1. k-Step Ahead Prediction Error Model

ARMAX model is ARMA plus eXogeneous signal model:

$$A(z)y(n) = B(z)u(n-k) + C(z)\xi(n)$$

u - input

y - output

ξ - white noise

k - delay

- A, B, C are polynomials in z^{-1}
- All delay is factored into k so the constant terms of A, B, C are not zero
- Constant terms of A and C are one (that is, A, C are monic)

2. k-Step Ahead Prediction Error Model

Recall

$$A(z)y(n) = B(z)u(n-k) + C(z)\xi(n)$$

- Any change in u can affect y only after k samples
- But white noise starts affecting the process right away
- Want to get the best estimate of the output so as to take corrective action, starting now

The above equation can be rewritten as,

$$A(z)y(n+j) = B(z)u(n+j-k) + C(z)\xi(n+j)$$

Want to predict output from $n+k$ onwards or for $n+j$, $j \geq k$

3. k-Step Ahead Prediction Error Model

$$A(z)y(n) = B(z)u(n-k) + C(z)\xi(n)$$

$$y(n+k) = \frac{B(z)}{A(z)}u(n) + \frac{C(z)}{A(z)}\xi(n+k)$$

If $C = A$, the best prediction model is,

$$\hat{y}(n+k|n) = \frac{B(z)}{A(z)}u(n)$$

If $C \neq A$, divide C by A as follows, with j to be specified:

$$\frac{C(z)}{A(z)} = E_j(z) + z^{-j}\frac{F_j(z)}{A(z)}$$

$$E_j(z) = e_{j,0} + e_{j,1}z^{-1} + \cdots + e_{j,j-1}z^{-(j-1)}$$

$$F_j(z) = f_{j,0} + f_{j,1}z^{-1} + \cdots + f_{j,\mathrm{d}F_j}z^{-\mathrm{d}F_j}$$

Noise has past and future terms, to be split

4. Splitting Noise into Past and Future

$$y(n+j) = \frac{B(z)}{A(z)}u(n+j-k) + \frac{C(z)}{A(z)}\xi(n+j)$$

$$y(n+j) = \frac{B(z)}{A(z)}u(n+j-k)$$

$$+ \Bigg((e_{j,0} + e_{j,1}z^{-1} + \cdots + e_{j,j-1}z^{-(j-1)})$$

$$+ z^{-j}\frac{f_{j,0} + f_{j,1}z^{-1} + \cdots + f_{j,\mathrm{d}F_j}z^{-\mathrm{d}F_j}}{A(z)}\Bigg)\xi(n+j)$$

$$\mathrm{II} = e_{j,0}\xi(n+j) + e_{j,1}\xi(n+j-1) + \cdots + e_{j,j-1}\xi(n+1)$$

All future terms.

$$\mathrm{III} = \left(f_{j,0} + f_{j,1}z^{-1} + \cdots + f_{j,\mathrm{d}F_j}z^{-\mathrm{d}F_j}\right)\xi(n)/A(z)$$

III term is known from previous measurements

5. Example: Splitting Noise into Past and Future

$$y(n+j) = \frac{u(n+j-2)}{1-0.6z^{-1}-0.16z^{-2}} + \frac{1+0.5z^{-1}}{1-0.6z^{-1}-0.16z^{-2}}\xi(n+j)$$

Split C into E_j and F_j, for $j = 2$:

$$\frac{1+0.5z^{-1}}{1-0.6z^{-1}-0.16z^{-2}} = (1+1.1z^{-1}) + z^{-2}\frac{0.82+0.176z^{-1}}{1-0.6z^{-1}-0.16z^{-2}}$$

Substitute it in the expression for $y(n+j)$, with $j = 2$:

$$\begin{aligned} y(n+2) = {} & \frac{1}{1-0.6z^{-1}-0.16z^{-2}}u(n) \\ & + (1+1.1z^{-1})\xi(n+2) \\ & + z^{-2}\frac{0.82+0.176z^{-1}}{1-0.6z^{-1}-0.16z^{-2}}\xi(n+2) \end{aligned}$$

Second term is unknown; Last term is known.

