and on the vector of delayed states w_{i_t} communicated to the agent $\mathcal{A}_i$ by the cooperating agents $\mathcal{G}_i = \{\mathcal{A}_j,\, j \in G_i\}$.

The system (7.1) under the action of the *RH* optimal control law can thus be rewritten as

$$x_{i_{k+1}} = \tilde{f}_i(x_{i_k}, w_{i_k}) \triangleq f_i(x_{i_k}, \kappa_i^{MPC}(x_{i_k}, w_{i_k}))\,,\ k \geq t,\ x_{i_t} = \bar{x}_i \tag{7.8}$$

It is worth noting that, from well-known results on RH control (see, for instance, [Mayne *et al.* 2000] and the references cited therein), there is $\kappa^{MPC}(0,0) = 0$ and hence $\tilde{f}_i(0,0) = 0$, that is, the origin is an equilibrium state for agent $\mathcal{A}_i$ when $w_{i_k} = 0,\, k \geq t$.

Assumption 7.1 *The solution of closed-loop (7.8) is continuous at* $\bar{x}_i = 0$ *and* $\mathbf{w}_i = 0$ *with respect to disturbances and initial conditions.*

7.3 Stability of the team of cooperating agents

The stability analysis of the team of cooperating agents will be carried out in three main steps. In Subsection 7.3.1 some basic results concerning the regional input–to–state stability properties of discrete–time systems will be stated and proved, extending the approach presented in Chapter 2 to this particular case. In Subsection 7.3.2, the regional stability results will be exploited referring to specific dynamic models (7.8) of $\mathcal{A}_i$. It will be indeed proven that each agent is regionally ISS with respect to the input represented by the delayed incoming information from its neighbors. Finally the team of cooperating agents will be considered in Subsection 7.3.3 as a single dynamic system resulting from a feedback interconnection of regionally ISS systems. Showing that both the elements of this interconnection are endowed with ISS-Lyapunov functions, will result in proving the asymptotic stability of the team of cooperating agents by resorting to appropriate small–gain conditions.

7.3.1 Regional ISS results

The regional ISS stability analysis will now be associated to the existence of a suitable Lyapunov function (in general, a-priori non smooth). Note that, differently from what stated in Chapter 2, in this case, in order to fit the considered problem, the ISS-Lyapunov function will be a function of both variables x and w.

Consider a discrete-time autonomous nonlinear dynamic system described by

$$x_{k+1} = \tilde{f}(x_k, w_k), \ k \geq 0, \tag{7.9}$$

where the map $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ is nonlinear possibly discontinuous, $x_k \in \mathbb{R}^n$ is the state, $w_k \in \mathbb{R}^q$ is an unknown disturbance. The transient of the system (7.9) with initial state $x_0 = \bar{x}$ and disturbance sequence $\mathbf{w}$ is denoted by $x(k, \bar{x}, \mathbf{w})$. This system is supposed to fulfill the following assumptions.

Assumption 7.2

1. *The origin of the system is an equilibrium point, i.e.* $\tilde{f}(0, 0) = 0$.

2. *The disturbance w is such that*

$$w \in \mathcal{W} \tag{7.10}$$

where $\mathcal{W}$ is a compact set containing the origin, with $\mathcal{W}^{sup}$ known.

Assumption 7.3 *The solution of (7.9) is continuous at $\bar{x} = 0$ and $\mathbf{w} = 0$ with respect to disturbances and initial conditions.*

Definition 7.2 (ISS-Lyapunov function in Ξ) *A function $V : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}_{\geq 0}$ is called an ISS-Lyapunov function in Ξ for system (7.9), if*

1. *Ξ is a compact RPI set including the origin as an interior point*

2. *there exist a compact set $\Omega \subseteq \Xi$ (including the origin as an interior point), and a pair of suitable $\mathcal{K}_\infty$-functions α_1, α_2 such that*

$$V(x, w) \geq \alpha_1(|x|), \ \forall x \in \Xi, \ \forall w \in \mathcal{W} \tag{7.11}$$

$$V(x, w) \leq \alpha_2(|x|) + \sigma_1(|w|), \ \forall x \in \Omega, \ \forall w \in \mathcal{W} \tag{7.12}$$

3. *there exist a suitable $\mathcal{K}_\infty$-function α_3 and some suitable $\mathcal{K}$-function σ_2 and σ_3 such that*

$$V(\tilde{f}(x, w_1), w_2) - V(x, w_1) \leq -\alpha_3(|x|) + \sigma_2(|w_1|) + \sigma_3(|w_2|) \tag{7.13}$$

for all $x \in \Xi$, and all $w_1, w_2 \in \mathcal{W}$

4. *there exist some suitable $\mathcal{K}_\infty$-function ζ and ρ (with ρ such that $(id - \rho)$ is a $\mathcal{K}_\infty$-function) and a suitable constant $c_\theta > 0$, such that, given a disturbance sequence $\mathbf{w} \in \mathcal{M}_\mathcal{W}$, there exists a nonempty compact set*

$\Theta_{\mathbf{w}} \subseteq I\Omega \triangleq \{x : x \in \Omega, |x|_{\delta\Omega} > c_\theta\}$ (including the origin as an interior point) defined as follows

$$\Theta_{\mathbf{w}} \triangleq \{x : V(x) \leq b(||\mathbf{w}||)\} \tag{7.14}$$

where $b \triangleq \alpha_4^{-1} \circ \rho^{-1} \circ \sigma_4$, with $\alpha_4 \triangleq \underline{\alpha}_3 \circ \overline{\alpha}_2^{-1}, \underline{\alpha}_3(s) \triangleq \min(\alpha_3(s/2), \zeta(s/2))$, $\overline{\alpha}_2(s) \triangleq \alpha_2(s) + \sigma_1(s)$, $\sigma_4(s) \triangleq \zeta(s) + \sigma_2(s) + \sigma_3(s)$. □

Remark 7.2 *Note that, in order to verify that $\Theta_{\mathbf{w}} \subseteq I\Omega$ for all $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$, one has to verify that*

$$\Theta \triangleq \{x : V(x) \leq b(\mathcal{W}^{sup})\} \subseteq I\Omega \tag{7.15}$$

□

A sufficient condition for regional ISS of system (7.9) can now be stated.

Theorem 7.1 *Suppose that Assumption 7.2 and 7.3 hold. If system (7.9) admits an ISS-Lyapunov function in Ξ, then it is ISS in Ξ and, for all disturbance sequences $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$, $\lim_{k\to\infty} |x(k, \bar{x}, \mathbf{w})|_{\Theta_{\mathbf{w}}} = 0$.* ■

To sum up, in this subsection an important sufficient condition for regional ISS of constrained systems of the form (7.9) has been stated and proved. In the next subsection, Theorem 7.1 will be exploited with reference to each agent $\mathcal{A}_i$ under the action of the local MPC control law.

7.3.2 Stability properties of the single agents

Consider a generic agent $\mathcal{A}_i$ whose dynamics is described by (7.1). By exploiting the results proved in Subsection 7.3.1, it will now shown that each agent $\mathcal{A}_i$, with $i = 1, 2, \ldots, M$ is regionally ISS with respect to the inputs represented by the information vectors w_{i_t} received from its cooperating agents at each time–step t. Clearly, in this context, each agent is considered as a "separate" dynamic system in the team, in the sense that the input vectors w_{i_t} are "external" variables that are assumed not to depend on the behavior of the other cooperating agents (i.e., at the present stage, the coupling between the agents is not directly taken into account). Let now introduce some further useful assumptions and definitions.

Assumption 7.4 *The design parameters V_{if} and $\mathcal{X}_{if}$ are such that, given an auxiliary control law κ_{if},*

1. $\mathcal{X}_{if} \subseteq \mathcal{X}_i$, $\mathcal{X}_{if}$ *closed,* $0 \in \mathcal{X}_{if}$
2. $\kappa_{if}(x) \in \mathcal{U}_i$, $|\kappa_{if}(x_i)| \leq \mathcal{L}_{\kappa_{if}} |x_i|$, $\mathcal{L}_{\kappa_{if}} > 0$, *for all* $x \in \mathcal{X}_{if}$
3. $|f_i(x_i, \kappa_{if}(x_i))| \leq \mathcal{L}_{f_{ic}} |x_i|$, $\mathcal{L}_{f_{ic}} > 0$, *for all* $x_i \in \mathcal{X}_{if}$
4. $f(x_i, \kappa_{if}(x_i)) \in \mathcal{X}_{if}$, *for all* $x_i \in \mathcal{X}_{if}$
5. *there exist a pair of suitable* $\mathcal{K}_\infty$*-functions* $\alpha_{V_{if}}$ *and* $\beta_{V_{if}}$ *such that* $\alpha_{V_f} < \beta_{V_{if}}$ *and*
$$\alpha_{V_{if}}(|x_i|) \leq V_{if}(x_i) \leq \beta_{V_{if}}(|x_i|)$$
6. $V_{if}(f_i(x_i, \kappa_{if}(x_i))) - V_{if}(x_i) \leq -l_i(x_i, \kappa_{if}(x_i)) - q_i(x_i, \tilde{w}_i) + \psi_i(|\tilde{w}_i|)$ *for all* $x_i \in \mathcal{X}_{if}$, *and all* $w_i \in \mathcal{W}_i$, *where* ψ_i *is a* $\mathcal{K}$*-function and* $\tilde{w}_i \triangleq (A_{w_i})^{N_{ip}-1} w_i$.

Assumption 7.5 *The partial cost function* l_i *is such that* $\underline{r}_i(|x_i|) \leq l_i(x_i, u_i)$, *for all* $x_i \in \mathcal{X}_i$, *and all* $u_i \in \mathcal{U}_i$ *where* $\underline{r}_i$ *is a* $\mathcal{K}_\infty$*-function. Moreover,* l_i *is Lipschitz with respect to* x_i *and* u_i *in* $\mathcal{X}_i \times \mathcal{U}_i$, *with Lipschitz constants denoted as* $\mathcal{L}_{l_i}$ *and* $\mathcal{L}_{l_{iu}}$, *respectively.*

Assumption 7.6 *The cooperation cost function q_i is such that $0 \leq q_i(x_i, w_i)$, for all $x_i \in \mathcal{X}_i$, and all $w_i \in \mathcal{W}_i$. Moreover q_i is Lipschitz with respect to x_i and w_i in $\mathcal{X}_i \times \mathcal{W}_i$, with Lipschitz constants denoted as $\mathcal{L}_{q_i}$ and $\mathcal{L}_{q_{iw}}$, respectively.* $\square$

In the following, let $\mathcal{X}_i^{MPC}$ denote the set of states for which a solution of the FHOCP problem exists.

Assumption 7.7 *Consider system (7.1). Let denote $\mathcal{X}_i^{\kappa_{if}}$ the set of states x_{i_t} for which*

$$\tilde{\mathbf{u}}_{i[t,t+N_{ic}-1]} \triangleq \mathrm{col}\left[\kappa_{if}(x_{i_t}), \kappa_{if}(x_{i_{t+1}}), \ldots, \kappa_{if}(x_{i_{t+N_{ic}-1}})\right]$$

is an admissible control sequence for the FHOCP and for which points 2 and 3 of Assumption 7.4 are satisfied. $\square$

In what follows, the optimal value of the performance index, i.e.

$$V_i(x_{i_t}, w_{i_t}) \triangleq J_i(x_{i_t}, w_{i_t}, \mathbf{u}^o_{i[t,t+N_{ic}-1]}, N_{ic}, N_{ip}) \tag{7.16}$$

is employed as an ISS-Lyapunov function for the closed-loop system formed by (7.1) and (7.7).

Assumption 7.8 *Suppose*[1] *that $L_{f_{ic}} \neq 1$ and let*

- $\Xi = \mathcal{X}_i^{MPC}$
- $\Omega = \mathcal{X}_i^{\kappa_{if}}$
- $\alpha_1(s) = \underline{r}_i(s)$

[1]The very special case $L_{f_{ic}} = 1$ can be trivially addressed by a few suitable modifications to the proof of Theorem 7.2.

- $\alpha_2(s) = (\mathcal{L}_{l_i} + \mathcal{L}_{l_{iu}}\mathcal{L}_{\kappa_{if}} + \mathcal{L}_{q_i})\frac{(\mathcal{L}_{f_{ic}})^{N_{ip}-1}}{\mathcal{L}_{f_{ic}-1}}s + \beta_{V_{if}}(\mathcal{L}_{f_{ic}}^{N_{ip}}s)$
- $\alpha_3(s) = \underline{r}_i(s)$
- $\sigma_1(s) = \mathcal{L}_{q_{iw}}\dfrac{(\alpha_{w_i})^{N_{ip}}-1}{\alpha_{w_i}-1}s$
- $\sigma_2(s) = \alpha_{w_i}\mathcal{L}_{q_{iw}}\dfrac{(\alpha_{w_i})^{N_{ip}}-1}{\alpha_{w_i}-1}s + \psi_i((\alpha_{w_i})^{N_{ip}-1}s)$
- $\sigma_3(s) = \mathcal{L}_{q_{iw}}\dfrac{(\alpha_{w_i})^{N_{ip}}-1}{\alpha_{w_i}-1}s$

The set $\mathcal{W}$ is such that the set Θ (depending from $\mathcal{W}^{sup}$), defined in (7.15), with function V given by (7.2), is contained in $I\Omega$. □

The main result can now be stated.

Theorem 7.2 *Under Assumptions 7.1, 7.4-7.8, the locally–controlled agent $\mathcal{A}_i$, $i = 1, \dots, M$, whose closed–loop dynamics are described by (7.8), subject to constraints (7.2), (7.3), and (7.4), is ISS with RPIA set $\mathcal{X}_i^{MPC}$.*

■

It is worth noting that, from the perspective of determining regionally ISS stabilizing control laws, a key aspect is the design of an auxiliary control law $\kappa_{if}(x_i)$ such that Assumption 7.4 holds. In this respect, under slightly more restrictive hypotheses on the agents' dynamic models and on the FH cost function, the following useful result is given.

Lemma 7.1 *Assume that $f_i \in \mathcal{C}^2$, and*

$$l_i(x_i, u_i) = x_i^\top Q_i x_i + u_i^\top R_i u_i$$

$$q_i(x_i, w_i) \leq x_i^\top \tilde{S}_i x_i^\top + \psi_i(|w_i|)$$

with Q_i, R_i, *and* $\tilde{S}_i$ *being positive definite matrices and* ψ_i *being a* $\mathcal{K}$-*function. Furthermore, suppose that there exists a matrix* K_i *such that* $A_{i_{cl}} = A_i + B_i K_i$ *is stable with*

$$A_i \triangleq \left.\frac{\partial f_i}{\partial x_i}\right|_{x_i=0;\, u_i=0}, \quad B_i \triangleq \left.\frac{\partial f_i}{\partial u_i}\right|_{x_i=0;\, u_i=0}$$

Let $\tilde{Q}_i \triangleq \beta_i(Q_i + K_i^\top R_i K_i + \tilde{S}_i)$ *with* $\beta_i > 1$, *and denote by* Π_i *the unique symmetric positive definite solution of the following Lyapunov equation:*

$$A_{i_{cl}}^\top \Pi_i A_{i_{cl}} - \Pi_i + \tilde{Q}_i = 0. \tag{7.17}$$

Then, there exist a constant $\Upsilon_i \in \mathbb{R}_{>0}$ *and a finite integer* $\bar{N}_p$ *such that for all* $N_p \geq \bar{N}_p$ *the final set* $\mathcal{X}_{if} \triangleq \{x_i \in \mathbb{R}^{n_i} :\ x_i \top \Pi_i x_i \leq \Upsilon_i\}$ *satisfies Assumption 7.4 with* $\kappa_{if}(x_i) = K_i x_i$, $V_{if} = x_i^\top \Pi_i x_i$. ■

In the next subsection, the stability analysis of the whole team of agents will be addressed.

7.3.3 Stability properties of the team of agents

In this subsection, the coupling effects due to the exchange of the delayed state information between the cooperating agents will be taken into account in the context of the stability analysis of the whole team of agents. In this respect, let consider the team $\mathcal{A} = \{\mathcal{A}_i,\ i = 1, ..., M\}$ where each cooperating agent $\mathcal{A}_i$ is controlled by the regionally ISS–stabilizing MPC control local law solving Problem 7.2.1 for each $i = 1, \ldots, M$. Therefore, one can write

$$
\begin{aligned}
x_{1_{t+1}} &= \tilde{f}_1(x_{1_t}, w_{1_t}) \triangleq f_1(x_{1_t}, \kappa_1^{MPC}(x_{1_t}, w_{1_t}))\,, \\
x_{2_{t+1}} &= \tilde{f}_2(x_{2_t}, w_{2_t}) \triangleq f_2(x_{2_t}, \kappa_2^{MPC}(x_{2_t}, w_{2_t}))\,, \\
&\vdots \\
x_{M_{t+1}} &= \tilde{f}_M(x_{M_t}, w_{M_t}) \triangleq f_M(x_{M_t}, \kappa_M^{MPC}(x_{M_t}, w_{M_t})).
\end{aligned}
$$

First of all, let rewrite the team of dynamical systems as a suitable interconnection of two composite systems. To this end, let

$$
x_t \triangleq \text{col}\,[\,x_{1_t}, \cdots, x_{M_t}\,], \quad w_t \triangleq \text{col}\,[\,w_{1_t}, \cdots, w_{M_t}\,]\,.
$$

Hence the following state equation can be written:

$$
x_{t+1} = \tilde{f}(x_t, w_t) \tag{7.18}
$$

where

$$
\tilde{f}(x_t, w_t) \triangleq \text{col}\left[\,\tilde{f}_1(x_{1_t}, w_{1_t}),\, \tilde{f}_2(x_{2_t}, w_{2_t}),\, \cdots,\, \tilde{f}_M(x_{M_t}, w_{M_t})\right]\,.
$$

Vector w_t can be easily characterized as the output of a system describing the delay dynamics of the information exchange process among the agents. For the sake of simplicity and without loss of generality, it is assumed that $\dim(w_{i_t}) \geq 1,\ i = 1, \ldots, M$, that is, it is assumed that each agent receives at least one delayed state information from another neighboring agent. First, let set $\Delta \triangleq \max\{\Delta_{ij},\ i,j = 1, ..., M,\ i \neq j\}$. Then, let introduce the state vector $z_t \triangleq \text{col}\,[\,\rho_{1_t}, \cdots, \rho_{\tau_t}, \cdots, \rho_{\Delta_t}\,]$, $z_t \in \mathbb{R}^{n_z}$, where $n_z \triangleq \dim(z_t)$ and where the variables ρ are introduced to store the delayed states; specifically $\rho_{1_{t+1}} = x_t$ and $\rho_{\tau_{t+1}} = \rho_{\tau-1_t}$, $\tau = 2, \ldots, \Delta$.

Hence, it follows that

$$\begin{cases} z_{t+1} = A\, z_t + B\, x_t \\ w_t \quad = C\, z_t \end{cases} \tag{7.19}$$

where

$$A = \begin{bmatrix} \emptyset & \cdots & \cdots & \cdots & \emptyset \\ I_1 & \emptyset & \cdots & \cdots & \emptyset \\ \emptyset & I_2 & \emptyset & \cdots & \emptyset \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \emptyset & \cdots & \cdots & I_{\Delta-1} & \emptyset \end{bmatrix}, \; B = \begin{bmatrix} I_0 \\ \emptyset \\ \vdots \\ \emptyset \end{bmatrix}, \; C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_M \end{bmatrix}$$

$$C_i = \begin{bmatrix} C_i(1) & \cdots & C_i(\tau) & \cdots & C_i(\Delta) \end{bmatrix}$$

$$C_i(\tau) = \begin{bmatrix} \delta_{i1}(\tau) & \emptyset & \cdots & \cdots & \emptyset \\ \emptyset & \delta_{i2}(\tau) & \emptyset & \cdots & \emptyset \\ \emptyset & \cdots & \delta_{i3}(\tau) & \cdots & \emptyset \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \emptyset & \cdots & \cdots & \emptyset & \delta_{iM}(\tau) \end{bmatrix}$$

All matrices I_τ , for $\tau = 0, ..., \Delta - 1$ are identity matrices of dimension $n_x \times n_x$, where $n_x \triangleq \dim(x_t)$ and $\delta_{ij}(\tau) \triangleq I$ is equal to the identity matrix of dimension n_j , $i, j = 1, ..., M$, $i \neq j$ only for τ corresponding to the delay associated with the information received by the i-th from the j-th agent; otherwise $\delta_{ij}(\tau) = 0$, thus there is no replication of information. It is worth noting that agent $\mathcal{A}_i$ does not get replicated information from agent $\mathcal{A}_j$, thus in matrix C the matrix $\delta_{ij}(\tau)$ is equal to the identity for only one value of τ.

