Complexity of Integration, Special Values, and Recent Developments

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Outline

Definitions

Integration of $f(x)$, in the sense of determining a formula $F(x)$ such that $F'(x) = f(x)$, or proving that no such $F(x)$ exists in a suitable class, is a process of differential algebra. There is then a question of whether this formula actually corresponds to a continuous function $\mathbf{R} \rightarrow \mathbf{R}$.
This is an important (and under-studied) question in terms of usability of the results, but a rather different one than we wish to consider here: see [JD93, Mul97].

Questions

Two questions can be asked.

1. What is the computational complexity of the integration process?
2. If $f(x, c_1, \ldots, c_k)$ is not integrable, for what special values of the c_i is it integrable?

Setting

In order to use differential algebra, the integrand f is written [Ris79] in a suitable field $K(x, \theta_1, \ldots, \theta_n)$ where each θ_i is transcendental over $K(x, \theta_1, \ldots, \theta_{i-1})$ with $K(x, \theta_1, \ldots, \theta_i)$ having the same field of constants as $K(x, \theta_1, \ldots, \theta_{i-1})$ and each θ_i being either:

l) a *logarithm* over $K(x, \theta_1, \ldots, \theta_{i-1})$, i.e. $\theta_i' = \eta_i'/\eta_i$ for $\eta_i \in K(x, \theta_1, \ldots, \theta_{i-1})$;

e) an *exponential* over $K(x, \theta_1, \ldots, \theta_{i-1})$, i.e. $\theta_i' = \eta_i'\theta_i$ for $\eta_i \in K(x, \theta_1, \ldots, \theta_{i-1})$.

Special cases (of Risch Structure Theorem)

This process may generate special cases: for example $\exp(a \log x)$ lives in such a $K(x, \theta_1, \theta_2)$ with

$\theta_1' = \frac{1}{x}$ (θ_1 corresponds to $\log x$) and

$\theta_2' = \frac{a}{x}\theta_2$ (θ_2 corresponds to $\exp(a \log x)$),

except when a is rational, when in fact we have x^a. However, this is generally not what is meant by the "special values" question, and in general we assume that parameters are not in exponents.

Q0 Is this really legitimate?

Rational Functions (1)

To illustrate these points, consider the following examples.

$$\int \frac{5x^4+60x^3+255x^2+450x+274}{x^5+15x^4+85x^3+225x^2+274x+120}\mathrm{d}x$$
$$= \log(x^5+15x^4+85x^3+225x^2+274x+120)$$
$$= \log(x+1)+\log(x+2)+\log(x+3)+\log(x+4)+\log(x+5) \tag{1}$$

is pretty straightforward, but adding 1 to the numerator gives

$$\int \frac{5x^4+60x^3+255x^2+450x+275}{x^5+15x^4+85x^3+225x^2+274x+120}\mathrm{d}x$$
$$= \frac{5}{24}\log(x^{24}+72x^{23}+\cdots+102643200000x+9331200000) \tag{2}$$

$$= \frac{25}{24}\log(x+1)+\frac{5}{6}\log(x+2)+\frac{5}{4}\log(x+3)+ \frac{5}{6}\log(x+4)+\frac{25}{24}\log(x+5) \tag{3}$$

We presumably want the second form!

Rational Functions (2)

Adding 1 to the denominator is pretty straightforward,

$$\int \frac{5x^4+60x^3+255x^2+450x+274}{x^5+15x^4+85x^3+225x^2+274x+121}\mathrm{d}x \\ = \log(x^5+15x^4+85x^3+225x^2+274x+121), \tag{4}$$

(but notice that the argument of the logarithm doesn't factor)

Rational Functions (3)

but adding 1 to both gives

$$\int \frac{5x^4+60x^3+255x^2+450x+275}{x^5+15x^4+85x^3+225x^2+274x+121}\mathrm{d}x$$
$$= 5\sum_{\alpha} \alpha \ln\Big(x+\frac{2632025698}{289}\alpha^4-\frac{2086891452}{289}\alpha^3+ \frac{608708804}{289}\alpha^2-\frac{4556915}{17}\alpha+\frac{3632420}{289}\Big), \tag{5}$$

where

$$\alpha = \mathrm{RootOf}\left(38569\,z^5-38569\,z^4+15251\,z^3-2981\,z^2+288\,z-11\right). \tag{6}$$

In the dense model, the complexity is (only) polynomial!

