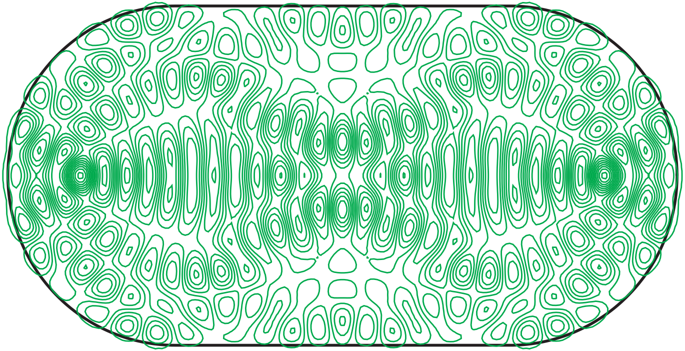


Quantum Chaos: what is it?

Steve Tomsovic

Washington State University, Pullman, WA USA

Today's introduction

I. Where is quantum chaos found?

II. What is quantum chaos?

III. Methods of analysis
   a) Semiclassical methods
   b) Random matrix theory
   c) Effective field theories

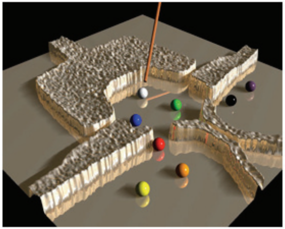
IV. What can we do with all this?

Sampling of where quantum chaos is found

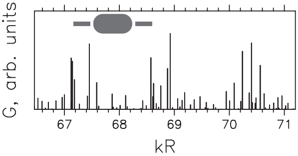
- Low-energy proton and slow neutron resonances
- Quantum dots
- Disordered electronic conductors
- Decoherence and fidelity studies
- The Dirac spectrum in non-Abelian gauge field backgrounds
- Quantum computation
- Riemann zeta function and the generalization, L-functions
- atomic and molecular spectra
- microwave-driven atoms
- optical resonators
- ultra-cold atoms in optical lattices
- Acoustics in crystals
- Long range ocean acoustics

A few illustrations

Quantum dot vs nucleus

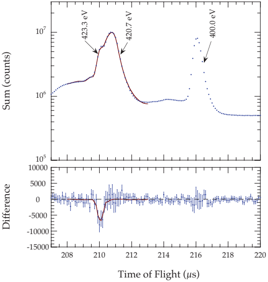


Christian Schönenberger group



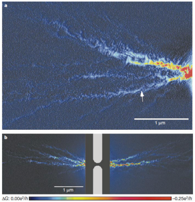
Narimanov et al.

Parity violation in the compound nucleus



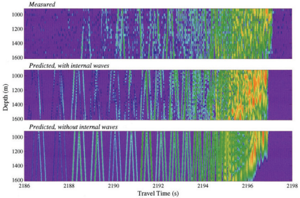
TRIPLE collaboration

Quantum point contact



Robert Westervelt group

Long range ocean acoustics



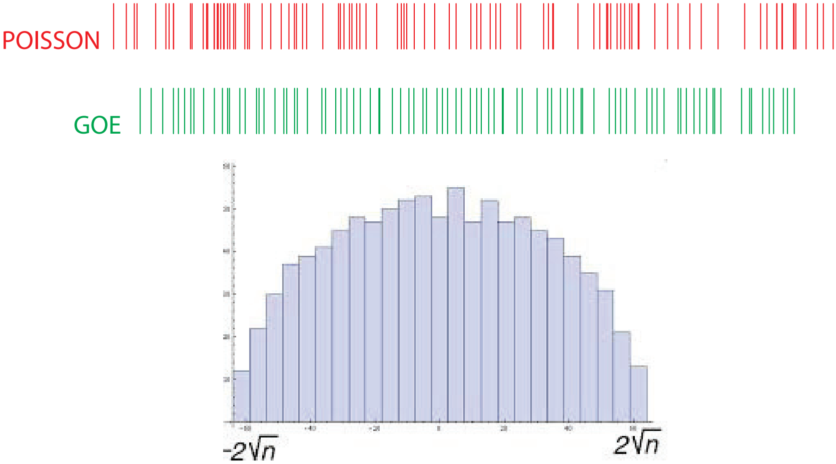
Acoustic engineering test

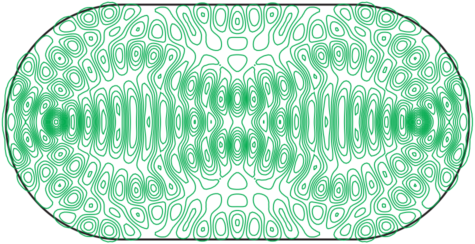
What is quantum chaos?[1]