Summing up, the overall state equation describing the dynamics of the team of agents can be written as a feedback interconnection between the systems described by the state equations (7.18) and (7.19). In the following, it will be shown that an ISS-Lyapunov function can be defined for each of these systems, which implies that both will turn out to be regionally ISS. After this step, the stability properties of the team of agents will be analyzed by resorting to *nonlinear small–gain theorem* arguments. First, let $\mathcal{W} \triangleq \mathcal{W}_1 \times \cdots \times \mathcal{W}_M$, $\mathcal{X} \triangleq \mathcal{X}_1 \times \cdots \times \mathcal{X}_M$, $\mathcal{X}^{\kappa_f} \triangleq \mathcal{X}_1^{\kappa_{1f}} \times \cdots \times \mathcal{X}_M^{\kappa_{Mf}}$, $\mathcal{X}^{MPC} \triangleq \mathcal{X}_1^{MPC} \times \cdots \times \mathcal{X}_M^{MPC}$, $\Theta \triangleq \Theta_1 \times \cdots \Theta_M$. The following intermediate result can now be proved.

Lemma 7.2 *Under Assumptions 7.1,7.4-7.8, dynamic systems (7.18) and (7.19) are provided with suitable ISS-Lyapunov functions $V(x_t, w_t)$ in $\mathcal{X}^{MPC}$ and $V_D(z_t)$ in $\mathbb{R}^{n_z}$, respectively.* ■

Now, from the proof of Lemma 7.2, from (7.44) it follows immediately that the ISS-Lyapunov function $V(x_t, w_t)$ satisfies

$$\begin{aligned} V(x_{t+1}, w_{t+1}) - V(x_t, w_t) \quad \leq \quad & -\alpha_4(V(x_t, w_t)) + \zeta(|w_t|) \\ & +\sigma_2(|w_t|) + \sigma_3(|w_{t+1}|) \end{aligned}$$

for all $x_t \in \mathcal{X}^{\kappa_f}$, and all w_t, $w_{t+1} \in \mathcal{W}$, where, given a suitable $\mathcal{K}_\infty$-function ζ as in point 4 of Definition 7.2, $\alpha_4(s) \triangleq \underline{\alpha}_3 \circ \overline{\alpha}_2^{-1}(s), \underline{\alpha}_3(s) \triangleq \min(\alpha_3(s/2), \zeta(s/2))$, $\overline{\alpha}_2(s) \triangleq \alpha_2(s) + \sigma_1(s)$, (see for details the Step 1 of the Proof of Theorem 7.1). Now, since from (7.19) $w_t = Cz_t$, one has

$|w_t| \le |z_t|$ and, by using (7.45),

$$\begin{aligned} V(x_{t+1}, w_{t+1}) - V(x_t, w_t) \le & -\alpha_4(V(x_t, w_t)) + \zeta(\alpha_{1D}^{-1}(V_D(z_t))) \\ & +\sigma_2(\alpha_{1D}^{-1}(V_D(z_t))) + \sigma_3(\alpha_{1D}^{-1}(V_D(z_{t+1}))) \\ \le & -\alpha_4(V(x_t, w_t)) + \zeta(\alpha_{1D}^{-1}(||V_D(\mathbf{z})_{[t+1]}||)) \\ & +\sigma_2(\alpha_{1D}^{-1}(||V_D(\mathbf{z})_{[t+1]}||)) \\ & +\sigma_3(\alpha_{1D}^{-1}(||V_D(\mathbf{z})_{[t+1]}||)) \\ = & -\alpha_4(V(x_t, w_t)) + \sigma_4(||V_D(\mathbf{z})_{[t+1]}||) \end{aligned} \tag{7.20}$$

for all $x_t \in \mathcal{X}^{\kappa_f}$, and all $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$, where $\sigma_4(s) \triangleq \zeta(\alpha_{1D}^{-1}(s)) + \sigma_2(\alpha_{1D}^{-1}(s)) + \sigma_3(\alpha_{1D}^{-1}(s))$.

Moreover, as far as the ISS-Lyapunov function $V_D(z_t)$ is concerned, from (7.46) it follows that

$$V_D(z_{t+1}) - V_D(z_t) \le -\alpha_{4D}(V_D(z_t)) + \sigma_{4D}(V(x_t, w_t)) \tag{7.21}$$

where, again, $\alpha_{4D}(s) \triangleq \alpha_{3D} \circ \alpha_{2D}^{-1}(s)$, whereas $\sigma_{4D}(s) \triangleq \sigma_D \circ \alpha_{1D}^{-1}(s)$.

Analogously to the proof of Theorem 7.1, given $e \in \mathbb{R}_{\ge 0}$, let $\mathcal{R}(e) \triangleq \{x : V(x, w) \le e, \forall w \in \mathcal{W}\}$. Let $\Psi \triangleq \{x : V(x) \le \bar{e} = \max_{\mathcal{R}(e) \subseteq \mathcal{X}^{\kappa_f}} e, \forall w \in \mathcal{W}\}$. It is clear that $\Psi \supseteq \Theta_{\mathbf{w}}$ and that Ψ is a RPI set. Since the region $\Theta_{\mathbf{w}}$ is reached asymptotically, the state will arrive in Ψ in a finite time, that is, there exists T_ψ such that $V(x_k, w_k) \le \bar{e}$, $\forall k \ge T_\psi$. Thanks to Remark 3.7 in [Jiang & Wang 2001], from (7.20) and (7.21) it follows that there exist some $\mathcal{KL}$-functions $\hat{\beta}$ and $\hat{\beta}_D$ such that

$$V(x_k, w_k) \le \max\{\hat{\beta}(V(x_t, w_t), k - t), \gamma_1(||V_D(z)_{[k]}||)\} \tag{7.22}$$

for all $x_t \in \Psi$, all $k \in \mathbb{Z}_{\geq 0}$, $k \geq t$, and

$$V_D(z_k) \leq \max\{\hat{\beta}_D(V_D(z_t), k-t), \gamma_2(||V(x,w)_{[k]}||)\}, \tag{7.23}$$

for all $x \in \mathcal{X}$ and all $k \in \mathbb{Z}_{\geq 0}$, $k \geq t$, where let define

$$\gamma_1 \triangleq \alpha_4^{-1} \circ \rho^{-1} \circ \sigma_4 \tag{7.24}$$

and

$$\gamma_2 \triangleq \alpha_{4D}^{-1} \circ \rho^{-1} \circ \sigma_{4D} \tag{7.25}$$

with ρ any $\mathcal{K}_\infty$-function such that $(\mathrm{id} - \rho) \in \mathcal{K}_\infty$.

Now, the following result about the stability properties of the team of cooperating agents can be proved.

Theorem 7.3 *Suppose that Assumptions 7.1 and 7.4-7.8 are verified. Moreover, assume that the following small gain condition holds:*

$$\gamma_1 \circ \gamma_2(s) < s. \tag{7.26}$$

with γ_1 and γ_2 given by (7.24) and (7.25) and argument s takes its values from a suitable subset of $\mathbb{R}_{\geq 0}$ according to inequalities (7.20)–(7.23). Then the team of cooperating agents described by the interconnected dynamic equations (7.18) and (7.19) is 0–AS in $\mathcal{X}^{MPC} \times \mathbb{R}^{n_z}$. ■

Remark 7.3 *It is worth noting that the small–gain condition (7.26) may turn out to be conservative in practice as it is typical of these kind of results. On the other hand, the generality of the problem makes it rather difficult to obtain tighter conditions without introducing more restrictive assumptions on the structure of the agents' dynamics and on the cost function. Indeed, for special classes of cooperative control problems, different conditions for* *the stability of the team of agents can be obtained. For instance, we recall that in [Dunbar & Murray 2006] stability has been shown for formation control of UAV's under different hypotheses as the knowledge of the neighbors dynamics, suitably fast information exchange and bounded error between the predicted and actuated state trajectories of each member of the team. As another example, stability of a set of decoupled systems is ensured in [Kevicky et al. 2004a], by assuming the knowledge of feasibility regions and a specific hierarchical design of the decentralized MPC control problem: the computations are shared by nodes with different priorities, which can impose their control decisions on the subordinate neighbors.*

Remark 7.4 *As expected, in the special case where the state equation (7.1) takes on a linear structure, the FH cost function (7.6) is quadratic, and no state and control constraints are present, more specialized and tight results can be found. In particular, the control law takes on an* explicit feedback–feedforward structure *and some interesting properties hold. The reader is referred to [Franco et al. 2006] for more details.*

7.4 Simulation results

In this section we will show some simulation results concerning a team of UAVs moving in $\mathbb{R}^2$ with nonlinear dynamics. Such a problem has been selected because of its reasonable simplicity so as to be able to ascertain the basic features and properties of the proposed cooperative control law. A team of $M = 3$ vehicles will be considered, whose continuous-time models and data are taken according to [Jin *et al.* 2004]:

$$\begin{aligned} m\ddot{x}_i &= -\mu_1\dot{x}_i + (u_{iR} + u_{iL})\cos(\theta_i), \\ m\ddot{y}_i &= -\mu_1\dot{y}_i + (u_{iR} + u_{iL})\sin(\theta_i), \\ J\ddot{\theta}_i &= -\mu_2\dot{\theta}_i + (u_{iR} - u_{iL})r_v. \end{aligned} \tag{7.27}$$

where $i = 1, 2, 3$. For simplicity, we assume that all the members of the team have the same physical parameters: the mass is $m = 0.75\,Kg$, the inertia is $J = 0.00316\,Kgm^2$, the linear friction coefficient is $\mu_1 = 0.15\,Kg/s$ and the rotational friction coefficient is $\mu_2 = 0.005\,Kgm^2/s$ and finally the radius of the vehicle is $r_v = 8.9cm$. The state vector of each agent will be from now on denoted as z_i, and is defined by considering the position and velocity in each direction of the plane, plus the orientation angle and rotational velocity $z_i \triangleq \mathrm{col}\,(\theta_i, \dot{\theta}_i, x_i, \dot{x}_i, y_i, \dot{y}_i)$, whereas the control vector is given by $u_i \triangleq \mathrm{col}\,(u_{iL}, u_{iR})$. The continuous-time models (7.27) are discretized with a sampling time $T = 0.1s$, thus obtaining suitable discrete-time models, where, at time t, the state vectors are denoted by z_{i_t} and the control vectors are denoted by u_{i_t}.

Remark 7.5 *In the following, the simulation trials will refer to the above approximated discrete-time model for mere illustration purposes and to show the effectiveness of the proposed cooperative control scheme. However, as shown in [Nesic & Teel 2004a], in some cases the control law that stabilizes the approximated discrete-time model may perform quite poorly when applied to the exact model. This is clearly an important issue (see the above reference for more details and the works [Nesic et al. 1999a, Nesic et al. 1999b] for the general case of control of nonlinear sampled-data systems). For a MPC algorithm where the continuous time evolution of the system is explicitly taken into account, while the optimization is performed with respect to* *a piece-wise constant control signal, see [Magni & Scattolini 2004].* □

The objective of the distributed cooperative controller is to reach a certain formation following a predefined desired trajectory, based on leader one, for each UAV. The desired trajectories have been chosen with constant velocities and null rotational velocity. At every time instant t, each agent solves a local Problem 7.2.1 with FH cost function

$$\begin{aligned} J_i = & \sum_{k=t}^{t+N_{ip}-1} \left(\|z_{i_k} - \bar{z}_{1_k} + d_{i1}\|^2_{Q_i} + \|u_{i_k} - \bar{u}_i\|^2_{R_i} \right) \\ & + \|z_{i_{t+N_{ip}}} - \bar{z}_{1_{t+N_{ip}}} + d_{i1}\|^2_{P_i} + \sum_{k=t}^{t+N_{ip}-1} \sum_{j \in G_i} \|z_{i_k} - \tilde{z}_{j_k} + d_{ij}\|^2_{S_{ij}} \end{aligned} \tag{7.28}$$

where $\bar{z}_{1_k}$ represents the desired trajectory of the leader while d_{ij} are the desired distance between agent i and agent j ($d_{ii} = 0,\ \forall i = 1, \ldots, M$). Hence the term $\bar{z}_{1_k} - d_{i1}$ represents the desired trajectory of the $i-th$ UAV. The values of d_{ij} are such that the three UAVs assume a triangle formation. The term $\bar{u}_i$ is the control vector necessary in order to maintain each UAV on the desired trajectory. For the information vector to take on a constant value within the prediction horizon, we let

$$\tilde{z}_{j_k} = (\bar{z}_{1_k} - d_{j1}) + (z_{j_{t-\Delta^{ij}}} - \bar{z}_{1_{t-\Delta^{ij}}} + d_{j1}) \,.$$

The delays have all been set to $\Delta_{ij} = \Delta = 5T$ and the communication topology is assumed to be stationary. Specifically, it is supposed that the leader does not receive any information from the other agents (hence $S_{1j} = 0,\ \forall j \in G_1$). Moreover agent 2 gets information from the leader and from agent 3 and, analogously, agent 3 gets information from the leader and from agent 2.

The values of the parameters used for the leader are $N_{1c} = N_{1p} = 5$, $Q_1 = 0.1 \cdot \text{diag}\,(1, 50, 1, 1, 1, 1)$, $R_1 = 0.01 \cdot \text{diag}\,(1, 1)$, and $S_{1j} = 0$, $\forall j \in G_1$. The lengths of horizons N_{1c}, N_{1p}, though quite small, are indeed sufficient for the leader to show a reasonably good tracking performance as it starts quite close to the desired trajectory. For the other agents, we consider the same values of the parameters, that is, we have $N_{ic} = 10$, $N_{ip} = 250$, $Q_i = 0.1 \cdot \text{diag}\,(1, 50, 1, 1, 1, 1)$, $R_i = 0.01 \cdot \text{diag}\,(1, 1)$, $S_{ij} = \text{diag}\,(0.1, 0.1, 1, 0.1, 1, 0.1)$, $\alpha_{w_i} = 0.96$, $i = 2, 3$. The matrices P_i are obtained, from the choice of Q_i, R_i and S^{ij}, by the auxiliary control law designed according to Lemma 7.1 using $\beta_i = 3$ and $\tilde{S}_{ij} = 2S_{ij}$, $i = 1, 2, 3$. The FHOCP is characterized by the constraints $u_{iLmin} \leq u_{1i_t} \leq u_{iRmax}$; $u_{iLmin} \leq u_{2i_t} \leq u_{iRmax}$, with $u_{iLmin} = 0$, $u_{iLmax} = 6$, $u_{iRmin} = 0$, and $u_{iRmax} = 6$, $i = 1, 2, 3$, where u_{1i_t} (u_{2i_t}) denotes the first (second) component of vector u_{i_t}. Moreover, the terminal constraints $\|z_{i_{t+N_{ip}}} - \bar{z}_{i_{t+N_{ip}}} + d_{i1}\|^2_{P_i} \leq \Upsilon_i$, $i = 1, 2, 3$, have been obtained numerically according to Lemma 7.1. The values of Υ_i are constant along the trajectories and are respectively $\Upsilon_1 = 0.3$ and $\Upsilon_i = 1.2$, $i = 2, 3$. These values are not comparable since the matrices P_i are different. The control necessary in order to maintain each UAV on the desired trajectory is $\bar{u}_{1i} = 1$, $\bar{u}_{2i} = 1$. The values of the desired distances between the agents are the following:

$$\begin{aligned} d_{12} = 16col(0, 0, &-\sin(\pi/3)\cos(\pi/4) - 0.5\cos(\pi/4), \\ &0, -\sin(\pi/3)\cos(\pi/4) + 0.5\cos(\pi/4), 0) \end{aligned}$$

$$\begin{aligned} d_{13} = 16col(0, 0, &-\sin(\pi/3)\cos(\pi/4) + 0.5\cos(\pi/4), \\ &0, -\sin(\pi/3)\cos(\pi/4) - 0.5\cos(\pi/4), 0) \end{aligned}$$

$$\begin{aligned} d_{21} = 16col(0, 0, &+\sin(\pi/3)\cos(\pi/4) + 0.5\cos(\pi/4), \\ &0, +\sin(\pi/3)\cos(\pi/4) - 0.5\cos(\pi/4), 0) \end{aligned}$$

$$d_{23} = 16col(0, 0, \cos(\pi/4), 0, -\cos(\pi/4), 0)$$

$$\begin{aligned} d_{31} = 16col(0, 0, &\sin(\pi/3)\cos(\pi/4) - 0.5\cos(\pi/4), \\ &0, +\sin(\pi/3)\cos(\pi/4) + 0.5\cos(\pi/4), 0) \end{aligned}$$

and

$$d_{32} = 16col(0, 0, -\cos(\pi/4), 0, \cos(\pi/4), 0)\,.$$

Moreover, the initial condition of the desired trajectory of the leader is:

$$\bar{z}_{1_0} = \mathrm{col}\left(\pi/4, 0, 0, \frac{1}{m}(\bar{u}_{1i} + \bar{u}_{2i})\cos(\pi/4), 0, \frac{1}{m}(\bar{u}_{1i} + \bar{u}_{2i})\sin(\pi/4)\right).$$

The entire desired leader's trajectory is obtained, starting from the initial conditions, holding constant the velocities. Finally, the initial conditions of the UAVs are $z_{1_0} = \bar{z}_{1_0}$, $z_{2_0} = \bar{z}_{2_0}$, $z_{3_0} = \bar{z}_{2_0} + 3.8 d_{23}$.

In Fig. 7.2, the team trajectories are reported in the two-dimensional space: the objective is to attain a triangle formation along a straight line of 45° as followers of the leader. The dotted lines depict the actual behavior of the agents. It is worth noting the cooperative behavior of the agents: in particular agent 2 (on the left of the leader) even if it starts on its trajectory ($z_{2_0} = \bar{z}_{2_0}$) it moves on the right in order to reach faster a better (with respect to cost (7.28)) formation with agent 3. Without the cooperative term in the cost function the trajectory of agent 2 would be a straight line.

In Fig. 7.3, the behaviors of the control variables $u_i \triangleq col(u_{iL}, u_{iR})$ of Agents 2 and 3 are shown. In particular, in Figs. 7.3(a) and 7.3(b) the behaviors of the first component of the control variables are plotted, whereas in Figs. 7.3(c) and 7.3(d), the difference between the first and the second components of the control variables are shown. This has been done to better appreciate the differences between the first and the second

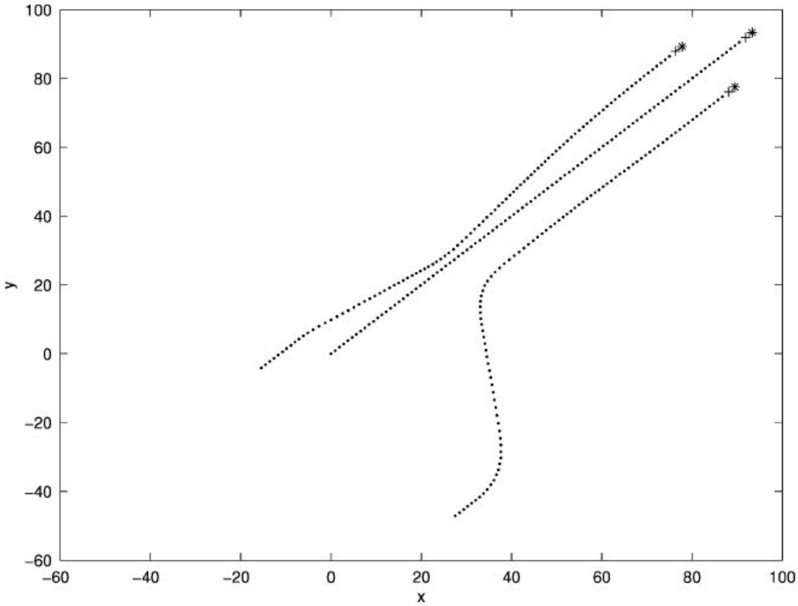


Figure 7.2: Team trajectories (dotted lines). The front of the vehicle is represented by the symbol '*' whereas the back of the vehicle is represented by the symbol '+'.

components of the control variables; actually, these differences are rather small due to the small magnitude of the variations of the orientation of the two agents. In Fig. 7.3, the dashed lines depict the constraints imposed on the control variables.