Rational Functions (4)

Nevertheless, we want an algorithm that generates, if not the "shortest" form, at least a short form, so (3) rather than (2). We also want running time "commensurate" with this, which implies that we should be in output-sensitive complexity territory.

The Trager–Rothstein resultant [Rot77, Tra76] seems to satisfy this.

- Q1 Formalise this.
- Q2 What about the sparse model?

Elementary Transcendental Functions

Here we have a decision procedure [Ris69]. The proof proceeds by induction on n: we suppose that we can:

a) "integrate in $K(x, \theta_1, \ldots, \theta_{n-1})$", i.e. given $g \in K(x, \theta_1, \ldots, \theta_{n-1})$, either write $\int g \mathrm{d}x$ as an elementary function over $K(x, \theta_1, \ldots, \theta_{n-1})$, or prove that no such elementary function exists;

b) "solve Risch differential equations in $K(x, \theta_1, \ldots, \theta_{n-1})$", i.e. given elements $F, g \in K(x, \theta_1, \ldots, \theta_{n-1})$ such that $\exp(F)$ is transcendental over $K(x, \theta_1, \ldots, \theta_{n-1})$ (with the same field of constants), solve $y' + F'y = g$ for $y \in K(x, \theta_1, \ldots, \theta_{n-1})$, or prove that no such y exists.

We then prove that (a) and (b) hold for $K(x, \theta_1, \ldots, \theta_n)$.

Logarithmic Functions (1)

If θ_n is logarithmic, the proof of part (a) is a straightforward exercise building on part (a) for $K(x, \theta_1, \ldots, \theta_{n-1})$: see, e.g. [DST93, §5.1]. Unintegrability manifests itself as the insolubility of certain equations, and any special values of the parameters will be found as special values rendering these equations soluble.

It is also straightforward (though as far as the author knows, not done) to prove that, if all θ_i are logarithmic, then the degree in each θ_i of the integral is no more than it is in the integrand, and that the denominator of the integral is a divisor of the denominator of the integrand. Hence, in the dense model, the integral is, apart from coefficient growth, not much larger than the integrand, and the compute cost is certainly polynomial.

Logarithmic Functions (2)

In a sparse model, the situation is very different.

$$\int \log^n x \mathrm{d}x = x \log^n x - nx \log^{n-1} x + \cdots \pm n!x,$$

so an integrand requiring $\Theta(\log n)$ bits can require $\Omega(n)$ bits for the integral. The same is true for $\int x^n \log^n x \mathrm{d}x$, but $\int x^n \log^n (x+1) \mathrm{d}x$ shows that $\Omega(n^2)$ bits can be required.

Q3 Is the problem even in EXPSPACE?

C3 It probably is.

Logarithmic Functions: Unintegrability

$$\int \left(x^4 (\ln (x+1))^2 - 2\,\frac{\ln(x+1)}{5\,x+5}\right) \ln (x) - \frac{137\,\ln(x+1)}{150\,x}\mathrm{d}x =$$
$$\frac{\left(30\,x^5 \ln(x) - 6\,x^5 - 6\right)(\ln(x+1))^2}{150} +$$
$$\frac{\ln(x+1)}{150}\Big[-12\,x^5 \ln (x) + 15\,x^4 \ln (x) - 20\,x^3 \ln (x) + 30\,x^2 \ln (x)$$
$$-60\,\ln (x)\,x + \frac{24\,x^5}{5} - \frac{27\,x^4}{4} + \frac{32\,x^3}{3} - 21\,x^2 + 72\,x - 137\,\ln (x)\Big]$$
$$+\frac{2\,x^5 \ln(x)}{125} - \frac{9\,x^4 \ln(x)}{200} + \frac{47\,x^3 \ln(x)}{450} - \frac{77\,x^2 \ln(x)}{300} + \frac{137\,\ln(x)x}{150} - \frac{6\,x^5}{625} +$$
$$\frac{61\,x^4}{2000} - \frac{2273\,x^3}{27000} + \frac{4903\,x^2}{18000} - \frac{15133\,x}{9000} + \frac{6913\,\ln(x+1)}{9000}$$

but any number other than 137 gives “unintegrable” after doing all this work , so “output-sensitive” isn't quite right.