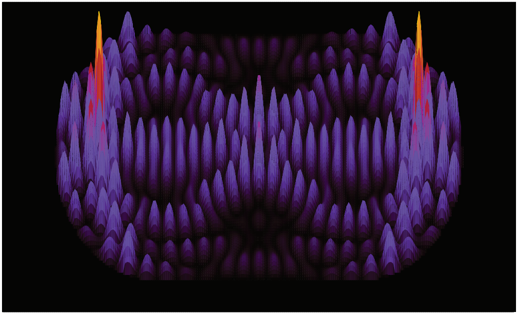
- "Quantum chaos is a branch of physics which studies how chaotic classical systems can be shown to be limits of quantum-mechanical systems." Martin C. Gutzwiller
- Quantum chaos is "the quantum mechanics of chaotic systems." Hans Jürgen Stöckmann
  "...an emerging science that is leading to the discovery of unfamiliar regimes of behavior in microscopic systems," M. V. Berry
- Common views: quantum chaos $\Rightarrow$ random matrix theory
  or: quantum chaos $\Rightarrow$ periodic orbit theory
- Quantum chaos is partly a new statistical mechanics, one not based on a thermodynamic limit, and partly an asymptotic analysis applied to non-integrable wave-mechanical systems.

[1] Wikipedia article is in dire need of some serious improvement

ENERGY LEVELS



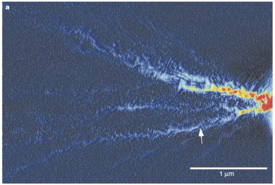




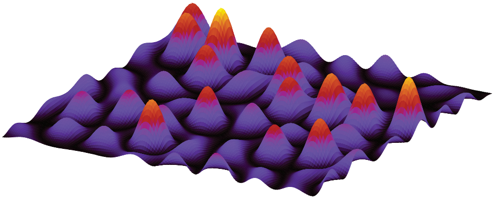
Stadium eigenstate

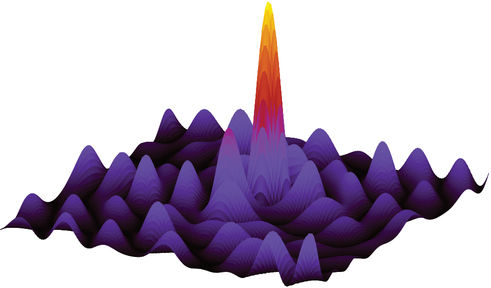
What are some basic eigenstate properties?

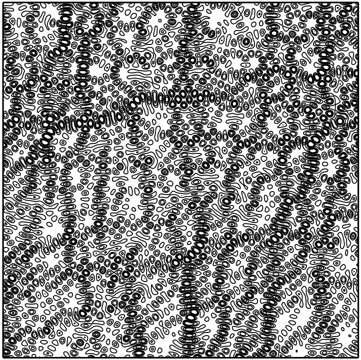
- Berry conjecture - asymptotically eigenstates of chaotic systems are like random waves
- Quantum ergodicity - probability densities associated with quantum eigenstates tend to uniform in a classical phase space (Schnirelman, Colin de Verdiére, Zelditch)
- Eigenstate scarring - short periodic orbits enhance features of chaotic quantum eigenstates (Heller)
- Coherent branching - scattering eigenstates or waves in weakly random media exhibit strong features along branches



Random waves



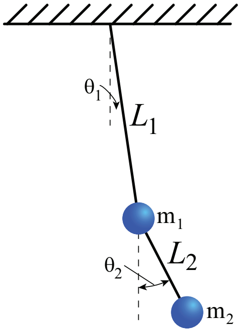


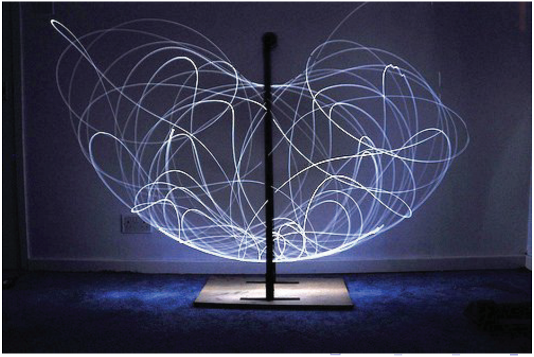


- Hamilton's phase space formulation of classical mechanics

$$\dot{\mathbf{q}} = \frac{\partial \mathcal{H}(\mathbf{q}, \mathbf{p}; t)}{\partial \mathbf{p}} \quad , \qquad \dot{\mathbf{p}} = -\frac{\partial \mathcal{H}(\mathbf{q}, \mathbf{p}; t)}{\partial \mathbf{q}}$$