7.5 Conclusions

In this chapter, the problem of designing cooperative control algorithms for a team of distributed agents with nonlinear discrete-time dynamics, and analyzing the stability properties of the overall closed-loop system has been investigated. The problem formulation is based on a decentralized MPC framework, where the dynamics of the distributed agents are linked

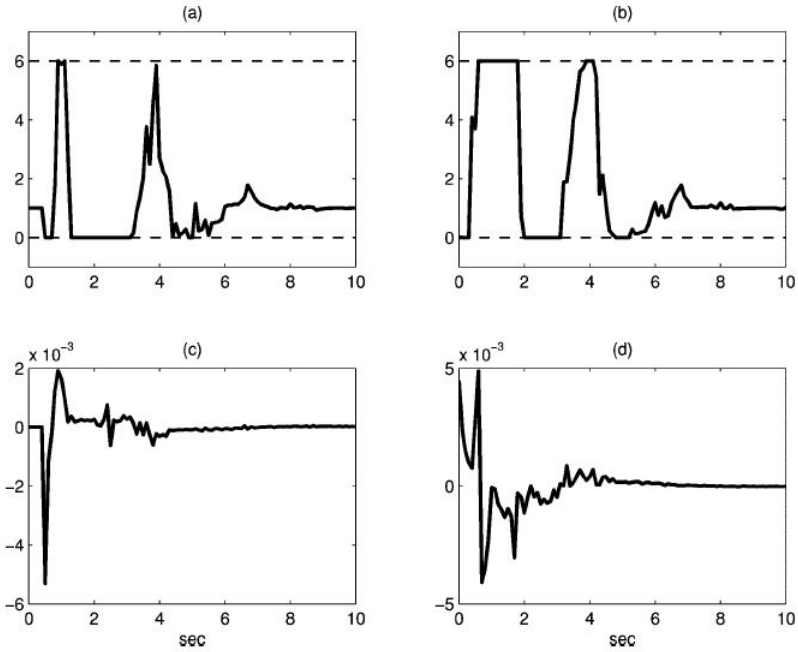


Figure 7.3: Behaviors of the control variables of Agents 2 and 3. (a) and (b) behaviors of the first component of the control variables. (c) and (d) difference between the first and the second components of the control variables. Dashed lines: control constraints.

by a cooperative cost function. Each agent uses locally computed control laws, which take into consideration delayed state information from neighboring agents. The resulting local control laws take the form of a feedback–feedforward structure, which is derived by a nonlinear MPC framework.

A key contribution is the general problem formulation, which allows the systematic derivation of rigorous stability results. The stability analysis is made possible by combining the stability properties of the MPC local control laws and ISS arguments. Finally, the team of cooperating agents is treated as a single dynamical system resulting from a feedback interconnection of regionally ISS systems, thus allowing the use of small-gain conditions to show asymptotic stability.

Despite the general formulation, there are some important issues requiring further investigation. Future research efforts will be devoted towards (i) considering the case where disturbances and uncertainties affect the communication between the agents of the team and (ii) addressing the robustness issue by generalizing the methodology to the case where optimality of the algorithm is not required at each time-stage.

Appendix

Proof of Theorem 7.1: The proof will be carried out in three steps.

Step 1. First, it is going to be shown that $\Theta_{\mathbf{w}}$ defined in (7.14) is a RPI set for system (7.9). From the definition of $\overline{\alpha}_2(s)$ it follows that $\alpha_2(|x|) + \sigma_1(|w|) \leq \overline{\alpha}_2(|x| + |w|)$. Therefore, $V(x, w) \leq \overline{\alpha}_2(|x| + |w|)$ and hence $|x| + |w| \geq \overline{\alpha}_2^{-1}(V(x, w))$, $\forall x \in \Omega$. Moreover (see [Limon *et al.* 2006a]):

$$\alpha_3(|x|) + \zeta(|w|) \geq \underline{\alpha}_3(|x| + |w|) \geq \alpha_4(V(x, w)) \tag{7.29}$$

where $\alpha_4(s) \triangleq \underline{\alpha}_3 \circ \overline{\alpha}_2^{-1}(s)$ is a $\mathcal{K}_\infty$-function. Then, let consider system (7.9) and the state transition from x_k to x_{k+1}:

$$\begin{aligned} V(\tilde{f}(x_k, w_k), w_{k+1}) - V(x_k, w_k) &\leq -\alpha_4(V(x_k, w_k)) + \zeta(|w_k|) \qquad (7.30) \\ &\quad +\sigma_2(|w_k|) + \sigma_3(|w_{k+1}|) \\ &\leq -\alpha_4(V(x_k, w_k)) + \sigma_4(||\mathbf{w}||) \end{aligned}$$

for all $x_k \in \Omega$, all $w_k, w_{k+1} \in \mathcal{W}$, and all $k \geq 0$, where $\sigma_4(s) = \zeta(s) + \sigma_2(s) + \sigma_3(s)$.

Assume now that $x_t \in \Theta_{\mathbf{w}}$. Then $V(x_t, w_t) \leq b(||\mathbf{w}||)$; this implies

$\rho \circ \alpha_4(V(x_t, w_t)) \le \sigma_4(||\mathbf{w}||)$. Without loss of generality, assume that $(\text{id} - \alpha_4)$ is a $\mathcal{K}_\infty$-function, otherwise take a "bigger" α_2 so that $\underline{\alpha}_3 < \overline{\alpha}_2$. Then

$$\begin{aligned} V(\tilde{f}(x_t, w_t), w_{t+1}) &\le (\text{id} - \alpha_4)(V(x_t, w_t)) + \sigma_4(||\mathbf{w}||) \\ &\le (\text{id} - \alpha_4)(b(||\mathbf{w}||)) + \sigma_4(||\mathbf{w}||) \\ &= -(\text{id} - \rho) \circ \alpha_4(b(||\mathbf{w}||)) + b(||\mathbf{w}||) \\ &\quad -\rho \circ \alpha_4(b(||\mathbf{w}||)) + \sigma_4(||\mathbf{w}||). \end{aligned}$$

From the definition of b, it follows that $\rho \circ \alpha_4(b(s)) = \sigma_4(s)$ and, owing to the fact that $(\text{id} - \rho)$ is a $\mathcal{K}_\infty$-function, it follows that

$$V(\tilde{f}(x_t, w_t), w_{t+1}) \le -(\text{id} - \rho) \circ \alpha_4(b(||\mathbf{w}||)) + b(||\mathbf{w}||) \le b(||\mathbf{w}||)\,.$$

By induction one can show that $V(\tilde{f}(x_{t+j-1}, w_{t+j-1}), w_{k+j}) \le b(||\mathbf{w}||)$ for all $j \in \mathbb{Z}_{\ge 0}$, that is $x_k \in \Theta_{\mathbf{w}}$, for all $k \ge t$. Hence $\Theta_{\mathbf{w}}$ is a RPI set for system (7.9).

Step 2. Now it is shown that, starting from $\Xi \setminus \Theta_{\mathbf{w}}$, the state tends asymptotically to $\Theta_{\mathbf{w}}$. Firstly, if $x_k \in \Omega \setminus \Theta_{\mathbf{w}}$, then

$$\rho \circ \alpha_4(V(x_k, w_k)) > \sigma_4(||\mathbf{w}||).$$

From the inequality (7.29), one has

$$\rho(\alpha_3(|x_k|) + \zeta(|w_k|)) > \sigma_4(||\mathbf{w}||).$$

On the other hand, $(\text{id} - \rho)$ is a $\mathcal{K}_\infty$-function, hence

$$\text{id}(s) > \rho(s), \ \forall s > 0$$

then

$$\begin{aligned}\alpha_3(|x_k|) + \zeta(||\mathbf{w}||) &> \alpha_3(|x_k|) + \zeta(|w_k|)\\ &> \rho(\alpha_3(|x_k|) + \zeta(|w_k|))\\ &> \sigma_4(||\mathbf{w}||) = \zeta(||\mathbf{w}||) + \sigma_2(||\mathbf{w}||) + \sigma_3(||\mathbf{w}||)\end{aligned}$$

for all $x_k \in \Omega \setminus \Theta_{\mathbf{w}}$, and all $w_k \in \mathcal{W}$, which, in turn, implies that

$$\begin{aligned}V(\tilde{f}(x_k, w_k), w_{k+1}) - V(x_k, w_k) &\leq -\alpha_3(|x_k|) + \sigma_2(||\mathbf{w}||) + \sigma_3(||\mathbf{w}||)\\ &< 0 \end{aligned}\tag{7.31}$$

for all $x_k \in \Omega \setminus \Theta_{\mathbf{w}}$, and all $w_k, w_{k+1} \in \mathcal{W}$. Moreover, in view of (7.14), there exists $\bar{c} > 0$ such that for all $x_1 \in \Xi \setminus \Omega$, there exists $x_2 \in \Omega \setminus \Theta_{\mathbf{w}}$ such that $\alpha_3(|x_2|) \leq \alpha_3(|x_1|) - \bar{c}$. Then from (7.31) it follows that

$$-\alpha_3(|x_1|) + \bar{c} \leq -\alpha_3(|x_2|) < -\sigma_2(||\mathbf{w}||) - \sigma_3(||\mathbf{w}||),$$

for all $x_1 \in \Xi \setminus \Omega$, and all $x_2 \in \Omega \setminus \Theta_{\mathbf{w}}$. Then

$$\begin{aligned}V(\tilde{f}(x_k, w_k), w_{k+1}) - V(x_k, w_k) &\leq -\alpha_3(|x_k|) + \sigma_2(||\mathbf{w}||) + \sigma_3(||\mathbf{w}||)\\ &< -\bar{c}\end{aligned}$$

for all $x_k \in \Omega \setminus \Theta_{\mathbf{w}}$, and all $w_k, w_{k+1} \in \mathcal{W}$. Hence, there exists T_1 such that

$$x(T_1, \bar{x}, \mathbf{w}) \in \Omega.$$

Therefore, starting from Ξ, the state will reach the region Ω in a finite time. If in particular $x(T_1, \bar{x}, \mathbf{w}) \in \Theta_{\mathbf{w}}$, the region $\Theta_{\mathbf{w}}$ is achieved in a finite time. Since $\Theta_{\mathbf{w}}$ is a RPI set, it is true that $\lim_{k\to\infty} |x(k, \bar{x}, \mathbf{w})|_{\Theta_{\mathbf{w}}} = 0$. Otherwise, if $x(T_1, \bar{x}, \mathbf{w}) \notin \Theta_{\mathbf{w}}$, $\rho \circ \alpha_4(V(x_k, w_k)) > \sigma_4(||\mathbf{w}||)$; moreover, from (7.30) one has

$$\begin{aligned} V(\tilde{f}(x_k, w_k), w_{k+1}) - V(x_k, w_k) &\leq -\alpha_4(V(x_k, w_k)) + \sigma_4(||\mathbf{w}||) \\ &= -(\mathrm{id} - \rho) \circ \alpha_4(V(x_k, w_k)) \\ &\quad -\rho \circ \alpha_4(V(x_k, w_k)) + \sigma_4(||\mathbf{w}||) \\ &\leq -(\mathrm{id} - \rho) \circ \alpha_4(V(x_k, w_k)) \\ &\leq -(\mathrm{id} - \rho) \circ \alpha_4 \circ \alpha_1(|x_k|) \end{aligned}$$

for all $x_k \in \Omega \setminus \Theta_{\mathbf{w}}$, and all $w_k, w_{k+1} \in \mathcal{W}$, where the last step is obtained using (7.11). Then, $\forall \varepsilon' > 0$, $\exists T_2(\varepsilon') \geq T_1$ such that

$$V(x_{T_2}, w_{T_2}) \leq \varepsilon' + b(||\mathbf{w}||)\,. \tag{7.32}$$

Therefore, starting from Ξ, the state will arrive close to $\Theta_{\mathbf{w}}$ in a finite time and in $\Theta_{\mathbf{w}}$ asymptotically. Hence $\lim_{k\to\infty} |x(k, \bar{x}, \mathbf{w})|_{\Theta_{\mathbf{w}}} = 0$.

Step 3: Finally, it is shown that system (7.9) is regionally ISS in Ξ. Given $e \in \mathbb{R}_{\geq 0}$, let $\mathcal{R}(e) \triangleq \{x : V(x, w) \leq e, \forall w \in \mathcal{W}\}$. Let $\Psi \triangleq \{x : V(x, w) \leq \bar{e} = \max_{\mathcal{R}(e) \subseteq \Omega} e, \forall w \in \mathcal{W}\}$. It is clear that $\Psi \supseteq \Theta_{\mathbf{w}}$ and that Ψ is a RPI set. Since the upper bound of $V(x, w)$ is known in $\Psi \subseteq \Omega$ then, using the same steps of the proof of Lemma 3.5 in [Jiang & Wang 2001], that also hold for discontinuous systems, it is possible to show that there exist a $\mathcal{KL}$-function $\hat{\beta}$ and a $\mathcal{K}$-function $\hat{\gamma}$ such that

$$V(x_k, w_k) \leq \max\{\hat{\beta}(V(x_t, w_t), k - t), \hat{\gamma}(||\mathbf{w}||)\},\ \forall k \geq t$$

for all $x_t \in \Upsilon$, all $\mathbf{w} \in \mathcal{M}_W$, where $\hat{\gamma} = \alpha_4^{-1} \circ \rho^{-1} \circ \sigma_4$. Hence

$$\begin{aligned} \alpha_1(|x_k|) &\leq \max\{\hat{\beta}(\alpha_2(|x_t|) + \sigma_1(|w_t|), k - t), \hat{\gamma}(||\mathbf{w}||)\}, \\ &\leq \max\{\hat{\beta}(2\alpha_2(|x_t|), k - t) + \hat{\beta}(2\sigma_1(|w_t|), k - t), \hat{\gamma}(||\mathbf{w}||)\}, \end{aligned}$$

for all $k \geq t$, all $x_t \in \Upsilon$, and all $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$. The last step is obtained considering that $\hat{\beta}(r+s,t) \leq \hat{\beta}(2r,t) + \hat{\beta}(2s,t)$ (see [Limon *et al.* 2006a]). Then

$$\begin{aligned}
|x_t| &\leq \max\{\hat{\beta}_1(\alpha_2(|x_t|), k-t) + \hat{\beta}_1(\sigma_1(|w_t|), k-t), \hat{\gamma}_1(||\mathbf{w}||)\} \\
&\leq \hat{\beta}_1(\alpha_2(|x_t|), k-t) + \hat{\beta}_1(\sigma_1(|w_t|), k-t) + \hat{\gamma}_1(||\mathbf{w}||) \\
&\leq \hat{\beta}_1(\alpha_2(|x_t|), k-t) + \hat{\beta}_1(\sigma_1(||\mathbf{w}||), k-t) + \hat{\gamma}_1(||\mathbf{w}||) \\
&\leq \hat{\beta}_2(|x_t|, k-t) + \hat{\gamma}_2(||\mathbf{w}||), \ \forall k \geq t \qquad (7.33)
\end{aligned}$$

for all $x_t \in \Upsilon$, and all $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$, where $\hat{\beta}_1(s,t) \triangleq \alpha_1^{-1} \circ \beta_1(2s,t)$ and $\hat{\beta}_2(s,t) \triangleq \hat{\beta}_1(\alpha_2(s),t)$ are $\mathcal{KL}$-functions while $\hat{\gamma}_1(s) \triangleq \alpha_1^{-1} \circ \hat{\gamma}(s)$ and $\hat{\gamma}_2(s) \triangleq \hat{\beta}_1(\sigma_1(||\mathbf{w}||), 0) + \hat{\gamma}_1(||\mathbf{w}||)$ are $\mathcal{K}$-functions. So, by (7.33) the system (7.9) is ISS in Υ. By Lemma 2.1, the regional ISS property in Υ is equivalent to UAG in Υ and LS. Using the fact that Υ is achieved in a finite time, UAG in Υ implies UAG in Ξ. Then UAG in Ξ and LS imply ISS in Ξ.

■

Proof of Theorem 7.2: First, by Assumption 7.4, for any $x_{i_t} \in \mathcal{X}_{if}$, the sequence

$$\tilde{\mathbf{u}}_{i[t,t+N_{ic}-1]} \triangleq \mathrm{col}\left[\kappa_{if}(x_{i_t}), \kappa_{if}(x_{i_{t+1}}), \ldots, \kappa_{if}(x_{i_{t+N_{ic}-1}})\right]$$

is an admissible control sequence for the FHOCP. Then, considering also Assumption 7.7, $\mathcal{X}_i^{MPC} \supseteq \mathcal{X}_i^{\kappa_{if}} \supseteq \mathcal{X}_{if}$. By Theorem 7.1, if system admits an ISS-Lyapunov function in $\mathcal{X}_i^{MPC}$, then it is ISS in $\mathcal{X}_i^{MPC}$. In this respect, in the following it will be shown that $V_i(x_{i_t}, w_{i_t}) \triangleq J_i(x_{i_t}, w_{i_t}, \mathbf{u}^o_{i[t,t+N_{ic}-1]}, N_{ic}, N_{ip})$ is an ISS-Lyapunov function in $\mathcal{X}_i^{MPC}$.

Moreover, in view of Assumptions 7.5-7.7 and point 5 of Assumption 7.4

$$\begin{aligned}
V_i(x_{i_t}, w_{i_t}) \quad &\leq \quad J_i(x_{i_t}, w_{i_t}, \tilde{\mathbf{u}}_{i[t,t+N_{ic}-1]}, N_{ic}, N_{ip}) \\
\leq \sum_{k=t}^{t+N_{ip}-1} &\left[\mathcal{L}_{l_i}|x_{i_k}| + \mathcal{L}_{l_{iu}}|\kappa_{if}\left(x_{i_k}\right)| + \mathcal{L}_{q_i}|x_{i_k}| + \mathcal{L}_{q_{iw}}|w_{i_k}|\right] + \beta_{V_{if}}(|x_{i_{t+N_{ip}}}|) \\
\leq \sum_{k=t}^{t+N_{ip}-1} &\left[(\mathcal{L}_{l_i} + \mathcal{L}_{l_{iu}}\mathcal{L}_{\kappa_{if}} + \mathcal{L}_{q_i})|x_{i_k}| + \mathcal{L}_{q_{iw}}|w_{i_k}|\right] + \beta_{V_{if}}(|x_{i_{t+N_{ip}}}|)
\end{aligned}$$

so that in view of point 3 of Assumption 7.4 and owing to (7.5), one has

$$\begin{aligned}
V_i(x_{i_t}, w_{i_t}) \leq &\sum_{k=t}^{t+N_{ip}-1} \left[(\mathcal{L}_{l_i} + \mathcal{L}_{l_{iu}}\mathcal{L}_{\kappa_{if}} + \mathcal{L}_{q_i})(\mathcal{L}_{f_{ic}})^{k-t}|x_{i_t}| + \mathcal{L}_{q_{iw}}(\alpha_{w_i})^{k-t}|w_{i_t}|\right] \\
&+\beta_{V_{if}}(\mathcal{L}_{f_{ic}})^{N_{ip}}|x_{i_t}|) \\
\leq &\ (\mathcal{L}_{l_i} + \mathcal{L}_{l_{iu}}\mathcal{L}_{\kappa_{if}} + \mathcal{L}_{q_i})\frac{(\mathcal{L}_{f_{ic}})^{N_{ip}} - 1}{\mathcal{L}_{f_{ic}} - 1}|x_{i_t}| \\
&+\mathcal{L}_{q_{iw}}\frac{(\alpha_{w_i})^{N_{ip}} - 1}{\alpha_{w_i} - 1}|w_{i_t}| + \beta_{V_{if}}((\mathcal{L}_{f_{ic}})^{N_{ip}}|x_{i_t}|)
\end{aligned}$$

Hence there exist two $\mathcal{K}_\infty$-functions α_{i_2} and σ_{i_1} such that the following upper bound is verified:

$$V_i(x_{i_t}, w_{i_t}) \leq \alpha_{i_2}(|x_{i_t}|) + \sigma_{i_1}(|w_{i_t}|),\ \forall x_{i_t} \in \mathcal{X}_i^{\kappa_{if}}, \forall w_{i_t} \in \mathcal{W}_i \tag{7.34}$$

The lower bound on $V_i(x_{i_t}, w_{i_t})$ is easily obtained using Assumption 7.5:

$$V_i(x_{i_t}, w_{i_t}) \geq \underline{r}_i(|x_{i_t}|),\ \forall x_{i_t} \in \mathcal{X}_i, \forall w_{i_t} \in \mathcal{W}_i \tag{7.35}$$

Now, in view of Assumption 7.4, it turns out that

$$\bar{\mathbf{u}}_{i[t+1,t+N_{ic}]} \triangleq \mathrm{col}\left[\mathbf{u}^o_{i[t+1,t+N_{ic}-1]}, \kappa_{if}(x_{i_{t+N_{ic}}})\right] \tag{7.36}$$

is an admissible (in general, suboptimal) control sequence for the FHOCP at time $t+1$ with cost

$$
\begin{aligned}
&J_i(x_{i_{t+1}}, w_{i_{t+1}}, \bar{\mathbf{u}}_{i[t+1,t+N_{ic}]}, N_{ic}, N_{ip}) \\
&\quad = V_i(x_{i_t}, w_{i_t}) - l_i(x_{i_t}, u^o_{i_t}) - q_i(x_{i_t}, w_{i_t}) + \sum_{k=t+1}^{t+N_{ip}-1} \Big[l_i(x_{i_k}, \bar{u}_{i_k}) + \\
&\qquad + q_i(x_{i_k}, A_{w_i}^{k-(t+1)} w_{i_{t+1}}) - l_i(x_{i_k}, u^o_{i_k}) - q_i(x_{i_k}, A_{w_i}^{k-t} w_{i_t}) \Big] \\
&\qquad + l_i(x_{i_{t+N_{ip}}}, \kappa_{if}(x_{i_{t+N_{ip}}})) + q_i(x_{i_{t+N_{ip}}}, A_{w_i}^{N_{ip}-1} w_{i_{t+1}}) \\
&\qquad + V_{if}(f_i(x_{i_{t+N_{ip}}}, \kappa_{if}(x_{i_{t+N_{ip}}}))) - V_{if}(x_{i_{t+N_{ip}}})
\end{aligned}
$$

Noting that, using Assumption 7.6

$$
\begin{aligned}
&q_i(x_{i_k}, A_{w_i}^{k-(t+1)} w_{i_{t+1}}) - q_i(x_{i_k}, A_{w_i}^{k-t} w_{i_t}) \\
&\quad \le \left| q_i(x_{i_k}, A_{w_i}^{k-(t+1)} w_{i_{t+1}}) - q_i(x_{i_k}, A_{w_i}^{k-t} w_{i_t}) \right| \\
&\quad \le L_{qw_i} \left| A_{w_i}^{k-(t+1)} w_{i_{t+1}} - A_{w_i}^{k-t} w_{i_t} \right| \\
&\quad = L_{qw_i} (\alpha_{w_i})^{k-(t+1)} \left| w_{i_{t+1}} - A_{w_i} w_{i_t} \right| \\
&\quad = L_{qw_i} (\alpha_{w_i})^{k-(t+1)} \left[\left| w_{i_{t+1}} \right| + \alpha_{w_i} \left| w_{i_t} \right| \right]
\end{aligned}
$$

and by using point 6 of Assumption 7.4, one obtains

$$
\begin{aligned}
&J_i(x_{i_{t+1}}, w_{i_{t+1}}, \bar{\mathbf{u}}_{i[t+1,t+N_{ic}]}, N_{ic}, N_{ip}) \\
&\quad \le V_i(x_{i_t}, w_{i_t}) - l_i(x_{i_t}, u^o_t) - q_i(x_{i_t}, w_{i_t}) + \psi_i(| (A_{w_i})^{N_{ip}-1} w_{i_t} |) \\
&\qquad + \sum_{k=t+1}^{t+N_{ip}-1} \left\{ L_{qw_i} (\alpha_{w_i})^{k-(t+1)} \left[\left| w_{i_{t+1}} \right| + \alpha_{w_i} \left| w_{i_t} \right| \right] \right\} \\
&\quad \le V_i(x_{i_t}, w_{i_t}) - l_i(x_{i_t}, u^o_t) - q_i(x_{i_t}, w_{i_t}) + \sigma_{i_2}(|w_{i_t}|) + \sigma_{i_3}(|w_{i_{t+1}}|)
\end{aligned}
$$

where $\sigma_{i_2}(|w_{i_t}|) \triangleq \alpha_{w_i} \mathcal{L}_{q_{iw}} \dfrac{(\alpha_{w_i})^{N_{ip}} - 1}{\alpha_{w_i} - 1} |w_{i_t}| + \psi_i((\alpha_{w_i})^{N_{ip}-1} |w_{i_t}|)$ and $\sigma_{i_3}(|w_{i_{t+1}}|) \triangleq \mathcal{L}_{q_{iw}} \dfrac{(\alpha_{w_i})^{N_{ip}} - 1}{\alpha_{w_i} - 1} |w_{i_{t+1}}|$ are $\mathcal{K}_\infty$-functions.