Exponential Functions (1)

Suppose $\theta_n = \exp(F)$. $\int g \exp(F) \mathrm{d}x = y \exp(F)$ where $y' + F'y = g$ (and can be nothing else if it is to be an elementary function).

Hence solving (a) in $K(x, \theta_1, \ldots, \theta_n)$ reduces (among other things) to solving (b) in $K(x, \theta_1, \ldots, \theta_{n-1})$.

The solution to (b) proceeds essentially by undetermined coefficients, which is feasible as $y' + F'y$ is linear in the unknown coefficients. Before we can start this, we need to answer two questions:

- what is the denominator of y, and
- what is the degree (number of unknown coefficients)?

Exponential Functions (2)

In general, the answers are obvious:

- if the denominator of g has an irreducible factor p of multiplicity k, y' will have the same, so the denominator of y will have a factor of (at most) p^{k-1}, and F' can only reduce this.
- Similarly, if g has degree d, y' will have degree at most d, so y will have degree $d+1$, and again F' can only reduce this.

The complication is when there is cancellation in $y' + F'y$, so that this has lower degree, or smaller denominator, than its summands. [Ris69] shows how to resolve this problem, and does not pay it much attention, not being interested in the complexity question.

Exponential Functions (3)

These come from “eccentric” integrands [Dav86]. For example

$$y' + \left(1 + \frac{5}{x}\right) y = 1 \tag{7}$$

has solution

$$y = \frac{x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120}{x^5}, \tag{8}$$

but this comes from

$$\int \exp(x + 5\log x)\mathrm{d}x, \tag{9}$$

which might be more clearly expressed as

$$\int x^5 \exp(x)\mathrm{d}x. \tag{10}$$

Exponential Functions (4)

However, the integrand in (9) has total degree 1, whereas that in (10) has total degree 6, consistent with the degrees in (8). The dense model is not applicable when we can move things into/out of the exponents at will.

We do have a result [Dav86, Theorem 4] which says that, provided $K(x, \theta_1, \ldots, \theta_n)$ is *exponentially reduced* (loosely speaking, doesn't allow "eccentric" integrands) then we have natural degree bounds on the solutions of (b) equations. As stated there, "this is far from being a complete bounds on integrals, but it does indicate that the worst anomalies cannot take place" here.

Q4 Is the problem even in EXPSPACE?

C4 It probably is (but less certain than C3!).

Exponential Functions: Special Values

These come in two kinds:

1. As in the logarithmic case, we can get proofs of unintegrability because certain equations are insoluble. For example $(x + a)\exp(-bx^2 + cx)$ is integrable if, and only if, $c = -2ab$, and this equation arises during the undetermined coefficients process.
2. More complicated are those that change the "exponentially reduced" nature of the integrand. For example, $\int \exp(x + a \log x)\mathrm{d}x$ does not have an elementary expression *except* when a is a non-negative integer, when we are in a similar position to (9). These values are similar to those that change the Risch Structure Theorem expression of the integrand.

Algebraic Functions (1) [Dav81, Tra84]

If $f \in K(x, y)$ where y is algebraic over $K(x)$, the integral, if it is elementary, has to have the form $v_0 + \sum c_i \log(v_i)$, where $v_0 \in K(x, y)$, the c_i are algebraic over K, and the $v_i \in L(x, y)$ where L is the extension of K by the c_i.
So far, the same as rational functions
The $\sum c_i \log(v_i)$ term represents the logarithmic singularities in $\int f \mathrm{d}x$, which come from the simple poles of f: in a power series world c_i would be the residue at the pole corresponding to v_i.
So the trick would seem to be to combine the poles to make v_i.

Algebraic Functions (2)

Hence an obvious algorithm would be

1. Compute all the residues r_j at all the corresponding poles p_j (which might include infinity, and which might be ramified: the technical term would be "place"). Assume $1 \leq j \leq m$.
2. Let c_i be a **Z**-basis for the r_j, so that $r_j = \sum \alpha_{i,j} c_i$.
3. For each c_i, let v_i be a function $\in L(x, y)$ with residue $\alpha_{i,j}$ at p_j for $1 \leq j \leq m$ (and nowhere else). The technical term for this residue/place combination is "divisor", and a divisor with a corresponding function v_i is termed a "principal divisor".
   * Returning "unintegrable" if we can't find such v_i.
4. Having determined the logarithms this way, find v_0 by undetermined coefficients.