- Poincaré (1899) proves existence of doubly asymptotic orbits - heteroclinic transport and chaos





Schrödinger equation

- The general Schrödinger equation is the basic starting point

$$i\hbar\frac{\partial\Psi(\mathbf{x};t)}{\partial t} = \mathcal{H}(\hat{\mathbf{x}},\hat{\mathbf{p}};t)\Psi(\mathbf{x};t)$$

- Green functions often provide useful, powerful techniques

$$i\hbar\frac{\partial K(\mathbf{x},\mathbf{x}';t)}{\partial t} = \mathcal{H}(\hat{\mathbf{x}},\hat{\mathbf{p}};t)K(\mathbf{x},\mathbf{x}';t)$$

- The Green function is given by the Feynman path integral

$$K(\mathbf{x},\mathbf{x}';t) = \int \exp\left[\frac{i}{\hbar}\int_0^t \mathcal{L}(\dot{\mathbf{x}},\mathbf{x};t)\mathrm{d}t\right]\mathcal{D}\mathbf{x}(t)$$

van Vleck - Gutzwiller propagator: fundamental link

- The bridge between the quantum and classical worlds is

$$K_{sc}(\mathbf{x},\mathbf{x}';t) = \left(\frac{1}{2\pi i\hbar}\right)^{\frac{d}{2}} \sum_j \left|\mathrm{Det}\left(\frac{\partial^2 S_j(\mathbf{x},\mathbf{x}';t)}{\partial\mathbf{x}\partial\mathbf{x}'}\right)\right|^{\frac{1}{2}} \mathrm{e}^{\frac{iS_j(\mathbf{x},\mathbf{x}';t)}{\hbar} - \frac{i\pi\nu_j}{2}}$$

  where j sums every classical orbit connecting $\mathbf{x}'$ to $\mathbf{x}$ in time t.
- Wavefunction propagation in terms of classical orbits comes by

$$\Psi(\mathbf{x};t) = \int_{-\infty}^{\infty} d\mathbf{x}'\, K_{sc}(\mathbf{x},\mathbf{x}';t)\Psi(\mathbf{x}') \quad S_j(\mathbf{x},\mathbf{x}';t) = \int_0^t \mathrm{d}t'\, \mathcal{L}(\dot{\mathbf{x}},\mathbf{x};t)$$

- $K_{sc}(\mathbf{x},\mathbf{x}';t)$ is the Feynman path integral by stationary phase

$K_{sc}(\mathbf{z}, \mathbf{z}'; t)$ - heteroclinic orbit summations

- Heteroclinic orbit summations capture all chaotic transport

$$\Gamma_{\mathbf{z}\mathbf{z}'}(t) = \sum_{\gamma} (\rho_{\mathbf{z}}, \rho_{\mathbf{z}'}(t))_{\gamma} \sim \sum_{\gamma} \int d\mathbf{q} d\mathbf{p}\, \rho_{\mathbf{z}}(\mathbf{q}, \mathbf{p}) \left[T_{\gamma} \rho_{\mathbf{z}'} \right] (\mathbf{q}, \mathbf{p}; t)$$

- T_{γ} is a classical local linear transformation
- Dirac derived the unitary transformation corresponding to a linear transformation
- The coherent state Green function has a heteroclinic orbit summation

$$K_{sc}(\mathbf{z}, \mathbf{z}'; t) \approx \sum_{\gamma} \langle \mathbf{z} | \mathbf{z}'(t) \rangle_{\gamma}$$

- This also follows from a more rigorous generalized Gaussian wave packet dynamics

Trace formulae

- Densities of states can be expressed as periodic orbit sums
- Gutzwiller derived the sum for fully chaotic systems

$$\rho(E) \sim \rho_W(E) + \frac{1}{\pi\hbar} \sum_{\mathrm{PO}} \frac{T}{r\ |\mathrm{Det}\,(M - \mathbf{1})|^{1/2}} \cos\left(\frac{S}{\hbar} - \sigma\frac{\pi}{2}\right)$$

- Berry-Tabor furnished the integrable system expression

$$\rho(E) \sim \rho_W(E) + \sum_{\mathbf{M}} \frac{T_{\mathbf{M}}}{\pi\hbar^{3/2} M_2^{3/2} \left|g_E''\right|^{1/2}} \cos\left(\frac{S_{\mathbf{M}}^0}{\hbar} - \frac{\eta_{\mathbf{M}}\pi}{2} - \frac{\pi}{4}\right)$$