Now, from inequality

$$V_i(x_{i_{t+1}}, w_{i_{t+1}}) \leq J_i(x_{i_{t+1}}, w_{i_{t+1}}, \bar{\mathbf{u}}_{i[t+1,t+N_{ic}]}, N_{ic}, N_{ip})$$

and by Assumption 7.5, it follows that

$$V_i(x_{i_{t+1}}, w_{i_{t+1}}) - V_i(x_{i_t}, w_{i_t}) \leq -\underline{r}_i(|x_{i_t}|) + \sigma_{i_2}(|w_{i_t}|) + \sigma_{i_3}(|w_{i_{t+1}}|) \quad (7.37)$$

for all $x_{i_t} \in \mathcal{X}_i$, and all $w_{i_t}, w_{i_{t+1}} \in \mathcal{W}_i$. Finally, in view of the admissible control sequence (7.36), it follows that $\mathcal{X}_i^{MPC}$ is a RPIA set for the closed loop (7.8). Therefore, by (7.34), (7.35), (7.37) and Assumption 7.8, the optimal cost $J_i(x_{i_t}, w_{i_t}, \mathbf{u}^o_{i[t,t+N_{ic}-1]}, N_{ic}, N_{ip})$ is an ISS-Lyapunov function for the closed-loop system (7.8) in $\mathcal{X}_i^{MPC}$ and hence the closed-loop system is ISS in $\mathcal{X}_i^{MPC}$. ■

Proof of Lemma 7.1: Owing to the smoothness of $\kappa_{if}(x_i)$ and the fact that it is a stabilizing control law and recalling that $0 \in \mathcal{X}_i$, $0 \in \mathcal{U}_i$, it follows that there exists $\bar{\Upsilon}_i \in (0, \infty)$ such that points 1, 2, and 3 of Assumption 7.4 are satisfied for $x_i^\top \Pi_i x_i \leq \bar{\Upsilon}_i$. point 5 is satisfied with $\alpha_{V_{if}}(|x_i|) = \lambda_{\min}(\Pi_i)|x_i|^2$, and $\beta_{V_{if}}(|x_i|) = \lambda_{\max}(\Pi_i)|x_i|^2$, where $\lambda_{\min}(\Pi_i)$ and $\lambda_{\max}(\Pi_i)$ denote the minimum and the maximum eigenvalues of Π_i, respectively. In order to prove point 6, letting

$$\Phi_i(x_i) \triangleq f_i(x_i, K_i x_i) - A_{i_{cl}} x_i$$

the inequality

$$\begin{aligned} f_i(x_i, K_i x_i)^\top \Pi_i f_i(x_i, K_i x_i) - x_i^\top \Pi_i x_i \quad &\leq \quad -x_i^\top \left(Q_i + K_i^\top R_i K_i \right) x_i \\ &\quad -q_i(x_i, w_i) + \psi_i(|w_i|) \end{aligned} \quad (7.38)$$

is equivalent to

$$
\begin{aligned}
&2\Phi_i(x_i)^\top \Pi_i A_{i_{cl}} x_i + \Phi_i(x_i)^\top \Pi_i \Phi_i(x_i) + x^\top A_{i_{cl}}^\top \Pi_i A_{i_{cl}} x_i - x_i^\top \Pi_i x_i \\
&\leq -x_i^\top \left(Q_i + K_i^\top R_i K_i\right) x_i - q_i(x_i, w_i) + \psi_i\left(|w_i|\right)
\end{aligned}
\tag{7.39}
$$

Indeed, from (7.17) it is easy to show that inequality (7.39) is equivalent to

$$
\begin{aligned}
&2\Phi_i(x_i)^\top \Pi_i A_{i_{cl}}^\top x_i + \Phi_i(x_i)^\top \Pi_i \Phi_i(x_i) \\
&\leq x_i^\top \tilde{Q}_i x_i - x_i^\top \left(Q_i + K_i^\top R_i K_i\right) x_i - q_i(x_i, w_i) + \psi_i\left(|w_i|\right)
\end{aligned}
\tag{7.40}
$$

Now, define $L_{r_i} \triangleq \sup_{x_i \in B_{r_i}} |\Phi_i(x_i)| \,/\, |x_i|$, where $B_{r_i} \triangleq \{x_i : |x_i| \leq r\}$ (Once chosen r, L_{r_i} does exist and takes on a finite value because $f_i \in \mathcal{C}^2$). Moreover by the assumption on $q_i(x_i, w_i)$ and the definition of $\tilde{Q}$ it follows that

$$
\begin{aligned}
&x_i^\top \tilde{Q}_i x_i - x_i^\top \left(Q_i + K_i^\top R_i K_i\right) x_i - q_i(x_i, w_i) + \psi_i\left(|w_i|\right) \\
&\geq x_i^\top \tilde{Q}_i x_i - x_i^\top \left(Q_i + K_i^\top R_i K_i\right) x_i - x_i^\top \tilde{S}_i x_i^\top \geq \gamma_i |x_i|^2,
\end{aligned}
$$

with $\gamma_i > 0$. Then, $\forall x_i \in B_{r_i}$, (7.40) is satisfied if

$$
\gamma_i |x_i|^2 \geq \{2 L_{r_i} |\Pi_i| |A_{i_{cl}}| + {L_{r_i}}^2 |\Pi_i|\} |x_i|^2 \tag{7.41}
$$

Hence, since $L_{r_i} \to 0$ as $r \to 0$, there exists $\Upsilon_i \in \left(0, \bar{\Upsilon}_i\right)$ such that inequality (7.41) holds $\forall x_i \in \mathcal{X}_{if}$, which implies that inequality (7.38) holds as well. Finally, there exists $\bar{N}_p$ such that for all $N_p \geq \bar{N}_p$, $\forall x_i \notin \mathcal{X}_{if}, \forall w_i \in$

W_i

$$\begin{aligned} V_{f_i}(f_i(x_i,\kappa_{if}(x_i))) - V_{if}(x_i) &\le -x_i^\top \left(Q_i + K_i^\top R_i K_i\right) x_i \\ &\quad -q_i(x_i,\tilde{w}_i) + \psi_i\left(|\tilde{w}_i|\right) \\ &< 0 \end{aligned}$$

where $\tilde{w}_i = (A_{w_i})^{N_{ip}-1} w_i$ so that point 4 of Assumption 7.4 is satisfied, too, thus ending the proof. ■

Proof of Lemma 7.2: Consider the ISS-Lyapunov function candidate

$$V(x_t,w_t) \triangleq \sum_{i=1}^{M} V_i(x_{i_t},w_{i_t})$$

for system (7.18).[2] From (7.34) and (7.35), it follows that

$$\sum_{i=1}^{M} \underline{r}_i(|x_{i_t}|) \le V(x_t,w_t) \le \sum_{i=1}^{M} \alpha_{i_2}(|x_{i_t}|) + \sum_{i=1}^{M} \sigma_{i_1}(|w_{i_t}|)$$

Clearly $|x_{i_t}| \le |x_t|$ and $|w_{i_t}| \le |w_t|$, $\forall i = 1,\dots,M$ and thus

$$\begin{aligned} V(x_t,w_t) &\le \sum_{i=1}^{M} \alpha_{i_2}(|x_{i_t}|) + \sum_{i=1}^{M} \sigma_{i_1}(|w_{i_t}|) \\ &\le \sum_{i=1}^{M} \alpha_{i_2}(|x_t|) + \sum_{i=1}^{M} \sigma_{i_1}(|w_t|) \\ &\le \alpha_2(|x_t|) + \sigma_1(|w_t|)\,, \end{aligned}$$

where $\alpha_2(|x_t|) \triangleq \sum_{i=1}^{M} \alpha_{i_2}(|x_t|)$ and $\sigma_1(|w_t|) \triangleq \sum_{i=1}^{M} \sigma_{i_1}(|w_t|)$.

[2]It is worth noting that, instead of the above definition of V, a weighted sum of Lyapunov functions could be used along the reasoning provided in [Khalil 2001] in the framework of composite systems.

Moreover $\sum_{i=1}^{M} |x_{i_t}| \leq \sum_{i=1}^{M} |x_t| = M|x_t|$. Then $|x_t| \geq \frac{1}{M}\sum_{i=1}^{M} |x_{i_t}|$ and $|x_t| \leq \sum_{i=1}^{M} |x_{i_t}|$. Now, recall that, for any $\mathcal{K}$ function γ, one has $\gamma\left(\sum_{i=1}^{M} a_i\right) \leq \sum_{i=1}^{M} \gamma(Ma_i)$ where $a_i > 0, i = 1, \ldots, M$ are arbitrarily chosen positive scalars). Therefore, considering the $\mathcal{K}$ function $\underline{r}_i$, for a generic $i \in \{1, \ldots, M\}$, one has

$$\underline{r}_i(|x_t|) \leq \underline{r}_i\left(\sum_{i=1}^{M} |x_{i_t}|\right) \leq \sum_{i=1}^{M} \underline{r}_i(M|x_{i_t}|) \leq \sum_{i=1}^{M} \underline{r}_i(M|x_t|)$$

and hence

$$\underline{r}_i\left(|x_t|/M\right) \leq \underline{r}_i\left(\frac{1}{M}\sum_{i=1}^{M} |x_{i_t}|\right) \leq \sum_{i=1}^{M} \underline{r}_i(|x_{i_t}|)\,.$$

Therefore, letting $\underline{r}\left(|x_t|\right) \triangleq \underline{r}_i\left(|x_t|/M\right)$ for an arbitrarily chosen index i, one has

$$\underline{r}(|x_t|) \leq V(x_t, w_t),\ \forall x_t \in \mathcal{X},\ \forall w_t \in \mathcal{W} \tag{7.42}$$

$$V(x_t, w_t) \leq \alpha_2(|x_t|) + \sigma_1(|w_t|),\ \forall x_t \in \mathcal{X}^{\kappa_f},\ \forall w_t \in \mathcal{W} \tag{7.43}$$

From (7.37) it follows that

$$\begin{aligned}
\Delta V \quad &\triangleq \quad \sum_{i=1}^{M} V_i(x_{i_{t+1}}, w_{i_{t+1}}) - \sum_{i=1}^{M} V_i(x_{i_t}, w_{i_t}) \\
&\leq \quad -\sum_{i=1}^{M} \underline{r}_i(|x_{i_t}|) + \sum_{i=1}^{M} \sigma_{i_2}(|w_{i_t}|) + \sum_{i=1}^{M} \sigma_{i_3}(|w_{i_{t+1}}|)
\end{aligned}$$

Moreover, note that

$$-\sum_{i=1}^{M} \underline{r}_i(|x_{i_t}|) \leq -\underline{r}_i\left(|x_t|/M\right)$$

and

$$\sum_{i=1}^{M} \sigma_{i_2}(|w_{i_t}|) \leq \sum_{i=1}^{M} \sigma_{i_2}(|w_t|)$$

and

$$\sum_{i=1}^{M} \sigma_{i_3}(|w_{i_{t+1}}|) \leq \sum_{i=1}^{M} \sigma_{i_3}(|w_{t+1}|)$$

Then, letting

$$\sigma_2(|w_t|) \triangleq \sum_{i=1}^{M} \sigma_{i_2}(|w_{i_t}|)$$

and

$$\sigma_3(|w_{t+1}|) \triangleq \sum_{i=1}^{M} \sigma_{i_3}(|w_{i_{t+1}}|)$$

it follows that

$$\Delta V \leq -\underline{r}\left(|x_t|\right) + \sigma_2(|w_t|) + \sigma_3(|w_{t+1}|) \tag{7.44}$$

for all $x_t \in \mathcal{X}$, and all $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$. Therefore, by (7.42), (7.43) and (7.44), $V(x_t, w_t)$ is an ISS-Lyapunov function in $\mathcal{X}^{MPC}$ for system (7.18) and hence this system is ISS in $\mathcal{X}^{MPC}$.

As far as system (7.19) is concerned (remember that this system describes the effects of the time–delays in the information exchange variables), the proof that it is ISS is obviously trivial since (7.19) is an asymptotically stable discrete-time linear system. Let only very briefly sketch some parts of the proof just for the purpose of introducing a few quantities that will be used subsequently. Following [Jiang & Wang 2001], a candidate ISS-

Lyapunov function for system (7.19) is $V_D(z_t) \triangleq z_t^\top P z_t$, where P is the positive definite solution of the Lyapunov equation $A^\top PA - P = -Q$ for a given symmetric positive-definite matrix Q. It can be immediately shown that

$$\alpha_{1D}(|z_t|) \leq V_D(z_t) \leq \alpha_{2D}(|z_t|) \tag{7.45}$$

where $\alpha_{1D}(s) \triangleq \lambda_{min}(P)s^2$ and $\alpha_{2D}(s) \triangleq \lambda_{max}(P)s^2$ ($\lambda_{min}(P)$ and $\lambda_{max}(P)$ denote the minimum and maximum eigenvalues of P, respectively). Moreover, defining $\Delta V_D \triangleq V_D(z_{t+1}) - V_D(z_t)$, it is straightforward to obtain

$$\Delta V_D \leq -\alpha_{3D}(|z_t|) + \sigma_D(|x_t|) \tag{7.46}$$

where $\alpha_{3D}(s) \triangleq \frac{1}{2}\lambda_{min}(Q)s^2$ and $\sigma_D(s) \triangleq \lambda_{max}(\frac{2B^\top PAA^\top PB}{\lambda_{min}(Q)} + B^\top PB)s^2$.

■

Proof of Theorem 7.3: If $\gamma_1 \circ \gamma_2(s) < s$, from (7.22) and (7.23) it follows that

$$\begin{aligned}
&V(x_k, w_k) \leq ||V(x,w)_{[k]}|| \\
&\leq \max\{\hat{\beta}(V(x_t,w_t),0), \gamma_1(\hat{\beta}_D(V_D(z_t),0), \gamma_1 \circ \gamma_2(||V(x,w)_{[k]}||)\} \\
&\leq \max\{\hat{\beta}(V(x_t,w_t),0), \gamma_1(\hat{\beta}_D(V_D(z_t),0)\}
\end{aligned}$$

for all $x_t \in \Psi$, and all $k \in \mathbb{Z}_{\geq 0}$, $k \geq t$ and

$$\begin{aligned}
&V_D(z_k) \leq ||V_D(z)_{[k]}|| \\
&\leq \max\{\hat{\beta}_D(V_D(z_t),0), \gamma_2(\hat{\beta}(V(x_t,w_t),0)), \gamma_2 \circ \gamma_1(||V_D(z)_{[k]}||)\} \\
&\leq \max\{\hat{\beta}_D(V_D(z_t),0), \gamma_2(\hat{\beta}(V(x_t,w_t),0))\}
\end{aligned}$$

for all $x_t \in \Psi$, and all $k \in \mathbb{Z}_{\geq 0}$, $k \geq t$ and hence $V(x_k, w_k)$, $V_D(z_k)$ are bounded by initial condition. By Lemma 3.13 in [Jiang & Wang 2001], an asymptotic gain from $V_D(z_k)$ to $V(x_k, w_k)$ is given by γ_1 whereas an asymptotic gain from $V(x_k, w_k)$ to $V_D(z_k)$ is given by γ_2. Hence:

$$\begin{aligned}\overline{\lim_{k\to\infty}} V(x_k, w_k) &\leq \overline{\lim_{k\to\infty}}[\alpha_4^{-1} \circ \rho^{-1} \circ \sigma_4(V_D(z_k))] \\ &= \overline{\lim_{k\to\infty}} \gamma_1(V_D(z_k))\end{aligned}$$

and

$$\begin{aligned}\overline{\lim_{k\to\infty}} V_D(z_k) &\leq \overline{\lim_{k\to\infty}}[\alpha_{4D}^{-1} \circ \rho^{-1} \circ \sigma_{4D}(V(x_k, w_k))] \\ &= \overline{\lim_{k\to\infty}} \gamma_2(V(x_k, w_k))\end{aligned}$$

Hence

$$\begin{aligned}\overline{\lim_{k\to\infty}} V(x_k, w_k) &\leq \overline{\lim_{k\to\infty}} \gamma_1(V_D(z_k)) \\ &\leq \overline{\lim_{k\to\infty}} \gamma_1 \circ \gamma_2(V(x_k, w_k))\end{aligned} \tag{7.47}$$

Again, the assumption that $\gamma_1 \circ \gamma_2(s) < s$ implies that

$$\overline{\lim_{k\to\infty}} V(x_k, w_k) = \overline{\lim_{k\to\infty}} V_D(z_k) = 0$$

Thus, the system is 0-AS in $\mathcal{X}^{MPC} \times \mathbb{R}^{n_z}$. ■

Chapter 8

MPC of Glycaemia in Type 1 Diabetic Patients

8.1 Introduction

Control systems theory has been extending into many fields and medicine is not an exception (see for example [Palerm 2003] and the references therein), although the progress has been slow in some cases due to particular challenges encountered by the inherent complexity of biological systems. Biological systems are highly non-linear and there is a large inter-individual variability. Furthermore, the human body is a system intrinsically time-variant.

A typical medical application is glycemic control for the type 1 (insulin-dependent) diabetic patient. Diabetes (medically known as diabetes mellitus) is the name given to disorders in which the body has trouble regulating its blood glucose, or blood sugar, levels. There are two major types of diabetes: type 1 diabetes and type 2 diabetes. Type 1 diabetes, also called juvenile diabetes or insulin-dependent diabetes, occurs when the body's immune system attacks and destroys beta cells of the pancreas. Beta cells normally produce insulin, a hormone that regulates, together with glucagon (an other hormone produced by alpha cells of the pancreas), the blood glucose concentration in the body, i.e. while insulin lowers it, glucagon increases it. The food intake in the body results in an increase in the glucose concentration. Insulin permits most of the body's cells to take up glucose from the blood and use it as energy. When the beta cells are destroyed, no insulin can be produced, and the glucose stays in the blood instead, where it can cause serious damage to all the organ systems of the body. For this reason, people with type 1 diabetes must take insulin in order to stay alive. This means undergoing multiple injections daily (normally coinciding with the meal times), or having insulin delivered through an insulin pump, and testing their blood sugar by pricking their fingers for blood six or more times a day [JDRF Website]. This method is approximate, merely an open-loop control with intermittently monitored glucose levels resulting in intermittently administered insulin doses that are manually administered by a patient using a written algorithm. Poor glycemic control leads to severe health complications, including blindness and kidney failure in the long term, and loss of consciousness and coma in the short term when hypoglycemic.