Algebraic Functions (3)

But it is possible that D_i is not a principal divisor, but that $2D_i$, or $3D_i$ or $\dots$ is principal.

In this case, we say that D_i is a torsion divisor, and the corresponding order is referred to as the torsion of the divisor.

If, say, $3D_i$ is principal with corresponding function v_i, then, although not in $L(x, y)$, $\sqrt[3]{v_i}$ corresponds to the divisor D_i, and we can use $c_i \log \sqrt[3]{v_i}$, or, more conveniently and fitting in with general theory, $\frac{c_i}{3} \log v_i$ as a contribution to the logarithmic part.

Algebraic Functions (4): Complexity

There are three main challenges with complexity theory for algebraic function integration.

1. It is far from clear what the “simplest” form of an integral of this form is. The choice of c_i is far from unique, and a “bad” choice of c_i may lead to large $\alpha_{i,j}$ and complicated v_i. Recall (3) rather than (2) for rational functions.
2. The r_j are algebraic numbers, and there are no known non-trivial bounds for the r_j, or the $\alpha_{i,j}$.
3. Very little known is about the torsion.

Hence it appears unrealistic to think of complexity bounds in the current state of knowledge.

Don't we know about the torsion?

Surely there's Mazur's bound [Maz77].

This does indeed show that, if the algebraic curve defined by y is elliptic (has genus 1) *and* the divisor is defined over $\mathbf{Q}$, then the torsion is at most 12.

The trouble is that this requires the divisor to be defined over $\mathbf{Q}$, and not just f.

For elliptic curves, a recent survey of the known bounds is given in [Sut12].

Special Values; two meis culpis (1)

In [Dav81]) we considered the question of whether $f(x,u)\mathrm{d}x$, an algebraic function of x, could have an elementary integral for specific values of u, even if the integral were not elementary. How might this happen?Transcendental u trivial.

1. The curve can change genus: look at the canonical divisor.
2. The [geometry of the] places at which residues occur can change: look at values of u for which numerator/denominator cancel, or roots coincide.
3. The dimension of the space of residues can collapse.
4. A divisor may be a torsion divisor for a particular value of u, even though it is not a torsion divisor in general. These cases can be detected ...
5. the algebraic part may be integrable for a particular u, though not in general. Hence the contradicting equation in `FIND_ALGEBRAIC_PART` collapses.

Special Values; two meis culpis (2)

As a potential example of case 3, consider

$$\frac{1}{x\sqrt{x^2+1}} + \frac{1}{x\sqrt{x^2+u^2}}$$

whose residues are $\pm 1, \pm u$ and therefore every rational u is apparently a special case.
[Dav81, Lemma 6, page 90] claims to prove that, if there are enough special u_i, the the general integral must also be integrable.

Special Values; two meis culpis (3)

[Mas16] observes that $\frac{x\,\mathrm{d}x}{(x^2-u^2)\sqrt{x^3-x}}$ is not elementarily integrable, but is integrable whenever the point $(u, ?)$ is of order at least three on the curve $y^2 = x^3 - x$, and this can be achieved infinitely often, at the cost of extending the number field. When $u = i$, $(i, 1-i)$ is of order 4 and we have

$$\int \frac{x\,\mathrm{d}x}{(x^2+1)\sqrt{x^3-x}} =$$
$$\frac{1+i}{16}\ln\left(\frac{x^2+(2+2\,i)\sqrt{x^3-x}+2\,ix-1}{x^2-(2+2\,i)\sqrt{x^3-x}+2\,ix-1}\right)$$
$$+\frac{1-i}{16}\ln\left(\frac{x^2+(2-2\,i)\sqrt{x^3-x}-2\,ix-1}{x^2-(2-2\,i)\sqrt{x^3-x}-2\,ix-1}\right)$$

Unfortunately neither Maple (2016) nor Mathematica (10.0) nor Reduce (build 3562) can actually integrate this elementarily.

Special Values; two meis culpis (4)

The assertion that the case of transcendental u is trivial, if true at all, is certainly not trivial, and probably false, if we also allow transcendental constants in f, for they and u can then "collide". [Mas16].

Summary

For a proper treatment

1. We need the sparse model, and output-sensitive complexity analysis
2. But this doesn't handle "unintegrable".
3. Special values wait for [MZ16].
4. Never forget to check that the output is a continuous function

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