6. Splitting Noise into Past and Future

$$Ay(n) = Bu(n-k) + C\xi(n)$$

$$\begin{aligned} y(n+j) &= \frac{B}{A}u(n+j-k) + \frac{C}{A}\xi(n+j) \\ &= \frac{B}{A}u(n+j-k) + \left[E_j + z^{-j}\frac{F_j}{A}\right]\xi(n+j) \\ &= \frac{B}{A}u(n+j-k) + \frac{F_j}{A}\xi(n) + E_j\xi(n+j) \end{aligned}$$

$$\begin{aligned} &= \frac{B}{A}u(n+j-k) + \frac{F_j}{A}\frac{Ay(n) - Bu(n-k)}{C} + E_j\xi(n+j) \\ &= \frac{B}{A}u(n+j-k) - \frac{F_jB}{AC}u(n-k) + \frac{F_j}{C}y(n) + E_j\xi(n+j) \\ &= \frac{B}{A}\left[1 - \frac{F_j}{C}z^{-j}\right]u(n+j-k) + \frac{F_j}{C}y(n) + E_j\xi(n+j) \end{aligned}$$

7. Splitting Noise into Past and Future

From the previous slide,

$$y(n+j) = \frac{B}{A}\left[1 - \frac{F_j}{C}z^{-j}\right]u(n+j-k) + \frac{F_j}{C}y(n) + E_j\xi(n+j)$$

$$\frac{C}{A} = E_j + z^{-j}\frac{F_j}{A} \Rightarrow \frac{C}{A} - z^{-j}\frac{F_j}{A} = E_j \Rightarrow \frac{C}{A}\left[1 - z^{-j}\frac{F_j}{C}\right] = E_j$$

$$y(n+j) = \frac{E_jB}{C}u(n+j-k) + \frac{F_j}{C}y(n) + E_j\xi(n+j)$$

Last term has only future terms. Hence, best prediction model:

$$\hat{y}(n+j|n) = \frac{E_jB}{C}u(n+j-k) + \frac{F_j}{C}y(n)$$

$\hat{}$ means estimate. $|n$ means "using measurements, available up to and including n".

8. Example: Splitting C/A into E_j and F_j

$$\frac{1+0.5z^{-1}}{1-0.6z^{-1}-0.16z^{-2}} = \frac{C}{A} = E_j + z^{-j}\frac{F_j}{A}$$

$$\begin{array}{r|llll}
 & 1 & +1.1z^{-1} & & \\
\cline{2-3}
1-0.6z^{-1}-0.16z^{-2} & 1 & +0.5z^{-1} & & \\
 & 1 & -0.6z^{-1} & -0.16z^{-2} & \\
\cline{2-3}
 & & +1.1z^{-1} & +0.16z^{-2} & \\
 & & +1.1z^{-1} & -0.66z^{-2} & -0.176z^{-3} \\
\cline{3-4}
 & & & +0.82z^{-2} & +0.176z^{-3}
\end{array}$$

$$\frac{1+0.5z^{-1}}{1-0.6z^{-1}-0.16z^{-2}} = (1+1.1z^{-1}) + z^{-2}\frac{0.82+0.176z^{-1}}{1-0.6z^{-1}-0.16z^{-2}}$$

9. Another Method to Split C/A into E_j and F_j

An easier method exists to solve

$$\frac{C}{A} = E_j + z^{-j}\frac{F_j}{A}$$

Cross multiply by A:

$$C = AE_j + z^{-j}F_j$$

- C, A, z^{-j} are known
- E_j, F_j are to be calculated.
- Think: How would you solve it?

10. Different Noise and Prediction Models: ARMAX

ARMAX Model :

$$\boldsymbol{Ay(n) = Bu(n-k) + C\xi(n)}$$

$$\boldsymbol{C = E_jA + z^{-j}F_j}$$

$$\boldsymbol{\hat{y}(n+j|t) = \frac{E_jB}{C}u(n+j-k) + \frac{F_j}{C}y(n)}$$

11. Different Noise and Prediction Models: ARIMAX

ARIMAX model with $\Delta = 1 - z^{-1}$:

$$Ay(n) = Bu(n-k) + \frac{C}{\Delta}\xi(n)$$

$$A\Delta y(n) = B\Delta u(n-k) + C\xi(n)$$

Recall ARMAX model:

$$Ay(n) = Bu(n-k) + C\xi(n)$$

Is the solution for ARMAX model useful?