The currently available intensive therapy can be contrasted to glucose management by a closed-loop system, which for control of blood glucose in patients with diabetes is known as an artificial pancreas. In this case, glucose levels are monitored continuously, which results in continuously infused insulin dosed according to a computerized algorithm without a need for patient input. Compared to currently applied intensive therapy, an artificial pancreas can potentially result in: 1) less glycemic variability; 2) less hypoglycemia; 3) less pain from pricking the skin to check the blood glucose and deliver insulin boluses; and 4) less overall patient effort. No artificial pancreas system is currently approved; however, devices that could become components of such a system are now becoming commercially available [Klonoff 2007].

The World Health Organization (WHO) estimates that in 2006 more than 180 million people worldwide have diabetes. This number is likely to more than double by 2030 [WHO Website], a substantial figure to warrant relevant research in this area.

8.2 Artificial Pancreas

Type 1 diabetes (T1DM) is caused by an absolute deficiency of insulin secretion. It includes cases primarily due to β cell destruction, and who are prone to ketoacidosis. These cases are those attributable to an autoimmune process, as well as those with β cell destruction for which no pathogenesis is known (i.e. idiopathic). People with this type of Diabetes Mellitus fully depend on exogenous insulin. It is this specific group which would benefit the most from closed-loop glucose regulation. Closed-loop control for regulating glycemia requires three components: glucose measurements, the control algorithm and a way to administer the insulin (see [Palerm 2003] for details). The administration of insulin, and type of insulin, is something that is relevant even with the current modalities of treatment. As such, the rest of this section covers the various options. Glucose sensing is dealt in the modeling section below. The two main modalities currently in clinical use are multiple daily injections (MDI) and continuous subcutaneous insulin infusion (CSII) with an externally worn infusion pump that operates in open-loop. The type of insulin used also varies, as there is the regular human insulin plus other variations that have been engineered to have specific absorption properties; some are quickly absorbed, while others are slow acting. This allows for a single shot to have an effect over several hours, in order to regulate fasting levels, or to quickly bring insulin levels up to coincide with a meal. Research towards the realization of an artificial pancreas is active since 1970. The first prototype commercialized was the so-called Biostator®. It adopted the intravenous (iv) route both for glucose monitoring and insulin infusion: an early version used a glucose infusion to counter-regulate insulin action and prevent hypoglycaemia. The control algorithm, based on Albisser *et al.* [Albisser *et al.* 1974], calculates current insulin infusion as a sigmoidal function of actual blood glucose level, mimicking β cell response observed in *in-vitro* experiments. The iv-iv approach is highly invasive, so it can't be used on a daily basis. However, this is the best approach in terms of performance because it minimizes the delay between insulin infusion and action. Recently new fast insulin analogues (e.g. Lispro insulin [Homko *et al.* 2003]) and less invasive glucose sensors have been introduced. These technological improvements have made more feasible the adoption of a subcutaneous route (sc-sc); however, there are still difficulties that justifies the development of specific and sophisticated control algorithms. Main issues are delays and the presence of important disturbances on glucose level like meals and physical activity. Furthermore, control can only act on insulin infusion, without counter-regulation: that increases the risk of hypoglycaemia. There is also the need of controller personalization: in fact, there is a large inter-individual and intra-individual variability in diabetic patients. Many solutions have been proposed in literature for this control problem, which are based on different approaches. PID (Proportional-Integral-Derivative) strategy is certainly one of the most simple and follow, partly as a result of studies that show the similarity with the β cell functionality [Steil *et al.* 2004]. Steil *et al.* [Steil *et al.* 2006] have proposed a controller based on PID with time-variant parameters to manage the different response between day and night. Marchetti *et al.* [Marchetti *et al.* 2006] use a PID switching strategy at meals with an algorithm that generates a time-variant setpoint (*reference governor*). Some authors have shown that integral action is fully responsible for the administration of excessive insulin in the postprandial period, with the result of generating numerous hypoglycaemia. For example, Shimoda *et al.* [Shimoda *et al.* 1997] have used an algorithm PD (Proportional-Derivative) in tests on diabetics. This work has also shown that the use of a rapid insulin analogue, known as Lispro, can obtain benefits comparable to those using regular insulin via the intravenous route. A pure feedback control, such as PID, is able to react against a disturbance only when it is manifested as a variation of the measured variable. In the case sc-sc, the presence of delays results in unsatisfactory performance. This has led to the study of solutions based on more sophisticated approaches, such as predictive control or MPC, which had already been applied successfully in other biomedical issues such as regulation of blood pressure [Gopinath *et al.* 1995] and of anesthesia [Linkens & Mahfouf 1994]. The predictive control uses a model to predict the dynamic of the system within a certain time and calculates the input through the optimization of a fitness function in that time interval. The main benefit of this strategy is to predict the evolution of blood glucose and, therefore, to be able to act on time by avoiding hypoglycaemic and hyperglycaemic events. This ability is very useful especially if the disturbances trend is known in advance, even if only partially. Another advantage is to explicitly consider the presence of constraints on the inputs and outputs. Parker *et al.* [Parker *et al.* 1999] use a linear predictive control, the simplest of that class, based on a model of the system identified by the step response. Hovorka *et al.* [Hovorka *et al.* 2004] have proposed a regulator that uses a nonlinear and time-variant model for the prediction, whose parameters are adjusted over time using a Bayesian estimate. This technique has the disadvantage compared to linear MPC, not to lead to a closed form for the control law: so the optimization problem must be solved on-line and can be very computationally expensive. Another approach that allows to solve the problem of constrained optimization off-line needs the use of a parametric programming algorithm (Dua *et al.*[Dua *et al.* 2006]).

8.2.1 Artificial Pancreas Project

In 2006, the Juvenile Diabetes Research Foundation (JDRF [JDRF Website]), the largest nonprofit foundation in the world who work for type 1 diabetics, has launched a project to accelerate the development and adoption of a device for the closed loop control of blood glucose. The objective of the project is to make available on the market within 5 years (i.e. by 2011) artificial pancreas technology and, subsequently, to promote its use on a large scale. To achieve these goals, JDRF has launched a campaign to expedite the approval of this device by the U.S. regulatory institutions (FDA, Food and Drug Administration) and to obtain coverage by health insurance. JDRF has created a consortium (the *Artificial Pancreas Consortium*) of British and Americans research centers that currently includes the University of Cambridge, the University of Colorado, Stanford, Yale, the Sansum research center and the University of Virginia. The University of Pavia collaborates in this project joined with the University of Virginia and the University of Padova (see Figure 8.1).

In this chapter, the feedback control of glucose concentration in Type-1 diabetic patients using s.c. insulin delivery and s.c. continuous glucose monitoring is considered. In particular, linear and nonlinear MPC controller synthesized on a recently developed nonlinear in-silico model of glucose metabolism (see [Magni *et al.* 2007b]) are proposed. The algorithms are tested on virtual patient populations and the performances are evaluated by using particular metrics described later on.

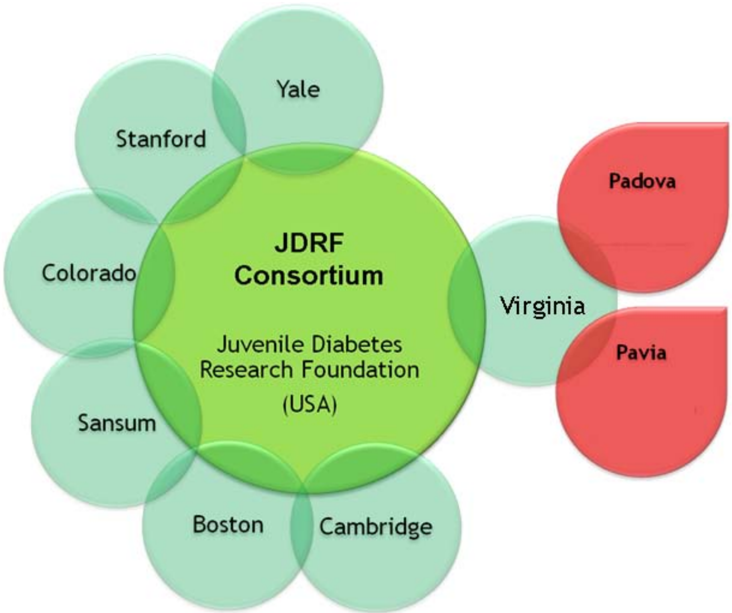


Figure 8.1: JDRF Consortium

The Food and Drug Administration (FDA) accepted the in-silico experiments, based on a metabolic simulator and virtual patient populations, as substitutes of animal tests. Currently, experiments using the developed linear model predictive controller are in progress at the Charlottesville and Padova hospitals.

8.3 Glucose Regulation

There are several topics in physiology that are related to an understanding of glucose regulation and the particulars that need to be considered in modeling and control. Three categories need to be considered: the gastrointestinal (GI) system as related to digestion and nutrient absorption, the hepatic function in metabolism and the endocrine pancreas with its secretion of insulin and glucagon as the main regulators of blood glucose level (see [Palerm 2003] for details).

The root source of glucose for the body is the food. As ingested, food cannot be absorbed; it thus requires both mechanical and chemical digestion. It is the job of the GI system to do this processing, and then to facilitate the absorption of nutrients. The GI tract is composed of hollow organs connected in series; these have sphincters at key locations in order to regulate flow. Associated with the GI system are organs and glands that secrete various substances into the hollow organs.

Carbohydrates, are present in food as disaccharides and monosaccharides (including glucose); since only monosaccharides are absorbed, carbohydrates require chemical digestion as well. Digestion duration varies with the composition of the meal, but usually it lasts about two hours. At end glucose is absorbed in the intestine, entering bloodstream.

In carbohydrate metabolism and the regulation of blood glucose levels the liver plays a central role. It receives blood from the portal circulation (coming from the stomach, small and large intestines, pancreas and spleen). Thus most nutrients (lipids are absorbed into the lymphatic circulation) and pancreatic hormones pass through the liver before continuing on.

The liver stores several substances, including carbohydrates. Depending on metabolic requirements these are either sequestered or released into the blood stream. Among these is glucose, which it can also synthesize. Given its central position, the liver plays a critical role in energy metabolism.

The two hormones with the largest role in glucose regulation are insulin and glucagon. Between meals, insulin levels are low and glucagon levels are high; given these conditions, the liver serves as a source of glucose. It does this by gluconeogenesis (synthesis of glucose from amino acids and lactate)

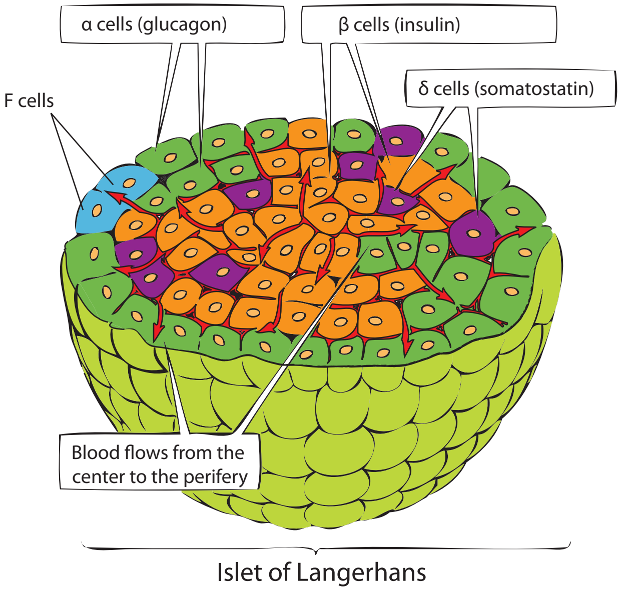


Figure 8.2: Structure and functionality of islet of Langerhans

and glycogenolysis (the breakdown of glycogen stores into glucose). Other monosaccharides are largely converted into glucose. In the postprandial period, insulin levels are higher. The liver then takes up glucose from the portal circulation; it either stores it by synthesizing glycogen, or breaks it down to pyruvate (glycolysis). Carbohydrates not oxidized nor stored as glycogen are metabolized to fat. Insulin and glucagon are secreted by the pancreas, in the *islets of Langerhans*, the endocrine glands.

Each islet contains four types of cells (see Figure 8.2). β cells (the most numerous) secrete insulin, proinsulin, C peptide and amylin. α cells secrete glucagon, δ cells secrete somatostatin and F cells secrete pancreatic polypeptide. These cells are influenced by external links (neural, cell-cell and humoral), as well as by other cells in the islet. Insulin's actions serve to replenish fuel reserves in muscle, liver and adipose tissue. During fasting, β cells secrete small amounts of insulin. With low insulin levels, lipids are mobilized from adipose tissue, and amino acids are mobilized from protein stores in muscle and other tissues. These lipids and amino acids are precursors for hepatic ketogenesys and gluconeogenesis, and are also oxidized as fuel.

Insulin secretion increases with feeding. Elevated levels of insulin reduce lipid and amino acid mobilization within fuel stores, which are also replenished by the enhanced uptake of carbohydrates, lipids and amino acids by insulin-sensitive target tissues. This action maintains plasma glucose within tight limits, assuring a constant supply of fuel for the central nervous system (CNS). Plasma glucose below 2-3 mM/l (36-54 mg/dl), hypoglycemia, results in confusion and seizures, and if not resolved, coma and eventually death. Severe hyperglycemia (plasma glucose above 30-40 mM) results in osmotic diuresis, leading to severe dehydration, hypertension and vascular collapse. Insulin synthesis and secretion in the β cells is stimulated by exposure to glucose, which is its main regulator.

Glucagon promotes hepatic glucose production; it does so by activating glycogenolysis and inhibiting glycogen synthesis and glycolysis. In the liver, glucagon stimulates fat oxidation as a source of energy. If the needs for energy of the liver are exceeded, fatty acids are only partially oxidized, forming ketone bodies. These keto acids can then be used by other tissues as fuel (besides glucose, keto acids are the only other energy source the brain will utilize). This is particularly important during fasting. Other hormones have minor effects, for example, somatostatin, secreted by δ cells in the pancreas and the hypothalamus, inhibits the secretion of several other hormones, including insulin and glucagon. It is still unclear if it has any paracrine effects on α or β cells.

8.4 Models for the Glucose-Insulin System

Mathematical models for the glucose-insulin system could be very useful for the research about automatic glycemic control. In fact, they permit the design and validation of control algorithms reducing the number of in-vivo experiments required.

Different models have been proposed in literature ranging in complexity from simple linear models like Ackerman *et al.* [Gatewood *et al.* 1968], via the classical minimal model of Bergmann *et al.* [Bergman & Urquhart 1971], to the non-linear models and inclusive models of Hovorka *et al.* [Hovorka *et al.* 2004] and Sorensen *et al.* [Sorensen 1985].

Recently, however, thanks to new types of OGTT[1] or *Mixed Meal* experiments using radioactive tracers and new techniques such as NMR[2] and PET[3], quantitative data have been obtained on physiological variables never seen before, such as the liver production or glucose muscular utilization. Thanks to this new database was possible to develop new models more complex and detailed than previous: in particular, the model developed by Dalla Man *et al.* [Dalla Man *et al.* 2007b] from the University of Padova (Italy).

8.4.1 Dalla Man *et al.* model for the diabetic patient

The development of this model was made possible thanks to a database of 204 healthy subjects, who was given a mixed meal containing glucose labeled with a radioactive tracer. At the same time, two tracer are ad-

[1] Oral Glucose Tolerance Test.
[2] Nuclear Magnetic Resonance.
[3] Positron Emission Tomography.

ministered intravenously: this triple-tracer technique [Basu *et al.* 2003] has enabled the flow measurement of glucose in the body. Without the use of this technique it is only possible to measure blood glucose levels and insulinemia, without being able to distinguish, for example, between glucose produced endogenously by the liver and glucose introduced with the meal. So it was possible to obtain direct measurements of the rate of appearance Ra, namely the glucose flow entering the bloodstream from the gastro-intestinal tract, endogenous glucose production EGP, renal extraction E and glucose utilization U. These new informations have enabled a qualitative leap in modeling, with the possibility of splitting the system into sub-units interconnected: for each unit is postulated a compartment model whose parameters are estimated using the available measures of inputs and outputs.

The model described in [Dalla Man *et al.* 2007b] refers to an healthy person. Therefore, it was necessary to make some structural changes to the model to reflect the absence in patients with type 1 diabetes of endogenous insulin production (see [Magni *et al.* 2007b]). The fundamental change concerns the replacement of the subsystem of the pancreatic β-cell with a unit that models the dynamics of a subcutaneous (sc) infusion of insulin.

The model, overall, may be seen as a MISO (Multi Input Single Output) with two inputs, the glucose introduced with the meal and insulin administered subcutaneously, and only one exit, the blood glucose. In the following sections the functional units are described in details. For an overview of interconnections between the units, see Figure 8.3.

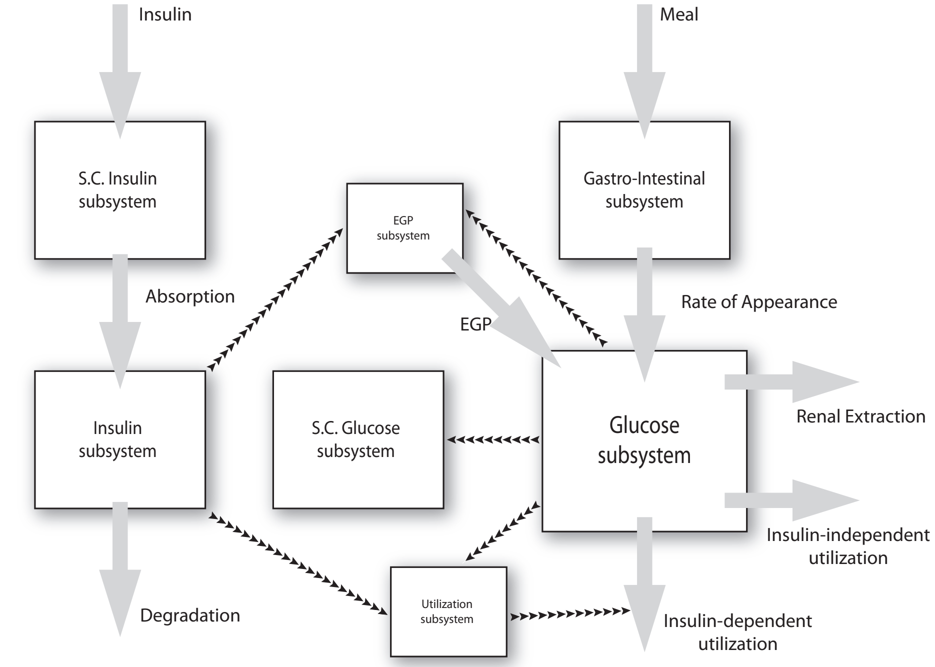


Figure 8.3: Interconnections between subsystems of Dalla Man *et al.* model [Dalla Man *et al.* 2007b] changed to cope with type 1 diabetic patients as described in [Magni *et al.* 2007b]

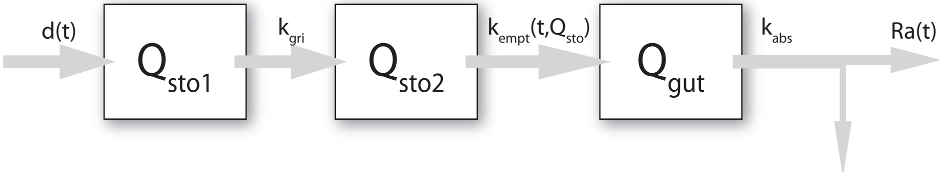


Figure 8.4: Schema of gastro-intestinal subsystem

Gastro-Intestinal subsystem

Gastro-Intestinal subsystem describes the glucose transfer coming from a meal through the gastro-intestinal system. It consists of three compartments (see Figure 8.4): two for the glucose in the stomach (solid Q_{sto1} and liquid Q_{sto2} (mg)) and one for the glucose in the intestinal tract Q_{gut} (mg). The glucose flow $d(t)$ (mg/min) coming from the meal is the input of the first compartment. The stomach digests the meal with grinding coefficient k_{gri}; then the chyme enters the intestine with fractional coefficient of transfer k_{empt}, that is a time-variant nonlinear function of total $Q_{sto} = Q_{sto1} + Q_{sto2}$ that will be later described; finally glucose is absorbed and enters the bloodstream. The *rate of appearance* Ra (mg/Kg/min) is a constant percentage f (about 90%) of the total flow leaving the intestine.

$$\begin{cases} \dot{Q}_{sto1}(t) = -k_{gri} Q_{sto1}(t) + d(t) \\ \dot{Q}_{sto2}(t) = -k_{empt}(t, Q_{sto}(t)) \cdot Q_{sto2}(t) + k_{gri} Q_{sto1}(t) \\ \dot{Q}_{gut}(t) = -k_{abs} Q_{gut}(t) + k_{empt}(t, Q_{sto}(t)) \cdot Q_{sto2}(t) \\ Q_{sto}(t) = Q_{sto1}(t) + Q_{sto2}(t) \\ Ra(t) = f \cdot k_{abs} \cdot Q_{gut}(t)/BW \end{cases} \tag{8.1}$$

In order to guarantee model identifiability, k_{gri} is fixed and equal to k_{max}. Furthermore, f is considered constant ($f = 0.9$).