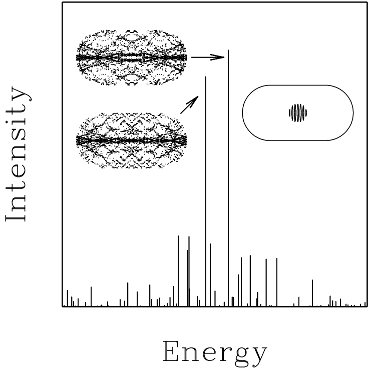
- Formulae properly interpolating the two expressions also exist

Eigenstate scarring - Heller 1984

- Periodic orbits and a linearized wave packet dynamics implies recurrences of states localized in these orbits' neighborhoods
- Their spreading is governed by the Lyapunov exponent and the period of the orbit:

  A lower bound for excess intensity in the neighborhood of the periodic orbit is given by

$$\frac{\mathcal{I}_{sc}}{\mathcal{I}_0} > \frac{2\pi}{\tau\lambda}$$



Random matrices

Imagine an N-dimensional Hermitian matrix, could be a Hamiltonian, but it has Gaussian random matrix elements,

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} & H_{13} & \cdots \\ H_{21} & H_{22} & H_{23} & \\ H_{31} & H_{32} & H_{32} & \\ \vdots & & & \ddots \end{pmatrix}$$

where ($j \neq k$ - diagonal elements get multiplied by a $\sqrt{2}$)

$$\beta = 1, \quad H_{jk} = x_{jk} \qquad \text{say } \rho(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$\beta = 2, \quad H_{jk} = x_{jk} + i x_{jk}^{(i)}$$

$$\beta = 4, \quad H_{jk} = x_{jk}\mathbf{1} + i\left[x_{jk}^{(1)}\sigma_1 + x_{jk}^{(2)}\sigma_2 + x_{jk}^{(3)}\sigma_3\right]$$

Random matrices II

Then, the joint probability density for the ensemble of such matrices can be written (no correlations)

$$\rho(\mathbf{H})\mathrm{d}\mathbf{H} = \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{\beta N}{2}\left(N-1+\frac{2}{\beta}\right)} \exp\left[-\frac{1}{4}\mathrm{Tr}\left(\mathbf{H}^2\right)\right]\mathrm{d}\mathbf{H}$$

We care more about the eigenvalues and eigenvectors of **H** than the individual matrix elements. We are really interested more in something like

$$\begin{aligned}\rho(\mathbf{H})\mathrm{d}\mathbf{H} &= \rho(\lambda, \mathbf{U})\mathrm{d}\lambda\mathrm{d}\mathbf{U} \\ &= \rho(\lambda)\mathrm{d}\lambda\mathrm{d}\mathbf{U}\end{aligned}$$

where

$$\lambda = \mathbf{U}\mathbf{H}\mathbf{U}^{-1}$$

Random matrices III

Suddenly, there are very strong correlations introduced by the variable change, notice the Vandermonde determinant,

$$\rho(\lambda)\mathrm{d}\lambda\mathrm{d}\mathbf{U} = \mathcal{C}_\beta \prod_{k>j=1}^{N} |\lambda_j - \lambda_k|^\beta \exp\left[-\frac{1}{4}\sum_{j=1}^{N}\lambda_j^2\right] \mathrm{d}\lambda\mathrm{d}\mathbf{U}$$

- There are two very well-known spectral correlations, level repulsion and spectral rigidity.
- $\beta = 0$ in the Vandermonde determinant would correspond to Poisson eigenvalue statistics.
- The eigenstates behave statistically like uniformly random vectors according to the Haar measures for the special orthogonal $SO(N)$, unitary $SU(N)$, or symplectic $Sp(2N, \mathbf{R})$ groups.

Disordered systems - nonlinear σ models

- An effective field theory for disordered systems with diffusive dynamics - Wegner 1979, Efetov 1982

$$
\begin{aligned}
\mathcal{Z}(\omega) &= \int \mathcal{D}\mathcal{Q} \exp\left(\mathcal{F}[\mathcal{Q};\omega]\right) \\
\mathcal{F}[\mathcal{Q};\omega] &= \frac{1}{L^d} \int \mathrm{d}^d r \mathrm{Tr} \left[\frac{1}{8} D(\nabla \mathcal{Q})^2 - \frac{i\omega\alpha}{4} \Lambda \mathcal{Q}\right]
\end{aligned}
$$

- Has to contain the physics of

  Anderson localization
  conduction in the metallic regime
  the metal-insulator transition and the mobility edge

Summary and Outlook

- There are justifications for taking a broader perspective of what quantum chaos is than has often occurred
- Quantum chaos provides a new perspective, which allows for new discoveries
- Quantum chaos appears in an immensely broad set of contexts
- The language of quantum chaos is very physical - like optics for Maxwell's equations
- There is still a great deal unknown about the interrelationships of the statistical and asymptotic analysis methods, and this statistical mechanics still has a long way to go