$$A \leftarrow A\Delta, \quad B \leftarrow B\Delta$$

$$C = E_j A\Delta + z^{-j}F_j$$

$$\hat{y}(n+j|n) = \frac{E_j B\Delta}{C}u(n+j-k) + \frac{F_j}{C}y(n)$$

12. Different Noise and Prediction Models: ARIX

Recall ARIMAX model from previous slide:

$$\boldsymbol{A\Delta y(n) = B\Delta u(n-k) + C\xi(n)}$$
$$\boldsymbol{\hat{y}(n+j|n) = \frac{E_j B\Delta}{C} u(n+j-k) + \frac{F_j}{C} y(n)}$$

ARIX model, obtained with $\boldsymbol{C = 1}$ in ARIMAX:

$$\boldsymbol{Ay(n) = Bu(n-k) + \frac{1}{\Delta}\xi(n)}$$
$$\boldsymbol{1 = E_j A\Delta + z^{-j} F_j}$$
$$\boldsymbol{\hat{y}(n+j|t) = E_j B\Delta u(n+j-k) + F_j y(n)}$$

13. Minimum Variance Control: Regulation

ARMAX Model:

$$\begin{aligned} Ay(n) &= Bu(n-k) + C\xi(n) \\ C &= E_j A + z^{-j} F_j \\ y(n+j) &= \frac{E_j B}{C} u(n+j-k) + \frac{F_j}{C} y(n) + E_j \xi(n+j) \end{aligned}$$

Minimum variance control: Minimize the variations in y at k:

$$y(n+k) = \frac{E_k B}{C} u(n) + \frac{F_k}{C} y(n) + E_k \xi(n+k)$$

To minimize $\mathscr{E}\left[y^2(n+k)\right]$. $\xi(n+k)$ is ind. of $u(n)$, $y(n)$

$$E_k B u(n) + F_k y(n) = 0$$

$$u(n) = -\frac{F_k}{E_k B} y(n)$$

14. Example: Minimum Variance Control

$$y(n) = \frac{0.5}{1 - 0.5z^{-1}} u(n-1) + \frac{1}{1 - 0.9z^{-1}} \xi(n)$$

$$\begin{aligned} A &= (1 - 0.5z^{-1})(1 - 0.9z^{-1}) \\ &= 1 - 1.4z^{-1} + 0.45z^{-2} \\ B &= 0.5(1 - 0.9z^{-1}) \\ C &= (1 - 0.5z^{-1}) \\ k &= 1 \\ C &= E_k A + z^{-k} F_k \\ 1 - 0.5z^{-1} &= E_1(1 - 1.4z^{-1} + 0.45z^{-2}) + z^{-1} F_1 \end{aligned}$$

Solving,

$$\begin{aligned} E_1 &= 1 \\ F_1 &= 0.9 - 0.45z^{-1} \end{aligned}$$

15. Example: Minimum Variance Control

$$
\begin{aligned}
B &= 0.5(1 - 0.9z^{-1}) \\
E_1 &= 1 \\
F_1 &= 0.9 - 0.45z^{-1} \\
u(n) &= -\frac{F_k}{E_k B} y(n) = -\frac{0.9 - 0.45z^{-1}}{0.5(1 - 0.9z^{-1})} y(n) \\
&= -0.9 \frac{2 - z^{-1}}{1 - 0.9z^{-1}} y(n) \\
\mathscr{E}\left[y^2(n+k)\right] &= \mathscr{E}\left[(E_k \xi(n+k))^2\right] = \mathscr{E}\left[(\xi(n+1))^2\right] \\
&= \sigma^2
\end{aligned}
$$

16. Minimum Variance Control for ARIX Model

Recall

$$A y(n) = B u(n-k) + \frac{1}{\Delta}\xi(n)$$
$$\hat{y}(n+j|n) = E_j B \Delta u(n+j-k) + F_j y(n)$$
$$1 = E_j A \Delta + z^{-j} F_j$$

Minimum variance control law is obtained by forcing $\hat{y}(n+j|n)$ to be zero:

$$E_k B \Delta u(n) = -F_k y(n)$$
$$\Delta u(n) = -\frac{F_k}{E_k B} y(n)$$

For nonminimum phase systems, use an alternate approach