Coefficient of gastric emptying k_{empt}

The coefficient of gastric emptying k_{empt} (1/min) is a time-variant nonlinear function of Q_{sto}

$$\begin{aligned} k_{empt}(t, Q_{sto}(t)) &= k_{max} + \frac{k_{max} - k_{min}}{2} \{\tanh[\alpha(Q_{sto}(t) - b \cdot D(t))] \\ &\quad - \tanh[\beta(Q_{sto}(t) - d \cdot D(t))]\} \end{aligned}$$

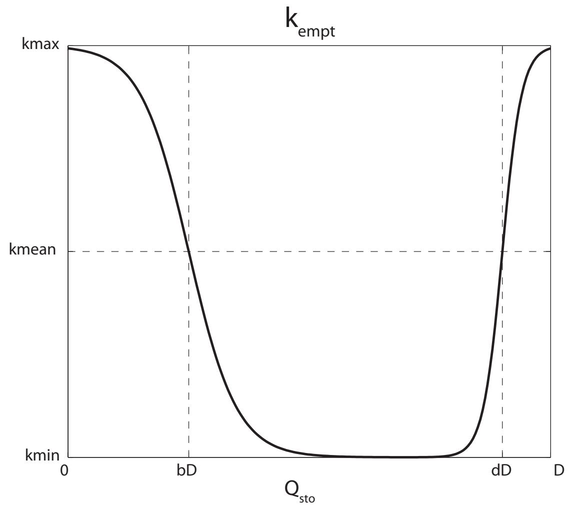


Figure 8.5: $k_{empt}(t, Q_{sto})$ function, where D is the total glucose quantity of the last meal

where

$$\alpha = \frac{5}{2D(t)(1-b)}, \quad \beta = \frac{5}{2D(t)d}$$

$$D(t) = \int_{t_i}^{t_f} d(t)dt$$

with t_i and t_f respectively start time and end time of the last meal, b, d, k_{max} and k_{min} model parameters (see Figure 8.5). For details, see [Dalla Man *et al.* 2006].

Glucose subsystem

Glucose subsystem model consists of two compartments (see Figure 8.6), one for plasma glucose G_p (mg/Kg) and one for the glucose on tissue G_t (mg/Kg). Glucose flows (mg/Kg/min) entering the first compartment are EGP coming from the liver and Ra coming from the gastro-intestinal tract. As outputs there are insulin-dependent utilization U_{id} and independent U_{ii}

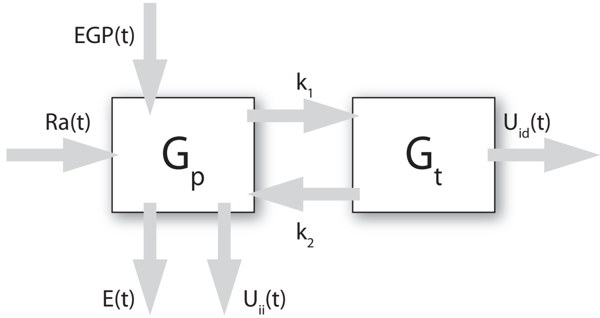


Figure 8.6: Schema of glucose subsystem

and the renal extraction E. Subsystem equations are

$$\begin{cases} \dot{G}_p(t) = -k_1 G_p(t) + k_2 G_t(t) + EGP(t) + Ra(t) - U_{ii} - E(t) \\ \dot{G}_t(t) = k_1 G_p(t) - k_2 G_t - U_{id}(t) \\ G(t) = G_p(t)/V_G \end{cases} \tag{8.2}$$

where V_G (dl/Kg) is the distribution volume of glucose, G (mg/dl) is the glycemia. Insulin-independent utilization U_{ii}, that occurs primarily in central nervous system, in this model is considered constant and equal to 1 mg/Kg/min.

Basal steady state, i.e. constant glycemia G_b (mg/dl), is characterized by the following equations

$$\begin{cases} -k_1 G_{pb} + k_2 G_{tb} + EGP_b + Ra_b - U_{ii} - E_b = 0 \\ k_1 G_{pb} - k_2 G_{tb} - U_{idb} = 0 \end{cases} \tag{8.3}$$

so, noting that at basal equilibrium $Ra_b = 0$,

$$EGP_b = U_{ii} + U_{idb} + E_b \tag{8.4}$$

Renal Extraction E

Renal extraction represents the glucose flow which is eliminated by the kidney, when glycemia exceeds a certain threshold k_{e2}

$$E(t) = \max(0, k_{e1} \cdot (G_p(t) - k_{e2}))$$

The parameter k_{e1} (1/min) represents renal glomerular filtration rate.

At basal one has

$$E_b = \max(0, k_{e1} \cdot (G_{pb} - k_{e2})) \tag{8.5}$$

Basal renal extraction is null in almost all patients.

Endogenous glucose production EGP

EGP comes from the liver, where a glucose reserve exists (glycogen). EGP is inhibited by high levels of glucose and insulin

$$EGP(t) = \max(0, EGP_b - k_{p2}(G_p(t) - G_{pb}) - k_{p3}(I_d(t) - I_b)) \tag{8.6}$$

where k_{p2} and k_{p3} are model parameters and I_d (pmol/l) is a delayed insulin signal, coming from the following dynamic system

$$\begin{cases} \dot{I}_1(t) = k_i I(t) - k_i I_1(t) \\ \dot{I}_d(t) = k_i I_1(t) - k_i I_d(t) \end{cases} \tag{8.7}$$

where I (pmol/l) is plasma insulin concentration or insulinemia and k_i (1/min) is a model parameter.

Insulin-dependent utilization U_{id}

It depends non-linearly from tissue glucose

$$U_{id}(t) = V_m(X(t)) \frac{G_t(t)}{K_m + G_t(t)} \tag{8.8}$$

where V_m (1/min) is a linear function of interstitial fluid insulin X (pmol/l)

$$V_m(X(t)) = V_{m0} + V_{mx}X(t)$$

which depends from insulinemia in the following way

$$\dot{X}(t) = -p_{2U}X(t) + p_{2U}(I(t) - I_b) \tag{8.9}$$

where p_{2U} (1/min) is called *rate* of insulin action on peripheral glucose.

Considering basal steady state

$$U_{idb} = V_{m0}\frac{G_{tb}}{K_m + G_{tb}} \tag{8.10}$$

and

$$\begin{aligned} G_{tb} &= (U_{ii} - EGP_b + k_1 G_{pb} + E_b)/k_2 \\ V_{m0} &= (EGP_b - U_{ii} - E_b) \cdot (K_m + Gtb)/G_{tb} \end{aligned} \tag{8.11}$$

Insulin subsystem

Insulin flow s, coming from the subcutaneous compartments, enters the bloodstream and is degradated in the liver and in the periphery

$$\begin{cases} \dot{I}_p(t) = -(m_2 + m_4)I_p(t) + m_1 I_l(t) + s(t) \\ \dot{I}_l(t) = -(m_1 + m_3)I_l(t) + m_2 I_p(t) \\ I(t) = I_p(t)/V_I \end{cases} \tag{8.12}$$

where V_I (l/Kg) is the distribution volume of insulin, m_1, m_2, m_3 e m_4 (1/min) are model parameters.

m_2, m_3, m_4 depend on m_1 in the following way

$$\begin{aligned} m_2 &= 0.6 \cdot \frac{CL_{ins}}{HE_b \cdot V_I \cdot BW} \\ m_3 &= m_1 \cdot \frac{HE_b}{1 - HE_b} \\ m_4 &= 0.4 \cdot \frac{CL_{ins}}{V_I \cdot BW} \end{aligned} \tag{8.13}$$

where HE_b (adimensional) is the basal hepatic insulin extraction, while CL_{ins} (l/min) is the insulin *clearance*. HE_b is considered constant and equal to 0.6.

At basal one has

$$\begin{cases} 0 = -(m_2 + m_4)I_{pb} + m_1 I_{lb} + s_b \\ 0 = -(m_1 + m_3)I_{lb} + m_2 I_{pb} \end{cases} \tag{8.14}$$

so

$$\begin{cases} I_{lb} = \dfrac{m_2}{m_1 + m_3} I_{pb} \\ s_b = (m_2 + m_4)I_{pb} - m_1 I_{lb} \end{cases} \tag{8.15}$$

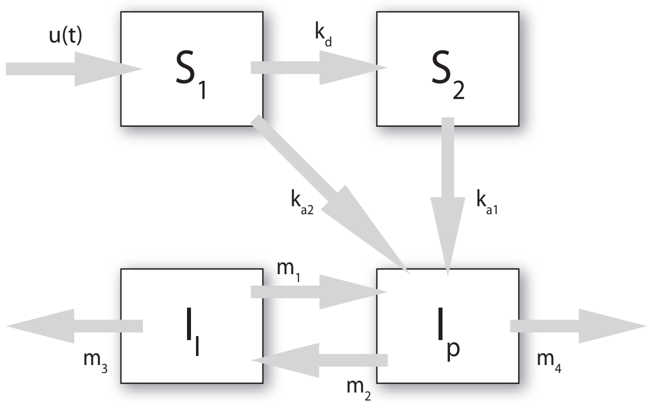


Figure 8.7: Schema of insulin subsystem

where $I_{pb} = I_b \cdot V_I$.

S.C. Insulin subsystem

Usually in diabetic patients, insulin is administered by subcutaneous injection. Insulin takes some time to reach the circulatory apparatus, unlike in the healthy subject in which the pancreas secretes directly into the portal vein. This delay is modeled here with a two compartments, S_1 and S_2 (pmol/Kg), which represent respectively polymeric and monomeric insulin in the subcutaneous tissue

$$\begin{cases} \dot{S}_1(t) = -(k_{a1} + k_d)S_1(t) + u(t) \\ \dot{S}_2(t) = k_d S_1(t) - k_{a2} S_2(t) \\ s(t) = k_{a1} S_1(t) + k_{a2} S_2(t) \end{cases} \tag{8.16}$$

where $u(t)$ (pmol/Kg/min) represents injected insulin flow, k_d is called degradation constant, k_{a1} and k_{a2} are absorption constants.

At basal one has

$$\begin{cases} 0 = -(k_{a1} + k_d)S_{1b} + u_b \\ 0 = k_d S_{1b} - k_{a2} S_{2b} \\ s_b = k_{a1} S_{1b} + k_{a2} S_{2b} \end{cases} \tag{8.17}$$

and solving the system

$$\begin{cases} S_{1b} = \dfrac{s_b}{k_d + k_{a1}} \\ S_{2b} = \dfrac{k_d}{k_{a1}} S_{1b} \\ u_b = s_b \end{cases} \tag{8.18}$$

The quantity u_b (pmol/min) represents insulin infusion to maintain diabetic patient at basal steady state.

	Stato	Description	Basal Value	Unit
x_1	Q_{sto1}	Solid glucose in stomach	0	mg
x_2	Q_{sto2}	Liquid glucose in stomach	0	mg
x_3	Q_{gut}	Glucose in intestine	0	mg
x_4	G_p	Plasma glucose	G_{pb}	mg/Kg
x_5	G_t	Glucose in tissue	G_{tb}	mg/Kg
x_6	I_1	Delayed insulin signal	I_b	$pmol/l$
x_7	I_d	Delayed insulin signal	I_b	$pmol/l$
x_8	X	Interstitial fluid insulin	0	$pmol/l$
x_9	I_l	Liver insulin	I_{lb}	$pmol/Kg$
x_{10}	I_p	Plasma insulin	I_{pb}	$pmol/Kg$
x_{11}	S_1	S.c. polymeric insulin	S_{1b}	$pmol/Kg$
x_{12}	S_2	S.c. monomeric insulin	S_{2b}	$pmol/Kg$
x_{13}	G_M	S.c. glucose	G_b	mg/dl

Table 8.1: State table for Dalla Man *et al.* model

S.C. Glucose subsystem

Subcutaneous glucose G_M (mg/dl) is, at steady state, highly correlated with plasma glucose; dynamically, instead, it follows the changes in plasma glucose with some delay. This dynamic was modeled with a system of the first order

$$\dot{G}_M(t) = -k_{sc}G_M(t) + k_{sc}G(t) \tag{8.19}$$

Model states and parameters

Final model consists of 13 states, reported in Table 8.1 with basal values.

Dalla Man *et al.* [Dalla Man *et al.* 2007b] model is uniquely defined by 25 independent parameters, and 5 constants reported in Table 8.2 and Table 8.3 respectively.

Parameter	**Unit**
BW	Kg
k_{abs}	min^{-1}
k_{max}	min^{-1}
k_{min}	min^{-1}
b	adimensional
d	adimensional
k_{a1}	min^{-1}
k_{a2}	min^{-1}
k_d	min^{-1}
m_1	min^{-1}
CL_{ins}	l/min
V_I	l/Kg
k_1	min^{-1}
k_2	min^{-1}
V_G	dl/Kg
EGP_b	$mg/Kg/min$
k_{p2}	min^{-1}
k_{p3}	min^{-1}
k_i	min^{-1}
I_b	$pmol/l$
K_m	mg/Kg
V_{mx}	min^{-1}
p_{2U}	min^{-1}
k_{sc}	min^{-1}
G_b	mg/dl

Table 8.2: Parameter table for Dalla Man *et al.* model.

Constant	**Unit**
f	adimensional
U_{ii}	$mg/Kg/min$
HE_b	adimensional
k_{e1}	min^{-1}
k_{e2}	mg/Kg

Table 8.3: Constant table for Dalla Man *et al.* model

8.5 Metrics to asses control performance

In evaluating the performance of a control algorithm one has to remember that its basic function is to mimic as best as possible the feature of the β-cells, which is to maintain glucose levels between 80 and 120 mg/dl in the face of disturbances such as meals or physical activity. A good algorithm should be able to maintain blood sugar low enough, as this reduces the long-term complications related to diabetes, but also must avoid even isolated episodes of hypoglycemia. A metric has to take into account these features: other metrics such as average blood sugar or measures related to it as HbA1c are not very significant.

Blood Glucose Index (BGI)

Blood Glucose Index is a metric proposed by Kovatchev *et al.* [Kovatchev *et al.* 2005], to evaluate the clinical risk related to a particular glycemic value

$$BGI(\cdot) = 10(g[\ln^a(\cdot) - b])^2 \tag{8.20}$$

where a, b and g are fixed constant.

In particular, these parameters are equal to

$$a = 1.44 \qquad b = 10.07 \qquad g = 0.75$$

The target G_0, i.e. the value whose risk is zero, is

$$G_0 = e^{\sqrt[a]{b}} = 148.31 \text{ mg/dl}$$

The BGI function is asymmetric, because hypoglycemia is considered

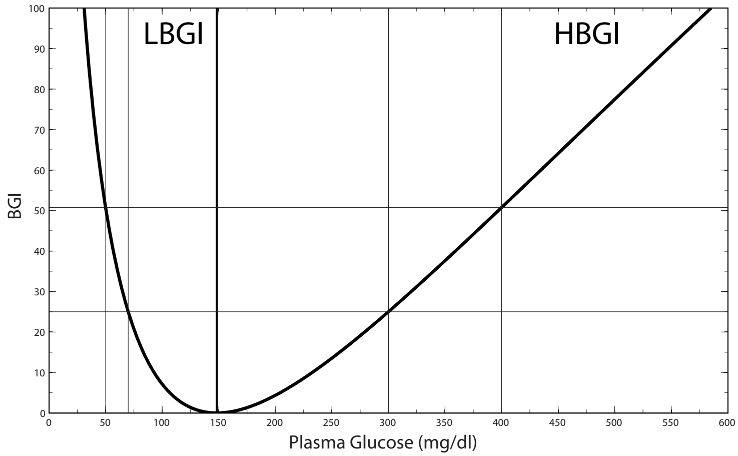


Figure 8.8: Blood Glucose Index

more dangerous than hyperglycemia (see Figure 8.8).

Starting from the BGI, two synthetic indices for sequences of n measurements of blood glucose can be defined

LBGI (Low BGI) measures hypoglycemic risk

$$LBGI = \frac{10}{n}\sum_{i=1}^{n} rl^2(G_i)$$

where $rl(\cdot) = \min(0, g[\ln^a(\cdot) - \ln^a(G_0)])$

HBGI (High BGI) measures hyperglycemic risk

$$HBGI = \frac{10}{n}\sum_{i=1}^{n} rh^2(G_i)$$

where $rh(\cdot) = \max(0, g[\ln^a(\cdot) - \ln^a(G_0)])$.

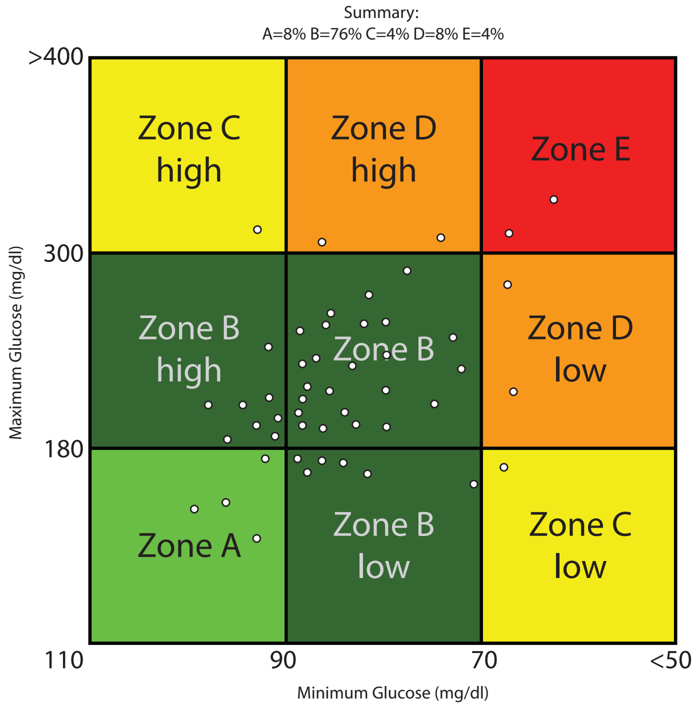


Figure 8.9: Control Variability Grid Analysis with summary

Control Variability Grid Analysis (CVGA)

The Control Variability Grid Analysis (CVGA) is a graphical representation of min/max glucose values in a population of patients either real or virtual. The CVGA provides a simultaneous assessment of the quality of glycemic regulation in all patients. As such it may play an important role in the tuning of closed-loop glucose control algorithms and also in the comparison of their performances. Assuming that for each subject a time series of measured Blood Glucose (BG) values over a specified time period (e.g. 1 day) is available, the CVGA is obtained as follows. For each subject a point is plotted whose X-coordinate is the minimum BG and whose Y-coordinate is the maximum BG within the considered time period (see Figure 8.9). Note that the X axis is reversed as it goes form 110 (left) to 50 (right) so that optimal regulation is located in lower left corner. The appearance of the overall plot is a cloud of points located in various regions of the plane. Different regions on the plane can be associated with different qualities of glycemic regulation. In order to classify subjects into categories, 9 rectangular zones are defined as follows

Zone A $(G_{max} < 180,\ G_{min} > 90)$: Accurate control.

Zone B high $(180 < G_{max} < 300,\ G_{min} > 90)$: Benign trend towards hyperglycemia.

Zone B low $(G_{max} < 180,\ 70 < G_{min} < 90)$: Benign trend towards hypoglycemia.

Zone B $(180 < G_{max} < 300,\ 70 < G_{min} < 90)$: Benign control.

Zone C high $(G_{max} > 300,\ G_{min} > 90)$: overCorrection of hypoglycemia.

Zone C low $(G_{max} < 180,\ G_{min} < 70)$: overCorrection of hyperglycemia

Zone D high $(G_{max} > 300,\ 70 < G_{min} < 90)$: failure to Deal with hyperglycemia.

Zone D low $(180 < G_{max} < 300,\ G_{min} < 70)$: failure to Deal with hypoglycemia.

Zone E $(G_{max} > 300,\ G_{min} < 70)$: Erroneous control.

A synthetic numeric assessment of the global level of glucose regulation in the population is given by the Summary of the CVGA.

8.6 Regulator design

8.6.1 Unconstrained LMPC

The LMPC regulator design needs to identify a linear-low order model which approximates the behavior of the nonlinear system. It will be used as the prediction model of the LMPC controller. In the unconstrained case, LMPC gives an explicit control law.

MPC model for prediction

Considering the difficulty of identifying an individualized model for each patient, we decided to use for prediction the mean Dalla Man *et al.* model, which is nonlinear and time-variant. Hence, some steps are required in order to obtain a LTI[4] model as required by the LMPC formulation.

Linearization

The mean Dalla Man *et al.* model with $k_{empt}(t, Q_{sto}) = k_{mean} = \frac{k_{min}+k_{max}}{2}$, fixed in in order to obtain a time-invariant model, can be synthetically described by

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), d(t)) \\ y(t) = Cx(t) \end{cases} \tag{8.21}$$

where $x(t) \in \mathbb{R}^{13}$ is the state vector of Dalla Man *et al.* model, $u(t) \in \mathbb{R}$ is the insulin flow injected for unit of weight (pmol/Kg/min), $d(t) \in \mathbb{R}$ is the CHO flow assumed orally (mg/min), the output $y(t) \in \mathbb{R}$ is the measurable variable, i.e. subcutaneous glucose $y = x_{13} = G_M(t)$ (mg/dl).

Linearizing the system around the basal steady-state point, the following

[4] Linear Time-Invariant

13th-order linear system, with two inputs and one output, is obtained

$$\begin{cases} \delta\dot{x}(t) = A\delta x(t) + B\delta u(t) + Md(t) \\ \delta y(t) = C\delta x(t) \end{cases} \tag{8.22}$$

where

$$\begin{cases} \delta x = x - x_b \\ \delta y = y - y_b \\ \delta u = u - u_b \end{cases}$$

where $y_b = G_b$ and A, B and M matrices are obtained as:

$$A = \frac{\delta f}{\delta x}\Big(x, u, d\Big)\Big|_{\text{basal}}, \quad B = \frac{\delta f}{\delta u}\Big(x, u, d\Big)\Big|_{\text{basal}}, \quad M = \frac{\delta f}{\delta d}\Big(x, u, d\Big)\Big|_{\text{basal}}$$

$$A \in \mathbb{R}^{13\times 13}, \quad B \in \mathbb{R}^{13\times 1}, \quad M \in \mathbb{R}^{13\times 1}$$

Discretization

After defining an appropriate sample time T_S, the system has been discretized

$$\begin{cases} \delta x(k+1) = A_D\delta x(k) + B_D\delta u(k) + M_D d(k) \\ \delta y(k) = C_D\delta x(k) \end{cases} \tag{8.23}$$

In order to calculate A_D, B_D, M_D and C_D matrices, consider the discrete-time system in Figure 8.10. It consists of a zero-order hold (ZOH), the continuous-time system and the sampler connected in series. The ZOH holds, for the whole time period of length T_S, a constant value equal to the last value of the input sequence:

$$\delta u(t) = \delta u^*(k), \qquad kT_S \le t < (k+1)T_S \tag{8.24}$$

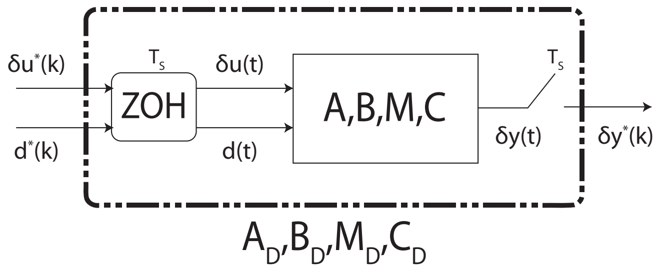


Figure 8.10: ZOH method for system discretization

$$d(t) = d^*(k), \qquad kT_S \leq t < (k+1)T_S \tag{8.25}$$

while the sampler gives as output $y^*(k)$ as:

$$y^*(k) = y(kT_S) \tag{8.26}$$

Considering continuous-time Lagrange equation:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + \int_0^t e^{A(t-\tau)}Md(\tau)d\tau, \quad t > 0 \tag{8.27}$$

and equations (8.24), (8.25) and (8.26) one has:

$$A_D = e^{AT_S}, \quad B_D = \int_0^{T_S} e^{A\eta}Bd\eta, \quad M_D = \int_0^{T_S} e^{A\eta}Md\eta, \quad C_D = C \tag{8.28}$$

Sample time T_S was fixed equal to 15 minutes.

Order reduction, balancing and Input-Output realization

In order to reduce the complexity of the system, a model order reduction and a balancing have been done, so obtaining a third order model described by the following transfer function

$$\begin{aligned}\Delta Y(z) &= C_t(zI - A_t)^{-1}B_t\Delta U(z) + C_t(zI - A_t)^{-1}M_tD(z) = \\ &= \frac{Num_B(z)}{Den(z)}\Delta U(z) + \frac{Num_M(z)}{Den(z)}D(z)\end{aligned} \tag{8.29}$$

with

$$\begin{aligned}Num_B(z) &= b_2z^2 + b_1z + b_0 \\ Num_M(z) &= m_2z^2 + m_1z + m_0 \\ Den(z) &= z^3 + a_2z^2 + a_1z + a_0\end{aligned} \tag{8.30}$$

Since, due to the balancing, the two transfer functions have the same denominator, the following input-output representation is obtained

$$\begin{aligned}\delta y(k+1) = &-a_2\delta y(k) - a_1\delta y(k-1) - a_0\delta y(k-2)+ \\ &+ b_2\delta u(k) + b_1\delta u(k-1) + b_0\delta u(k-2)+ \\ &+ m_2d(k) + m_1d(k-1) + m_0d(k-2)\end{aligned} \tag{8.31}$$

Note that, by using an input-output representation of the system, the necessity of an observer is avoided.

Finally, system (8.29) can be given in the following state-space (non-minimal) representation

$$\begin{cases}\xi(k+1) = A_{IO}\xi(k) + B_{IO}\delta u(k) + M_{IO}d(k) \\ \delta y(k) = C_{IO}\xi(k)\end{cases} \tag{8.32}$$

where

$$\xi(k) = \begin{bmatrix} \delta y(k) \\ \delta y(k-1) \\ \delta y(k-2) \\ \delta u(k-1) \\ \delta u(k-2) \\ d(k-1) \\ d(k-2) \end{bmatrix}, \quad B_{IO}(k) = \begin{bmatrix} b_2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad M_{IO}(k) = \begin{bmatrix} m_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{8.33}$$

$$A_{IO} = \begin{bmatrix} -a_2 & -a_1 & -a_0 & b_1 & b_0 & m_1 & m_0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{8.34}$$

$$C_{IO} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{8.35}$$

Control law computation

In order to derive the LMPC control law the following quadratic discrete-time cost function is considered

$$\begin{aligned} J(x(k), u(\cdot), k) \quad &= \quad \sum_{i=0}^{N-1} \left[q\Big(\delta y^0(k+i) - \delta y(k+i)\Big)^2 + r\Big(u(k+i)\Big)^2 \right] \\ &\quad + s\Big(\delta y^0(k+N) - \delta y(k+N)\Big)^2 \end{aligned} \tag{8.36}$$

where N is the optimization horizon, $y^o(k)$ the desired output at time k, while q, r and s are positive scalars.

The evolution of the system can be re-written in a compact way as follows

$$\mathcal{Y}(k) = \mathcal{A}_c x(k) + \mathcal{B}_c \mathcal{U}(k) + \mathcal{M}_c \mathcal{D}(k) \tag{8.37}$$

where, considering (8.32)

$$\mathcal{Y}(k) = \begin{bmatrix} \delta y(k+1) \\ \vdots \\ \delta y(k+N) \end{bmatrix}, \ \mathcal{U}(k) = \begin{bmatrix} \delta u(k) \\ \vdots \\ \delta u(k+N-1) \end{bmatrix}, \ \mathcal{D}(k) = \begin{bmatrix} d(k) \\ \vdots \\ d(k+N-1) \end{bmatrix}$$

$$\mathcal{A}_c = \begin{bmatrix} C_{IO}A_{IO} \\ C_{IO}A_{IO}^2 \\ \vdots \\ C_{IO}A_{IO}^{N-1} \\ C_{IO}A_{IO}^{N} \end{bmatrix}$$

$$\mathcal{B}_c = \begin{bmatrix} C_{IO}B_{IO} & 0 & \cdots & 0 & 0 \\ C_{IO}A_{IO}B_{IO} & C_{IO}B_{IO} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{IO}A_{IO}^{N-2}B_{IO} & C_{IO}A_{IO}^{N-3}B_{IO} & \cdots & C_{IO}B_{IO} & 0 \\ C_{IO}A_{IO}^{N-1}B_{IO} & C_{IO}A_{IO}^{N-2}B_{IO} & \cdots & C_{IO}A_{IO}B_{IO} & C_{IO}B_{IO} \end{bmatrix}$$

$$\mathcal{M}_c = \begin{bmatrix} C_{IO}M_{IO} & 0 & \cdots & 0 & 0 \\ C_{IO}A_{IO}M_{IO} & C_{IO}M_{IO} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{IO}A_{IO}^{N-2}M_{IO} & C_{IO}A_{IO}^{N-3}M_{IO} & \cdots & C_{IO}M_{IO} & 0 \\ C_{IO}A_{IO}^{N-1}M_{IO} & C_{IO}A_{IO}^{N-2}M_{IO} & \cdots & C_{IO}A_{IO}M_{IO} & C_{IO}M_{IO} \end{bmatrix} \tag{8.38}$$

Matrices $\mathcal{A}_c, \mathcal{B}_c$ and $\mathcal{M}_c$ are derived using the discrete time Lagrange formula

$$\xi(k+i) = A_{IO}^{i}\xi(k) + \sum_{j=0}^{i-1} A_{IO}^{i-j-1}\Big(B_{IO}\delta u(k+j) + M_{IO}d(k+j)\Big), \quad i > 0 \tag{8.39}$$

In this way, the cost function becomes

$$\begin{aligned} J(x(k), \mathcal{U}(k), k) &= \\ &= (\delta y^0(k) - \delta y(k))^\top q(\delta y^0(k) - \delta y(k)) + \\ &+ (\mathcal{Y}^0(k) - \mathcal{Y}(k))^\top \mathcal{Q}(\mathcal{Y}^0(k) - \mathcal{Y}(k)) + \\ &+ \mathcal{U}^\top(k)\mathcal{R}\mathcal{U}(k) \end{aligned} \tag{8.40}$$

In the unconstrained case, the solution of the optimization problem is

$$\begin{aligned} \mathcal{U}^0(k) &= \Big(\mathcal{B}_c^\top \mathcal{Q}\mathcal{B}_c + \mathcal{R}\Big)^{-1} \mathcal{B}_c^\top \mathcal{Q}\Big(\mathcal{Y}^0(k) - \mathcal{A}_c\xi(k) - \mathcal{M}_c\mathcal{D}(k)\Big) = \\ &= K_0\mathcal{Y}^0(k) - K_x\xi(k) - K_D\mathcal{D}(k) \end{aligned} \tag{8.41}$$

where

$$\begin{cases} K_0 = \Big(\mathcal{B}_c^\top \mathcal{Q}\mathcal{B}_c + \mathcal{R}\Big)^{-1} \mathcal{B}_c' \mathcal{Q} \\ K_x = \Big(\mathcal{B}_c^\top \mathcal{Q}\mathcal{B}_c + \mathcal{R}\Big)^{-1} \mathcal{B}_c^\top \mathcal{Q}\mathcal{A}_c \\ K_D = \Big(\mathcal{B}_c^\top \mathcal{Q}\mathcal{B}_c + \mathcal{R}\Big)^{-1} \mathcal{B}_c^\top \mathcal{Q}\mathcal{M}_c \end{cases} \tag{8.42}$$

Finally, following the Receding Horizon approach the control law is given by

$$\delta u^0(k) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \mathcal{U}^0(k)$$

In order to tune just one parameter, in (8.36), r has been fixed at 1 and $s = q$. Moreover, matrices $\mathcal{Q}$ and $\mathcal{R}$ are defined as follows

$$\mathcal{Q} = qI_1, \quad \mathcal{R} = I_2$$

where I_1 and I_2 are identity matrices of appropriate dimensions. In order to complete the regulator definition, it is necessary to define the optimization horizon, the scalar q and how to generate the vectors $\delta\mathcal{Y}^0(k)$, $\xi(k)$ e $\mathcal{D}(k)$ that appear in (8.41).

Optimization horizon

The optimization horizon has been fixed for each virtual patient at 240 minutes (4 hours). Since sample time is 15 min, $N = 240/T_S = 16$ sampling times.

Generation of state vector $\xi(k)$

In order to generate vector $\xi(k)$ it is useful to define the following dynamic system, whose aim is to store the previously measured values of inputs and outputs

$$\begin{cases} \psi(k+1) = A_{DIO}\psi(k) + B_{DIO}\begin{bmatrix} \delta y_m(k) & \delta u_m(k) & d_m(k) \end{bmatrix}^\top \\ \xi(k) = C_{DIO}\psi(k) + D_{DIO}\begin{bmatrix} \delta y_m(k) & \delta u_m(k) & d_m(k) \end{bmatrix}^\top \end{cases} \tag{8.43}$$

where

$$\psi(k) = \begin{bmatrix} \delta y(k-1) \\ \delta y(k-2) \\ \delta y(k-3) \\ \delta u(k-1) \\ \delta u(k-2) \\ d(k-1) \\ d(k-2) \end{bmatrix}, \; B_{DIO}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; D_{DIO}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.44}$$

$$A_{DIO} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad C_{DIO} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{8.45}$$

Note that the inputs of such a system are *measured* variables, that are used as inputs of the regulator, that means the measured values of δy, δu and d.

The warm-up procedure of the regulator consists in store the first three measures of this variables in order to initialize the state vector ψ.

Generation of vector $\delta\mathcal{Y}^0(k)$

The glycemia reference vector y^0, has been considered constant at 112 mg/dl. Hence

$$\delta\mathcal{Y}^0(k) = (y^0 - y_b) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 112 - y_b \\ \vdots \\ 112 - y_b \end{bmatrix} \tag{8.46}$$

Generation of vector $\mathcal{D}(k)$

Vector $\mathcal{D}(k)$ represents the meals prediction along the optimization horizon. This means that the i-th element of vector $\mathcal{D}(k)$ is equals to the amount of carbohydrates (mg/min) that is supposed to be absorbed at time instant $k+i-1$. This information, that can be called *meal announcement*, is very useful since permits to react in advance with respect to distrubances on the controlled variable. This is one of the key advantage of model predictive control with respect to other kind of feedback regulators such as the PID.

Even if it is unrealistic to know in advance the real amount and time of future meals, we can suppose to know a *nominal diet* of the patient, that can be used in order to make the vector of predictable disturbances along the optimization horizon

$$\mathcal{D}(k) = \begin{bmatrix} diet(k) \\ \vdots \\ diet(k+N-1) \end{bmatrix} \tag{8.47}$$

8.6.2 NMPC

Unconstrained LMPC with a quadratic cost function provides an explicit control law. The simplicity of this solution is of great advantage in order to implement the controller on real patients.

In the following a nonlinear model predictive control is proposed. The goal is to evaluate the (upper) performance limit of a predictive control algorithm in the blood glucose control problem. In order to do this, we consider the best unrealistic situation:

1. the nonlinear model used in the synthesis of the NMPC describes exactly the system

2. all the states of the systems are measurables. A state feedback NMPC is proposed (we don't use an observer).

The model used in the synthesis of the NMPC is the full nonlinear continuous time model previously described in Section 8.4.1.

Nonlinear predictive control allows the use of nonlinear cost functions. We have decided to adopt a cost function related to the LBGI and HBGI indexes previously described in Section 8.5. In order to minimize the control energy used along the optimization horizon, the cost function presents also a term depending from the control action.

The Finite Horizon Optimal Control Problem consists in minimizing, with respect to control sequence $u_1(t_k), \ldots, u_N(t_k)$ ($t_k = kT_S$ with T_S sampling time period), the following nonlinear cost function with horizon N

$$
\begin{aligned}
& J(x_{t_k}, u_1(t_k), \cdots, u_N(t_k)) \qquad (8.48) \\
&= \int_{t_k}^{t_{k+N}} \left\{ 10q(g((ln(G_p(\tau)))^a - b))^2 + ru(\tau)^2 \right\} d\tau
\end{aligned}
$$

Note that a full hybrid solution is developed, i.e. the control inputs $u_i(t_k)$ are constrained to be piecewise constants, $u(\tau) = u_i(t_k)$, $\tau \in [t_{k+i-1}, t_{k+i})$, $i \in [1, \ldots, N]$, see [Magni & Scattolini 2004].

The values of g, a and b adopted in this case are

$$g = 0.75, \qquad a = 1.44, \qquad b = 9.02$$

These values have been changed with respect to the original ones of the risk indexes in order to consider the fact that the cost function is a compromise between state and control requirements.

As for the LMPC, the optimization horizon has been fixed at $N = 16$ sampling times. In order to complete the regulator definition, it is necessary to define the scalars q and r. In order to tune just one of them, q has been fixed at 1.

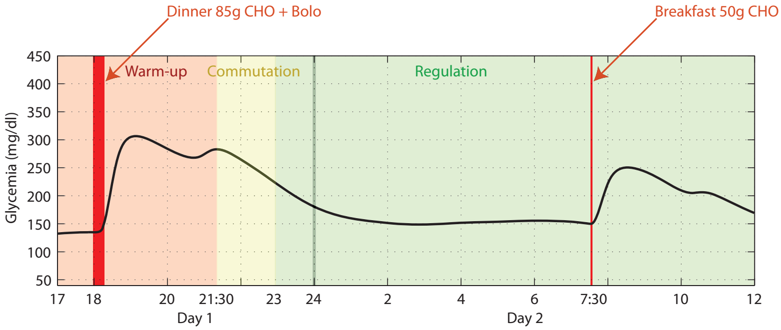


Figure 8.11: Nominal scenario for adult patient

8.7 Clinical Scenario

An in silico trial consisting of 100 patients is used to assess the performances of LMPC and NMPC controllers. The performance are tested on a clinical scenario. It is a specific of all the parameters which characterize a given experiment, for example the simulation duration, the meal profile, open-loop insulin therapy, switching time to closed loop and start time of regulation phase.

8.7.0.1 Nominal scenario

Nominal scenario consists of the following phases (see figure 8.11):

1. Patient enters at 17.00 of Day 1. Simultaneously data gathering for warm-up phase starts. Microinfusor is programmed to inject basal insulin for the specific patient.

2. At 18.00 of Day 1 patient has a meal of about 15 minutes of duration. The meal contains 85g of carbohydrates. Simultaneously patient receives an insulin bolus δ according to his personal in-

sulin/carbohydrate ratio (*CHO ratio* o CR), that is

$$\delta = 85 \cdot CR$$

3. Control loop is closed at 21.30 of Day 1. At this time the *switching* phase starts. In this phase the metrics for evaluation are not calculated.
4. At 23.00 of Day 1 *regulation* phase starts.
5. At 7.30 of Day 2, patient has breakfast containing 50g of CHO. Meal duration is considered about 2 minutes.
6. Experiment finishes at 12.00 of Day 2. Patient is discharged and restarts his normal insulinic therapy.

8.8 Control design procedure

LMPC and NMPC have respectively one parameter that must be tuned for each patient in order to achieve good performances in term of glycemia regulation. Figure 8.12 explains the control design procedure. Given a model, linear in the LMPC case, nonlinear in the NMPC, the controllers are synthesized by fixing all the parameters of their designs. Then, once the control laws are determined, these are applied to a subject realistic model of each patient. The test consists in the nominal scenario previously described. The CVGA, described in Section 8.5, is a very useful tool in order to evaluate the outcome of the test. However, it does not provide a numerical values that permits to quantify the quality of the performance. Basing on the CVGA, a good performance index is the norm-infinite of the distance, in CVGA coordinates, from the lower left corner of the grid. Note

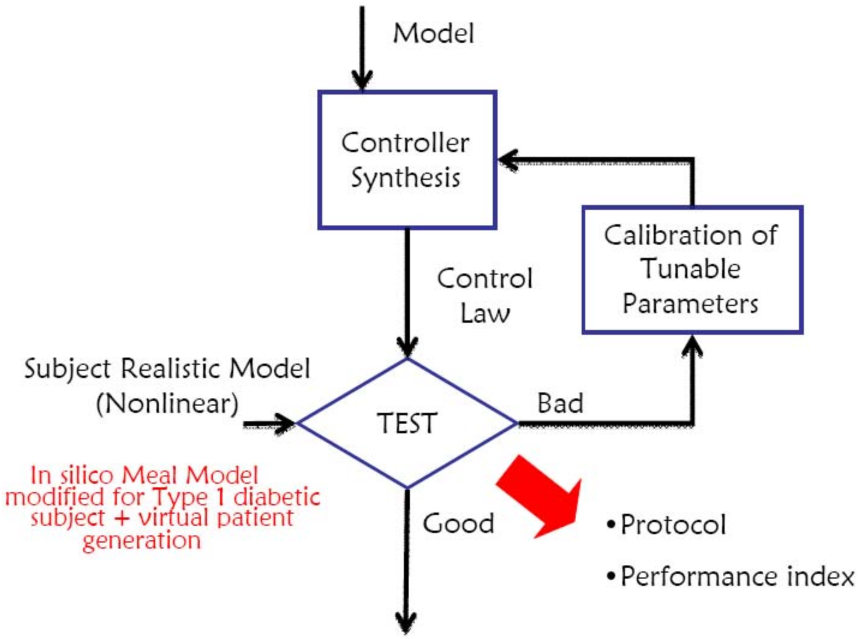


Figure 8.12: Control design procedure

that the lines of level of this performance index are squared (see Figure 8.13). Whenever the results obtained are not satisfactory, tunable parameters of the regulators (that means q or r) are changed. In particular, a calibration procedure has been developed in order to obtain the best value of parameter q (or r), for each patient (see [Tessera 2007] for details). Figure 8.14 show an example of calibration curve for a patient in the LMPC case. An increase in the value of parameter q causes a decrease in both maximum and minimum blood glucose (the reverse happens in NMPC case by changing parameter r). The arrow points the optimal value of the index at which an optimal q is associated.

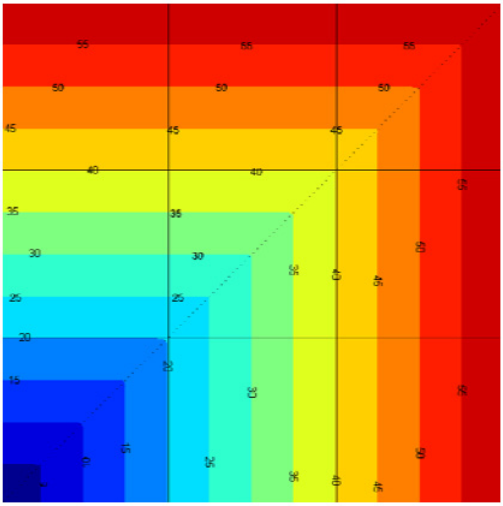


Figure 8.13: Performance index: norm-infinite of the distance from the lower left corner of the CVGA

8.9 Results

In this section the results obtained by applying LMPC and NMPC to the 100 virtual patient population are shown. In particular, the CVGA of Figure 8.15 show the comparison between the NMPC with r calibrated for each patient and the NMPC with an unique r for all the population. It is worth to note that clearly the first solution gives better performances even if also the second one provide good results in terms of patients contained in the good zones (A and B).

Figure 8.16 show the comparison between NMPC and LMPC. The parameters q and r have been individually tuned. It is worth to note that LMPC uses a mean model of the population for the synthesis of the regulator of each patient. Differently, NMPC is in the best situation: for each patient, the model used in the synthesis is exactly the nonlinear model used in the test. The figure show that, in terms of CVGA, the unrealistic NMPC

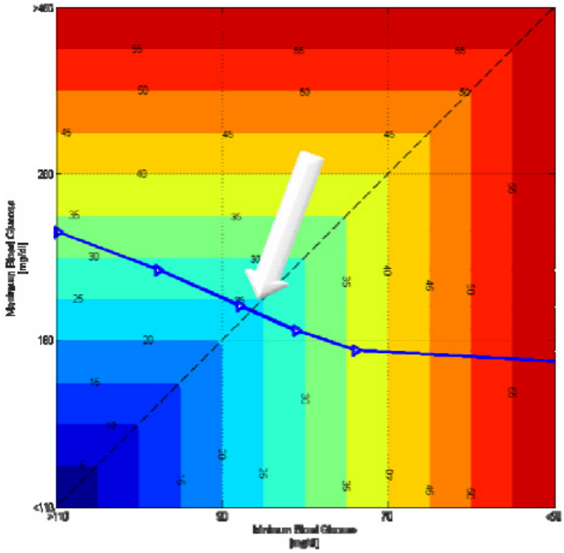


Figure 8.14: Example of calibration curve in LMPC case

improves the performances, but this improvement is not very significant. The real advantage of the NMPC is shown in Figure 8.17 and regards the insulin profile. Comparing with the LMPC one it is smoother in spite of a more or less equivalent glycaemia regulation.

8.10 Conclusions

In this chapter, the feedback control of glucose concentration in Type-1 diabetic patients using s.c. insulin delivery and s.c. continuous glucose monitoring is considered. In particular, a linear model predictive control based and a nonlinear state feedback model predictive control (MPC) synthesized on a recently developed nonlinear in-silico model of glucose metabolism ([Magni *et al.* 2007b]) have been proposed. In order to assess the performances of the proposed algorithm against interindividual variability, an in silico trial with 100 virtual patients was carried out. The simulation ex-

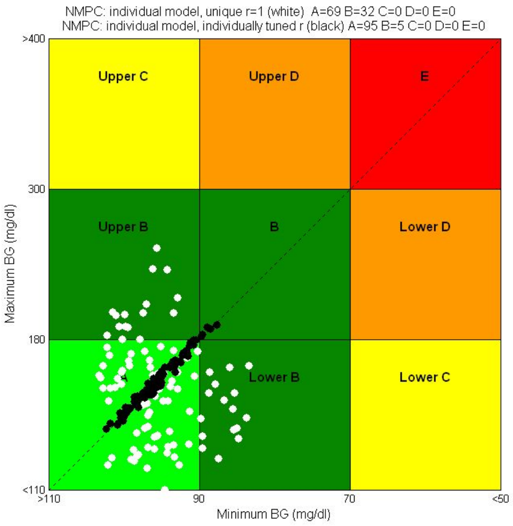


Figure 8.15: Comparison between NMPC with r individually tuned and r unique for all the patients

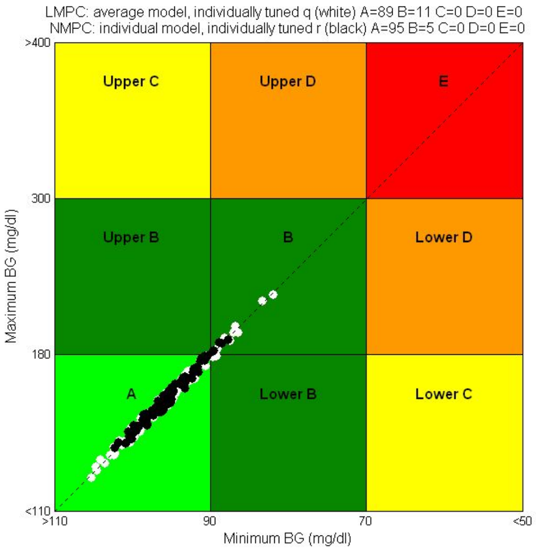


Figure 8.16: Comparison between LMPC with q individually tuned and NMPC with r individually tuned for all the patients

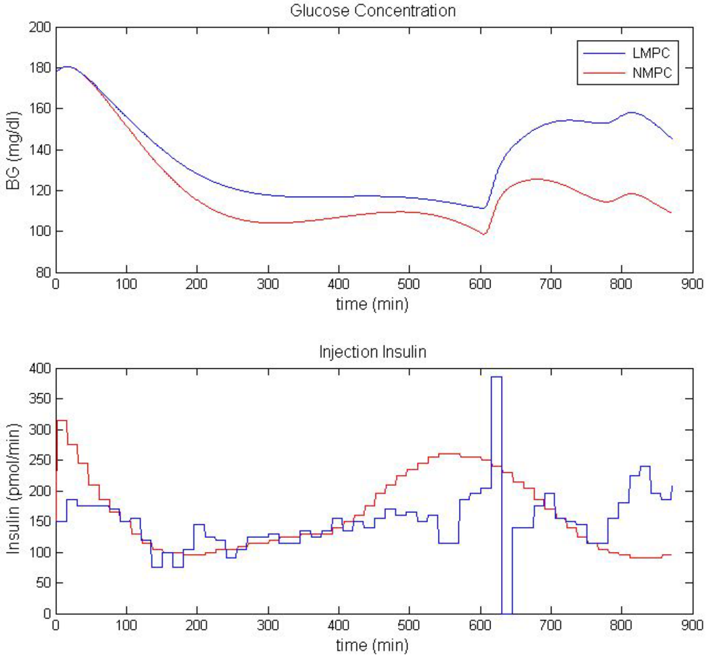


Figure 8.17: Comparison between LMPC and NMPC: glucose and insulin profiles of a particular patient

periments highlight the increased effectiveness of the meal announcement signal with respect to the linear MPC due to a more accurate nonlinear model. Moreover, one of the main advantages of a nonlinear approach is the possibility to use a nonlinear cost function based on the risk index defined in [Kovatchev *et al.* 2005]. The obtained results encourage a deeper investigation along this direction. In particular, the following assumptions should be removed: perfect knowledge of the model parameters and state availability. An on-line optimization problem could also be a limitation for a real application. It is worth to note that in 2006, Juvenile Diabetes Research Foundation has launched a project to accelerate the development and adoption of a device for the closed loop control of blood glucose. University of Pavia collaborates in this project joined with the University of Virginia and the University of Padova. Currently, experiments using the developed linear model predictive controller are in progress at the Charlottesville and Padova hospitals.

CHAPTER 9

Appendix of the Thesis

This section provides the main notations and definitions used in the thesis. Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$ and $\mathbb{Z}_{\geq 0}$ denote the real, the non-negative real, the integer and the non-negative integer sets of numbers, respectively.

The Euclidean norm is denoted as $|\cdot|$ while $|\cdot|_\infty$ denotes the infinity-norm.

Given a matrix $A \in \mathbb{R}^{n \times n}$, let denote with $\bar{\sigma}(A)$ the maximum singular value of A.

Given a signal w, let $\mathbf{w}_{[t_1,t_2]}$ be a signal sequence defined from time t_1 to time t_2. In order to simplify the notation, when it is inferrable from the context, the subscript of the sequence is omitted. The set of sequences w, whose values belong to a compact set $\mathcal{W} \subseteq \mathbb{R}^m$ is denoted by $\mathcal{M}_\mathcal{W}$, while $\mathcal{W}^{sup} \triangleq \sup_{w \in \mathcal{W}}\{|w|\}$, $\mathcal{W}^{inf} \triangleq \inf_{w \in \mathcal{W}}\{|w|\}$. Moreover $\|\mathbf{w}\| \triangleq \sup_{k \geq 0}\{|w_k|\}$ and $\|\mathbf{w}_{[\tau_1,\tau_2]}\| \triangleq \sup_{\tau_1 \leq k \leq \tau_2}\{|w_k|\}$ where w_k denotes the values that the sequence w takes in correspondence to the index k.

The symbol id represents the identity function from $\mathbb{R}$ to $\mathbb{R}$, while $\gamma_1 \circ \gamma_2$ is the composition of two functions γ_1 and γ_2 from $\mathbb{R}$ to $\mathbb{R}$.

Given a set $\mathcal{A} \subseteq \mathbb{R}^n$, $|\zeta|_\mathcal{A} \triangleq \inf\{|\eta - \zeta|, \eta \in \mathcal{A}\}$ is the point-to-set distance from $\zeta \in \mathbb{R}^n$ to $\mathcal{A}$ while, given two sets $\mathcal{A} \in \mathbb{R}^n$, $\mathcal{B} \in \mathbb{R}^n$, $dist(\mathcal{A}, \mathcal{B}) \triangleq \inf\{|\zeta|_\mathcal{A}, \zeta \in \mathcal{B}\}$.

The difference between two given sets $\mathcal{A} \subseteq \mathbb{R}^n$ and $\mathcal{B} \subseteq \mathbb{R}^n$ with $\mathcal{B} \subseteq \mathcal{A}$, is denoted by $\mathcal{A} \setminus \mathcal{B} \triangleq \{x : x \in \mathcal{A}, x \notin \mathcal{B}\}$. Given a closed set $\mathcal{A} \subseteq \mathbb{R}^n$, $\partial\mathcal{A}$ denotes the boundary of $\mathcal{A}$ while $\text{int}(\mathcal{A})$ denotes the interior of $\mathcal{A}$.

Given two sets $\mathcal{A}\in\mathbb{R}^n$, $\mathcal{B}\in\mathbb{R}^n$, then the Pontryagin difference set $\mathcal{C}$ is defined as $\mathcal{C}=\mathcal{A} \backsim \mathcal{B}\triangleq\{x\in\mathbb{R}^n : x+\xi\in\mathcal{A}, \forall\xi\in\mathcal{B}\}$, while the Minkowski sum set is defined as $\mathcal{S}=\mathcal{A}\oplus\mathcal{B}\triangleq\{x\in\mathbb{R}^n : \exists\xi\in\mathcal{A}, \eta\in\mathcal{B}, x=\xi+\eta\}$.

Given a vector $\eta \in \mathbb{R}^n$ and a positive scalar $\rho \in \mathbb{R}_{>0}$, the closed ball centered in η and of radius ρ , is denoted as $\mathcal{B}(\eta, \rho)\triangleq\{\xi\in\mathbb{R}^n : |\xi-\eta|\leq\rho\}$. The shorthand $\mathcal{B}(\rho)$ is used when the ball is centered in the origin.

For $x, y \in \mathbb{R}^n$, $x \geq y \iff x_i \geq y_i,\ i = 1, \cdots, n$ and $x \ngeq y$ means the negation of $x \geq y$.

Definition 9.1 ($\mathcal{K}$-function) *A function $\gamma : \mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ (or a "$\mathcal{K}$-function") if it is continuous, positive definite and strictly increasing.*

Definition 9.2 ($\mathcal{K}_\infty$-function) *A function $\gamma : \mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ if it is a $\mathcal{K}$-function and $\gamma(s) \to +\infty$ as $s \to +\infty$.*

Definition 9.3 ($\mathcal{KL}$-function) *A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if, for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$, for each fixed $s \geq 0$, $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$.*

Definition 9.4 (Upper limit) *Given a bounded function $s : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$, the upper limit is defined as*

$$\overline{\lim}_{t\to\infty} s(t) \triangleq \inf_{t\geq 0} \sup_{\tau\geq t} s(\tau). \qquad \square$$

Consider a discrete-time autonomous nonlinear model

$$x_{k+1} = F(x_k, d_k, w_k),\ k \geq 0 \tag{9.1}$$

where $x_k \subseteq R^n$ is the system state, $d_k \in \mathcal{D}(x) \subseteq \mathbb{R}^p$ models a class of uncertainty which depends on the state (for each x the set $\mathcal{D}(x)$ is closed), while $w_k \in \mathcal{W} \subseteq \mathbb{R}^q$ models a class of disturbance. The transient of the system (9.1) with initial state $x_0 = \bar{x}$ and disturbance sequences $\mathbf{d}$ and $\mathbf{w}$ is denoted by $x(k, \bar{x}, \mathbf{d}, \mathbf{w})$.

Definition 9.5 (RPI set) *A set $\Xi \subseteq \mathbb{R}^n$ is a Robust Positively Invariant (RPI) set for the system (9.1), if $F(x, d, w) \in \Xi$ for all $x \in \Xi$, all $d \in \mathcal{D}(x)$, and all $w \in \mathcal{W}$.*

Definition 9.6 (0-AS in Ξ) *Given a compact set $\Xi \subset \mathbb{R}^n$ including the origin as an interior point, the system (9.1) with $\mathbf{d} = 0$ and $\mathbf{w} = 0$ is said to be 0-AS in Ξ, if Ξ is positively invariant and if there exists a $\mathcal{KL}$-function β such that*

$$|x(k, \bar{x}, 0, 0)| \leq \beta(|\bar{x}|, k), \ \forall k \geq 0, \ \forall \bar{x} \in \Xi. \tag{9.2}$$

Definition 9.7 (AG in Ξ) *: Given a compact set $\Xi \subset \mathbb{R}^n$ including the origin as an interior point, the system (9.1) satisfies the Asymptotic Gain (AG) property in Ξ, if Ξ is a RPI set for (9.1) and if there exists a $\mathcal{K}_\infty$-function γ_{AG} such that for all initial values $\bar{x} \in \Xi$, all $\mathbf{d} \in \mathcal{D}$ and all $\mathbf{w} \in \mathcal{W}$, one has*

$$\overline{\lim}_{k \to \infty} |x(k, \bar{x}, \mathbf{d}, \mathbf{w})| \leq \gamma_{AG}(||\mathbf{w}||).$$

Definition 9.8 (ISS in Ξ with respect to $\mathcal{A}$) *Given a compact set $\Xi \subset \mathbb{R}^n$ including the origin as an interior point, the system (9.1) with $\mathbf{d} \in \mathcal{M}_{\mathcal{D}}$ and $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$, is said to be ISS in Ξ with respect to set $\mathcal{A}$ (compact-ISS), if Ξ is a RPI set for (9.1) and if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ_2 such that*

$$|x(k, \bar{x}, \mathbf{d}, \mathbf{w})|_{\mathcal{A}} \quad \leq \quad \beta(|\bar{x}|_{\mathcal{A}}, k) + \gamma_2(||\mathbf{w}||)$$

for all $\bar{x} \in \Xi$ *and* $k \geq 0$.

Definition 9.9 (LpS) *System (9.1) with* $\mathbf{d} \in \mathcal{M}_{\mathcal{D}}$ *and* $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$ *satisfies the LpS (Local practical Stability) property if there exists a constant* $c \geq 0$ *such that for each* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that*

$$|x(k, \bar{x}, \mathbf{d}, \mathbf{w})| \leq c + \varepsilon$$

for all $k \geq 0$, *all* $|\bar{x}| \leq \delta$ *and all* $|w_k| \leq \delta$.

Definition 9.10 (LS with respect to $\mathcal{A}$) *System (9.1) with* $\mathbf{d}_1 \in \mathcal{M}_{\mathcal{D}}$ *and* $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$ *satisfies the LS (Local Stability) property with respect to set* $\mathcal{A}$, *if for each* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that*

$$|x(k, \bar{x}, \mathbf{d}, \mathbf{w})|_{\mathcal{A}} \leq \varepsilon$$

for all $k \geq 0$ *for all* $|\bar{x}|_{\mathcal{A}} \leq \delta$ *and all* $|w_k| \leq \delta$.

Definition 9.11 (UpAG in Ξ) *Given a compact set* $\Xi \subset \mathbb{R}^n$ *including the origin as an interior point, system (9.1) with* $\mathbf{d} \in \mathcal{M}_{\mathcal{D}_1}$ *and* $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$ *satisfies the UpAG (Uniform practical Asymptotic Gain) property in* Ξ, *if* Ξ *is a RPI set for (9.1) and if there exist a constant* $c \geq 0$ *and a* $\mathcal{K}$*-function* γ_2 *such that for each* $\varepsilon > 0$ *and* $\nu > 0$, $\exists T = T(\varepsilon, \nu)$ *such that*

$$|x(k, \bar{x}, \mathbf{d}, \mathbf{w})| \leq \gamma_2(||\mathbf{w}||) + c + \varepsilon$$

for all $\bar{x} \in \Xi$ *with* $|\bar{x}| \leq \nu$, *and all* $k \geq T$. *If the origin of system (9.1) is an equilibrium point and* $c = 0$, *the system is said to satisfy the UAG (Uniform Asymptotic Gain) property.*

Definition 9.12 (UAG in Ξ with respect to $\mathcal{A}$) *Given a compact set*

$\Xi \subset \mathbb{R}^n$ including the origin as an interior point, system (9.1) with $\mathbf{d} \in \mathcal{M}_{\mathcal{D}}$ and $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$ satisfies the UAG (Uniform Asymptotic Gain) property in Ξ, if Ξ is a RPI set for (9.1) and if there exist a $\mathcal{K}$-function γ_2 such that for each $\varepsilon > 0$ and $\nu > 0$, $\exists T = T(\varepsilon, \nu)$ such that

$$|x(k, \bar{x}, \mathbf{d}, \mathbf{w})|_{\mathcal{A}} \leq \gamma_2(||\mathbf{w}||) + \varepsilon$$

for all $\bar{x} \in \Xi$ with $|\bar{x}|_{\mathcal{A}} \leq \nu$, and all $k \geq T$. $\square$

Consider now a discrete-time nonlinear model

$$x_{k+1} \quad = \quad f(x_k, u_k, d_k, w_k), \;\; k \geq 0 \tag{9.3}$$

where $x_k \in \mathcal{X} \subseteq R^n$ is the system state, $u_k \in \mathcal{U} \subseteq \mathbb{R}^m$ is the current control vector, $d_k \in \mathcal{D}_1(x, u) \subseteq \mathbb{R}^p$ models a class of uncertainty which depends on the state and the control input (for each x and u the set $\mathcal{D}_1(x, u)$ is closed), while $w_k \in \mathcal{W} \subseteq \mathbb{R}^q$ models a class of disturbance.

Definition 9.13 (RPIA set) *Consider system (9.3) with $x_0 = \bar{x}$. Given a control law $u = \kappa(x)$, $\Xi \subseteq \mathcal{X}$ is a Robust Positively Invariant Admissible (RPIA) set for the closed-loop system (9.3) with $u_k = \kappa(x_k)$, if $\bar{x} \in \Xi$ implies $x_k \in \Xi$ and $\kappa(x_k) \in \mathcal{U}$, for all $d_k \in \mathcal{D}_1(x, u)$, all $w_k \in \mathcal{W}$ and all $k \geq 0$.*

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